Automatica 100 (2019) 260–273

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Distributed stabilization of Korteweg–de Vries–Burgers equation in the presence of input delay[☆]

Wen Kang[a,b,*], Emilia Fridman[b]

[a] School of Automation and Electrical Engineering, University of Science and Technology Beijing, China
[b] Department of Electrical Engineering-Systems, Tel Aviv University, Israel

ARTICLE INFO

Article history:
Received 2 January 2018
Received in revised form 6 July 2018
Accepted 1 November 2018

Keywords:
Korteweg–de Vries–Burgers equation
Time delay
Lyapunov–Krasovskii method
Distributed control
LMI

ABSTRACT

We consider distributed stabilization of 1-D Korteweg–de Vries–Burgers (KdVB) equation in the presence of constant input delay. The delay may be uncertain, but bounded by a known upper bound. On the basis of spatially distributed (either point or averaged) measurements, we design a regionally stabilizing controller applied through distributed in space shape functions. The existing Lyapunov–Krasovskii functionals for heat equation that depend on the state derivative are not applicable to KdVB equation because of the third order partial derivative. We suggest a new Lyapunov–Krasovskii functional that leads to regional stability conditions of the closed-loop system in terms of linear matrix inequalities (LMIs). By solving these LMIs, an upper bound on the delay that preserves regional stability can be found, together with an estimate on the set of initial conditions starting from which the state trajectories of the system are exponentially converging to zero. This estimate includes a priori Lyapunov-based bounds on the solutions of the open-loop system on the initial time interval of the length of delay. Numerical examples illustrate the efficiency of the method.

1. Introduction

The Korteweg–de Vries–Burgers (KdVB) equation has been derived in fluid mechanics to describe a model of long waves in shallow water in a rectangular channel in which the effects of dispersion, dissipation and nonlinearity are present (see e.g. Demiray, 2004). Without the diffusion term, the KdVB equation becomes Korteweg–de Vries (KdV) equation, which has been proposed as a model of waves on shallow water surfaces.

In recent years, control of KdV and KdVB equations has attracted a lot of attention, owing to their potential applications in physics. Regional boundary stabilization of KdV equation via the backstepping method by the state feedback and output feedback controllers has been studied in Cerpa and Coron (2013) and Marx and Cerpa (2018) respectively. In Feng and Kawahara (2000) travelling wave solutions of unstable KdVB equation were considered. Global boundary stabilization of KdVB equation was studied in Balogh and Krstic (2000) and Liu and Krstic (2002). Distributed global stabilization of KdVB equation was considered in Kalantarov and Titi (2017). For practical implementation of controllers for KdVB equation, it is important to ensure their robustness with respect to small input delay. This paper introduces distributed stabilizing controllers for KdVB equation in the presence of uncertain input delays.

In the past decade, stabilization of systems described by PDEs or a cascade of ODE–PDE subject to state or input/output time-delay has been studied. In Krstic (2009) input delay compensation was suggested. Delay-independent stabilization of heat equations with state delay was considered in Hashimoto and Krstic (2016) and Kang and Fridman (2017). Robustness with respect to small input/output time-varying delay of semilinear heat equation with globally Lipschitz nonlinearity was studied in Fridman and Bar Am (2013) and Fridman and Blighovsky (2012), where distributed control under point and averaged measurements was introduced. In Fridman and Bar Am (2013) and Fridman and Blighovsky (2012) global stability conditions were derived by using Lyapunov–Krasovskii functionals that depended on the state derivative via the descriptor method (Fridman, 2001, 2014). In Azouani and Titi (2014) similar control laws for reaction–diffusion equation were proposed, however, effects of input delay were not studied.

☆ This work was supported by Israel Science Foundation (Grant No 1128/14), National Natural Science Foundation of China (Grant No. 61803026, 61873028), Fundamental Research Funds for the Central Universities (Grant No. FRF-TP-18-032A1) and China Postdoctoral Science Foundation (Grant No. 2018M640065). Some preliminary results under sampled in space measurements were presented in Kang and Fridman (2018). Wen Kang's work was done when she was with Tel Aviv University. The material in this paper was not presented at any conference. This paper was recommended for publication in revised form by Associate Editor Rafael Vazquez under the direction of Editor Miroslav Krstic.

* Corresponding author at: School of Automation and Electrical Engineering, University of Science and Technology Beijing, China.
E-mail addresses: kangwen@amss.ac.cn (W. Kang), emilia@eng.tau.ac.il (E. Fridman).

https://doi.org/10.1016/j.automatica.2018.11.025
0005-1098/© 2018 Elsevier Ltd. All rights reserved.

