structure, the behavior of the curve is extremely complicated. The numbers written inside each region indicate the respective number of RHP roots. The shaded region indicates the set of controller parameters that stabilizes the given plant (i.e., number of RHP roots $= 0$).

Fig. 3 depicts the three dimensional stability region in the controller parameter space. The two-dimensional sections of the region are for $-1.25 \leq x_3 \leq 1.375$. The figure shows that the region gets smaller as x_3 approaches $+1.375$ and -1.25.

IV. Concluding Remarks

An effective computational procedure to determine all first-order controllers that stabilize a given discrete-time LTI system is described in this note. The determination of the complete set requires fixing x_3, obtaining the stabilizing set in the $x_1 - x_2$ plane and repeating this solution by sweeping over all x_3. In practice, the range of x_3 may be predetermined based on other design considerations and a method involving sweeping over *only* one parameter is quite practical, given the available computational power. As in [4], the result can also be extended to simultaneous stabilization, maximal delay tolerance and maximally deadbeat designs. A control scheme that optimizes a given optimality criteria and the attainment of other performance criteria over these sets is currently under investigation. We anticipate the parameter plane techniques developed in [11] to be useful in this context.

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Parameter Dependent Stability and Stabilization of Uncertain Time-Delay Systems

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***Abstract*—A new robust delay dependent stability test is introduced that determines the asymptotic stability of linear systems with state delays. The parameters of the system are not exactly given. They are known to reside in a given polytope. The test provides an efficient sufficient condition for the stability of the system over the uncertainty polytope. This condition is parameter dependent and it therefore improves previous results that were derived using a single Lyapunov–Krasovskii functional. The stability test is readily extended to provide a criterion for robust stabilization via state-feedback.**

***Index Terms*—Lyapunov–Krasovskii functionals, parameter dependent stability, time-delay systems.**

I. Introduction

Efficient conditions for the stability and the stabilizability via state-feedback control of linear systems with time-delay have been introduced recently [1]. These conditions are expressed via linear matrix inequalities (LMIs) that are tuned by a scalar parameter. The conditions provide delay dependent sufficient conditions for the stability of the system and for a specific value of the tuning parameter they provide the condition for delay-independent stability. The LMIs obtained are affine in the matrices of the system's state space model. This affinity enables the consideration of systems with uncertain parameters. Assuming this uncertainty is of the polytopic type [2], conditions have been derived in [1] also for the quadratic stability and stabilizability of the time-delayed system over the entire uncertainty polytope.

Quadratic stability conditions are conservative [2]. Many attempts have been made in the past few years to reduce the latter conservatism by seeking parameter dependent techniques for determining the stability of systems with polytopic type uncertainties. These techniques assign a different Lyapunov function to each vertex of the uncertainty polytope and due to the convexity of the resulting LMIs the parameter dependent condition is obtained [3]. One of the most efficient parameter dependent condition for the stability of linear systems with polytopic uncertainty has been recently introduced [4]. This condition applies to systems without delay and it outperforms other known method for robust stability determination. Unfortunately, its application to synthesis is limited since the inequalities obtained by this method contain bilinear terms in the decision variables and some iterations should be used to derive, say, the stabilizing state-feedback. The iteration procedure requires an initial state-feedback gain matrix which stabilizes the system over the entire uncertainty polytope, and this gain matrix in not easy to come by.

The problem of verifying the stability and the stabilizability of time-delay systems with polytopic parameter uncertainty has been treated in the past by applying quadratic stability arguments [1], [5]. Recently, another method which is based on parameter varying Lyapunov–Krasovskii functionals has been introduced [6]. This method is based on an improved first model transformation (see [7] and [8] for different model transformations) and it applies the gridding method to cope with the parameter uncertainty.

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In this note, we develop efficient, delay dependent, conditions for the stability and the stabilizability of systems with time delay. These conditions are based on the descriptor model transformation of [1]. They extend the idea of the stability conditions of [4] to systems with time delay and apply the resulting criteria to the augmented descriptor model of the system. Sufficient conditions are obtained for the stability of the system that improves the corresponding quadratic stability results. These conditions are LMIs that are tuned by a scalar parameter. A specific value of this parameter leads to a corresponding delay-independent criterion for stability. The stabilizability problem is then solved by assigning a special structure to the decision variables in the relevant LMIs. This structure allows the derivation of the stabilizing gain by applying standard methods for the solution of LMIs.

