Received: 2 May 2016 | Revised: 6 September 2016 | Accepted: 15 May 2017

DOI: 10.1002/rnc.3862

SHORT COMMUNICATION

WILEY

Decentralized networked control of discrete-time systems with local networks

Dror Freirich | Emilia Fridman

School of Electrical Engineering, Tel-Aviv University, Tel-Aviv, 69978, Israel

Correspondence
Emilia Fridman, School of Electrical Engineering, Tel-Aviv University, Tel-Aviv 69978, Israel.
Email: emilia@eng.tau.ac.il

Funding information
Israel Science Foundation, Grant/Award Number: 1128/14 ; Kamea Fund of Israel

Summary

This paper considers discrete-time large-scale networked control systems with multiple local communication networks connecting sensors, controllers, and actuators. The local networks operate asynchronously and independently of each other in the presence of variable sampling intervals, transmission delays, and scheduling protocols (from sensors to controllers). The time-delay approach that was recently suggested to decentralized stabilization of large-scale networked systems in the continuous time is extended to H_∞ decentralized control in the discrete time. An appropriate Lyapunov-Krasovskii method is presented that leads to efficient LMI conditions for the exponential stability and l_2-gain analysis of the closed loop large-scale system. Differences from the continuous-time results are discussed. A numerical example of decentralized control of 2 coupled cart-pendulum systems illustrates the efficiency of the results.

KEYWORDS

decentralized control, large-scale systems, l_2-gain, networked control systems, scheduling protocols, time delay

1 | INTRODUCTION

Networked control systems (NCS) are systems with spatially distributed sensors, actuators, and controller nodes, which exchange data over a communication data channel.[1] In the case of large-scale or spatially distributed systems, local groups of sensors and actuators may be located far apart, leading to a *decentralized networked control design*. Here, the design may be based on local controllers and networks.

Decentralized control of continuous-time large-scale systems with independent networks was initiated in Heemels et al[2,3] via the hybrid-system approach. To manage with large communication delays (that may be larger than sampling intervals) in the presence of scheduling protocols from sensors to actuators, the *time-delay* approach to continuous-time decentralized NCS was suggested in Freirich and Fridman.[4] The results of Freirich and Fridman[4] were confined to stability analysis.

In the discrete-time, stability analysis of large-scale nonlinear systems with application to a power system network was developed in Gielen and Lazar.[5] Self-triggered control of discrete-time systems was studied in Zhang et al.[6] The time-delay approach to discrete-time NCS was developed in Liu and Fridman,[7] where the scheduling protocols from sensors to controllers were considered. The results were confined to the case of 1-plant and 2-sensor nodes.

The goal of this paper is to extend the time-delay approach to decentralized stabilization and H_∞ control of large-scale discrete-time systems with multiple local communication networks connecting sensors, controllers, and actuators. Local networks operate asynchronously and independently of each other in the presence of variable sampling intervals, transmission delays and scheduling protocols from sensors to controllers.

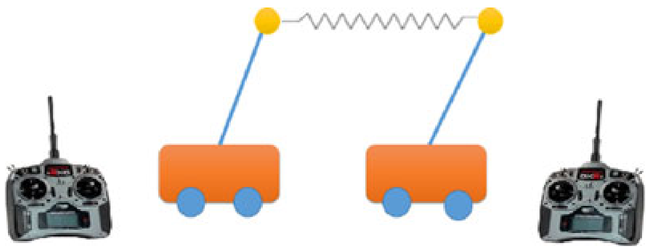

FIGURE 1 Two coupled inverted pendulums [Colour figure can be viewed at wileyonlinelibrary.com]

The main challenge is in the stability and performance analysis of the closed-loop discrete-time NCS under try-once-discard (TOD) scheduling protocol.[8] Note that as in the continuous-time case,[4,9] the closed-loop system in the discrete-time is modeled as a hybrid large-scale system with delays. However, its structure is different, where the same state equation appears also in the reset system. This leads to a different from the continuous-time case Lyapunov technique (see Liu and Fridman[7] for the case of one plant). We present a novel Lyapunov functional candidate explaining that its continuous counterpart leads to simplified LMIs comparatively to Freirich and Fridman.[4] Under the round robin (RR) protocol, the closed-loop system is modeled as a system with multiple delays, and the stability conditions naturally extend the conditions of Freirich and Fridman[4] to the discrete-time case. Extension to l_2-gain analysis considers both, independent and coupled disturbances. A numerical example of decentralized control of 2 coupled inverted pendulums with local networks (see Figure 1) illustrates the efficiency of the results.

Notation: Throughout the paper the superscript "T" stands for matrix transposition, $\mathcal{R}^n$ denotes the n dimensional Euclidean space with vector norm $|\cdot|$. $l_2(\mathcal{R}^n)$ is the space of sequences $x(t) \in \mathcal{R}^n, \quad t \in [0, \infty)$ with $\sum_{t=0}^{\infty} \|x(t)\|^2 < \infty$. $\mathcal{R}^{n\times m}$ is the set of all $n \times m$ real matrices, and the notation $P > 0$, for $P \in \mathcal{R}^{n\times n}$ means that P is symmetric and positive definite. The symmetric elements of the symmetric matrix will be denoted by $*$. $\mathcal{Z}_+$ and $\mathcal{N}$ denote the set of nonnegative integers and positive integers, respectively. *TOD* stands for *try-once-discard* and *LKF* stands for Lyapunov-Krasovskii functional. *MATI* and *MAD* denote maximum allowable transmission interval and maximum allowable delay, respectively. By δ_{ij}, we denote the Kronecker delta meaning $\delta_{nm} = 0, n \neq m$ and $\delta_{nn} = 1$ $(n, m \in \mathcal{N})$. Throughout this paper the letter "j" (subscript or superscript) will stand for a subsystem index, while subscript "i" will stand for sensor index.

2 | PROBLEM FORMULATION AND PRELIMINARIES

2.1 | Problem formulation

Consider the large-scale system with M interconnected subsystems (see Figure 2).

$$\begin{aligned} x_j(t+1) &= A_j x_j(t) + B_j u_j(t) + D_j w(t) + \sum\nolimits_{l\neq j} F_j^l x_l(t), \\ y_{ij}(t) &= C_{ij} x_j(t) \in \mathcal{R}^{n_i^j}, \quad i = 1 \ldots N_j \quad t \in \mathcal{Z}_+, \end{aligned} \tag{1}$$

where $x_j(t) \in \mathcal{R}^{n^j}$ is the state, $w(t) \in l_2(\mathcal{R}^{n_w})$ is a perturbation, and F_j^l are the coupling matrices. The jth subsystem has N_j local sensors and a local controller, and $y_j(t) = [y_{1j}^T(t) \ \ldots \ y_{N_j j}^T(t)]^T \in \mathcal{R}^{n_y^j}$ is the local measurement vector.

The jth subsystem is assumed to have an independent sequence of sampling instants

$$0 = s_0^j < s_1^j < \cdots < s_k^j < \cdots, \quad \lim_{k\to\infty} s_k^j = \infty$$

with bounded sampling intervals $1 \leqslant s_{k+1}^j - s_k^j \leqslant MATI_j$. At each sampling instant s_k^j, one of the sensors i_k^* is being chosen by a *scheduling protocol*, and its output $y_{i_k^* j}(s_k^j)$ is being transmitted via a local sensor network to the local controller node. We assume that the local network is independent of other subsystems' networks.

Suppose that data loss is not possible and that the transmission of the information over the networks from sensors to actuators (through controller) is subject to a variable round-trip delay η_k^j. Then $t_k^j = s_k^j + \eta_k^j$ is the updating time instant of the subsystem input $u_j(t)$. Communication delay is assumed to be bounded $\eta_k^j \in [\eta_m^j, \eta_M^j]$, where $\eta_M^j \equiv MAD_j$. We will assume that an old sample cannot get to the same destination (same controller or same actuator) after the most recent one. Suppose that the controllers and the actuators are event driven (in the sense that they update their outputs as soon as they receive a new sample).

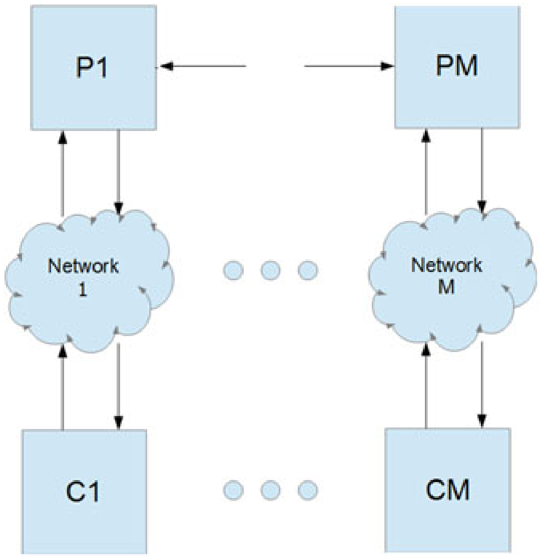

FIGURE 2 Decentralized networked control system with plants P1, … , PM and controllers C1, … , CM [Colour figure can be viewed at wileyonlinelibrary.com]

Similar to Freirich and Fridman[4], we assume the following:

A1 There exist M gain matrices $K_j = \begin{bmatrix} K_{1j} & \dots & K_{N_j j} \end{bmatrix}$, $K_{ij} \in \mathcal{R}^{m^j \times n_i^j}$ such that the matrices $A_j + B_j K_j C_j$ are Schur, where $C_j = \begin{bmatrix} C_{1j}^T & \cdots & C_{N_j j}^T \end{bmatrix}^T$.

