Robust Sampled-Data H_∞ Control of Linear Singularly Perturbed Systems

Emilia Fridman

***Abstract*—State-feedback H_∞ control problem for linear singularly perturbed systems with norm-bounded uncertainties is studied. The fast variables are sampled with fast rates, while for the slow variables both cases of slow and of fast sampling are considered. The recent "input delay" approach to sampled-data control is applied, where the closed-loop system is represented as a continuous one with time-varying input delay. Linear matrix inequalities (LMIs) for solution of H_∞ control problem are derived via input-output approach to stability and L_2-gain analysis of time-delay systems. A numerical example illustrates the efficiency of the method.**

***Index Terms*—H_∞ control, linear matrix inequality (LMI), sampled-data control, singularly perturbed systems, time-delay.**

I. Introduction

Singular perturbations in control systems often occur due to the presence of small "parasitic" parameters, such as small masses, small time-delays. The main objective of singular perturbation methods is to alleviate the difficulties caused by the high dimensionality and the ill-conditioning that results from the interaction of slow and fast dynamical modes. Decomposition of the full-order problem to the ε-independent reduced-order slow and fast subproblems was started with the classical Tikhonov theorem on the asymptotic behavior of the solution to the initial value problem [19] and developed further to composite controller design [2], [15] (see a survey [17] for recent references). A LMI approach to linear singularly perturbed systems was introduced in [6], [9].

Two main approaches have been used to the sampled-data robust control. The first one is based on the lifting technique [1], [21] in which the problem is transformed to equivalent finite-dimensional discrete problem. This approach was applied to sampled-data nonlinear singularly perturbed systems, where the composite controller with the fast sampling in the fast variables was suggested [4]. The second approach is based on the representation of the system in the form of hybrid discrete/continuous model. This approach leads to necessary and sufficient conditions for stability and L_2-gain analysis in the form of differential equations (or inequalities) with jumps and it was applied to sampled-data H_∞ control of linear singularly perturbed systems [18], where the slow sampled-data controller was designed. The above approaches do not work in the cases with uncertain sampling times or uncertain system matrices.

A new "input delay" approach to sampled-data control has been suggested recently in [7]. By this approach, a digital control law is represented as a delayed control as follows:

$$u(t) = u_d(t_k) = u_d(t-(t-t_k)) = u_d(t-\tau(t)) \qquad t_k \le t < t_{k+1} \quad \tau(t) = t - t_k \tag{1}$$

where u_d is a discrete-time control signal and the time-varying delay $\tau(t) = t - t_k$ is piecewise linear with derivative $\dot{\tau}(t) = 1$ for $t \ne t_k$. Moreover, $\tau \le t_{k+1} - t_k$. The solution to the problem is found then by solving the problem for a continuous-time system with uncertain but bounded (by the maximum sampling interval) time-varying delay in the control input via Lyapunov technique. Given $h > 0$, the conditions obtained are robust with respect to different samplings with the only requirement that the maximum sampling interval is not greater than h.

Stability of singularly perturbed systems with a *constant* delay h has been studied in two cases: 1) h is proportional to ε (*small* delay), and 2) ε and h are independent. The first case, being less general than the second one, is encountered in many publications (see, e.g., [11], [10], and the references therein). The second case has been studied in the frequency domain [16]. A Lyapunov-based approach to the problem leading to LMIs has been introduced in [6] for the general case of independent delay and ε. In the case of constant delay, it was shown [6], that the necessary condition for robust stability of singularly perturbed system for all small enough values of singular perturbation parameter $\varepsilon > 0$ is the delay-independent stability of the fast subsystem, which is rather restrictive. The same is true for systems with uncertain and bounded time-varying delays, where constant delay is just a particular case of delay. Therefore, it is natural to design a delayed state-feedback controller with a small delay in the fast variable $\varepsilon\tau(t)$. This corresponds to the fast sampling of fast variables considered in [4].

In this note, we solve the state-feedback sampled-data H_∞-control problem by applying the input delay approach to sampled-data control and by developing the input-output approach to singularly perturbed time-delay systems. The input-output approach was introduced for regular systems with constant delays in [13] and further developed in [12] (see also references therein), where it was generalized to the time-varying delays with the delay derivative less than $q < 1$. Recently, the input–output approach has been developed to L_2-gain analysis of regular systems with time-varying bounded delays without any constraints on the delay derivative [8]. It is the objective of the present note to develop this approach to singularly perturbed systems with time-varying delay. Two controller designs are considered: 1) With the fast sampling in the fast variables and the slow one in the slow variables, and 2) with the fast sampling in both variables.

Notation: Throughout this note, the superscript "T" stands for matrix transposition, $\mathcal{R}^n$ denotes the n-dimensional Euclidean space with vector norm $\|\cdot\|$, $\mathcal{R}^{n\times m}$ is the set of all $n \times m$ real matrices, and the notation $P > 0$, for $P \in \mathcal{R}^{n\times n}$ means that P is symmetric and positive definite. The symmetric elements of the symmetric matrix will be denoted by $*$. L_2 is the space of square integrable functions $v : [0,\infty) \to C^n$ with the norm $\|v\|_{L_2} = [\int_0^\infty \|v(t)\|^2 dt]^{1/2}$.

II. Problem Formulation

Given the following system:

$$E_\varepsilon \dot{x}(t) = (A + H\Delta F_0)x(t) + (B_1 + H\Delta F_1)w(t) + (B_2 + H\Delta F_2)u(t) \tag{2}$$

$$z(t) = Cx(t) + D_{12}u(t) \tag{3}$$

where $x(t) = \mathrm{col}\{x_1(t), x_2(t)\}, x_1(t) \in \mathcal{R}^{n_1}, x_2(t) \in \mathcal{R}^{n_2}$ is the system state vector, $u(t) \in \mathcal{R}^{\ell}$ is the control input, $w(t) \in \mathcal{R}^q$ is the exogenous disturbance signal, and $z(t) \in \mathcal{R}^p$ is the state combination (objective function signal) to be attenuated. The matrix E_ε is given by

$$E_\varepsilon = \begin{bmatrix} I_{n_1} & 0 \\ 0 & \varepsilon I_{n_2} \end{bmatrix} \tag{4}$$

where $\varepsilon > 0$ is a small parameter.