II. Robust Stability

Consider the following system with time-varying delay:

$$\dot{x}(t) = A_0 x(t) + A_1 x(t - \tau(t)), \quad x(t) = \phi(t), \ t \in [-h, 0] \quad (1)$$

where $x(t) \in \mathcal{R}^n$, A_0 and A_1 are constant $n \times n$-matrices and ϕ is a continuously differentiable initial function. We assume that $\tau(t)$ is a differentiable function, satisfying for all $t \geq 0$

$$0 \leq \tau(t) \leq h, \qquad \dot{\tau}(t) \leq d. \quad \text{(2a,b)}$$

We are looking for a stability criterion that depends on h and d. We consider, for simplicity, only one delay in (1) but all the results are easily generalized to the case of any finite number of delays.

The solution to (1) satisfies, for $t \geq h$, the following descriptor system:

$$\dot{x}(t) = y(t), \qquad y(t) = [A_0 + A_1]x(t) - A_1 \int_{t-\tau(t)}^{t} y(s)ds. \quad (3)$$

Conversely, if the pair $\{x(t), y(t+\theta)\}$, $t \geq 0$, $\theta \in [-h, 0]$ satisfies (3) for $t \geq 0$, then $x(t)$ satisfies (1) for $t \geq h$. Note that the descriptor system (3) has no impulsive solutions since in the second equation of (3) $y(t)$ is multiplied by the nonsingular matrix I [10]. Hence, (3) and (1) are equivalent from the point of view of stability, i.e., they are stable or unstable simultaneously.

Denoting

$$\bar{x}(t) = \mathrm{col}\{x(t), \ y(t)\}$$

the following Lyapunov–Krasovskii functional is applied:

$$V(t) = \bar{x}^T(t) E P \bar{x}(t) + V_2 + V_3 \quad (4)$$

where

$$E = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$$
$$P = \begin{bmatrix} P_1 & 0 \\ P_2 & P_3 \end{bmatrix}$$
$$P_1 = P_1^T > 0 \quad \text{(5a-c)}$$

and

$$V_2 = \int_{-h}^{0} \int_{t+\theta}^{t} y^T(s) R y(s) ds d\theta$$
$$V_3 = \int_{t-\tau(t)}^{t} x^T(\tau) S x(\tau) d\tau.$$

The following result has been obtained in [13].

Lemma 1: System (1) is asymptotically stable if there exist $2n \times 2n$ matrices: P of the form (5b, c) and Z, a $n \times 2n$ matrix Y and $n \times n$ matrices S and R that satisfy the following LMIs:

$$\Gamma = \begin{bmatrix} P^T \tilde{A}_0 + \tilde{A}_0^T P + \left[\begin{smallmatrix} I \\ 0 \end{smallmatrix}\right] Y + Y^T & \\ \times [I \quad 0] + \begin{bmatrix} S & 0 \\ 0 & hR \end{bmatrix} + hZ & Y^T - P^T \left[\begin{smallmatrix} 0 \\ A_1 \end{smallmatrix}\right] \\ * & -S(1-d) \end{bmatrix} < 0$$

and

$$\begin{bmatrix} R & Y \\ * & Z \end{bmatrix} > 0 \quad \text{(6a,b)}$$

where

$$\tilde{A}_0 = \begin{bmatrix} 0 & I \\ A_0 & -I \end{bmatrix}. \quad (7)$$

The last lemma provides an efficient sufficient condition for the asymptotic stability of (1) in the case where the parameters of A_0 and A_1 are known.

In the case where the matrices of the system are not exactly known, we denote

$$\Omega = [A_0 \quad A_1]$$

and assume that $\Omega \in \mathcal{C}o\{\Omega_j, j = 1, \ldots N\}$, namely

$$\Omega = \sum_{j=1}^{N} f_j \Omega_j \quad \text{for some} \quad 0 \leq f_j \leq 1, \ \sum_{j=1}^{N} f_j = 1 \quad (8)$$

where the N vertices of the polytope are described by

$$\Omega_j = \left[A_0^{(j)} \quad A_1^{(j)}\right].$$

In order to guarantee the stability of (1) over the entire polytope, one can use the result of Lemma 1 by applying the same matrix P for all the points in the polytope and solving (6a,b) for the N vertices only. A quadratic stability type criterion is then obtained which may be quite conservative. In order to allow a P that depends on the parameters of the system, we apply the following.