Denoted by

$$\hat{y}_j\left(s_k^j\right) = \left[\hat{y}_{1j}^T\left(s_k^j\right) \ \dots \ \hat{y}_{N_j j}^T\left(s_k^j\right)\right]^T \in \mathcal{R}^{n_y^j}, \tag{2}$$

the most recent output information submitted to the scheduling protocol of the jth subsystem (ie, the most recent information at the jth controller side) at the sampling instant s_k^j. Then under **A1** the static output-feedbacks are given by

$$u_j(t) = \sum_{i=1}^{N_j} K_{ij}\hat{y}_{ij}\left(s_k^j\right), \quad t_k^j \leqslant t < t_{k+1}^j. \tag{3}$$

We assume $u_j(t) = 0$ for $t < t_0^j$.

Define the initial time

$$T_0 = max_j t_0^j. \tag{4}$$

Definition 1. (Exponential stability)
A closed-loop system (1), (3) with $w(t) \equiv 0$ is *exponentially stable*(with a decay rate $\lambda \in [0, 1)$) if there exists $\beta > 0$ such that for any initial condition x_0 the following inequality holds

$$\|x(t)\|^2 \leqslant \beta \|x\|_{T_0}^2 \lambda^{t-T_0}$$

for $t > T_0$, where $x(t) = col\{x_1(t), \dots, x_M(t)\} \in \mathcal{R}^n$ and $\|x\|_{T_0}^2 \equiv \sum_{t=0}^{T_0} \|x(t)\|^2$.

In this paper, sufficient conditions are given for exponential stability of the unperturbed closed-loop system, as well as for l_2-gain analysis of the perturbed one.

3 | STABILITY OF DECENTRALIZED NETWORKED CONTROL UNDER TOD

Consider the system (1), (6) with M locally controlled subsystems, where the jth subsystem is under TOD scheduling. In this section, we study the case where $w(t) \equiv 0$.

Following Liu et al,[9] denote the error states $\{e_{ij}, i = 1 \dots N_j\}$ by

$$e_{ij}(t) = \hat{y}_{ij}\left(s^j_{k-1}\right) - y_{ij}\left(s^j_k\right), \quad t^j_k \leqslant t < t^j_{k+1}, \tag{5}$$

and the error-state vectors

$$e_j(t) = col\{e_{1j}(t) \dots e_{N_j j}(t)\}, \quad j = 1...M.$$

Using this notation the feedback (3) can be presented as

$$u_j(t) = \sum_{i=1}^{N_j} K_{ij}C_{ij}x_j(s_k) + \sum_{\substack{i=1 \\ i \neq i^*_k}}^{N_j} K_{ij}e_{ij}(t), \quad t^j_k \leqslant t < t^j_{k+1}. \tag{6}$$

Denote

$$e_{ij}[k] \equiv e_{ij}\left(t^j_k\right), \quad k = 0, 1, \dots \tag{7}$$

for the error-states in order to better express their piecewise-constant nature.

In TOD scheduling, the transmitted measurement sent at each instance is chosen according to (weighted) error discriminant function

$$i^{j*}_k = argmax_i \quad e^T_{ij}[k]Q_{ij}e_{ij}[k], \tag{8}$$

with some weight matrices Q_{ij}. Conditions for choosing weight matrices will be given below. For simplicity, we will omit the subsystem index j in i^{j*}_k throughout the rest of the paper. The error state vector is given by

$$e_{ij}\left(t^j_{k+1}\right) = \begin{cases} -C_{ij}\left[x_j\left(s^j_{k+1}\right) - x_j\left(s^j_k\right)\right], & i = i^*_k \\ e_{ij}\left(t^j_k\right) - C_{ij}\left[x_j\left(s^j_{k+1}\right) - x_j\left(s^j_k\right)\right], & else \end{cases}, \tag{9}$$

with $e_{ij}\left(t^j_0\right) = -C_{ij}x\left(s^j_0\right)$.

The jth subsystem closed loop is then given by

$$\begin{aligned} &x_j(t+1) = A_j x(t) + A_{1j}x_j(t - \tau_j(t)) + \sum_{i \neq i^*_{k,j}} B_{ij}e_{ij}(t) + \sum\nolimits_{l \neq j} F^l_j x_l(t), \\ &e_j(t+1) = e_j(t), \quad t^j_k \leqslant t \leqslant t^j_{k+1} - 2, \end{aligned} \tag{10}$$

with the reset equations (ie, equations at reset times $t = t^j_{k+1} - 1$)

$$\begin{aligned} x_j\left(t^j_{k+1}\right) =& A_j x_j\left(t^j_{k+1} - 1\right) + A_{1j}x_j\left(s^j_k\right) + \sum\nolimits_{i \neq i^*_k} B_{ij}e_{ij}\left(t^j_k\right) + \sum\nolimits_{l \neq j} F^l_j x_l\left(t^j_{k+1} - 1\right), \\ e_{ij}\left(t^j_{k+1}\right) =& (1 - \delta_{i^*_k})e_{ij}\left(t^j_k\right) - C_{ij}[x_j\left(s^j_{k+1}\right) - x_j\left(s^j_k\right)], \quad k \in \mathcal{Z}_+, \\ e_{ij}\left(t^j_0\right) =& - C_{ij}x_j\left(s^j_0\right), \end{aligned} \tag{11}$$

where $A_{1j} = B_j K_j C_j$ and $\tau_j(t) = t - s^j_k$ is the subsystem delay. We have

$$\begin{aligned} &\tau_j(t) \leqslant t^j_{k+1} - 1 - s^j_k \leqslant MAD_j + MATI_j - 1 \equiv \tau^j_M \\ &\tau_j(t) \geqslant t^j_k - s^j_k \geqslant \eta^j_m \geqslant 0. \end{aligned} \tag{12}$$

Remark 1. Note that in the discrete time domain, the equation for $x_j(t)$ at reset times $t = t^j_k - 1$ is given by the same difference equation as for $t \in [t^j_{k-1}, t^j_k - 2]$. This is different from the continuous-time case, where $x_j\left(t^{j-}_k\right) = x_j\left(t^j_k\right)$.

We define a functional $V_j(t)$ for the jth subsystem. Consider the following LKF:

$$V_j(t) = x^T_j(t)P_j x_j(t) + \tilde{V}_j(t) + V_Q(t,k) + V_G(t,k), \quad t^j_k \leqslant t \leqslant t^j_{k+1} - 1, \tag{13}$$

where $\lambda \in (0,1)$ and

$$\begin{aligned}
V_Q(t,k) &= \lambda^{t-t_k^j}\sum_{i=1}^{N_j} e_{ij}^T(t)Q_{ij}e_{ij}(t) - \sum_{i\neq i_k^*}\left(t-t_k^j\right)e_{ij}^T(t)U_{ij}e_{ij}(t),\\
V_G(t,k) &= \sum_{s=s_k^j}^{t-1}\sum_{i=1}^{N_j}\lambda^{t-s-1}z_j^T(s)C_{ij}^TG_{ij}C_{ij}z_j(s),\\
\tilde V_j(t) &= V_{0j}(t)+V_{1j}(t),\\
V_{0j}(t) &= \sum_{s=t-\eta_m^j}^{t-1}\lambda^{t-s-1}x_j^T(s)S_{0j}x_j(s)+\eta_m^j\sum_{\theta=1}^{\eta_m}\sum_{s=t-\theta}^{t-1}\lambda^{t-s-1}z_j^T(s)R_{0j}z_j(s),\\
V_{1j}(t) &= \sum_{s=t-\tau_M^j}^{t-\eta_m-1}\lambda^{t-s-1}x_j^T(s)S_{1j}x_j(s)+h_j\sum_{\theta=\eta_m^j+1}^{\tau_M^j}\sum_{s=t-\theta}^{t-1}\lambda^{t-s-1}z_j^T(s)R_{1j}z_j(s),\\
z_j(t) &= x_j(t+1)-x_j(t),\quad h_j=(\tau_M^j-\eta_m^j).
\end{aligned}\tag{14}$$

Here, $x_j^T(t)P_jx_j(t)+\tilde V_j(t)$ is a standard LKF for the stability of systems with interval delays,[10] $V_Q(t,k)$ is constructed to deal with the error states that appear in (10). The term $V_G(t,k)$ allows to compensate the delayed terms $C_{ij}[x_j\left(s_{k+1}^j\right)-x_j\left(s_k^j\right)]$ in the reset (11).