In the present work, we introduce a distributed control of KdVB equation under point or averaged measurements in the presence of constant input delay. It should be noticed that without delay, sufficient LMI conditions for global stabilization of KdVB equation are the same as for diffusion–reaction equations derived in Fridman and Bar Am (2013) and Fridman and Blighovsky (2012). However, with delay the situation becomes more complicated since the existing Lyapunov–Krasovskii functionals from Fridman and Bar Am (2013), Fridman and Blighovsky (2012) and Fridman and Orlov (2009) that depend on the state derivative are not applicable because of the third spatial derivative in KdVB equation. We introduce a novel Lyapunov functional that depends on the state only and leads to regional stability conditions of the closed-loop system in terms of LMIs. By solving these LMIs, an upper bound on delay that preserves regional stability can be found together with an estimate on the domain of attraction (i.e. on the set of initial conditions starting from which the state trajectories of the system are exponentially converging). As suggested for the case of ODEs in Liu and Fridman (2014), our estimate on the domain of attraction is based on the Lyapunov-based bounds of the solutions on the initial time interval (of the length of delay), where the system is open-loop. Some preliminary results under sampled in space measurements were presented in Kang and Fridman (2018).

The structure of the paper is as follows. Problem formulation is given in Section 2. Section 3 is devoted to the existence and uniqueness of the solution for the closed-loop system. A preliminary result on regional stability of the delayed KdVB equation via Lyapunov–Krasovskii method is presented in Section 4. The main results are given in Section 5, where distributed on the whole domain finite-dimensional controllers are constructed under the point or averaged measurements and distributed in space shape functions. We consider also controllers distributed on subdomains under measurements that are averaged over the same subdomains. Regional exponential stability conditions for the closed-loop system in terms of LMIs are presented and a bound on the domain of attraction is found. Section 6 contains numerical examples to illustrate the effectiveness of the proposed feedback control. Finally, some conclusions are drawn in the last section and some proofs are given in the Appendix.

Notation. Throughout the paper, $L^2(0,1)$ stands for the Hilbert space of square integrable scalar functions $u(x)$ on $(0,1)$ with the corresponding norm $\|u\|_{L^2} = [\int_0^1 u^2(x)dx]^{\frac{1}{2}}$. The Sobolev space $H^k(0,1)$ with $k \in \mathbb{Z}$ is defined as $H^k(0,1) = \{u : D^\alpha u \in L^2(0,1),\ \forall\, 0 \le |\alpha| \le k\}$ with norm $\|u\|_{H^k} = \{\sum_{0\le|\alpha|\le k}\|D^\alpha u\|_{L^2}^2\}^{\frac{1}{2}}$. $C^k(\Omega)$ is the space of functions k times continuously differentiable on ($k \ge 1$ is an integer). Moreover, $C^\infty(\Omega) = \cap_k C^k(\Omega)$. $C_c(\Omega)$ is the space of continuous functions on Ω with compact support in Ω, i.e., which vanish outside some compact set $K \subset \Omega$. Moreover, $C_c^\infty(\Omega) = C_c(\Omega) \cap C^\infty(\Omega)$.

2. Problem formulation

We consider the following KdVB equation involving both instability and dissipation under periodic boundary conditions:

$$\begin{cases} z_t(x,t) + z(x,t)z_x(x,t) - \beta z_{xx}(x,t) - \lambda z(x,t) \\ +z_{xxx}(x,t) = \sum_{j=1}^{N} b_j(x)u_j(t-h),\ 0<x<1,\ t\ge 0, \\ z(0,t) = z(1,t),\ z_x(0,t) = z_x(1,t), \\ z_{xx}(0,t) = z_{xx}(1,t), \\ z(x,0) = z_0(x), \end{cases} \tag{2.1}$$

where $x \in (0,1)$, β and λ are positive constants, $z(x,t)$ is the state of KdVB equation, $u_j(t) \in \mathbb{R}$, $(j = 1,2,\ldots,N)$ are the control inputs, and $h > 0$ is a constant delay. The delay may be unknown, but bounded by a known bound $\bar{h} > 0$. The above model is inspired by Feng and Kawahara (2000) (see (2), (4b)). For $\lambda = 0$, the open-loop system (2.1) has constant solutions, whereas for $\lambda > 0$ the open-loop system may become unstable (see the examples in Section 6).