Lemma 2: The LMIs of Lemma 1 are satisfied for a specific point in Ω iff there exist $2n \times 2n$ matrices: P, of the form (5b, c), G, $\bar{G}$, Z, H and $\bar{Q}$, a $n \times 2n$ matrix Y and $n \times n$ matrices S and R that satisfy (6b) and

$$\begin{bmatrix} \Gamma & \left[\begin{smallmatrix} P^T - G^T - \tilde{A}_0^T H \\ 0 \end{smallmatrix}\right] & \left[\begin{smallmatrix} P^T - \bar{G}^T \\ [0 \ \ A_1^T] \bar{Q} \end{smallmatrix}\right] \\ * & -H - H^T & 0 \\ * & * & -\bar{Q} - \bar{Q}^T \end{bmatrix} < 0. \quad (9)$$

Proof: Obviously, if (9) is satisfied (6a) follows. In order to show the inverse direction, namely that a solution to (6a) implies that there exists a solution also to (9) we choose $H = \bar{Q} = \rho I_{2n}$ and $P = G = \bar{G}$. It clearly follows that if $\Gamma = W < 0$ there exists $0 < \rho \ll 1$ for which

$$\rho \begin{bmatrix} \tilde{A}_0^T \tilde{A}_0 & 0 \\ 0 & A_1^T A_1 \end{bmatrix} < -2W.$$

□

Denoting

$$\Delta = \begin{bmatrix} I_{2n} & 0 & 0 & 0 \\ 0 & I_n & 0 & 0 \\ -\tilde{A}_0 & 0 & I_{2n} & 0 \\ 0 & \left[\begin{smallmatrix} 0 \\ A_1 \end{smallmatrix}\right] & 0 & I_{2n} \end{bmatrix}$$

we multiply both sides of (9) by Δ^T and Δ, from the left and the right, respectively, and obtain that (9), and therefore also (6a), are equivalent to the LMI shown in (10) at the bottom of the next page.

Remark 1: In the case where $h = 0$, the result obtained in (10) and (6b) is similar, for $\bar{G} = P$, $Y = [0\ A_1^T]P$, $S \to 0$ and $R, \bar{Q} \to 0$, to the stability condition that was obtained in [4]. The latter was obtained using positive real arguments that are not directly applicable to systems with time delay.

The LMI obtained in (10) is affine in the decision variables P, Z, G, $\bar{G}$, H, $\bar{Q}$, Y, S, and R. The matrix P does not multiply there any of the system matrices. Thus, one can apply this LMI on the vertices of Ω allowing P, Y, S, and R to be parameter dependent. Since (10) is affine in the system matrices (A_0 and A_1), a solution, if exists, to the set of the LMI's that are written for all the vertices of Ω will guarantee the asymptotic stability of the system (1) over the entire polytope and for all $\tau(t)$ that satisfy (2). Considering the following structures:

$$G_j = \begin{bmatrix} G_1^{(j)} & G_2^{(j)} \\ G_3 & G_4 \end{bmatrix}$$
$$\bar{G}_j = \begin{bmatrix} \bar{G}_1^{(j)} & \bar{G}_2^{(j)} \\ \bar{G}_3 & \bar{G}_4 \end{bmatrix}$$
$$H_j = \begin{bmatrix} H_1^{(j)} & H_2 \\ H_3^{(j)} & H_4 \end{bmatrix}$$

and

$$\bar{Q} = \begin{bmatrix} \bar{Q}_1^{(j)} & \bar{Q}_2 \\ \bar{Q}_3^{(j)} & \bar{Q}_4 \end{bmatrix} \tag{11a-d}$$

the latter result is summarized in the following theorem.

Theorem 1: System (1) is asymptotically stable for all A_0 and A_1 that reside in Ω of (8) and for all delays $\tau(t)$ that satisfy (2) if there exist P_j, G_j, $\bar{G}_j$, H_j, $\bar{Q}_j$, of the form (5b, c) and (11a–d), respectively, and Z_j in $\mathcal{R}^{2n\times 2n}$, $Y_j \in \mathcal{R}^{n\times 2n}$, S_j and R_j in $\mathcal{R}^{n\times n}$, $j = 1, \ldots, N$, that satisfy the LMIs shown in (12a–c) at the bottom of the page.

In the case where A_1 is known, a simpler condition is obtained taking $\bar{G}_j$ to be fully parameter dependent, $\bar{G}_j = P_j$, and $\bar{Q}_j = \rho I_{2n}$, where ρ tends to zero. The result obtained is given by the following.

Corollary 1: In the case where the uncertainty occurs only in A_0 and $\tau(t)$, the system (1) is asymptotically stable if there exist P_j, G_j, H_j, of the form (5b, c) and (11a, c), respectively, and Z_j in $\mathcal{R}^{2n\times 2n}$, $Y_j \in \mathcal{R}^{n\times 2n}$, S_j and R_j in $\mathcal{R}^{n\times n}$, $j = 1, \ldots, N$, that satisfy (12b) and (13), as shown at the bottom of the page, where $\tilde{A}_0^{(j)}$ is defined in (12c).