Manuscript received March 14, 2005; revised August 2, 2005. Recommended by Associate Editor L. Glielmo. This work was supported by the Kamea Fund of Israel.

The author is with the Department of Electrical Engineering-Systems, Tel Aviv University, Tel-Aviv 69978, Israel (e-mail: emilia@eng.tau.ac.il).

Digital Object Identifier 10.1109/TAC.2005.864194

Denote $n \triangleq n_1 + n_2$. The matrices $A, B_1, B_2, F_0, F_1, F_2, H, C$ and D_{12} are constant matrices of appropriate dimensions. The matrices in (2) and (3) have the following structures:

$$A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \quad H = \begin{bmatrix} H_1 & H_2 \\ H_3 & H_4 \end{bmatrix} \quad F_0 = \begin{bmatrix} F_{01} & F_{02} \\ F_{03} & F_{04} \end{bmatrix}$$
$$B_i = \begin{bmatrix} B_{i1} \\ B_{i2} \end{bmatrix} \quad C = [C_1 \; C_2] \quad F_i = \begin{bmatrix} F_{i1} \\ F_{i2} \end{bmatrix}, \qquad i = 1, 2. \quad (5)$$

We do not require A_4 to be nonsingular. Such a system is a *nonstandard* singularly perturbed system [14]. In the case of singular A_4 open-loop system (2) with $\varepsilon = 0$ has index more than one and possesses an impulse solution [3].

The uncertain time-varying matrix $\Delta(t) = \begin{bmatrix} \Delta_1(t) & \Delta_2(t) \\ \Delta_3(t) & \Delta_4(t) \end{bmatrix}$ satisfies the inequality

$$\Delta^T(t)\Delta(t) \le I_n, \qquad t \ge 0. \quad (6)$$

We are looking for a piecewise-constant control law of two forms.

1) A multiple (slow/fast) rate state-feedback

$$u(t) = u_s(t) + u_f(t) \quad u_s(t) = K_1 x_1(t_k), \; t_k \le t < t_{k+1}$$
$$u_f(t) = K_2 x_2(\varepsilon t_k) \quad \varepsilon t_k \le t < \varepsilon t_{k+1} \quad (7)$$

where $0 = t_0 < t_1 < \cdots < t_k < \cdots$ and $0 = \varepsilon t_0 < \varepsilon t_1 < \cdots < \varepsilon t_k < \cdots$ are the slow and the fast sampling instants and $\lim_{k\to\infty} t_k = \infty$.

2) A single (fast) rate state-feedback $u(t) = \bar{K}x(\varepsilon t_k), \; \varepsilon t_k \le t < \varepsilon t_{k+1}$, where $0 = \varepsilon t_0 < \varepsilon t_1 < \cdots < \varepsilon t_k < \cdots$ are the fast sampling instants and $\lim_{k\to\infty} t_k = \infty$.

Given $\gamma > 0$ our objective is to find a piecewise constant controller which internally stabilizes the system and leads to L_2-gain less than γ. The latter means that the following inequality

$$J = \|z\|_{L_2}^2 - \gamma^2 \|w\|_{L_2}^2 < 0 \quad (8)$$

holds for $x(0) = 0$ and for all nonzero $w \in L_2$.

We represent a piecewise-constant control law as a continuous-time control with a time-varying piecewise-continuous (continuous from the right) delay $\tau(t) = t - t_k$ as given in (1), corresponding to the slow sampling, and with small delay $\varepsilon\tau(t) = \varepsilon(t - t_k)$, corresponding to the fast sampling. We will thus look for state-feedback controllers of two forms

$$u(t) = K \begin{bmatrix} x_1(t - \tau(t)) \\ x_2(t - \varepsilon\tau(t)) \end{bmatrix} \quad K = [K_1 \; K_2] \quad (9)$$

and

$$u(t) = Kx(t - \varepsilon\tau(t)). \quad (10)$$

We assume that

A1) $t_{k+1} - t_k \le h \forall k \ge 0$.

From A1 it follows that $\tau(t) \le h$ since $\tau(t) \le t_{k+1} - t_k$.

To guarantee that for all small enough $\varepsilon > 0$ the full-order system is stabilizable-detectable we assume [20].

A2) Both pencils $[sE_0 - A; B_2]$ and $[sE_0 - A^T; C^T]$ are of full row rank for all s with nonnegative real parts, where E_0 is given by (4) with $\varepsilon = 0$.

A3) The triple $\{A_4, \; B_{22}, \; C_2\}$ is stabilizable-detectable.

III. Multiple Rate H_∞ Control

A. Input–Output Model

Substituting (9) into (2), we obtain the following closed-loop system:

$$E_\varepsilon \dot{x}(t) = (A + H\Delta F_0)x(t) + (B_2 + H\Delta F_2)K \times \begin{bmatrix} x_1(t - \tau(t)) \\ x_2(t - \varepsilon\tau(t)) \end{bmatrix} + (B_1 + H\Delta F_1)w(t)$$
$$z(t) = Cx(t) + D_{12}K \begin{bmatrix} x_1(t - \tau(t)) \\ x_2(t - \varepsilon\tau(t)) \end{bmatrix}. \quad (11)$$

We will further consider (11) as the system with uncertain and bounded delay $\tau(t) \in [0, h]$.