As in Azouani and Titi (2014), Fridman and Bar Am (2013), Fridman and Blighovsky (2012) and Lunasin and Titi (2017), we assume that the points $0 = x_0 < x_1 < \cdots < x_N = 1$ divide $[0,1]$ into N intervals $\Omega_j = [x_{j-1}, x_j)$ and the width of each interval is upper bounded by some constant Δ:

$$0 < x_j - x_{j-1} = \Delta_j \le \Delta.$$

The control inputs $u_j(t)$ enter (2.1) through the shape functions $b_j(x)$ such that

$$\begin{cases} b_j(x) = 0,\ x \notin \Omega_j, \\ b_j(x) = 1,\ otherwise, \end{cases} j = 1,\ldots,N. \tag{2.2}$$

Such shape functions correspond to actuation covering all the domain $[0,1]$.

We assume further that sensors provide either point

$$y_j(t) = z(\bar{x}_j, t),\quad \bar{x}_j = \frac{x_{j-1}+x_j}{2},\ t>0, \tag{2.3}$$

or averaged

$$y_j(t) = \frac{\int_{x_{j-1}}^{x_j} z(\zeta,t)d\zeta}{\Delta_j},\ t>0 \tag{2.4}$$

measurements of the state. In Section 5.2 we will also consider actuation and measurement concentrated on subdomains of $[0,1]$.

Our main objective is to design for (2.1) a regionally stabilizing distributed controller

$$u_j(t) = \begin{cases} -\mu y_j(t),\ j = 1,\ldots,N, & t>0, \\ 0, & t \le 0, \end{cases} \tag{2.5}$$

where μ is a positive constant to be determined later, and $y_j(t)$ is given by (2.3) or (2.4).

Denote the characteristic function of the time interval $[0,h]$ by $\chi_{[0,h]}(t)$. Under the control law (2.5), the closed-loop system becomes

$$\begin{cases} z_t(x,t) + z(x,t)z_x(x,t) - \beta z_{xx}(x,t) - \lambda z(x,t) + z_{xxx}(x,t) \\ = -\mu \sum_{j=1}^{N} b_j(x)(1-\chi_{[0,h]}(t))[z(x,t-h) - f_j(x,t-h)], \\ \qquad 0<x<1,\ t\ge 0, \\ z(0,t) = z(1,t),\ z_x(0,t) = z_x(1,t), \\ z_{xx}(0,t) = z_{xx}(1,t), \\ z(x,0) = z_0(x), \end{cases} \tag{2.6}$$

where for (2.3)

$$f_j(x,t-h) = \int_{\bar{x}_j}^{x} z_\zeta(\zeta,t-h)d\zeta, \tag{2.7}$$

and for (2.4)

$$f_j(x,t-h) = \frac{\int_{x_{j-1}}^{x_j}[z(x,t-h) - z(\zeta,t-h)]d\zeta}{\Delta_j}. \tag{2.8}$$

3. Well-posedness of (2.6) subject to (2.7) or (2.8)

Define

$$\begin{cases} H^1_{per}(0,1) = \{g \in H^1(0,1) : g(0) = g(1)\}, \\ \|g\|^2_{H^1_{per}} = P_1 \int_0^1 g^2(x)dx + P\int_0^1 [g'(x)]^2 dx. \end{cases} \tag{3.1}$$

Here P_1 and P are positive constants that are related to the Lyapunov–Krasovskii functional (see (4.2) below) that will be used for stability analysis. It is obvious that $H^1_{per}(0,1)$ is a subspace of Sobolev space $H^1(0,1)$. Moreover, the norm $\|\cdot\|_{H^1_{per}(0,1)}$ is equivalent to $\|\cdot\|_{H^1(0,1)}$.

Definition 3.1 (*Larkin, 2004; Robinson, 2001*). Given $T>0$, a function $z\in C(0,T;H^1_{per}(0,1))$ such that $z_t\in L^\infty(0,T;L^2(0,1))\cap L^2(0,T;H^1_{per}(0,1))$ is said to be a weak solution of the boundary value problem (2.6) subject to (2.7) or (2.8) on $[0,T]$ if for every ϕ from $H^1_{per}(0,1)$, the function $\int_0^1 z(\zeta,t)\phi(\zeta)d\zeta$ is absolutely continuous on $[0,T]$ and the relation

$$\begin{aligned}&\frac{d}{dt}\int_0^1 z(\zeta,t)\phi(\zeta)d\zeta+\int_0^1 z(\zeta,t)z_\zeta(\zeta,t)\phi(\zeta)d\zeta\\&+\beta\int_0^1 z_\zeta(\zeta,t)\phi_\zeta(\zeta)d\zeta-\lambda\int_0^1 z(\zeta,t)\phi(\zeta)d\zeta\\&-\int_0^1 z_{\zeta\zeta}(\zeta,t)\phi_\zeta(\zeta)d\zeta\\&=-\mu\sum_{j=1}^N\int_{x_{j-1}}^{x_j}(1-\chi_{[0,h]}(t))[z(\zeta,t-h)-f_j(\zeta,t-h)]\\&\times\phi(\zeta)d\zeta\end{aligned}\tag{3.2}$$

holds for almost all $t\in[0,T]$.