Remark 2. In the continuous-time case, an LKF candidate V_j for the stability analysis of the resulting jth hybrid subsystem under TOD protocol was suggested in Liu and Fridman[9] (for the uncoupled case, where $F_j^l=0, l\neq j$) and was modified in Freirich and Fridman[4] for coupled (large-scale) systems leading to simplified stability conditions. Thus, the stability of uncoupled jth subsystem was guaranteed in Freirich and Fridman[4] if the following derivative condition holds $\dot V_j+2\alpha V_j\leqslant 0$ for some $\alpha>0$ along the continuous-time dynamics, and if V_j does not grow in the reset times. Differently from this, in the discrete-time case the stability of the uncoupled jth subsystem can be guaranteed if $V_j(t+1)-\lambda V_j(t)\leqslant 0$ with some $\lambda\in(0,1)$ for all t (including the reset times $t=t_k^j-1$).[7] Note that the Lyapunov-Krasovskii method of Liu and Fridman[7] was confined to the case of 2 sensor nodes $N_j=2$, and only partial x-stability was established. As in the continuous-time case,[4] an extension of results to large-scale systems is not a trivial result that will be presented in Proposition 1 and Theorem 1.

Remark 3. In the discrete-time, by introducing the term $V_Q(t,k)$ of (14), we avoid the positive terms containing $\|\sqrt{Q_{i_k^*}}e_{i_k^*}\|^2$ in the upper bound on $V(t+1)-\lambda V(t)$ for $t\neq t_k-1$. The Q_{ij}-terms of V_Q in (14) is a discrete-time and modified version of the corresponding terms in Freirich and Fridman[4]

$$V_{conQ}(t)=\sum\nolimits_{i=1}^{N_j}e_{ij}^T(t)Q_{i,j}e_{ij}(t)-2\alpha\left(t-t_k^j\right)e_{i_k^*j}^T(t)Q_{i_k^*,j}e_{i_k^*j}(t),$$

where the negative term $-2\alpha(t_k^j-t)e_{i_k^*j}^T(t)Q_{i_k^*,j}e_{i_k^*j}(t)$ allows to compensate a positive term of the form $2\alpha e_{i_k^*j}^T(t)Q_{i_k^*,j}e_{i_k^*j}(t)$ that arises in $\dot V_j(t)+2\alpha V_j(t)$.

The continuous-time counterpart of Q-terms given by (14) has a form

$$V_{cQ}(t)=e^{-2\alpha(t-t_k)}\sum\nolimits_{i=1}^{N_j}e_{ij}^T(t)Q_{i,j}e_{ij}(t).\tag{15}$$

The term (15) improves and simplifies the condition for $\dot V+2\alpha V$ of Freirich and Fridman[4] based on V_{conQ}:

$$\begin{aligned}
&\dot V_{cQ}(t)+2\alpha V_{cQ}(t)=0\leqslant,\\
&\dot V_{conQ}(t)+2\alpha V_{conQ}(t)=\sum\nolimits_{i\neq i_k^*}e_{ij}^T(t)Q_{i,j}e_{ij}(t).
\end{aligned}$$

However, in the jump condition $t=t_{k+1}^j$, these functionals lead to different results:

$$V_{cQ}\left(t_{k+1}^{j+}\right)-V_{cQ}\left(t_{k+1}^{j-}\right)=\sum\nolimits_{i=1}^{N_j}e_{ij}^T\left(t_{k+1}^j\right)Q_{i,j}e_{ij}\left(t_{k+1}^j\right)-e^{-2\alpha\left(t_{k+1}^j-t_k^j\right)}\sum\nolimits_{i=1}^{N_j}e_{ij}^T\left(t_k^j\right)Q_{i,j}e_{ij}\left(t_k^j\right),$$

whereas

$$V_{conQ}\left(t_{k+1}^{+j}\right) - V_{conQ}\left(t_{k-1}^{+j}\right) = \sum\nolimits_{i=1}^{N_j} e_{ij}^T\left(t_{k+1}^j\right) Q_{i,j} e_{ij}\left(t_{k+1}^j\right) - \sum\nolimits_{i=1}^{N_j} e_{ij}^T\left(t_k^j\right) Q_{i,j} e_{ij}\left(t_k^j\right) \\ +2\alpha(t_{k+1}^j - t_k^j) e_{ij}^T\left(t_k^j\right) Q_{i,j} e_{ij}\left(t_k^j\right)_{|i=i_k^*}.$$

Hence, V_{cQ} may lead to different from Freirich and Fridman[4] (complementary) results. Note that in the example from Freirich and Fridman[4], V_{cQ} does not change the numerical result, but $Q_{i,j}$ terms are eliminated from part of LMIs (simplifying them).

Proposition 1. *Consider the jth closed-loop subsystem (10), (11). Given parameters* $0 < \lambda < 1$ *and* $\varepsilon \leqslant 1 - \lambda(\sum_{j\neq l}\varepsilon_{jl} \leqslant \varepsilon, l = 1 \dots M)$ *and matrices* $P_l > 0, \quad l = 1 \dots M$, *let there exist matrices* $S_{0j}, R_{0j}, S_{1j}, R_{1j}, Q_{ij}, U_{ij}, G_{ij} > 0$, *and* W_j *that satisfy LMIs,*

$$\Omega_j = \begin{bmatrix} R_{1j} & W_j \\ * & R_{1j} \end{bmatrix} \geqslant 0 \tag{16}$$

and

$$\begin{aligned} &\Sigma_i^j - \tilde{\Upsilon}_i^j + \tilde{E}_{ij2}^T G_j \tilde{E}_{ij2} < 0, \\ &\Psi_i^j + [I \quad 0]^T U_{ij} [I \quad 0] < 0, \\ &i = 1, \dots, N_j, \end{aligned} \tag{17}$$

where

$$\begin{aligned} &\Sigma_i^j = \tilde{E}_{ij1}^T P_j \tilde{E}_{ij1} + \tilde{E}_{ij2}^T H_j \tilde{E}_{ij2} - \tilde{E}_{ij3}^T \lambda^{\tau_M^j} \Omega_j \tilde{E}_{ij3} - \tilde{E}_{ij4}^T \lambda^{\eta_m^j} R_{0j} \tilde{E}_{ij4} + \tilde{E}_{ij5}, \\ &h_j = \tau_M^j - \eta_m^j, \quad \mathcal{F}^j = [F_j^1 \cdots F_j^{l\neq j} \cdots F_j^M], H_j = \eta_m^{j2} R_{0j} + h_j^2 R_{1j}, \quad G_j = \sum_{i=1}^{N_j} C_{ij}^T G_{ij} C_{ij}, \\ &E_{ij1} = \left[A_j \quad \begin{bmatrix}0 & 1 & 0\end{bmatrix} \otimes A_{1j} \quad \Theta_{ij}\right], \quad E_{ij2} = \left[\left(A_j - I_{n^j}\right) \begin{bmatrix}0 & 1 & 0\end{bmatrix} \otimes A_{1j} \quad \Theta_{ij}\right], \quad \Theta_{ij} = \left[B_{1j} \cdots B_{r\neq ij} \cdots B_{N_j j}\right], \\ &E_{ij3} = \left[0_{2n^j \times n^j} \begin{bmatrix} 1-1 & 0 \\ 0 & 1-1 \end{bmatrix} \otimes I_{n^j} \; 0_{2n^j \times (n_y^j - n_i^j)}\right], \quad E_{ij4} = [I_{n^j} - I_{n^j} \quad 0], \\ &E_{i5} = diag\{S_{0j} - \lambda P_j, -(S_{0j} - S_{1j})\lambda^{\eta_m^j}, 0_{n^j}, -S_{1j}\lambda^{\tau_M^j}, 0\}, \\ &\Psi_i^j = \begin{bmatrix} \lambda h_j U_{ij} + \left(1 - \frac{N_j \lambda^{h_j+1}}{N_j - 1}\right) Q_{ij} & Q_{ij} \\ * & Q_{ij} - \frac{\lambda^{\tau_M^j}}{h_j+1} G_{ij} \end{bmatrix}, \\ &\Upsilon_i^j = diag\{0_{4n^j}, U_{1j}, \cdots, U_{r\neq i,j}, \cdots, U_{N_j j}\}, \\ &\tilde{E}_{ij1} = [E_{ij1} \quad \mathcal{F}^j], \quad \tilde{E}_{ij2} = [E_{ij2} \quad \mathcal{F}^j], \quad \tilde{E}_{ij4} = [E_{ij4} \quad 0 \cdot \mathcal{F}^j], \quad \tilde{E}_{ij3} = \left[E_{ij3} \quad 0 \cdot \begin{bmatrix} \mathcal{F}^j \\ \mathcal{F}^j \end{bmatrix}\right], \\ &\tilde{E}_{ij5} = diag\{E_{ij5}, -\mathcal{P}^j\}, \quad \mathcal{P}^j = diag\{\varepsilon_{jl} P_l, l \neq j\}, \quad \tilde{\Upsilon}_i^j = diag\{\Upsilon_i^j, 0_{n-n_j}\}. \end{aligned} \tag{18}$$

Then V_j *given by (13) satisfies the following inequalities along (10), (11)* (*with some* $\beta_j > 0$)*:*

$$V_j(t) \geqslant \beta_j(\|x_j(t)\|^2 + \|e_j(t)\|^2), \quad t \geqslant t_0^j, \tag{19}$$

$$V_j(t+1) - \lambda V_j(t) \leqslant \sum\nolimits_{l\neq j} \varepsilon_{jl} x_l^T(t) P_l x_l(t), \quad t \geqslant t_0^j. \tag{20}$$

Proof. We prove first the positivity condition 19. Note that

$$t_{k+1}^j - t_k^j - 1 = (t_{k+1}^j - 1 - s_k^j) - \eta_k^j \leqslant \tau_M^j - \eta_m^j = h_j. \tag{21}$$