We represent (11) in the form

$$E_\varepsilon \dot{x}(t) = (A + B_2K + H\Delta(F_0 + F_2K))x(t) - (B_2 + H\Delta F_2)K \times \begin{bmatrix} \int_{-\tau(t)}^0 \dot{x}_1(t+s)\,ds \\ \int_{-\varepsilon\tau(t)}^0 \dot{x}_2(t+s)\,ds \end{bmatrix} + (B_1 + H\Delta F_1)w(t)$$
$$z(t) = (C + D_{12}K)x(t) - D_{12}K \begin{bmatrix} \int_{-\tau(t)}^0 \dot{x}_1(t+s)\,ds \\ \int_{-\varepsilon\tau(t)}^0 \dot{x}_2(t+s)\,ds \end{bmatrix}. \quad (12)$$

We follow the idea of [13] and [12] to embed the perturbed system (12) into a class of systems with additional inputs and outputs, the stability of which guarantees the stability of (12). Consider the following forward system:

$$E_\varepsilon \dot{x}(t) = (A + B_2K)x(t) + hB_2Kv(t) + B_1w(t) + Hv_3(t)$$
$$z(t) = (C + D_{12}K)x(t) + hD_{12}Kv(t)$$
$$y(t) = E_\varepsilon \dot{x}(t) = (A + B_2K)x(t) + hB_2Kv(t) + B_1w(t) + Hv_3(t)$$
$$y_3(t) = (F_0 + F_2K)x(t) + hF_2Kv(t) + F_1w(t) \quad \text{(13a-d)}$$

where

$$v(t) = \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix} \quad y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$

with feedback

$$v_1(t) = -\frac{1}{h}\int_{-\tau(t)}^0 y_1(t+s)\,ds$$
$$v_2(t) = -\frac{1}{\varepsilon h}\int_{-\varepsilon\tau(t)}^0 y_2(t+s)\,ds \quad v_3(t) = \Delta y_3(t). \quad (14)$$

Note that for $h \to 0$ the above model (13), (14) corresponds to the closed-loop system (2) with the continuous state-feedback $u(t) = Kx(t)$.

Assume that $y_i(t) = 0, \forall t \le 0, i = 1, 2, 3$. The following holds for $n_i \times n_i$-matrices $R_i > 0, i = 1, 2$ and a scalar $r > 0$ [12]

$$\|\sqrt{R_i}v_i\|_{L_2} \le \|\sqrt{R_i}y_i\|_{L_2}, \qquad i = 1, 2 \quad \|\sqrt{r}v_3\|_{L_2} \le \|\sqrt{r}y_3\|_{L_2}. \quad (15)$$

For $\varepsilon \to 0$ inequality (15) is valid and y_2 given by (13c) vanishes. Thus, for $\varepsilon \to 0$ (13), (14) is the input–output model, which corresponds to the descriptor system without delay in x_2

$$E_0 \dot{x}(t) = (A + H\Delta F_0)x(t) + (B_1 + H\Delta F_1)w(t) + (B_2 + H\Delta F_2)u(t)$$
$$u(t) = K_1x_1(t_k) + K_2x_2(t) \quad t \in [t_k, t_{k+1})$$
$$0 \le t_{k+1} - t_k \le h. \quad (16)$$

Remark 3.1: Descriptor system can be destabilized by arbitrary fast sampling in the fast variable of the state-feedback even if the system is stable under continuous-time state-feedback. Consider the following simple example:

$$E_0\dot{x}(t) = \begin{bmatrix}-1 & 0\\ 1 & 1\end{bmatrix}x(t) + \begin{bmatrix}0\\1\end{bmatrix}u(t), \qquad x(t)\in R^2. \tag{17}$$

It is clear that the closed-loop system is stable with the continuous state-feedback $u(t) = -2x_2(t)$, while it is unstable with $u(t) = -2x(t_k), t \in [t_k, t_{k+1})$, for any sampling t_k. Really, the resulting closed-loop triangular system is stable if equation $x_2(t)+u(t)=0$ is stable. However, this equation in the sampled-data case $x_2(t) = 2x_2(t_k)$, $t\in[t_k,t_{k+1})$ is unstable.

B. L_2-Gain Analysis

Consider the Lyapunov function $V(t) = x^T(t)E_\varepsilon P_\varepsilon x(t)$, where P_ε has the structure of

$$P_\varepsilon = \begin{bmatrix}P_1 & \varepsilon P_2^T\\ P_2 & P_3\end{bmatrix}, \qquad P_1>0, \quad P_3>0. \tag{18}$$

Note that P_ε is chosen to be of the form of (18) (as, e.g., in [20]), such that for $\varepsilon = 0$, the function V with $E_\varepsilon = E_0$ and $P_\varepsilon = P_0$, corresponds to the descriptor case).

Given $\varepsilon > 0$, from (15) it follows that the following condition along (13a):

$$\begin{aligned}\mathcal{W} &\stackrel{\Delta}{=} \dot{V}(t) + h\sum_{i=1}^{2} y_i^T(t)R_iy_i(t) + r\|y_3(t)\|^2\\ &\quad - h\sum_{i=1}^{2}v_i^T(t)R_iv_i(t) - r\|v_3(t)\|^2 + \|z(t)\|^2 - \gamma^2\|w(t)\|^2\\ &< -\alpha(\|x(t)\|^2+\|u(t)\|^2+\|w(t)\|^2), \qquad \alpha>0 \end{aligned}\tag{19}$$

guarantees the internal stability of (11) and that L_2-gain of (11) less than γ. Moreover, since $y(t)$ depends on $\dot{x}(t)$, we consider the derivative condition $\dot{V}(t) \le -\beta(\|x(t)\|^2 + \|\dot{x}(t)\|^2), \beta>0$. Such derivative condition corresponds to the descriptor model transformation introduced in [5].

Given $n\times n$-matrices

$$\Phi_j = \begin{bmatrix}\Phi_{j1} & 0\\ \Phi_{j2} & \Phi_{j3}\end{bmatrix}, \qquad j=2,3, \quad \Phi_{j1}\in R^{n_1\times n_1}$$
$$\Phi_{j3}\in R^{n_2\times n_2} \tag{20}$$

denote

$$\mathcal{P}_\varepsilon = \begin{bmatrix}P_\varepsilon & 0\\ \Phi_2 & \Phi_3\end{bmatrix}. \tag{21}$$

We have, similarly to [5], the first equation shown at the bottom of the page.