The following well-posedness result is proved in the Appendix:

Proposition 3.1. *Assume that the initial value $z_0\in H^3(0,1)\cap H^1_{per}(0,1)$ satisfies the compatible conditions:*

$$z_0'(0)=z_0'(1),\ z_0''(0)=z_0''(1).\tag{3.3}$$

Then, for all $T>0$, there exists a unique (weak) solution to the system (2.6) subject to (2.7) or (2.8) from the class

$$\begin{aligned}&z\in C(0,T;H^1_{per}(0,1)),\\&z_t\in L^\infty(0,T;L^2(0,1))\cap L^2(0,T;H^1_{per}(0,1)).\end{aligned}$$

4. Regional stability of the delayed KdVB equation

In order to derive stability conditions for the closed-loop system (2.6), we first establish stability conditions for (2.6) with $f_j=0$:

$$\begin{cases}z_t(x,t)+z(x,t)z_x(x,t)-\beta z_{xx}(x,t)-\lambda z(x,t)+z_{xxx}(x,t)\\=-\mu(1-\chi_{[0,h]}(t))z(x,t-h),\ 0<x<1,\ t\ge 0,\\z(0,t)=z(1,t),\ z_x(0,t)=z_x(1,t),\\z_{xx}(0,t)=z_{xx}(1,t),\\z(x,0)=z_0(x).\end{cases}\tag{4.1}$$

Remark 4.1. Simple and efficient delay-dependent stability conditions that lead to LMIs for ODEs and heat equations are usually derived via Lyapunov–Krasovskii functionals V that depend on the state derivative (on z_t for heat equation) (Fridman, 2014; Fridman & Orlov, 2009). However, such functionals for (4.1) would lead to positive term hz_t^2 in $\dot V$. By substituting z_t from (4.1) we will arrive at positive term $h\|z_{xxx}\|^2_{L_2}$ that cannot be compensated by using negative terms like $-\|z\|^2_{L_2}$, $-\|z_x\|^2_{L_2}$ or $-\|z_{xx}\|^2_{L_2}$.

To avoid using Lyapunov–Krasovskii functionals that depend on z_t, we introduce a novel augmented Lyapunov–Krasovskii functional that depends on the state only:

$$V(t)=V_{aug}+V_P+V_R+V_Q+V_S,\tag{4.2}$$

where

$$\begin{aligned}V_{aug}&=\int_0^1\theta^T\begin{bmatrix}P_1&P_2\\P_2&P_3\end{bmatrix}\theta dx,\quad \theta=\begin{bmatrix}z(x,t)\\\int_{t-h}^t z(x,s)ds\end{bmatrix},\\V_P&=P\int_0^1 z_x^2(x,t)dx,\\V_R&=R\int_0^1\int_{t-h}^t e^{-2\delta(t-s)}(s+h-t)z^2(x,s)dsdx,\\V_Q&=Q\int_0^1\int_{t-h}^t e^{-2\delta(t-s)}z^2(x,s)dsdx,\\V_S&=S\int_0^1\int_{t-h}^t e^{-2\delta(t-s)}(s+h-t)z_x^2(x,s)dsdx.\end{aligned}$$

Here $P>0$, $R>0$, $Q>0$, and $S>0$. Moreover, we assume that for some constant $\gamma>0$ the following LMI holds:

$$\begin{bmatrix}P_1-\gamma&P_2\\ *&P_3+Q\frac{e^{-2\delta h}}{h}\end{bmatrix}>0.\tag{4.3}$$

Due to Jensen's inequality (Gu, Kharitonov, & Chen, 2003)

$$\begin{aligned}&Q\int_0^1\int_{t-h}^t e^{-2\delta(t-s)}z^2(x,s)dsdx\\&\ge Qe^{-2\delta h}\int_0^1\int_{t-h}^t z^2(x,s)dsdx\\&\ge Q\frac{e^{-2\delta h}}{h}\int_0^1\left[\int_{t-h}^t z(x,s)ds\right]^2dx\end{aligned}$$