We have

$$
\begin{aligned}
V_Q(t,k) &\geqslant \lambda^{t^j_{k+1}-1-t^j_k}\sum_{i=1}^{N_j} e_{ij}^T(t)Q_{ij}e_{ij}(t) - (t^j_{k+1}-1-t^j_k)\sum_{i\neq i_k^*} e_{ij}^T(t)U_{ij}e_{ij}(t)\\
&\geqslant \lambda^{h_j}\sum_{i=1}^{N_j} e_{ij}^T(t)Q_{ij}e_{ij}(t) - h_j\sum_{i\neq i_k^*} e_{ij}^T(t)U_{ij}e_{ij}(t),\\
&t^j_k \leqslant t \leqslant t^j_{k+1}-1.
\end{aligned}
\tag{22}
$$

From the second inequality (17) it follows that

$$
\lambda h_j U_{ij} < \left[\frac{N_j\lambda^{h_j+1}}{N_j-1}-1\right]Q_{ij} < \lambda^{h_j+1}\left[\frac{N_j}{N_j-1}-1\right]Q_{ij} = \frac{\lambda^{h_j+1}}{N_j-1}Q_{ij} \leqslant \lambda^{h_j+1}Q_{ij}.
$$

The latter inequality together with (22) imply

$$
V_Q(t,k) \geqslant \beta_j\|e_j(t)\|^2, \quad t^j_k \leqslant t \leqslant t^j_{k+1}-1
$$

for some $\beta_j > 0$, yielding (19).

We prove next inequality (20). We find

$$
\begin{aligned}
V_{0j}(t+1)-\lambda V_{0j}(t) &\leqslant \eta_m^{j2} z_j^T(t)R_{0j}z_j(t)\\
&\quad + x_j^T(t)S_{0j}x_j(t) - \lambda^{\eta_m^j}x_j^T\left(t-\eta_m^j\right)S_{0j}x_j\left(t-\eta_m^j\right)\\
&\quad - \eta_m^j\sum_{\theta=1}^{\eta_m^j}\lambda^{\theta}z_j^T(t-\theta)R_{0j}z_j(t-\theta),\ V_{1j}(t+1)-\lambda V_{1j}(t)\\
&\leqslant h_j^2 z_j^T(t)R_{1j}z_j(t) + \lambda^{\eta_m^j}x_j^T\left(t-\eta_m^j\right)S_{1j}x_j\left(t-\eta_m^j\right) - \lambda^{\tau_M^j}x_j^T\left(t-\tau_M^j\right)S_{1j}x_j\left(t-\tau_M^j\right)\\
&\quad - h_j\sum_{\theta=\eta_m^j+1}^{\tau_M^j}\lambda^{\theta}z_j^T(t-\theta)R_{1j}z_j(t-\theta).
\end{aligned}
\tag{23}
$$

By using Jensen's inequality, we obtain

$$
\eta_m^j\sum_{\theta=1}^{\eta_m^j}\lambda^{\theta}z_j^T(t-\theta)R_{0j}z_j(t-\theta) \geqslant \lambda^{\eta_m^j}\|\sqrt{R_{0j}}\left(x_j(t)-x_j\left(t-\eta_m^j\right)\right)\|^2 = \begin{bmatrix}\eta_j(t)\\ \xi_{ij}(t)\\ X_j^c(t)\end{bmatrix}^T \tilde{E}_{ij4}^T\lambda^{\eta_m^j}R_{0j}\tilde{E}_{ij4}\begin{bmatrix}\eta_j(t)\\ \xi_{ij}(t)\\ X_j^c(t)\end{bmatrix}.
\tag{24}
$$

Under (16) the following holds[11]:

$$
\begin{aligned}
h_j\sum_{\theta=\eta_m^j+1}^{\tau_M^j}\lambda^{\theta}z_j^T(t-\theta)R_{1j}z_j(t-\theta) &\geqslant \lambda^{\tau_M^j}\begin{bmatrix} x_j\left(t-\eta_m^j\right)-x_j(t-\tau_j(t))\\ x_j(t-\tau_j(t))-x_j\left(t-\tau_M^j\right)\end{bmatrix}^T\Omega_j\begin{bmatrix} x_j\left(t-\eta_m^j\right)-x_j(t-\tau_j(t))\\ x_j(t-\tau_j(t))-x_j\left(t-\tau_M^j\right)\end{bmatrix}\\
&= \begin{bmatrix}\eta_j(t)\\ \xi_{ij}(t)\\ X_j^c(t)\end{bmatrix}^T \tilde{E}_{ij3}^T\lambda^{\tau_M^j}\Omega_j\tilde{E}_{ij3}\begin{bmatrix}\eta_j(t)\\ \xi_{ij}(t)\\ X_j^c(t)\end{bmatrix}.
\end{aligned}
\tag{25}
$$

Denote

$$
\eta_j(t) = col\left\{x_j(t), x_j\left(t-\eta_m^j\right), x_j(t-\tau_j(t)), x_j\left(t-\tau_M^j\right)\right\}
$$

$$
X_j^c(t) = col\{x_l(t), l\neq j\}, \quad \xi_{ij}(t) = col\{e_{rj}(t), r\neq i\},
$$

$$
\sigma_{ij}[k] = C_{ij}\left(x_j\left(s^j_{k+1}\right)-x_j\left(s^j_k\right)\right).
$$

Employing the relations

$$z_j(t) = \tilde{E}_{i_k^* j2} \begin{bmatrix} \eta_j(t) \\ \xi_{i_k^* j}(t) \\ X_j^c(t) \end{bmatrix}, \quad x_j(t+1) = \tilde{E}_{i_k^* j1} \begin{bmatrix} \eta_j(t) \\ \xi_{i_k^* j}(t) \\ X_j^c(t) \end{bmatrix} \tag{26}$$

and

$$\sum\nolimits_{l \neq j} \varepsilon_{jl} x_l^T(t) P_l x_l(t) = X_j^{cT}(t) \mathcal{P}^j X_j^c(t), \tag{27}$$

from (23)-(25), we obtain

$$\begin{aligned} &x_j^T(t+1) P_j x_j(t+1) + \tilde{V}_j(t+1) - \lambda(x_j(t)^T P_j x_j(t) + \tilde{V}_j(t)) \\ &-\sum\nolimits_{l \neq j} \varepsilon_{jl} x_l^T(t) P_l x_l(t) \leqslant \begin{bmatrix} \eta_j^T(t) & \xi_{i_k^* j}^T(t) & X_j^{cT}(t) \end{bmatrix} \Sigma_{i_k^*}^j \begin{bmatrix} \eta_j(t) \\ \xi_{i_k^* j}(t) \\ X_j^c(t) \end{bmatrix}, \\ &t_k \leqslant t \leqslant t_{k+1} - 1. \end{aligned} \tag{28}$$

Consider $t_k^j \leqslant t \leqslant t_{k+1}^j - 2$. From (21) and definition of $V_Q(t)$,

$$\begin{aligned} V_Q(t+1,k) - \lambda V_Q(t,k) &= \left(-1 + (\lambda - 1)\left(t - t_k^j\right)\right) \sum_{i \neq i_k^*} \|\sqrt{U_{ij}} e_{ij}(t)\|^2 \\ &\leqslant -\sum_{i \neq i_k^*} \|\sqrt{U_{ij}} e_{ij}(t)\|^2 \leqslant -\begin{bmatrix} \eta_j^T(t) & \xi_{i_k^* j}^T(t) \end{bmatrix} \Upsilon_{i_k^*}^j \begin{bmatrix} \eta_j(t) \\ \xi_{i_k^* j}(t) \end{bmatrix} = -\begin{bmatrix} \eta_j(t) \\ \xi_{i_k^* j}(t) \\ X_j^c(t) \end{bmatrix}^T \tilde{\Upsilon}_{i_k^*}^j \begin{bmatrix} \eta_j(t) \\ \xi_{i_k^* j}(t) \\ X_j^c(t) \end{bmatrix}. \end{aligned} \tag{29}$$

Note that the terms $|\sqrt{Q_{ij}} e_{ij}(t)|^2$ vanish in Equation 29 due to multiplication of them by $\lambda^{t - t_k^j}$.
Therefore,

$$V_j(t+1) - \lambda V_j(t) - \sum\nolimits_{l \neq j} \varepsilon_{jl} x_l^T(t) P_l x_l(t) \leqslant \begin{bmatrix} \eta_j^T(t) & \xi_{i_k^* j}^T(t) & X_j^{cT}(t) \end{bmatrix} \left(\Sigma_{i_k^*}^j - \tilde{\Upsilon}_{i_k^*}^j\right) \begin{bmatrix} \eta_j(t) \\ \xi_{i_k^* j}(t) \\ X_j^c(t) \end{bmatrix} + z_j^T(t) G_j z_j(t), \quad t_k^j \leqslant t \leqslant t_{k+1}^j - 2.$$