Thus, along the trajectories of (13) we obtain

$$\mathcal{W} \le \zeta^T(t)\bar{\Gamma}\zeta(t) + h\sum_{i=1}^{2}y_i^T(t)R_iy_i(t) + r\|y_3(t)\|^2 + \|z(t)\|^2 \tag{22}$$

where $\zeta(t) = \mathrm{col}\{x(t), E_\varepsilon\dot{x}(t), v(t), v_3(t), w(t)\}$ and

$$\bar{\Gamma} = \begin{bmatrix}\Gamma_\varepsilon & h\mathcal{P}_\varepsilon^T\begin{bmatrix}0\\B_2K\end{bmatrix} & \mathcal{P}_\varepsilon^T\begin{bmatrix}0\\H\end{bmatrix} & \mathcal{P}_\varepsilon^T\begin{bmatrix}0\\B_1\end{bmatrix}\\ * & -\begin{bmatrix}hR_1 & 0\\ 0 & hR_2\end{bmatrix} & 0 & 0\\ * & * & -rI_n & 0\\ * & * & * & -\gamma^2I_q\end{bmatrix}$$
$$\Gamma_\varepsilon = \mathcal{P}_\varepsilon^T\begin{bmatrix}0 & I_n\\ A+B_2K & -I_n\end{bmatrix} + \begin{bmatrix}0 & A^T+K^TB_2^T\\ I_n & -I_n\end{bmatrix}\mathcal{P}_\varepsilon. \tag{23a,b}$$

By applying Schur complements to the term $h\sum_{i=1}^{2}y_i^T(t)R_iy(t) + r\|y_3(t)\|^2 + \|z(t)\|^2$ we conclude that (19) is satisfied if (24), as shown at the bottom of the page, holds.

Denote by $\Xi_\varepsilon, \varepsilon\ge 0$ the matrix in the left-hand side of (24). If $\Xi_0 < 0$, i.e., (24) is feasible for $\varepsilon = 0$, then for the same values of P_1, P_2, P_3, R, Φ_2 and Φ_3 the full-order LMI (24) is feasible for small enough values of ε, since $\Xi_\varepsilon = \Xi_0 + \varepsilon M$, where M is some constant matrix. Hence, $\Xi_0 < 0$ implies (19) for small enough ε.

We thus proved the following.

Lemma 3.1:

i) Given $\gamma>0, h>0$ and $m\times n$-matrix K, (11) is internally stable and has L_2-gain less than γ for all small enough $\varepsilon>0$ and $0\le\tau(t)\le h$, if there exist $n_1\times n_1$ matrices $P_1>0, R_1>0, \Phi_{21}, \Phi_{31}, n_2\times n_2$ matrices $P_3>0, R_2>0, \Phi_{23}, \Phi_{33}, n_1\times n_2$-matrices $P_2, \Phi_{22}, \Phi_{32}$ and a scalar $r>0$ such that LMI (24) is feasible for $\varepsilon=0$, where $\mathcal{P}_0$ and Γ_0 are given by (18), (20), (21), and (23b).

ii) Given $\varepsilon>0, \gamma>0, h>0$ and $m\times n$-matrix K, (11) is internally stable and has L_2-gain less than γ for all $0\le\tau(t)\le h$, if there exist $n_1\times n_1$ matrices $P_1>0, R_1>0, \Phi_{21}$,

$$\begin{aligned}\dot{V}(t) &= 2x^T(t)P_\varepsilon^TE_\varepsilon\dot{x}(t)\\ &= 2\begin{bmatrix}x(t)\\ E_\varepsilon\dot{x}(t)\end{bmatrix}^T\mathcal{P}_\varepsilon^T\begin{bmatrix}E_\varepsilon\dot{x}(t)\\ (A+B_2K)x(t)+hB_2Kv(t)+B_1w(t)+Hv_3(t)-E_\varepsilon\dot{x}(t)\end{bmatrix}\end{aligned}$$

$$\begin{bmatrix}\Gamma_\varepsilon & h\mathcal{P}_\varepsilon^T\begin{bmatrix}0\\B_2K\end{bmatrix} & \mathcal{P}_\varepsilon^T\begin{bmatrix}0\\H\end{bmatrix} & \mathcal{P}_\varepsilon^T\begin{bmatrix}0\\B_1\end{bmatrix} & \begin{matrix}r(F_0+F_2K)^T\\0\end{matrix} & \begin{matrix}0\\hR\end{matrix} & \begin{matrix}C^T+K^TD_{12}^T\\0\end{matrix}\\ * & -hR & 0 & 0 & hrK^TF_2^T & 0 & hK^TD_{12}^T\\ * & * & -rI_n & 0 & 0 & 0 & 0\\ * & * & * & -\gamma^2I_q & rF_1^T & 0 & 0\\ * & * & * & * & -rI_n & 0 & 0\\ * & * & * & * & * & -hR & 0\\ * & * & * & * & * & * & -I_p\end{bmatrix} < 0$$
$$R = \begin{bmatrix}R_1 & 0\\ 0 & R_2\end{bmatrix} \tag{24}$$

$\Phi_{31}, n_2 \times n_2$ matrices $P_3 > 0, R_2 > 0, \Phi_{23}, \Phi_{33}, n_1 \times n_2$-matrices $P_2, \Phi_{22}, \Phi_{32}$ and a scalar $r > 0$ such that LMI (24) is feasible and $E_\varepsilon P_\varepsilon > 0$, where $\mathcal{P}_\varepsilon$ and Γ_ε are given by (18), (20), (21), and (23b).