LMI (4.3) guarantees the positivity of the Lyapunov–Krasovskii functional:

$$V(t)\ge\gamma\int_0^1 z^2(x,t)dx+P\int_0^1 z_x^2(x,t)dx.\tag{4.4}$$

Similar augmentations of Lyapunov–Krasovskii functionals were considered for ODEs in Seuret and Gouaisbaut (2013), but such functionals there depended on state derivatives. Note that when differentiating P_2-term, we arrive at z_{xx} term (see (4.25) below). Therefore, we choose V with non-zero V_P that leads to H^1 exponential stability. In $\dot V_P$ we have $-\|z_{xx}\|^2_{L^2}$ term (see (4.26) below) that allows to compensate z_{xx} term in (4.25). However, the last term in (4.26) is non-quadratic, leading to regional exponential stability. In order to find domain of attraction we derive stability conditions in terms of matrix inequalities that are affine in z ($\Xi<0$, $\Lambda<0$ where Ξ and Λ are given by (4.10) and (4.11) respectively). We find a bound on initial conditions that guarantees the bound $\max_{x\in[0,1]}|z(x,t)|\le C_1$ for all $t\ge 0$ for all solutions starting from these initial conditions.

The following extension of Sobolev inequality will be useful:

Lemma 4.1. *Let $z(x)\in H^1(0,1)$ be a scalar function. Then for all $\Gamma>0$*

$$\max_{x\in[0,1]}z^2(x)\le(1+\Gamma)\int_0^1 z^2(x)dx+\frac{1}{\Gamma}\int_0^1 z_x^2(x)dx.$$

Proof. Since $z(\cdot)\in H^1(0,1)$ implies $z(\cdot)\in C[0,1]$ (cf. Brezis, 2011; Robinson, 2001), by mean value theorem, there exists $c\in(0,1)$ such that

$$z(c)=\int_0^1 z(x)dx.$$

Then, by integration by parts and further application of Jensen's and Young's inequalities, for all $x_1 \in [0,1]$ we have

$$\begin{aligned} z^2(x_1) &= z^2(c) + 2\int_c^{x_1} z(x)z_x(x)dx \\ &= \left[\int_0^1 z(x)dx\right]^2 + 2\int_c^{x_1} z(x)z_x(x)dx \\ &\le \int_0^1 z^2(x)dx + \Gamma\int_0^1 z^2(x)dx + \frac{1}{\Gamma}\int_0^1 z_x^2(x)dx \\ &\le (1+\Gamma)\int_0^1 z^2(x)dx + \frac{1}{\Gamma}\int_0^1 z_x^2(x)dx \;\forall \Gamma > 0. \quad \square \end{aligned}$$

Remark 4.2. Differently from Theorem 8.8 of Brezis (2011), Lemma 4.1 gives the explicit upper bound of $\|z\|_{C[0,1]}$ and applies the Young inequality with a general constant $\Gamma > 0$.

Lemma 4.2. *Consider the system (4.1). Given positive scalars $\mu > \lambda$, h, δ and positive tuning parameters $\delta_1 > \lambda$, $C > 0$ and $C_1 > 0$. Let there exist scalars $\gamma > 0$, $\Gamma > 0$, $P_1 > 0$, $P_2 < 0$, $P > 0$, $Q > 0$, $R > 0$, $S > 0$, $P_3 \in \mathbb{R}$, and $\nu \in \mathbb{R}$ such that the LMIs (4.3) and*

$$\Xi\big|_{z=C_1} \le 0,\ \Xi\big|_{z=-C_1} \le 0, \tag{4.5}$$

$$\Lambda\big|_{z=C_1} \le 0,\ \Lambda\big|_{z=-C_1} \le 0, \tag{4.6}$$

$$\Phi\big|_{z=C_1} \le 0,\ \Phi\big|_{z=-C_1} \le 0, \tag{4.7}$$

$$\gamma \ge 1+\Gamma,\ \begin{bmatrix} -P & 1 \\ 1 & -\Gamma \end{bmatrix} < 0, \tag{4.8}$$

hold, where

$$\Xi = \begin{bmatrix} -2P_1(\delta_1-\lambda) & 0 & \nu \\ * & -2P_1\beta - 2P(\delta_1-\lambda)+2\nu & Pz \\ * & * & -2P\beta \end{bmatrix}, \tag{4.9}$$

$$\Phi = \begin{bmatrix} \phi_{11} & -P_1\mu - P_2 & 0 & \nu & \phi_{15} & 0 \\ * & -Qe^{-2\delta h} & 0 & P\mu & -P_2\mu - P_3 & 0 \\ * & * & \phi_{33} & Pz & -P_2 z & -P_2\beta \\ * & * & * & -2P\beta & 0 & P_2 \\ * & * & * & * & \phi_{55} & 0 \\ * & * & * & * & * & \phi_{66} \end{bmatrix}, \tag{4.10}$$