Substituting into the latter inequality 26, we see that the first inequality 17 implies the inequality 20 for $t_k^j \leqslant t \leqslant t_{k+1}^j - 2$.
Consider now the case of reset times, where $t = t_{k+1}^j - 1$. Using notation 7, we have

$$\begin{aligned} &V_Q\left(t_{k+1}^j, k+1\right) - \lambda V_Q\left(t_{k+1}^j - 1, k\right) \\ &\leqslant \sum_{i=1}^{N_j} \left[e_{ij}^T[k+1] Q_{ij} e_{ij}[k+1] - e_{ij}^T[k] \lambda^{t_{k+1}^j - t_k^j} Q_{ij} e_{ij}[k]\right] + \lambda h_j \sum_{i \neq i_k^*} e_{ij}^T[k] U_{ij} e_{ij}[k]. \end{aligned} \tag{30}$$

Exploiting the reset Equation 9, we find

$$\sum_{i=1}^{N_j} e_{ij}^T[k+1] Q_{ij} e_{ij}[k+1] = \sigma_{i_k^* j}^T[k] Q_{i_k^* j} \sigma_{i_k^* j}^T[k] + \sum_{i \neq i_k^*} (e_{ij}[k] + \sigma_{ij}[k])^T Q_{ij} (e_{ij}[k] + \sigma_{ij}[k]). \tag{31}$$

Since under TOD

$$|\sqrt{Q_{i_k^* j}} e_{i_k^* j}(t)|^2 \geqslant \frac{1}{N_j - 1} \sum\nolimits_{i \neq i_k^*} |\sqrt{Q_{i,j}} e_{ij}(t)|^2, \tag{32}$$

we arrive at

$$\begin{aligned} \sum_{i=1}^{N_j} e_{ij}^T[k] Q_{ij} e_{ij}[k] &= e_{i_k^* j}^T[k] Q_{i_k^* j} e_{i_k^* j}^T[k] + \sum_{i \neq i_k^*} e_{ij}[k]^T Q_{ij} e_{ij}[k] \\ &\geqslant \sum_{i \neq i_k^*} e_{ij}[k]^T \left(1 + \frac{1}{N_j - 1}\right) Q_{ij} e_{ij}[k]. \end{aligned} \tag{33}$$

Substituting Equations 31 and 33 into Equation 30 and using Equation 21, we obtain

$$\begin{aligned} &V_Q\left(t^j_{k+1},k+1\right)-\lambda V_Q\left(t^j_{k+1}-1,k\right)\\ &\leqslant \sum_{i\neq i^*_k} e^T_{ij}[k]\left[1-\lambda^{h_j+1}(1+\frac{1}{N_j-1})\right]Q_{ij}e_{ij}[k]+\lambda h_j\sum_{i\neq i^*_k} e^T_{ij}[k]U_{ij}e_{ij}[k]+2\sum_{i\neq i^*_k} e^T_{ij}[k]Q_{ij}\sigma_{ij}[k]\\ &\quad+\sum_{i\neq i^*_k}\sigma_{ij}[k]^TQ_{ij}\sigma_{ij}[k]+\sigma^T_{i^*_kj}[k]Q_{i^*_kj}\sigma^T_{i^*_kj}[k]. \end{aligned} \tag{34}$$

For V_G, consider 2 cases. In the case, where $t=t^j_{k+1}-1\leqslant s^j_k$, we have $t^j_k=s^j_k,\quad t^j_{k+1}=s^j_{k+1}$. Thus,

$$\begin{aligned} V_G\left(t^j_{k+1},k+1\right)-\lambda V_G\left(t^j_{k+1}-1,k\right)=0&\\ &=\sum_{i=1}^{N_j} z^T_j\left(t^j_{k+1}-1\right)C^T_{ij}G_{ij}C_{ij}z_j\left(t^j_{k+1}-1\right)-\sum_{i=1}^{N_j} z^T_j\left(s^j_k\right)C^T_{ij}G_{ij}C_{ij}z_j\left(s^j_k\right)\\ &\leqslant \sum_{i=1}^{N_j} z^T_j(t)C^T_{ij}G_{ij}C_{ij}z_j(t)-\frac{\lambda^{\tau^j_M}}{h_j+1}\sum_{i=1}^{N_j}\sigma^T_{ij}[k]G_{ij}\sigma_{ij}[k]. \end{aligned} \tag{35}$$

Here, the latter inequality is due to Jensen's inequality.

Otherwise, $t^j_{k+1}-1>s^j_k$, and similarly to Equation 35, we have

$$\begin{aligned} &V_G\left(t^j_{k+1},k+1\right)-\lambda V_G\left(t^j_{k+1}-1,k\right)\\ &\qquad\leqslant \sum_{i=1}^{N_j} z^T_j\left(t^j_{k+1}-1\right)C^T_{ij}G_{ij}C_{ij}z_j\left(t^j_{k+1}-1\right)-\sum_{s=s^j_k}^{s^j_{k+1}-1}\sum_{i=1}^{N_j}\lambda^{t^j_{k+1}-s-1}z^T_j(s)C^T_{ij}G_{ij}C_{ij}z_j(s)\\ &\qquad\leqslant \sum_{i=1}^{N_j} z^T_j(t)C^T_{ij}G_{ij}C_{ij}z_j(t)-\frac{\lambda^{\tau^j_M}}{h_j+1}\sum_{i=1}^{N_j}\sigma^T_{ij}[k]G_{ij}\sigma_{ij}[k]. \end{aligned} \tag{36}$$

Summarizing, we obtain

$$\begin{aligned} &V_Q\left(t^j_{k+1},k+1\right)+V_G\left(t^j_{k+1},k+1\right)-\lambda(V_Q\left(t^j_{k+1}-1,k\right)+V_G\left(t^j_{k+1}-1,k\right))\\ &\leqslant z_j(t)^TG_jz_j(t)+\sum_{i\neq i^*_k}\begin{bmatrix}e_{ij}[k]\\ \sigma_{ij}[k]\end{bmatrix}^T\Psi^j_i\begin{bmatrix}e_{ij}[k]\\ \sigma_{ij}[k]\end{bmatrix}+\begin{bmatrix}e_{i^*_kj}[k]\\ \sigma_{i^*_kj}[k]\end{bmatrix}^T\left(Q_{i^*_kj}-\frac{\lambda^{\tau^j_M}}{h_j+1}G_{i^*_kj}\right)\begin{bmatrix}e_{i^*_kj}[k]\\ \sigma_{i^*_kj}[k]\end{bmatrix}, \end{aligned} \tag{37}$$

and together with Equation 28

$$\begin{aligned} &V_j\left(t^j_{k+1}\right)-\lambda V_j\left(t^j_{k+1}-1\right)-\textstyle\sum_{l\neq j}\varepsilon_{jl}\|\sqrt{P_l}x_l(t_{k+1}-1)\|^2\\ &\leqslant\left[\eta^T_j(t)\quad \xi^T_{i^*_kj}(t)\quad X^{cT}_j(t)\right]\Sigma^j_{i^*_k}\begin{bmatrix}\eta_j(t)\\ \xi_{i^*_kj}(t)\\ X^c_j(t)\end{bmatrix}+z_j(t)^TG_jz_j(t)\\ &+\sum_{i\neq i^*_k}\begin{bmatrix}e_{ij}[k]\\ \sigma_{ij}[k]\end{bmatrix}^T\Psi^j_i\begin{bmatrix}e_{ij}[k]\\ \sigma_{ij}[k]\end{bmatrix}+\begin{bmatrix}e_{i^*_kj}[k]\\ \sigma_{i^*_kj}[k]\end{bmatrix}^T\left(Q_{i^*_kj}-\frac{\lambda^{\tau^j_M}}{h_j+1}G_{i^*_kj}\right)\begin{bmatrix}e_{i^*_kj}[k]\\ \sigma_{i^*_kj}[k]\end{bmatrix}. \end{aligned} \tag{38}$$

From the second inequality (Equation 17), it follows that

$$Q_{i^*_kj}-\frac{\lambda^{\tau^j_M}}{h_j+1}G_{i^*_kj}<0.$$

Therefore,

$$V_j\left(t^j_{k+1}\right) - \lambda V_j\left(t^j_{k+1}-1\right) - \sum\nolimits_{l\neq j}\varepsilon_{jl}\|\sqrt{P_l}x_l(t_{k+1}-1)\|^2 \leqslant \begin{bmatrix}\eta_j^T(t) & \xi^T_{i_k^*j}(t) & X_j^{cT}(t)\end{bmatrix}\Sigma^j_{i_k^*}\begin{bmatrix}\eta_j(t)\\ \xi_{i_k^*j}(t)\\ X_j^c(t)\end{bmatrix} + z_j(t)^TG_jz_j(t) + \sum_{i\neq i_k^*}\begin{bmatrix}e_{ij}[k]\\ \sigma_{ij}[k]\end{bmatrix}^T\Psi_i^j\begin{bmatrix}e_{ij}[k]\\ \sigma_{ij}[k]\end{bmatrix}. \tag{39}$$

Note that

$$\begin{bmatrix}\eta_j^T(t) & \xi^T_{i_k^*j}(t) & X_j^{cT}(t)\end{bmatrix}\tilde{\Upsilon}^j_{i_k^*}\begin{bmatrix}\eta_j(t)\\ \xi_{i_k^*j}(t)\\ X_j^c(t)\end{bmatrix} - \sum_{i\neq i_k^*}e_{ij}^T[k]U_{ij}e_{ij}[k] = 0. \tag{40}$$

So, by adding Equation 40 to Equation 39, we arrive at

$$\begin{aligned}V_j\left(t^j_{k+1}\right) - \lambda V_j\left(t^j_{k+1}-1\right) - \sum\nolimits_{l\neq j}\varepsilon_{jl}\|\sqrt{P_l}x_l(t_{k+1}-1)\|^2 &\leqslant \begin{bmatrix}\eta_j^T(t) & \xi^T_{i_k^*j}(t) & X_j^{cT}(t)\end{bmatrix}\left(\Sigma^j_{i_k^*}-\tilde{\Upsilon}^j_{i_k^*}\right)\begin{bmatrix}\eta_j(t)\\ \xi_{i_k^*j}(t)\\ X_j^c(t)\end{bmatrix} + z_j(t)^TG_jz_j(t)\\ &+ \sum_{i\neq i_k^*}\begin{bmatrix}e_{ij}[k]\\ \sigma_{ij}[k]\end{bmatrix}^T\Psi_i^j\begin{bmatrix}e_{ij}[k]\\ \sigma_{ij}[k]\end{bmatrix} + \sum_{i\neq i_k^*}e_{ij}^T[k]U_{ij}e_{ij}[k].\end{aligned} \tag{41}$$

Hence, taking into account Equation 26, the inequality 17 implies Equation 20 for $t = t^j_{k+1} - 1$. □

We are in a position to formulate the stability result.