The previous results provide delay-dependent sufficient conditions for the stability of (1) over the entire uncertainty polytope. The corresponding delay-independent result is obtained by substituting in Theorem 1 $R_j = \rho I_n$, $Z_j = \rho I_{2n}$ and $Y_j = 0$, and taking the limit where the positive scalar ρ tends to zero. We then obtain the following.

Corollary 2: System (1) is asymptotically stable for all A_0 and A_1 that reside in Ω of (8), independently of delay length, if there exist P_j, G_j, $\bar{G}_j$, H_j, $\bar{Q}_j$, of the form (5b, c) and (11a–d), respectively, in $\mathcal{R}^{2n\times 2n}$, and $S_j \in \mathcal{R}^{n\times 2n}$, $j = 1, \ldots, N$, that satisfy (12a), where $R_j = 0$, $Z_j = 0$ and $Y_j = 0$.

The aforementioned results correspond to time-varying delays satisfying (2). The delay-dependent/rate-independent results for delays $0 \le \tau(t) \le h$ which may be fastly varying follow from Theorem 1 by taking $S_j = 0$, $Y_j = [0\ A_1^{(j)}]\bar{G}_j$. We obtain the following.

Corollary 3: System (1) is asymptotically stable for all A_0 and A_1 that reside in Ω of (8) and for all delays $\tau(t)$ that satisfy $0 \le \tau(t) \le h$ if there exist P_j, G_j, $\bar{G}_j$, H_j, $\bar{Q}_j$, of the form (5b, c) and (11a–d), respectively, Z_j in $\mathcal{R}^{2n\times 2n}$ and R_j in $\mathcal{R}^{n\times n}$, $j = 1, \ldots, N$ that satisfy (12a) with deleted second column and second row and (12b), where $S_j = 0$, $Y_j = [0\ A_1^{(j)}]\bar{G}_j$.

$$\begin{bmatrix} G^T\tilde{A}_0 + \tilde{A}_0^T G + \begin{bmatrix} I \\ 0 \end{bmatrix} Y + Y^T [I \quad 0] + \begin{bmatrix} S & 0 \\ 0 & hR \end{bmatrix} + hZ & Y^T - \bar{G}^T \begin{bmatrix} 0 \\ A_1 \end{bmatrix} & P^T - G^T + \tilde{A}_0^T H^T & P^T - \bar{G}^T \\ * & -S(1-d) & 0 & -\begin{bmatrix} 0 & A_1^T \end{bmatrix} \bar{Q}^T \\ * & * & -H - H^T & 0 \\ * & * & * & -\bar{Q} - \bar{Q}^T \end{bmatrix} < 0. \tag{10}$$

$$\begin{bmatrix} G_j^T\tilde{A}_0^{(j)} + \tilde{A}_0^{(j)T} G_j + \begin{bmatrix} I \\ 0 \end{bmatrix} Y_j + Y_j^T [I \quad 0] + \begin{bmatrix} S_j & 0 \\ 0 & hR_j \end{bmatrix} + hZ_j & Y_j^T - \bar{G}_j^T \begin{bmatrix} 0 \\ A_1^{(j)} \end{bmatrix} & P_j^T - G_j^T + \tilde{A}_0^{(j)T} H_j^T & P_j^T - \bar{G}_j^T \\ * & -S_j(1-d) & 0 & -\begin{bmatrix} 0 & A_1^{(j)T} \end{bmatrix} \bar{Q}_j^T \\ * & * & -H_j - H_j^T & 0 \\ * & * & * & -\bar{Q}_j - \bar{Q}_j^T \end{bmatrix} < 0 \tag{12a}$$

$$\begin{bmatrix} R_j & Y_j \\ * & Z_j \end{bmatrix} > 0, \qquad j = 1, \ldots, N \tag{12b}$$

where

$$\tilde{A}_0^{(j)} = \begin{bmatrix} 0 & I \\ A_0^{(j)} & -I \end{bmatrix}. \tag{12c}$$

$$\begin{bmatrix} G_j^T\tilde{A}_0^{(j)} + \tilde{A}_0^{(j)T} G_j + \begin{bmatrix} I \\ 0 \end{bmatrix} Y_j + Y_j^T [I \quad 0] + \begin{bmatrix} S_j & 0 \\ 0 & hR_j \end{bmatrix} + hZ_j & Y_j^T - P_j^T \begin{bmatrix} 0 \\ A_1 \end{bmatrix} & P_j^T - G_j^T + \tilde{A}_0^{(j)T} H_j^T \\ * & -S_j(1-d) & 0 \\ * & * & -H_j - H_j^T \end{bmatrix} < 0$$
$$j = 1, \ldots, N \tag{13}$$

III. Stabilization

We consider the system

$$\dot{x}(t) = A_0x(t) + A_1x(t-\tau(t)) + Bu(t), \quad x(t) = \phi(t) \quad t \in [-h, 0] \tag{14}$$

where $x(t) \in \mathcal{R}^n$ is the state, $u(t) \in \mathcal{R}^m$ is the control input, A_0, A_1 and B are constant matrices of the appropriate dimensions, and ϕ is a continuously differentiable initial function. We assume that the delay $\tau(t)$ satisfies (2).