$$\Lambda = \begin{bmatrix} \Lambda_{11} & -P_2 & 0 & \nu & \Lambda_{15} & 0 \\ * & -Qe^{-2\delta h} & 0 & 0 & -P_3 & 0 \\ * & * & \Lambda_{33} & Pz & -P_2 z & -P_2\beta \\ * & * & * & -2P\beta & 0 & P_2 \\ * & * & * & * & \Lambda_{55} & 0 \\ * & * & * & * & * & \Lambda_{66} \end{bmatrix}, \tag{4.11}$$

$$\begin{aligned} &\phi_{11} = 2P_2 + 2P_1\lambda + Rh + Q + 2\delta P_1, \\ &\Lambda_{11} = \phi_{11} - 2\delta_1 P_1, \\ &\phi_{15} = \Lambda_{15} = P_3 + 2\delta P_2 + P_2\lambda, \\ &\phi_{33} = -2P_1\beta + 2P\lambda + 2\nu + Sh + 2\delta P, \\ &\Lambda_{33} = \phi_{33} - 2\delta_1 P, \\ &\phi_{55} = \Lambda_{55} = -Re^{-2\delta h}\frac{1}{h} + 2\delta P_3, \\ &\phi_{66} = \Lambda_{66} = -Se^{-2\delta h}\frac{1}{h}. \end{aligned} \tag{4.12}$$

Denote

$$\begin{aligned} M = \max \;& \left\{\left(P_1 + 2P_2 h + P_3 h^2 + (\frac{Rh^2}{2} + Qh)\right)P_1^{-1}, \right. \\ & \left. (P + \frac{Sh^2}{2})P^{-1}\right\} + (e^{2\delta_1 h} - 1). \end{aligned} \tag{4.13}$$

If

$$MC^2 < C_1^2, \tag{4.14}$$

then for any initial state $z_0 \in H^3(0,1) \cap H^1_{per}(0,1)$ satisfying the compatible conditions (3.3) with $\|z_0\|_{H^1_{per}} < C$, system (4.1) possesses a unique (weak) solution in the sense that for any $T > 0$, $z(x,t) \in C(0,T; H^1_{per}(0,1))$. Moreover, the solution of (4.1) satisfies

$$V(t) \le Me^{-2\delta(t-h)}\left[P_1\int_0^1 z_0^2(x)dx + P\int_0^1 [z_0'(x)]^2 dx\right] \tag{4.15}$$

for all $t \ge h$.

Proof. We divide the proof into three parts.

Step 1: By arguments of Fridman (2014) and Liu and Fridman (2014), we first derive a simple bound on $V(h)$ in terms of z_0 such that $V(h) < C_1^2$.

Since the solution to (4.1) does not depend on the values of $z(x,t)$ for $t < 0$, we redefine the initial condition to be a function:

$$z(x,t) = z_0(x),\ \ t \le 0. \tag{4.16}$$

Due to (4.16), we have

$$V_{aug}(0) = [P_1 + 2P_2 h + P_3 h^2]\int_0^1 z_0^2(x)dx.$$

Then

$$\begin{aligned} V(0) \;\; &\le \left[P_1 + 2P_2 h + P_3 h^2 + \frac{Rh^2}{2} + Qh\right]\int_0^1 z_0^2(x)dx \\ &\quad + (P + \frac{Sh^2}{2})\int_0^1 [z_0'(x)]^2 dx. \end{aligned} \tag{4.17}$$

We consider

$$V_0(t) = P_1\int_0^1 z^2(x,t)dx + P\int_0^1 z_x^2(x,t)dx. \tag{4.18}$$

Given $\delta > 0$. Assume that there exists $\delta_1 > 0$ such that along (4.1)

$$\dot V_0(t) - 2\delta_1 V_0(t) \le 0,\ \ t \in [0,h], \tag{4.19}$$

$$\dot V(t) + 2\delta V(t) - 2\delta_1 V_0(t) \le 0,\ \ t \in [0,h], \tag{4.20}$$

then

$$V_0(t) \le e^{2\delta_1 t}V_0(0),\ \ t \in [0,h], \tag{4.21}$$

$$V(t) \le e^{-2\delta t}V(0) + (e^{2\delta_1 t} - 1)V_0(0),\ \ t \in [0,h]. \tag{4.22}$$