Theorem 1. *Consider the large-scale system* (*Equations 10 and 11*), $j = 1 \dots M$. *Given tuning parameters* $0 < \lambda < 1$ *and* $\varepsilon \leqslant 1-\lambda(\sum_{j\neq l}\varepsilon_{jl} \leqslant \varepsilon, l = 1 \dots M)$, *let there exist matrices* $P_j, S_{0j}, R_{0j}, S_{1j}, R_{1j}, Q_{ij}, U_{ij}, G_{ij} > 0$, *and* W_j *that satisfy Equations 16 and 17 for all* $j = 1 \dots M$. *Then the system* (*Equations 10 and 11*) *is exponentially stable with a decay rate* $\lambda + \varepsilon$.

Proof. From Proposition 1, Equations 16 and 17 imply the inequality 19 for $j = 1 \dots M$. Then there exist positive $\{\beta_j\}_{j=1}^M$ such that

$$V_j(t) \geqslant x_j^T(t)P_jx_j(t) + \beta_j\|e_j(t)\|^2. \tag{42}$$

Consider now the *LKF*

$$V(t) = \sum_{j=1}^M V_j(t). \tag{43}$$

Since $x_j^T(t)P_jx_j(t) \leqslant V_j(t)$ and $\sum_{j\neq l}\varepsilon_{jl} \leqslant \varepsilon, l = 1 \dots M$, summing Equation 20 over $j = 1 \dots M$ for $t \geqslant T_0$, we obtain

$$V(t+1) - (\lambda+\varepsilon)V(t) \leqslant 0$$

that implies

$$V(t) \quad \leqslant \quad (\lambda+\varepsilon)^{t-T_0}V(T_0), \quad t > T_0. \tag{44}$$

Inequalities (Equation 20) for $j = 1...M$ yield

$$V_j(T_0) \leqslant \lambda^{T_0-t_0^j}V_j\left(t_0^j\right) + \sum_{s=t_0^j}^{T_0}\sum_{l\neq j}\lambda^{T_0-s}\varepsilon_{jl}x_l^T(t)P_lx_l(t).$$

Then there exist some constants C_j such that

$$V_j(T_0) \leqslant V_j\left(t_0^j\right) + \frac{1}{2}C_j\|x\|_{T_0}^2 \leqslant C_j\|x\|_{T_0}^2, \forall j.$$

Hence, for some constant C, we have

$$V(T_0) \leqslant C\|x\|_{T_0}^2,$$

implying (for some $\beta^- > 0$)

$$\beta^-(\|x(t)\|^2 + \sum_{j=1}^{M} \|e_j(t)\|^2) \leqslant V(t) \leqslant C\|x\|_{T_0}^2(\lambda+\varepsilon)^{(t-T_0)}$$

with $\lambda + \varepsilon < 1$. □

Remark 4. Note that differently from Freirich and Fridman,[4] the LMIs (Equation 17) are not affine in the system matrices. This is due to substitution of Equation 26 into the positive terms $z_j^T(t)H_jz_j^T(t), z_j^T(t)G_jz_j^T(t)$, and $x_j^T(t+1)P_jx_j(t+1)$ of $V_j(t+1) - \lambda V_j(t)$. However, by applying Schur complements to these terms, one can arrive at equivalent to Equation 17 LMIs that are affine in the systems matrices. Hence, the result of Theorem 1 is applicable to the case of system matrices from an uncertain time-varying polytope, where one can solve the LMIs (Equation 17) simultaneously for all the vertices of the polytope applying the same decision matrices.

4 | L_2-GAIN ANALYSIS OF THE LARGE-SCALE SYSTEM

Consider now the large-scale system (Equation 1) under the controller (Equation 6), where all subsystems are orchestrated by TOD protocol. The closed loop is then given by

$$x_j(t+1) = A_jx(t) + A_{1j}x_j(t-\tau_j(t)) + \sum_{i\neq i_{k,j}^*} B_{ij}e_{ij}[k] + \sum\nolimits_{l\neq j} F_j^l x_l(t) + D_jw_j(t), t_k^j \leqslant t \leqslant t_{k+1}^j - 1, \quad j = 1 \dots M, \tag{45}$$

where $w(t) = col\{w_j(t), j = 1 \dots M\} \in l_2([T_0,\infty), R^{n_w})$ is a disturbance. Let $Z(t) = col\{Z_j(t), j = 1 \dots M\}$ be the controlled output, where

$$Z_j(t) = \Lambda_{1j}x_j(t) + \Lambda_{2j}u_j(t) \in \mathcal{R}^{n_z}.$$

Given $\gamma > 0$, define the following performance index

$$J = \sum_{t=t_0}^{\infty}\left[\sum_j Z_j^T(t)Z_j(t) - \gamma^2 w^T(t)w(t)\right].$$

Definition 2. The closed-loop large-scale system (Equation 45) with initial time $T_0 = max_j\{t_0^j\}$ is said to have an induced l_2-gain less than γ, if

$$J < V(T_0), \quad \forall w \in l_2(\mathcal{R}^{n_w}), \quad w \neq 0$$

holds for V given by Equation 43.

It is well known (see, eg, Fridman[10]) that $J < V(T_0)$ if for some $\alpha > 0$, the following condition holds along Equation 45

$$V(t+1) - V(t) + Z^T(t)Z(t) - \gamma^2 w^T(t)w(t) \leqslant -\alpha[\|x(t)\|^2 + \|w(t)\|^2], \quad t \geqslant T_0. \tag{46}$$

Lemma 1. *Consider $\{V_j(t)\}_{j=1}^M$ given by Equation 13. Let there exist positive tuning parameters $\varepsilon < 1$ and $\{\varepsilon_{jl}\}_{j,l=1}^M$ such that $\sum_{j\neq l}\varepsilon_{jl} \leqslant \varepsilon, l = 1 \dots M$ and*

$$V_j(t+1) - (1-\varepsilon)V_j(t) + Z_j^T(t)Z_j(t) - \gamma^2 w_j^T(t)w_j(t) \leqslant \sum\nolimits_{l\neq j}\varepsilon_{jl}x_l^T(t)P_l x_l(t) \quad (47)$$

for $t \geqslant T_0$, then Equation 46 holds along Equation 45 with $V(t) = \sum_{j=1}^M V_j(t)$.

Taking into account Equation 42, the result of Lemma 1 follows from summation in Equation 47. By extending derivations of Proposition 1, we arrive at the following LMI conditions that guarantee Equation 47.

Theorem 2. *Given $\gamma > 0$, consider the closed-loop system (Equation 45) with a tuning parameter $0 < \varepsilon < 1(\sum_{j\neq l}\varepsilon_{jl} \leqslant \varepsilon, l = 1 \dots M)$. Let there exist matrices $P_j, S_{0j}, R_{0j}, S_{1j}, R_{1j}, Q_{ij}, U_{ij}, G_{ij} > 0$ and W_j such that Equation 16 holds for all $j = 1 \dots M$, and*

$$\begin{gathered}\Sigma_i^j - \tilde{\Upsilon}_i^j + \tilde{E}_{ij2}^T G_j \tilde{E}_{ij2} + \zeta_{ij}^T\zeta_{ij} < 0, \Psi_i^j + [I \quad 0]^T U_{ij}[I \quad 0] < 0, \\ i = 1, \dots, N_j, \quad j = 1, \dots, M,\end{gathered} \quad (48)$$