We consider the state-feedback control law

$$u(t) = Kx(t) \tag{15}$$

where K is a constant matrix, and we address the problem of finding K that asymptotically stabilizes the closed-loop obtained by applying (15) to (14).

Replacing A_0 in Lemma 1 by $A_0 + BK$, a nonlinear inequality is obtained. It follows, however, from the requirement of $0 < P_1$, and the fact that in (6) $-(P_3 + P_3^T)$ must be negative definite, that P is nonsingular. Defining

$$P^{-1} = Q = \begin{bmatrix} Q_1 & 0 \\ Q_2 & Q_3 \end{bmatrix} \quad \text{and} \quad \bar{\Delta} = \mathrm{diag}\{Q,\ I_n\} \tag{16a-b}$$

we multiply (6a) by $\bar{\Delta}^T$ and $\bar{\Delta}$, on the left and on the right, respectively and (6b), on the left and on the right, by $\mathrm{diag}\{R^{-1}, Q^T\}$ and $\mathrm{diag}\{R^{-1}, Q\}$, respectively. Applying Schur formula to the emerging quadratic term in Q, denoting $\bar{S} = S^{-1}$, $\bar{Z} = Q^TZQ$ and $\bar{R} = R^{-1}$ and choosing $Y = \varepsilon A_1^T[0\ I]P$, where ε is a tuning scalar, we obtain, similarly to [13], the following.

Lemma 3: System (14) is asymptotically stabilized by the controller of (15), for all the delays $\tau(t)$ that satisfy (2), if for some scalar ε there exist $2n \times 2n$ matrices: Q of the form (16a) and $\bar{Z}$, a $m \times n$ matrix $\bar{Y}$ and $n \times n$ matrices $\bar{S}$ and $\bar{R}$ that satisfy the LMIs shown in (17a)–(18) at the bottom of the page. The feedback gain K is then given by $K = \bar{Y}Q_1^{-1}$. Denoting

$$\hat{A} = \bar{A} + \begin{bmatrix} 0 \\ I \end{bmatrix} BK[I \quad 0]$$

we obtain similarly to the derivation of Lemma 2 that (17a) is satisfied iff there exist $2n \times 2n$ matrices: Q of the form (16), G, Z and H and $n \times n$ matrices $\bar{S}$ and $\bar{R}$ that satisfy

$$\begin{bmatrix} \bar{\Gamma} & \left[\begin{smallmatrix} Q^T - G^T - \hat{A}H^T \\ 0 \end{smallmatrix}\right] \\ * & -H - H^T \end{bmatrix} < 0. \tag{19}$$

Denoting

$$\hat{\Delta} = \begin{bmatrix} I_{2n} & 0 & 0 \\ 0 & I_{2n} & 0 \\ -\hat{A}^T & 0 & I_{2n} \end{bmatrix}$$

we multiply both sides of (19) by $\hat{\Delta}^T$ and $\hat{\Delta}$, from the left and the right, respectively and obtain that (19), and therefore also (17a), are equivalent to the LMI shown in (20) at the bottom of the page. The latter inequality is nonlinear since products of K by G and H are obtained. In order to obtain a linear inequality with a tuning parameter ε we seek G and H of a special structure. We choose

$$G = \begin{bmatrix} G_1 & 0 \\ G_2 & G_3 \end{bmatrix} \quad H = \begin{bmatrix} \alpha G_1 & 0 \\ H_2 & H_3 \end{bmatrix} \tag{21a,b}$$

where α is some positive scalar. Substituting the latter in (17) and denoting $\bar{Y} = KG_1$, we obtain the following.

Lemma 4: System (14) is asymptotically stabilized by the controller of (15), for all the delays $\tau(t)$ that satisfy (2), if for some positive scalars ε and α there exist $2n \times 2n$ matrices: Q, G and H, of the form (16a), (21a) and (21b), respectively, and $\bar{Z}$, a $m \times n$ matrix $\bar{Y}$ and $m \times n$ matrices $\bar{S}$ and $\bar{R}$ that satisfy (17b) and the LMI shown in (22a) and (22b) at the bottom of the next page. The state-feedback gain is then obtained by

$$K = \bar{Y}G_1^{-1}. \tag{23}$$

Note that the inverse of G_1 exists because of (21b) and the positive definiteness of $H + H^T$ in (22a).