If (24) is feasible for $\varepsilon = 0$, then the *slow* (descriptor) system (16) is internally stable and has L_2-gain less than γ. Moreover, the *fast* LMI, shown in (25) at the bottom of the page, is feasible. The latter LMI guarantees that the *fast*

$$\begin{aligned}\dot{x}_2(t) &= (A_4 + H_4\Delta_4 F_{04})x_2(t) + (B_{12} + H_4\Delta_4 F_{12})w(t)\\ &\quad + (B_{22} + H_4\Delta_4 F_{22})u(t)\\ u(t) &= K_2 x_2(t_k), \qquad t \in [t_k, t_{k+1}), \quad 0 \le t_{k+1} - t_k \le h\end{aligned} \tag{26}$$

system is internally stable and has L_2-gain less than γ. Thus the feasibility of ε-independent LMI (24), where $\varepsilon = 0$, implies that the fast subproblem is solvable by a sampled-data controller, while the slow subproblem is solvable by a mixed controller (continuous in the fast variable and sampled-data in the slow one).

C. State-Feedback Design

Our objective now is to find K. In order to obtain an LMI in (24) we have to restrict ourselves to the case of block-diagonal $\Phi_2 = \operatorname{diag}\{\Phi_{21}, \Phi_{23}\}$ and to $\Phi_3 = \rho\Phi_2$, where $\rho \neq 0$ is a scalar parameter. Note that Φ_2 is nonsingular due to the fact that the only matrix which can be negative definite in the second block on the diagonal of (24) is $-\rho(\Phi_2 + \Phi_2^T)$. Defining

$$\begin{aligned}\Psi &= \Phi_2^{-1} = \operatorname{diag}\left\{\Phi_{21}^{-1} \quad \Phi_{23}^{-1}\right\}\\ \bar{P} &= \Psi^T P_0 \Psi \quad \bar{R} = \Psi^T R \Psi \quad \bar{r} = r^{-1}\end{aligned}$$

and $Y = K\Psi$, multiplying LMI (24) by $\operatorname{diag}\{\Psi, \Psi, \Psi, I_n, I_q, I_n, \Psi, I_p\}$ and its transpose, from the right and the left, respectively, we obtain the LMI with a tuning parameter ρ, as shown in (27) at the bottom of the page. Note that $\bar{P}$ and $\bar{R}$ have the same, block-triangular and block-diagonal structures, as P_0 and R correspondingly.

Theorem 3.1: Given $\gamma > 0$, consider the system of (2) and the multirate state-feedback law of (9). Assume A1–A3.

i) The state-feedback (9) internally stabilizes (2) and guarantees L_2-gain less than γ for all small enough $\varepsilon \ge 0$, if for some prescribed scalar $\rho \neq 0$ there exist $n_1 \times n_1$-matrices $\bar{P}_1 > 0$, $\bar{R}_1 > 0, \Psi_1, n_2 \times n_2$-matrices $\bar{P}_3 > 0, \bar{R}_2 > 0, \Psi_3$, an $n_1 \times n_2$-matrix $\bar{P}_2$, a $p \times n$-matrix Y and a scalar $\bar{r} > 0$ such that LMI (27) with
$$\Psi = \begin{bmatrix}\Psi_1 & 0\\ 0 & \Psi_3\end{bmatrix} \quad \bar{P} = \begin{bmatrix}\bar{P}_1 & 0\\ \bar{P}_2 & \bar{P}_3\end{bmatrix} \quad \bar{R} = \begin{bmatrix}\bar{R}_1 & 0\\ 0 & \bar{R}_2\end{bmatrix} \tag{28}$$
is feasible. The state-feedback ε-independent gain is given by $K = Y\Psi^{-1}$.

ii) The gain $K = [K_1\ K_2]$ obtained in i) solves the slow (16) and the fast (26) subproblems.

iii) Given $\varepsilon > 0$ the gain obtained in i) internally stabilizes (2) and guarantees L_2-gain less than γ if there exist $n_1 \times n_1$ matrices $P_1 > 0, R_1 > 0, \Phi_{21}, \Phi_{31}, n_2 \times n_2$ matrices $P_3 > 0, R_2 > 0$, $\Phi_{23}, \Phi_{33}, n_1 \times n_2$-matrices $P_2, \Phi_{22}, \Phi_{32}$ and a scalar $r > 0$ such that LMI (24) is feasible and $E_\varepsilon P_\varepsilon > 0$, where P_ε is given by (18).

Example 3.1: [18] Consider (11) with

$$A_0 = \begin{bmatrix}2 & 1\\ -1 & -2\end{bmatrix} \quad B_2 = \begin{bmatrix}2\\ 2\end{bmatrix} \quad B_1 = \begin{bmatrix}1\\ 3\end{bmatrix}$$
$$C = \begin{bmatrix}\begin{pmatrix}2 & 1\\ 1 & 3\end{pmatrix}^{\frac{1}{2}}\\ 0 \quad 0\end{bmatrix} \quad D_{12} = \begin{bmatrix}0\\ 0\\ 1\end{bmatrix} \tag{29}$$

where $H = 0$. Given $\gamma = 3$ and the uniform sampling $t_{k+1} - t_k = 0.1$, it was shown in [18] that the slow state-feedback $u(t) = -1.1618x_1(t_k),\ t \in [t_k, t_{k+1})$ solves the H_∞-control problem for the full-order system for all small enough $\varepsilon > 0$. Moreover, the slow controller can not achieve $\gamma < 2.85$.

$$\begin{bmatrix}\Gamma_f & h\mathcal{P}_f^T\begin{bmatrix}0\\ B_{22}K_2\end{bmatrix} & \mathcal{P}_f^T\begin{bmatrix}0\\ H_4\end{bmatrix} & \mathcal{P}_f^T\begin{bmatrix}0\\ B_{12}\end{bmatrix} & \begin{matrix}r(F_{04}+F_{22}K_2)^T\\ 0\end{matrix} & \begin{matrix}0\\ hR_2\end{matrix} & \begin{matrix}C_2^T + K_2^T D_{12}^T\\ 0\end{matrix}\\ * & -hR_2 & 0 & 0 & hrK_2^T F_{22}^T & 0 & hK_2^T D_{12}^T\\ * & * & -rI_{n_2} & 0 & 0 & 0 & 0\\ * & * & * & -\gamma^2 I_q & rF_{12}^T & 0 & 0\\ * & * & * & * & -rI_{n_1} & 0 & 0\\ * & * & * & * & * & -hR_2 & 0\\ * & * & * & * & * & * & -I_p\end{bmatrix} < 0$$
$$\Gamma_f = \mathcal{P}_f^T\begin{bmatrix}0 & I_{n_2}\\ A_4 + B_{22}K_2 & -I_{n_2}\end{bmatrix} + \begin{bmatrix}0 & A_4^T + K_2^T B_{22}^T\\ I_{n_2} & -I_{n_2}\end{bmatrix}\mathcal{P}_f$$
$$\mathcal{P}_f = \begin{bmatrix}P_3 & 0\\ \Phi_{23} & \Phi_{33}\end{bmatrix} \tag{25}$$