where $\lambda = 1 - \varepsilon$ and

$$\begin{aligned}
\Sigma_i^j &= \tilde{E}_{ij1}^T P_j \tilde{E}_{ij1} + \tilde{E}_{ij2}^T H_j \tilde{E}_{ij2} - \tilde{E}_{ij3}^T \lambda^{\tau_M}\Omega_j \tilde{E}_{ij3} - \tilde{E}_{ij4}^T \lambda^{\eta_m^j} R_{0j}\tilde{E}_{ij4} + \tilde{E}_{ij5}, \\
h_j &= \tau_M^j - \eta_m^j, \quad \mathcal{F}^j = [F_j^1 \cdots F_j^{l\neq j} \cdots F_j^M], \\
\tilde{E}_{ij1} &= [E_{ij1} \quad \mathcal{F}^j \quad D_j], \quad \tilde{E}_{ij2} = [E_{ij2} \quad \mathcal{F}^j \quad D_j], \\
\tilde{E}_{ij4} &= [E_{ij4} \quad 0\cdot\mathcal{F}^j \quad 0\cdot D_j], \quad \tilde{E}_{ij3} = \left[E_{ij3} \quad 0\cdot\begin{bmatrix}\mathcal{F}^j & D_j \\ \mathcal{F}^j & D_j\end{bmatrix}\right], \\
\tilde{E}_{ij5} &= diag\{E_{ij5}, -\mathcal{P}^j, -\gamma^2 I_{n_w^j}\}, \quad \mathcal{P}^j = diag\{\varepsilon_{jl}P_{l\neq j}\}, \\
\tilde{\Upsilon}_i^j &= diag\{\Upsilon_i^j, 0_{n-n_j}, 0_{n_w^j}\}, \quad \mathcal{K}_i^j = row\{K_{rj}, r \neq i\} \\
\zeta_{ij} &= [\Lambda_{1j} \quad [0 \quad 1 \quad 0]\otimes\Lambda_{2j}K_jC_j \quad \Lambda_{2j}\mathcal{K}_i^j \quad 0]
\end{aligned} \quad (49)$$

with notations given by Equation 18. Then the large-scale system (Equation 45) has an induced l_2-gain less than γ, where $V(t) = \sum_j V_j(t)$ and $V_j(t)$ are given by Equation 13.

Remark 5. In the general case where disturbance is common for all subsystems (ie, $w_j(t) \equiv w(t), j = 1 \dots M$), we modify condition 47 as

$$V_j(t+1) - (1-\varepsilon)V_j(t) + Z_j^T(t)Z_j(t) - w^T(t)\Gamma_j w(t) \leqslant \sum\nolimits_{l\neq j}\varepsilon_{jl}x_l^T(t)P_l x_l(t),$$

where $n_w \times n_w$ positive definite matrices Γ_j are subject to $\gamma^2 I_{n_w} \geqslant \sum_j \Gamma_j$. Then Theorem 2 still holds if for every j, $\gamma^2 I_{n_w^j}$ in Equation 49 is replaced by Γ_j. Here, $\{\Gamma_j\}_{j=1}^M$ are additional decision variables of LMIs. The resulting bound γ_{Rem} on the l_2-gain is given by $\gamma_{Rem}^2 = \gamma_{Rem}^2(\Gamma_j) = \gamma^2$. Simpler LMIs, where $\gamma^2 I_{n_w^j}$ in Equation 49 is replaced by (common for all j)$\gamma^2 I_{n_w}/M$ may lead to a larger bound $\gamma_{Rem}^2 = \gamma_{Rem}^2(\gamma) = \gamma^2$ (as shown in the example below).

5 | ABOUT DECENTRALIZED CONTROL UNDER RR PROTOCOL

Consider next stabilization of Equation 1 ($w_j = 0$), where RR scheduling protocol orchestrates the transmitted measurements sent at each instance from sensors to a controller for every subsystem. Under RR scheduling, transmitted sensor measurement is chosen in a periodic manner. In this case the control law (6) can be presented as

$$u_j(t) = \sum_{i=1}^{N_j} K_{ij}C_{ij}x_j(t - \tau_j(t)),$$

where $\tau_j(t)$ are piecewise-linear delays with known bounds

$$\eta_m^j \leqslant \tau_j(t) \leqslant N_j \cdot MATI_j + \eta_M^j - 1 = \tau_M^j.$$

The closed-loop system is then given by

$$x_j(t+1) = A_j x_j(t) + \sum_{i=1}^{N_j} A_{ij} x_j(t-\tau_j(t)) + \sum_{l\neq j} F_j^l x_l(t), \quad t \geqslant t_{N_j}^j, \quad j = 1, \dots M, \tag{50}$$

where $A_{ij} = B_j K_{ij} C_{ij}$. We use Lyapunov functional of the form

$$V_j(t) = x_j^T(t) P_j x_j(t) + V_0^j(t) + \sum_{i=1}^{N_j} V_{1i}^j(t),$$

where

$$\begin{aligned} V_0^j(t) = & \sum_{s=t-\eta_m^j}^{t-1} \lambda^{t-s-1} x_j^T(s) S_{0j} x_j(s) + \sum_{s=t-\tau_M^j}^{t-\eta_m^j-1} \lambda^{t-s-1} x_j^T(s) S_{1j} x_j(s) \\ & + \eta_m^j \sum_{\theta=1}^{\eta_m^j} \sum_{s=t-\theta}^{t-1} \lambda^{t-s-1} z_j^T(s) R_{0j} z_j(s), \\ V_{1i}^j(t) = & (\tau_M^j - \eta_m^j) \sum_{\theta=\eta_m^j+1}^{\tau_M^j} \sum_{s=t-\theta}^{t-1} \lambda^{t-s-1} z_j^T(s) R_{1ij} z_j(s). \end{aligned}$$

By using arguments of Theorem 1, we arrive at the following result:

Theorem 3. *Consider the closed-loop large-scale system (Equation 50). Given tuning parameters* $0 < \lambda < 1$ *and* $\varepsilon \leqslant 1 - \lambda$ *(*$\sum_{j\neq l} \varepsilon_{jl} \leqslant \varepsilon, l = 1 \dots M$*), let there exist matrices* $P_j, S_{0j}, R_{0j}, S_{1j}, R_{1ij} > 0$*, and* $W_{ij} (i = 1 \dots N_j, j = 1 \dots M)$ *that satisfy*

$$\begin{aligned} \begin{bmatrix} R_{1ij} & W_{ij} \\ * & R_{1ij} \end{bmatrix} \geqslant 0, \quad i = 1, \dots, N_j, \\ \Sigma^j < 0, \quad j = 1, \dots, M, \end{aligned} \tag{51}$$

where

$$\begin{aligned} \Sigma^j &= E_{1j}^T P_j E_{1j} + E_{2j}^T H_j E_{2j} - \sum_{i=1}^{N_j} E_{3ij}^T \lambda^{\tau_M^j} \Omega_{ij} E_{3ij} - E_{4j}^T \lambda^{\eta_m^j} R_{0j} E_{4j} + E_{5j}, \\ \mathcal{F}^j &= [F_j^1 \cdots F_j^{l\neq j} \cdots F_j^M], \quad H_j = \eta_m^{j\,2} R_{0j} + (\tau_M^j - \eta_m^j)^2 \sum_{i=1}^{N_j} R_{1ij}, \\ E_{1j} &= \left[\left[100 \right] \otimes A_j A_{ij} \cdots A_{N_j j} \mathcal{F}^j \right], E_{2j} = \left[\left[100 \right] \otimes \ (A_j - I_{n^j}) \, A_{ij} \cdots A_{N_j j} \mathcal{F}^j \right], \\ E_{3ij} &= \left[\begin{bmatrix} 0 \\ 0 \end{bmatrix} \otimes A_j \quad \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \otimes I_{n^j} \quad \chi_1^i \ \dots \ \chi_{N_j}^i \quad \begin{bmatrix} 0 \\ 0 \end{bmatrix} \otimes \mathcal{F}^j \right], \quad \chi_r^i = \delta_{i,r} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \otimes I_{n^j}, \\ E_{4j} &= [I_{n^j} - I_{n^j} \quad 0], \quad E_{5j} = diag\{S_{0j} - \lambda P_j, -(S_{0j} - S_{1j}) \lambda^{\eta_m^j}, -S_{1j} \lambda^{\tau_M^j}, 0_{n_y^j}, -\mathcal{P}^j\}, \\ \mathcal{P}^j &= diag\{\varepsilon_{jl} P_l, l \neq j\}. \end{aligned}$$

Then the closed-loop system (Equation 50) is exponentially stable with a decay rate $\lambda + \varepsilon$*, where* $T_0^j = t_{N_j}^j$.

Derivation of LMIs consists of arguments similar to Proposition 1, where
$\eta_j(t) = col\{x_j(t), x_j\left(t - \eta_m^j\right) x_j\left(t - \tau_M^j\right), x_j(t - \tau_1^j(t)), \dots, x_j(t - \tau_{N_j}^j(t))\}$.

Remark 6. In the numerical example below, the results that follow from Theorem 3 (under RR protocol) are less conservative than the results based on Theorem 1. However, the improvement is achieved on the account of the numerical complexity:

- the conditions of Theorem 1 possess LMIs of $N_j n + (4N_j + 7)n^j + 4n_y^j$ total rows and $5T_{n^j} + n^{j^2} + 3\sum_{i=1}^{N_j} T_{n_i^j}$ scalar decision variables for each subsystem j, where $T_n = \frac{n^2+n}{2}$ is the nth triangular number;
- the conditions of Theorem 3 possess LMIs of $(6 + 4N_j)n^j + n$ total rows and $(4 + N_j)T_{n^j} + N_j n^{j^2}$ scalar decision variables for each subsystem j.

Comparatively to LMIs of Theorem 1, the number of lines in LMIs of Theorem 2 is enlarged by $N_j n_w^j$, whereas the number of scalar decision variables remains the same.