The last lemma was derived for a single system (14) with known parameters. As such, it provides no improvement to the result of Lemma 3. The merit of (22) lies in its ability to cope with uncertain systems.

In the case where the matrices of the system are not exactly known, we denote

$$\bar{\Omega} = [A_0 \quad A_1 \quad B]$$

and assume that $\bar{\Omega} \in \mathcal{C}o\{\Omega_j, j = 1, \ldots \bar{N}\}$, namely

$$\bar{\Omega} = \sum_{j=1}^{\bar{N}} f_j\bar{\Omega}_j \quad \text{for some} \quad 0 \le f_j \le 1,\ \sum_{j=1}^{\bar{N}} f_j = 1 \tag{24}$$

where the $\bar{N}$ vertices of the polytope are described by

$$\bar{\Omega}_j = \left[A_0^{(j)} \quad A_1^{(j)} \quad B^{(j)}\right].$$

$$\bar{\Gamma} = \begin{bmatrix} Q^T\bar{A}^T + \bar{A}Q + \left[\begin{smallmatrix} I \\ 0 \end{smallmatrix}\right]\bar{Y}^TB^T[0 \quad I] + \left[\begin{smallmatrix} 0 \\ I \end{smallmatrix}\right]B\bar{Y}[I \quad 0] + h\bar{Z} & \left[\begin{smallmatrix} 0 \\ (\varepsilon-1)A_1 \end{smallmatrix}\right]\bar{S} & Q^T \\ * & -\bar{S}(1-d) & 0 \\ * & * & -\mathrm{diag}\{\bar{S},\ h^{-1}\bar{R}\} \end{bmatrix} < 0 \tag{17a}$$

and

$$\begin{bmatrix} \bar{R} & \bar{R}\,[0 \quad \varepsilon A_1^T\,] \\ * & \bar{Z} \end{bmatrix} > 0 \tag{17b}$$

where

$$\bar{A} = \begin{bmatrix} 0 & I \\ A_0 + \varepsilon A_1 & -I \end{bmatrix}. \tag{18}$$

$$\begin{bmatrix} G^T\hat{A}^T + \hat{A}G + h\bar{Z} & \left[\begin{smallmatrix} 0 \\ (\varepsilon-1)A_1 \end{smallmatrix}\right]\bar{S} & Q^T & Q^T - G^T + \hat{A}H \\ * & -\bar{S}(1-d) & 0 & 0 \\ * & * & -\mathrm{diag}\{\bar{S},\ h^{-1}\bar{R}\} & 0 \\ * & * & * & -H^T - H \end{bmatrix} < 0. \tag{20}$$

In order to guarantee the stability of (14) over the entire polytope, one can use the result of Lemma 3 by applying the same matrix Q for all the points in the polytope and solving (17a,b) for the $\bar{N}$ vertices only. A quadratic stabilizability type criterion is then obtained which may be quite conservative. In order to allow a Q that depends on the parameters of the system, we apply Lemma 4 and define the structures

$$G_j = \begin{bmatrix} G_1 & 0 \\ G_2^{(j)} & G_3^{(j)} \end{bmatrix} \quad H_j = \begin{bmatrix} \alpha G_1 & 0 \\ H_2^{(j)} & H_3^{(j)} \end{bmatrix}. \tag{25a,b}$$

We obtain the following.

Theorem 2: Consider the system (14) and assume that its parameters lie in the polytope $\bar{\Omega}$. The system is asymptotically stabilized, over the entire polytope, by the controller of (15), for all the delays $\tau(t)$ that satisfy (2a,b), if for some tuning positive scalar parameters ε and α there exist $2n \times 2n$ matrices: Q_j, G_j and H_j, of the form (16a), and (25a,b), respectively, and $\bar{Z}_j$, $j = 1, \ldots, \bar{N}$, a $m \times n$ matrix $\bar{Y}$ and $n \times n$ matrices $\bar{R}$ and $\bar{S}$ that satisfy the LMIs (26a–c) shown at the bottom of the page. The state-feedback gain that stabilizes the system over the uncertainty polytope is then given by (23).

Remark 2: In the case where A_1 is known, the matrices R and S in (26a, b) can also be parameter dependent.

Remark 3: In the case where a state-feedback controller exists that quadratically stabilizes the system, it can be obtained as a special case of the last theorem by taking $G_j = Q_j = Q$, $H_j \to 0$, and $\alpha \to 0$.