Substituting (4.17) into the inequality (4.22), together with the condition (4.14), we have

$$\begin{aligned} V(h) \le \;& \left[\left(P_1 + 2P_2 h + P_3 h^2 + \frac{Rh^2}{2} + Qh\right)\int_0^1 z_0^2(x)dx \right. \\ & \left. + (P + \frac{Sh^2}{2})\int_0^1 [z_0'(x)]^2 dx\right] e^{-2\delta h} \\ & + (e^{2\delta_1 h} - 1)\left[P_1\int_0^1 z_0^2(x)dx + P\int_0^1 [z_0'(x)]^2 dx\right] \\ & \le M\left[P_1\int_0^1 z_0^2(x)dx + P\int_0^1 [z_0'(x)]^2 dx\right] < C_1^2, \end{aligned} \tag{4.23}$$

if

$$\|z_0\|^2_{H^1_{per}} = P_1\int_0^1 z_0^2(x)dx + P\int_0^1 [z_0'(x)]^2 dx < C^2. \tag{4.24}$$

Step 2: We show next the LMIs (4.5) and (4.6) guarantee that (4.19) and (4.20) hold.

Differentiating V_{aug} along (4.1), where (3.2) is used with $\phi = z$ and $t \in [0, h]$, we obtain

$$\begin{aligned}
\dot V_{aug} &= 2P_1\int_0^1 z(x,t)z_t(x,t)dx \\
&+2P_2\int_0^1 z_t(x,t)\int_{t-h}^t z(x,s)dsdx \\
&+2P_2\int_0^1 z(x,t)[z(x,t)-z(x,t-h)]dx \\
&+2P_3\int_0^1\int_{t-h}^t z(x,s)ds[z(x,t)-z(x,t-h)]dx \\
&= -2P_1\beta\int_0^1 z_x^2(x,t)dx + 2P_1\lambda\int_0^1 z^2(x,t)dx \\
&- 2P_2\int_0^1 z(x,t)z_x(x,t)\int_{t-h}^t z(x,s)dsdx \\
&- 2P_2\beta\int_0^1 z_x(x,t)\int_{t-h}^t z_x(x,s)dsdx \\
&+2P_2\int_0^1 z_{xx}(x,t)\int_{t-h}^t z_x(x,s)dsdx \\
&+2P_2\lambda\int_0^1 z(x,t)\int_{t-h}^t z(x,s)dsdx \\
&+2P_2\int_0^1 z(x,t)[z(x,t)-z(x,t-h)]dx \\
&+2P_3\int_0^1\int_{t-h}^t z(x,s)ds[z(x,t)-z(x,t-h)]dx.
\end{aligned} \tag{4.25}$$

We have

$$\begin{aligned}
\dot V_P &= 2P\int_0^1 z_x(x,t)z_{xt}(x,t)dx = -2P\int_0^1 z_{xx}(x,t)z_t(x,t)dx \\
&= -2P\beta\int_0^1 z_{xx}^2(x,t)dx + 2P\lambda\int_0^1 z_x^2(x,t)dx \\
&+2P\int_0^1 z_{xx}(x,t)z_x(x,t)z(x,t)dx
\end{aligned} \tag{4.26}$$

and

$$\dot V_Q + 2\delta V_Q = Q\int_0^1 z^2(x,t)dx - Q\int_0^1 e^{-2\delta h}z^2(x,t-h)dx. \tag{4.27}$$

Further by applying Jensen's inequality we obtain

$$\begin{aligned}
&\dot V_R + 2\delta V_R = Rh\int_0^1 z^2(x,t)dx \\
&-R\int_0^1\int_{t-h}^t e^{-2\delta(t-s)}z^2(x,s)dsdx \\
&\le Rh\int_0^1 z^2(x,t)dx - Re^{-2\delta h}\frac{1}{h}\int_0^1\left[\int_{t-h}^t z(x,s)ds\right]^2 dx, \\
&\dot V_S + 2\delta V_S = Sh\int_0^1 z_x^2(x,t)dx \\
&-S\int_0^1\int_{t-h}^t e^{-2\delta(t-s)}z_x^2(x,s)dsdx \\
&\le Sh\int_0^1 z_x^2(x,t)dx - Se^{-2\delta h}\frac{1}{h}\int_0^1\left[\int_{t-h}^t z_x(x,s)ds\right]^2 dx.
\end{aligned} \tag{4.28}$$

Additionally,

$$2\nu\left[\int_0^1 z(x,t)z_{xx}(x,t)dx + \int_0^1 z_x^2(x,t)dx\right] = 0 \quad \forall \nu \in \mathbb{R}. \tag{4.29}$$

We add to $\dot V(t)+2\delta V(t)$ the left-hand side of (4.29). Then, by taking into account (4.25)–(4.28) we arrive at

$$\dot V(t) + 2\delta V(t) - 2\delta_1 V_0(t) \le \int_0^1 \psi^\top(x,t)\Lambda\psi(x,t)dx,$$
$$t \in [0,h], \tag{4.30}$$

where

$$\begin{aligned}
\psi(x,t) = \ &\mathrm{col}\{z(x,t), z(x,t-h), z_x(x,t), z_{xx}(x,t), \\
&\int_{t-h}^t z(x,s)ds, \int_{t-h}^t z_x(x,s)ds\},
\end{aligned} \tag{4.31}$$

and Λ is given by (4.11).