$$\begin{bmatrix}\Sigma_1 & \Sigma_2 & hB_2Y & \bar{r}H & B_1 & \Psi^T F_0^T + Y^T F_2^T & 0 & \Psi^T C^T + Y^T D_{12}^T\\ * & -\rho(\Psi+\Psi^T) & h\rho B_2 Y & \bar{r}\rho H & \rho B_1 & 0 & h\bar{R} & 0\\ * & * & -h\bar{R} & 0 & 0 & hY^T F_2^T & 0 & hY^T D_{12}^T\\ * & * & * & -\bar{r}I_n & 0 & 0 & 0 & 0\\ * & * & * & * & -\gamma^2 I_q & F_1^T & 0 & 0\\ * & * & * & * & * & -\bar{r}I_n & 0 & 0\\ * & * & * & * & * & * & -h\bar{R} & 0\\ * & * & * & * & * & * & * & -I_p\end{bmatrix} < 0$$
$$\Sigma_1 = A\Psi + \Psi^T A^T + B_2 Y + Y^T B_2^T,\ \Sigma_2 = \bar{P}^T - \Psi + \rho\Psi^T A^T + \rho Y^T B_2^T \tag{27}$$

Consider the uncertain system (11), (29) with $H = I_2, F_0 = 0.1 \cdot I_2, F_1 = F_2 = [0.1\ 0.1]^T$. Applying Theorem 3.1 with the smaller $\gamma = 2.8$, the same $h = 0.1$ and choosing $\rho = -0.1$, we find that the multirate controller (9) with ε-independent gain $K = [-2.4407\ -0.5788]$ leads to L_2-gain less than 2.8 for all small enough $\varepsilon > 0$ and all the samplings with $t_{k+1} - t_k \leq 0.1$. By applying Lemma 3.1 to the resulting closed-loop system for $h = 0.1$ and for different values of $\varepsilon > 0$ we verify that this gain leads the full-order system to L_2-gain less than 2.8 for all $0 < \varepsilon \leq 0.49$ and for all the samplings $0 \leq t_{k+1} - t_k \leq 0.1$. The possibility to treat the uncertain system, as well as to check the solvability of the H_∞ control problem for given values of h and ε, are the advantages of the LMI approach.

IV. Fast Sample-Rate H_∞ Control

Substituting (10) into (2), we obtain the following closed-loop system:

$$\begin{aligned} E_\varepsilon \dot{x}(t) &= (A + H\Delta F_0)x(t) + (B_2 + H\Delta F_2)Kx(t - \varepsilon\tau(t)) \\ &\quad + (B_1 + H\Delta F_1)w(t) \\ z(t) &= Cx(t) + D_{12}Kx(t - \varepsilon\tau(t)). \end{aligned} \tag{30}$$

Similarly to the previous section we introduce the forward system:

$$\begin{aligned} E_\varepsilon \dot{x}(t) &= (A + B_2K)x(t) + hB_2K\bar{v}(t) + B_1w(t) + Hv_3(t) \\ z(t) &= (C + D_{12}K)x(t) + hD_{12}K\bar{v}(t) \\ y(t) &= E_\varepsilon \dot{x}(t) = (A + B_2K)x(t) + hB_2K\bar{v}(t) \\ &\quad + B_1w(t) + Hv_3(t) \\ y_3(t) &= (F_0 + F_2K)x(t) + hF_2K\bar{v}(t) + F_1w(t) \end{aligned} \tag{31a-e}$$

where

$$\bar{v}(t) = \begin{bmatrix} \varepsilon v_1(t) \\ v_2(t) \end{bmatrix} \quad y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$

with feedback

$$\begin{aligned} v_1(t) &= -\frac{1}{\varepsilon h}\int_{-\varepsilon\tau(t)}^{0} y_1(t+s)\,ds \\ v_2(t) &= -\frac{1}{\varepsilon h}\int_{-\varepsilon\tau(t)}^{0} y_2(t+s)\,ds,\ v_3(t) = \Delta y_3(t). \end{aligned} \tag{32}$$

Inequalities (15) are valid here. The only difference with the previous case that v_1 in (31) is multiplied by ε, which leads to the full-order LMI for stability and L_2-gain shown in (33) at the bottom of the page. Setting $\varepsilon = 0$ into (33) and applying the Schur complements to the row and the column with the only nonzero (diagonal) element hR_1 (and thus deleting also the row and the column containing hR_1) we obtain the ε-independent LMI shown in (34) at the bottom of the page.