Remark 4: Inequalities (26a, b) can be combined into one inequality, thus, reducing the computational burden involved in solving these inequalities. It follows from (26b) that $\bar{Z}_j > \varepsilon^2 [0 \; A_1^{(j)T}]^T \; \bar{R} [0 \; A_1^{(j)T}]$. Since $\bar{Z}_j$ appears only in M_j in (26a), one may replace the smallest possible $\bar{Z}_j$ by $\varepsilon^2 [0 \; A_1^{(j)T}]^T \; \bar{R} [0 \; A_1^{(j)T}] + \bar{\varepsilon} I_{2n}$ where $\bar{\varepsilon}$ is a small positive scalar that tends to zero.

Similarly to the derivation of Corollary 2 we obtain from Theorem 2 the following delay-independent result.

Corollary 4: Consider (14) and assume that its parameters lie in the polytope $\bar{\Omega}$. The system is asymptotically stabilized, over the entire polytope, by the controller of (15), for all the delays $\tau(t)$ that satisfy (2b), if for some tuning positive scalar parameter α there exist $2n \times 2n$ matrices: Q_j, G_j and H_j, of the form (16a), (25a,b), respectively, $j = 1, \ldots, \bar{N}$, a $m \times n$ matrix $\bar{Y}$ and a $n \times n$ matrix $\bar{S}$, that satisfy the LMIs shown in (27a, b) at the bottom of the page. The state-feedback gain is then given by (23).

Remark 5: The corresponding delay-dependent/rate-independent result follows from Theorem 2 by taking $\varepsilon = 1$ and deleting rows and columns containing $\bar{S}$.

IV. Examples

A. Stability

The time delayed uncertain system of (14) is considered where

$$A_0 = \begin{bmatrix} 0 & -0.12 + 12\rho \\ 1 & -0.465 - \rho \end{bmatrix} \quad \text{and} \quad A_1 = \begin{bmatrix} -0.1 & -0.35 \\ 0 & 0.3 \end{bmatrix}$$

$$\begin{bmatrix} M & \left[\begin{smallmatrix} 0 \\ (\varepsilon-1)A_1 \end{smallmatrix}\right]\bar{S} & Q^T & Q^T - G^T + \bar{A}H + \left[\begin{smallmatrix} 0 \\ \alpha I \end{smallmatrix}\right] B\bar{Y}[I \quad 0] \\ * & -\bar{S}(1-d) & 0 & 0 \\ * & * & -\mathrm{diag}\{\bar{S},\ h^{-1}\bar{R}\} & 0 \\ * & * & * & -H^T - H \end{bmatrix} < 0 \tag{22a}$$

where

$$M = G^T \bar{A}^T + \bar{A}G + \begin{bmatrix} I \\ 0 \end{bmatrix} \bar{Y}^T B^T [0 \quad I] + \begin{bmatrix} 0 \\ I \end{bmatrix} B\bar{Y}[I \quad 0] + h\bar{Z}. \tag{22b}$$

$$\begin{bmatrix} M_j & \left[\begin{smallmatrix} 0 \\ (\varepsilon-1)A_1^{(j)} \end{smallmatrix}\right]\bar{S} & Q_j^T & Q_j^T - G_j^T + \bar{A}^{(j)}H_j + \left[\begin{smallmatrix} 0 \\ \alpha B^{(j)} \end{smallmatrix}\right][\bar{Y} \quad 0] \\ * & -\bar{S}(1-d) & 0 & 0 \\ * & * & -\mathrm{diag}\{\bar{S}, h^{-1}\bar{R}\} & 0 \\ * & * & * & -H_j^T - H_j \end{bmatrix} < 0 \tag{26a}$$

$$\begin{bmatrix} \bar{R} & \bar{R}\left[0 \quad \varepsilon A_1^{(j)T}\right] \\ * & \bar{Z}_j \end{bmatrix} > 0, \qquad j = 1, \ldots, \bar{N} \tag{26b}$$

where

$$M_j = G_j^T \bar{A}^{(j)T} + \bar{A}^{(j)} G_j + \begin{bmatrix} \bar{Y}^T \\ 0 \end{bmatrix} \left[0 \quad B^{(j)T}\right] + \begin{bmatrix} 0 \\ B^{(j)} \end{bmatrix} [\bar{Y} \quad 0] + h\bar{Z}_j. \tag{26c}$$

$$\begin{bmatrix} \bar{M}_j & \left[\begin{smallmatrix} 0 \\ A_1^{(j)} \end{smallmatrix}\right]\bar{S} & Q_j^T \left[\begin{smallmatrix} I \\ 0 \end{smallmatrix}\right] & Q_j^T - G_j^T + \bar{A}^{(j)}H_j + \left[\begin{smallmatrix} 0 \\ \alpha B^{(j)} \end{smallmatrix}\right][\bar{Y} \quad 0] \\ * & -\bar{S}(1-d) & 0 & 0 \\ * & * & -\bar{S} & 0 \\ * & * & * & -H_j^T - H_j \end{bmatrix} < 0$$