Remark 7. There is a trade-off between enlarging the decay rate and the values of MATI/MAD. The choice of a decay rate λ close to 1 enlarges the values of MATI/MAD. Moreover, fast convergence of subsystems without coupling, where $\lambda \ll 1$ if $F_j^l = 0$, allows stronger coupling.

6 | EXAMPLE: 2 COUPLED INVERTED PENDULUMS

Consider the example of 2 coupled inverted pendulums on carts (Figure 1), under the scenario of decentralized networked control. We discretize the continuous-time model of Borgers and Heemels[2] with the sampling time $T_s = 10^{-4}$. Here $M = 2$, $N_j = 2$ or $N_j = 4 (j = 1, 2)$. The resulting system matrices of Equations 1 and 3 are given by

$$
\begin{aligned}
A_j = A = I_4 + 10^{-4} \cdot &\begin{bmatrix} 0 & 1 & 0 & 0 \\ 2.9156 & 0 & -0.0005 & 0 \\ 0 & 0 & 0 & 1 \\ -1.6663 & 0 & 0.0002 & 0 \end{bmatrix}, \\
B_j^T = B^T = 10^{-4} \cdot &\begin{bmatrix} 0 - 0.0042 & 0 & 0.0167 \end{bmatrix}, \\
F_1^2 = F_2^1 = A_{12} = 10^{-4} \cdot &\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0.0011 & 0 & 0.0005 & 0 \\ 0 & 0 & 0 & 0 \\ -0.0003 & 0 & -0.0002 & 0 \end{bmatrix}, \\
K_1 = [k_{11} \quad k_{21} \quad k_{31} \quad k_{41}] &= [11396 \quad 7196.2 \quad 573.96 \quad 1199.0], \\
K_2 = [k_{12} \quad k_{22} \quad k_{32} \quad k_{42}] &= [29241 \quad 18135 \quad 2875.3 \quad 3693.9].
\end{aligned}
$$

In the case of $N_j = 2$, we consider

$$
C_{1j} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad C_{2j} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},
$$
$$
K_{1j} = [k_{1j} \quad k_{2j}], \quad K_{2j} = [k_{3j} \quad k_{4j}], \quad j = 1, 2.
$$

In the case of $N_j = 4$, $C_{1j}, \ldots, C_{4j}$ are the rows of I_4 and $K_{1j}, \ldots, K_{4j}$ are the entries of K_j.

We start with the stability analysis and apply conditions of Theorems 1 and 3. Choosing the tuning parameters $\lambda = e^{-3\times10^{-6}}, \varepsilon_{12} = \varepsilon_{21} = 1 - \lambda$, we achieve maximal delays preserving the stability that are given in Table 1. For constant communication delays $MAD_1 = MAD_2 = 1$, $MATI_1$, and $MATI_2$ are given in Table 2. It is seen that in this example, the results under TOD are more conservative (see Remark 7).

TABLE 1 Maximal values of τ_M^1, τ_M^2 preserving stability

$\eta_m^1 = \eta_m^2$	$N_1 = N_2$	**Theorem 1 (TOD)**		**Theorem 3 (RR)**	
		τ_M^1	τ_M^2	τ_M^1	τ_M^2
1	2	102	39	339	133
1	4	29	10	327	130

Abbreviations: RR, round robin; TOD, try-once-discard.

TABLE 2 $MATI_1, MATI_2$ preserving stability for $MAD = 1$

$\eta_m^1 = \eta_m^2$	$N_1 = N_2$	Theorem 1 (TOD) $MATI_1$	Theorem 1 (TOD) $MATI_2$	Theorem 3 (RR) $MATI_1$	Theorem 3 (RR) $MATI_2$
1	2	101	38	169	66
1	4	28	9	81	32

Abbreviations: RR, round robin; TOD, try-once-discard.

TABLE 3 Results of Freirich and Fridman[4]: max. value of τ_M^j preserving stability ($j = 1, 2$) for $\eta_m^1 = \eta_m^2 = 10^{-4}$

#Sensors(N_j)	2		4	
	τ_M^1	τ_M^2	τ_M^1	τ_M^2
Freirich and Fridman[4] (TOD)	0.0103	0.004	0.003	0.001

Abbreviation: TOD, try-once-discard.

TABLE 4 Minimal values of γ^2_{Rem}

$N_1 = N_2$	$\eta_m^1 = \eta_m^2$	τ_M^1	τ_M^2	$\gamma^2_{Rem}(\Gamma_j)$	$\gamma^2_{Rem}(\gamma)$
2	1	2	2	0.0262	0.0482
4	1	2	2	0.0461	0.0791

Table 3 illustrates that in this example under TOD, the discrete-time results with $T_s = 10^{-4}$ are similar to the continuous-time ones obtained in Freirich and Fridman.[4] The same is true for the results under RR.

Consider now the perturbed model of 2 coupled inverted pendulums given as above, where each system input is perturbed such that

$$u_j(t) = \tilde{u}_j(t) + v_j(t) + \frac{9}{10} v_{3-j}(t), \quad j = 1, 2.$$

Here, $\tilde{u}_j(t)$ are the controllers as considered above, and v_1 and $v_2 \in l_2$ are the perturbations. The system is now given by Equation 45 where $w_1^T(t) = w_2^T(t) = \begin{bmatrix} v_1^T(t) & v_2^T(t) \end{bmatrix}$ and

$$D_1 = \begin{bmatrix} B_1 & \frac{9}{10} B_1 \end{bmatrix}, \quad D_2 = \begin{bmatrix} \frac{9}{10} B_2 & B_2 \end{bmatrix}.$$

The controlled output is given by

$$Z_1(t) = [500 \ \ 0 \ \ 0 \ \ 0] \cdot x_1(t), \quad Z_2(t) = [100 \ \ 0 \ \ 0 \ \ 0] \cdot x_2(t).$$

Choosing $\lambda, \varepsilon_{12}$, and ε_{21} as above, we find bounds on l_2-gain by using Remark 5 with different Γ_j, denoted by $\gamma_{Rem}(\Gamma_j)$, and common $\gamma^2 I_{n_w}/M$, denoted by $\gamma_{Rem}(\gamma)$, (see Table 4). Here, for $N_j = 2$, the following gain matrices are obtained:

$$\Gamma_1 = diag\{0.0224, 0.0216\}, \quad \Gamma_2 = diag\{0.0038, 0.0045\},$$

and for $N_j = 4$,

$$\Gamma_1 = diag\{0.04, 0.0388\}, \quad \Gamma_2 = diag\{0.0061, 0.0073\}.$$

It is seen that the bounds on γ_{Rem} achieved by using different Γ_j are less restrictive. However, the corresponding LMIs possess more decision variables.

7 | CONCLUSIONS

In this paper, a time-delay approach has been developed for the decentralized networked control of large-scale discrete-time systems with local networks, where asynchronous variable sampling intervals, large variable communication delays and scheduling protocols from sensors to controllers were taken into account. The proposed Lyapunov-Krasovskii method leads to efficient LMI conditions for the exponential stability and l_2-gain analysis of the closed-loop large-scale system.

ACKNOWLEDGEMENTS

This work was partially supported by Israel Science Foundation (grant no 1128/14) and by Kamea Fund of Israel.

REFERENCES

1. Antsaklis P, Baillieul J. Special issue on technology of networked control systems. *Proc IEEE*. 2007;95(1):5-8.
2. Borgers D, Heemels WPMH. Stability analysis of large-scale networked control systems with local networks: a hybrid small-gain approach. *CST report*. 2014.
3. Heemels WPMH, Borgers D, van de Wouw N, Nesic D, Teel A. Stability analysis of nonlinear networked control systems with asynchronous communication: a small-gain approach. In *Proceedings of the 52th IEEE Conference on Decision and Control*. Florence; 2013:4631-4637.
4. Freirich D, Fridman E. Decentralized networked control of systems with local networks: a time-delay approach. *Autom*. 2016;69:201-209.
5. Gielen RH, Lazar M. On stability analysis methods for large-scale discrete-time systems. *Autom*. 2015;55:66-72.
6. Zhang J, Zhao D, Zheng WX. Output feedback control of discrete-time systems with self-triggered controllers. *Int J Robust Nonlinear Control*. 2015;25(18):3698-3713.
7. Liu K, Fridman E. Discrete-time network-based control under scheduling and actuator constraints. *Int J Robust Nonlinear Control*. 2015;25:1816-1830.
8. Walsh GC, Beldiman O, Bushnell LG. Asymptotic behavior of nonlinear networked control systems. *IEEE Trans Autom Control*. 2001;46(7): 1093-1097.
9. Liu K, Fridman E, Hetel L. Networked control systems in the presence of scheduling protocols and communication delays. *SIAM J Control Optim*. 2015;53(4):1768-1788.
10. Fridman E. *Introduction to time-delay systems: analysis and control*, Birkhauser, Systems and Control: Foundations and Applications. Basel; 2014.
11. Park PG, Ko J, Jeong C. Reciprocally convex approach to stability of systems with time-varying delays. *Autom*. 2011;47:235-238.

How to cite this article: Freirich D, Fridman E. Decentralized networked control of discrete-time systems with local networks. *Int J Robust Nonlinear Control*. 2018;28:365–380. https://doi.org/10.1002/rnc.3862