Feasibility of (34) implies feasibility of (33) for small enough $\varepsilon > 0$ and $R_1 > 0$. LMI (34) implies the same fast LMI (25) and the fast problem (26), while the slow LMI has a form shown in (35) at the bottom of the page, and corresponds to the *slow* problem with a continuous-time state-feedback

$$\begin{aligned} E_0\dot{x}(t) &= (A + H\Delta F)x(t) + (B_1 + H\Delta F_1)w(t) \\ &\quad + (B_2 + H\Delta F_2)u(t) \quad u(t) = Kx(t). \end{aligned} \tag{36}$$

$$\begin{bmatrix} \Gamma_\varepsilon & h\mathcal{P}_\varepsilon^T \begin{bmatrix} 0 & \\ \varepsilon B_2K_1 & B_2K_2 \end{bmatrix} & \mathcal{P}_\varepsilon^T \begin{bmatrix} 0 \\ H \end{bmatrix} & \mathcal{P}_\varepsilon^T \begin{bmatrix} 0 \\ B_1 \end{bmatrix} & \begin{matrix} r(F_0 + F_2K)^T \\ 0 \end{matrix} & \begin{matrix} 0 \\ hR \end{matrix} & \begin{matrix} C^T + K^TD_{12}^T \\ 0 \end{matrix} \\ * & -hR & 0 & 0 & hr\begin{bmatrix} \varepsilon K_1^T \\ K_2^T \end{bmatrix} F_2^T & 0 & h\begin{bmatrix} \varepsilon K_1^T \\ K_2^T \end{bmatrix} D_{12}^T \\ * & * & -rI_n & 0 & 0 & 0 & 0 \\ * & * & * & -\gamma^2 I_q & rF_1^T & 0 & 0 \\ * & * & * & * & -rI_n & 0 & 0 \\ * & * & * & * & * & -hR & 0 \\ * & * & * & * & * & * & -I_p \end{bmatrix} < 0$$
$$R = \mathrm{diag}\{R_1, R_2\} \tag{33}$$

$$\begin{bmatrix} \Gamma_0 & h\mathcal{P}_0^T \begin{bmatrix} 0 \\ B_2K_2 \end{bmatrix} & \mathcal{P}_0^T \begin{bmatrix} 0 \\ H \end{bmatrix} & \mathcal{P}_0^T \begin{bmatrix} 0 \\ B_1 \end{bmatrix} & \begin{matrix} r(F_0 + F_2K)^T \\ 0 \end{matrix} & \begin{matrix} 0 \\ hR_2 \end{matrix} & \begin{matrix} C^T + K^TD_{12}^T \\ 0 \end{matrix} \\ * & -hR_2 & 0 & 0 & hrK_2^TF_2^T & 0 & hK_2^T D_{12}^T \\ * & * & -rI_n & 0 & 0 & 0 & 0 \\ * & * & * & -\gamma^2 I_q & rF_1^T & 0 & 0 \\ * & * & * & * & -rI_n & 0 & 0 \\ * & * & * & * & * & -hR_2 & 0 \\ * & * & * & * & * & * & -I_p \end{bmatrix} < 0 \tag{34}$$

$$\begin{bmatrix} \Gamma_0 & \mathcal{P}_0^T \begin{bmatrix} 0 \\ H \end{bmatrix} & \mathcal{P}_0^T \begin{bmatrix} 0 \\ B_1 \end{bmatrix} & \begin{matrix} r(F_0 + F_2K)^T \\ 0 \end{matrix} & \begin{matrix} C^T + K^TD_{12}^T \\ 0 \end{matrix} \\ * & -rI_n & 0 & 0 & 0 \\ * & * & -\gamma^2 I_q & rF_1^T & 0 \\ * & * & * & -rI_n & 0 \\ * & * & * & * & -I_p \end{bmatrix} < 0 \tag{35}$$

$$\begin{bmatrix} \Sigma_1 & \Sigma_2 & hB_2Y_2 & \bar{r}H & B_1 & \Psi^T F_0 + Y^T F_2^T & 0 & \Psi^T C^T + Y^T D_{12}^T \\ * & -\rho(\Psi+\Psi^T) & h\rho B_2 Y_2 & \bar{r}\rho H & \rho B_1 & 0 & h\bar{R}_2 & 0 \\ * & * & -h\bar{R}_2 & 0 & 0 & hY_2^T & 0 & hY_2^T D_{12}^T \\ * & * & * & -\bar{r}I_n & 0 & 0 & 0 & 0 \\ * & * & * & * & -\gamma^2 I_q & F_1^T & 0 & 0 \\ * & * & * & * & * & -\bar{r}I_n & 0 & 0 \\ * & * & * & * & * & * & -h\bar{R}_2 & 0 \\ * & * & * & * & * & * & * & -I_p \end{bmatrix} < 0$$

$$\Sigma_1 = A\Psi + \Psi^T A^T + B_2 Y + Y^T B_2^T \quad \Sigma_2 = \bar{P}^T - \Psi + \rho \Psi^T A^T + \rho Y^T B_2^T \quad Y = [Y_1 \; Y_2] \qquad (37a,b)$$

Note that singularly perturbed systems with small delay are usually decomposed into the nondelayed slow subsystem and the delayed fast one (see, e.g., [10]).

Multiplying (34) by $\mathrm{diag}\{\Psi, \Psi_3, \Psi, \bar{r}I_n, I_q, \bar{r}I_n, \Psi_3, I_p\}$ and its transpose, on the left and on the right, respectively, we obtain the following ε-independent LMI with a tuning parameter ρ; see (37a,b), as shown at the top of the page.

Theorem 4.1: Given $\gamma > 0$, consider the system of (2) and the fast-rate state-feedback law of (10). Assume A1–A3.

i) The state-feedback (10) internally stabilizes (2) and guarantees L_2-gain less than γ for all small enough $\varepsilon \geq 0$, if for some prescribed scalar $\rho \neq 0$ there exist $n_1 \times n_1$ matrices $\bar{P}_1 > 0, \Psi_1, n_2 \times n_2$ matrices $\bar{P}_3 > 0$, $\bar{R}_2 > 0, \Psi_3$, an $n_1 \times n_2$-matrix $\bar{P}_2$, a $p \times n$ matrix Y and a scalar $r > 0$ such that LMI (37) with Ψ and $\bar{P}$ given by (28) is feasible. The state-feedback ε-independent gain is given by $K = Y\Psi^{-1}$.

ii) The gain $K = [K_1 \; K_2]$ obtained in i) solves the slow (36) and the fast (26) subproblems.

iii) Given $\varepsilon > 0$ the state-feedback (10) with K from i) internally stabilizes (2) and guarantees L_2-gain less than γ if there exist $n_1 \times n_1$ matrices $P_1 > 0, R_1 > 0, \Phi_{21}, \Phi_{31}, n_2 \times n_2$ matrices $P_3 > 0, R_2 > 0, \Phi_{23}, \Phi_{33}, n_1 \times n_2$-matrices $P_2, \Phi_{22}, \Phi_{32}$ and a scalar $r > 0$ such that LMI (33) is feasible and $E_\varepsilon P_\varepsilon > 0$, where P_ε is given by (18).