$$j = 1, \ldots, \bar{N} \tag{27a}$$

where

$$\bar{M}_j = G_j^T \bar{A}^{(j)T} + \bar{A}^{(j)} G_j + \begin{bmatrix} \bar{Y}^T \\ 0 \end{bmatrix} \left[0 \quad B^{(j)T}\right] + \begin{bmatrix} 0 \\ B^{(j)} \end{bmatrix} [\bar{Y} \quad 0]. \tag{27b}$$

and where $|\rho| \leq 0.035$. The time delay is assumed to be constant, namely $d = 0$ in (2b) and bounded by h as in (2a). It is first realized that the uncertainty polytope possesses two vertices that correspond to $\rho = 0.035$ and $\rho = -0.035$. Applying the methods of [6], [9], and [11] for $h > 0$, no quadratic stability could be verified by LMI Toolbox of Matlab. Applying the method of [1], which implies the quadratic stability of the system, a maximum value of $h = 0.782$ is obtained. Solving the LMIs of Theorem 1 asymptotic stability is guaranteed for all delays that are less or equal to $h = 0.863$. Note that this is the maximum value of h that was obtained in one of the vertices by [6], [11], and [13] (by [9] the corresponding value is $h = 0.454$).

B. Stabilization

The time delayed uncertain system of (14) is considered where

$$A_0 = 0 \quad A_1 = \begin{bmatrix} 0 & 1 \\ -1 + g_1 & -.5 \end{bmatrix} \quad B = \begin{bmatrix} -1 + g_2 \\ 1 \end{bmatrix}$$

where $|g_1| \leq 0.53$ and $|g_2| \leq 1.7$. The time delay is assumed to be constant, namely $d = 0$ in (2b) and bounded by h as in (2a). The uncertainty in this system is described by a polytope with four vertices. It is verified that for $h = 0$ the system is not quadratically stabilizable. Therefore, by the delay-dependent methods of [12], [13] and of Lemma 3 the quadratically stabilizing controller can not be found for $h > 0$. By Theorem 2 it is found, using $\varepsilon = 1$ and $\alpha = 0.1$, that a solution to the stabilization problem over the polytope is achieved for $h \leq 0.2$. The state-feedback gain that solves the problem for $h = 0.2$ is $K = [0.0329 \;\; -0.1016]$. The result was obtained for $\varepsilon = 1$ and thus the derived state-feedback design stabilizes the system also when the time-delay is fastly varying but bounded by 0.2. The asymptotic stability of the resulting closed-loop is checked and verified by applying Lemma 1.

Remark 6: Recently, an example of chatter during milling has been introduced [6]. Applying a stability criterion, which depends on the cutting stiffness as a parameter, the maximum delay between successive passes of the blades has been determined which still guarantees the stability of the cutting system for all possible angular position of the blades. It has been found there that for stiffness k less than 0.27 the system is asymptotically stable for all delays. Applying Corollary 2 of the present note the delay-independent stability of the system is guaranteed for k less than 0.44. A maximum allowable delay of 0.53 is found using Theorem 1 for $k = 0.45$. The corresponding allowable delay in [6] is less and is equal to 0.41 in one of the vertices.

V. Conclusion

Efficient parameter dependent stability and stabilizability sufficient conditions are obtained for continuous time linear systems with time-varying delays. These conditions are delay dependent and they are expressed in terms of LMIs that are tuned by a scalar parameter. The delay-independent conditions are obtained from the former conditions by taking the tuning parameter to be zero. The efficiency of the proposed method is demonstrated by simple examples which show the advantage of the derived condition in comparison to other methods.

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An LPD Approach to Robust H_2 and H_∞ Static Output-Feedback Design

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***Abstract*—A linear parameter dependent approach for designing a constant output feedback controller for a linear time-invariant system with uncertain parameters that achieves a minimum bound on either the H_2 or the H_∞ performance level is introduced. Assuming that the uncertain parameters reside in a given polytope a parameter dependent Lyapunov function is described which enables the derivation of the required constant gain via a solution of a set of linear matrix inequalities that correspond to the vertices of the uncertainty polytope.**

***Index Terms*—H_∞ control, linear parameter dependent (LPD) design, robust static output control.**

I. Introduction

The static output-feedback problem has attracted the attention of many in the past [1]–[5]. The main advantage of the static output-feedback is the simplicity of its implementation and the ability it provides

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