Example 4.1: Consider the uncertain system (11), (29) with $H = I_2$, $F_0 = 0.1 \cdot I_2, F_1 = F_2 = [0.1 \; 0.1]^T$. We find by Theorem 4.1, where $\rho = -0.1$, that the ε-independent fast-rate controller (10), where $K = [-3.0049 \; -0.5954]$, leads to L_2-gain less than 2.6 (which is less than 2.8 achieved by the multi-rate controller (9)) for all small enough $\varepsilon > 0$ and all the samplings with $t_{k+1} - t_k \leq 0.1$. Moreover, this gain leads the full-order system to L_2-gain less than 2.6 for all the samplings $0 \leq t_{k+1} - t_k \leq 0.1$ and for all $0 < \varepsilon \leq 0.49$.

V. CONCLUSION

Sampled-data state-feedback H_∞ control problem for singularly perturbed system with norm-bounded uncertainties has been solved via input delay approach to sampled-data control. The only assumption on the sampling that the distance between the sequel sampling times is not greater than some $h > 0$. Two kinds of controllers have been designed (both with the fast sampling in the fast variables): the multirate state-feedback (slow rate in the slow variables) and the fast-rate state-feedback. The ε-independent gains of the controllers are found from ε-independent LMIs. ε-dependent LMIs are derived which give sufficient conditions for the solvability of the full-order system. An illustrative example shows that the fast-rate controller leads to better performance, than the multirate one. The tradeoff is in the fast sampling of the slow variables.

REFERENCES

[1] B. Bamieh, J. Pearson, B. Francis, and A. Tannenbaum, "A lifting technique for linear periodic systems," *Syst. Control Lett.*, vol. 17, pp. 79–88, 1991.

[2] J. Chow and P. Kokotovic, "A decomposition of near-optimum regulators for systems with slow and fast modes," *IEEE Trans. Autom. Control*, vol. AC-21, no. 5, pp. 701–705, Oct. 1976.

[3] L. Dai, *Singular Control Systems*. Berlin, Germany: Springer-Verlag, 1989.

[4] M. Djemai, J.-P. Barbot, and H. Khalil, "Digital multirate control for a class of nonlinear singualrly perturbed systems," *Int. J. Control*, vol. 72, no. 10, pp. 851–865, 1999.

[5] E. Fridman, "New Lyapunov-Krasovskii functionals for stability of linear retarded and neutral type systems," *Syst. Control Lett.*, vol. 43, pp. 309–319, 2001.

[6] ——, "Effects of small delays on stability of singularly perturbed systems," *Automatica*, vol. 38, no. 5, pp. 897–902, 2002.

[7] E. Fridman, A. Seuret, and J.-P. Richard, "Robust sampled-data stabilization of linear systems: An input delay approach," *Automatica*, vol. 40, pp. 1441–1446, 2004.

[8] E. Fridman and U. Shaked, "Input-output approach to stability and L_2-gain analysis of systems with time-varying delays," in *Proc. 44th Conf. on Decision and Control*, Seville, Spain, 2005, pp. 7175–7180.

[9] G. Garcia, J. Daafouz, and J. Bernussou, "The infinite time near optimal decentralized regulator problem for singularly perturbed systems: A convex optimization approach," *Automatica*, vol. 38, pp. 1397–1406, 2002.

[10] V. Glizer and E. Fridman, "H_∞ control of linear singularly perturbed systems with small state delay," *J. Math. Anal. Appl.*, vol. 250, pp. 49–85, 2000.

[11] A. Halanay, "An invariant surface for some linear singularly perturbed systems with time lag," *J. Diff. Equat.*, vol. 2, pp. 33–46, 1966.

[12] K. Gu, V. Kharitonov, and J. Chen, *Stability of Time-Delay Systems*. Boston, MA: Birkhäuser, 2003.

[13] Y.-P. Huang and K. Zhou, "Robust stability of uncertain time-delay systems," *IEEE Trans. Autom. Control*, vol. 45, no. 11, pp. 2169–2173, Nov. 2000.

[14] H. K. Khalil, "Feedback control of nonstandard singularly perturbed systems," *IEEE Trans. Autom. Control*, vol. 34, no. 10, pp. 1052–1060, Oct. 1989.

[15] P. Kokotovic, H. Khalil, and J. O'Reilly, *Singular Perturbation Methods in Control: Analysis and Design*. New York: Academic, 1986.

[16] D. W. Luse, "Multivariable singularly perturbed feedback systems with time delay," *IEEE Trans. Autom. Control*, vol. AC-32, no. 11, pp. 990–994, Nov. 1987.

[17] D. S. Naidu, "Singular perturbations and time scales in control theory and applications: An overview," *Dyna. Contin., Discrete Impul. Syst. (DCDIS) Series B: Appl. Algorithms*, vol. 9, no. 2, pp. 233–278, 2002.

[18] Z. Pan and T. Basar, "H-infinity optimal control for singularly perturbed systems with sampled state measurements," in *Advances in Dynamic Games and Applications*, T. Basar and A. Haurie, Eds. Boston, MA: Birkhäuser, 1994, vol. 1, pp. 23–55.

[19] A. Tikhonov, "Systems of differential equations containing small parameters multiplying some of derivatives," *Mathematica Sborniki*, vol. 31, pp. 575–586, 1952.

[20] H. Xu and K. Mizukami, "Infinite-horizon differential games of singularly perturbed systems: A unified approach," *Automatica*, vol. 33, pp. 273–276, 1997.

[21] Y. Yamamoto, "New approach to sampled-data control systems—A function space method," in *Proc. 29th Conf. Decision and Control*, Honolulu, HI, 1990, pp. 1882–1887.