Similarly, differentiating $V_0(t)$ along (4.1) and adding (4.29), we have

$$\dot V_0(t) - 2\delta_1 V_0(t) = \int_0^1 \xi^\top(x,t)\Xi\xi(x,t)dx, \ \ t \in [0,h], \tag{4.32}$$

where $\xi(x,t) = \mathrm{col}\{z(x,t), z_x(x,t), z_{xx}(x,t)\}$ and Ξ is given by (4.9).

As in Selivanov and Fridman (2017), first we assume that

$$z(x,t) \in (-C_1, C_1) \ \ \forall x \in [0,1], \ \ \forall t \in [0,h]. \tag{4.33}$$

From (4.30) and (4.32), it follows that if $\Xi \le 0$ and $\Lambda \le 0$ for all $z \in (-C_1, C_1)$, then (4.19) and (4.20) hold. Matrices Ξ and Λ given by (4.9) and (4.11) are affine in z. Thus, $\Xi \le 0$ and $\Lambda \le 0$ for all $z \in (-C_1, C_1)$ if LMIs (4.5) and (4.6) are feasible. Therefore, (4.5) and (4.6) guarantee that (4.19) and (4.20) hold.

We prove next (4.33). If the LMIs (4.3) and (4.8) are feasible, then Lemma 4.1 and Schur complement theorem lead to

$$\begin{aligned}
\max_{0\le x\le 1}|z(x,t)|^2 &\le (1+\Gamma)\int_0^1 z^2(x,t)dx \\
&+\frac{1}{\Gamma}\int_0^1 [z_x(x,t)]^2dx \\
&\le \gamma\int_0^1 z^2(x,t)dx + P\int_0^1 [z_x(x,t)]^2dx. \\
&\le P_1\int_0^1 z^2(x,t)dx + P\int_0^1 [z_x(x,t)]^2dx \\
&= V_0(t) \qquad \forall t \in [0,h].
\end{aligned} \tag{4.34}$$

Moreover, from (4.4), we have

$$\begin{aligned}
\max_{0\le x\le 1}|z(x,t)|^2 &\le \gamma\int_0^1 z^2(x,t)dx + P\int_0^1 [z_x(x,t)]^2dx \\
&\le V(t) \qquad \forall t \in [0,h].
\end{aligned} \tag{4.35}$$

Therefore, it is sufficient to show that

$$V_0(t) < C_1^2, \qquad V(t) < C_1^2 \tag{4.36}$$

for all $t \in [0,h]$.

Indeed, for $t = 0$, the inequalities (4.36) hold. From the definition of M, we obtain $M > e^{2\delta_1 h} > 1$. Since $\|z_0\|_{H^1_{per}} < C$ and $MC^2 < C_1^2$ are satisfied, we obtain $V_0(0) < C^2 < C_1^2$. On the other hand, from (4.17) we have $V(0) \le MV_0(0) < MC^2 < C_1^2$.

Let (4.36) be false for some $h^* \in (0,h]$. Then

$$V_0(h^*) \ge C_1^2 > C^2 > V_0(0),$$

or

$$V(h^*) \ge C_1^2 > V(0).$$

Since V_0 and V are continuous, there must exist $t^* \in (0,h^*]$ such that

$$V_0(t) < C_1^2 \ \forall t \in [0,t^*) \text{ and } V_0(t^*) = C_1^2, \tag{4.37}$$

or

$$V(t) < C_1^2 \ \forall t \in [0, t^*) \text{ and } V(t^*) = C_1^2. \tag{4.38}$$

The first relation of (4.37), together with the feasibility of (4.5), guarantees that $\dot V_0(t) - 2\delta_1 V_0(t) \le 0$ on $[0, t^*)$. Therefore, $V_0(t^*) \le e^{2\delta_1 t^*} V_0(0) \le e^{2\delta_1 h} C^2 < MC^2 < C_1^2$. This contradicts the second relation of (4.37). On the other hand, the first relation of (4.38), together with the feasibility of (4.6), guarantees that $\dot V(t) + 2\delta V(t) - 2\delta_1 V_0(t) \le 0$ on $[0, t^*)$. Hence, (4.22) holds and $V(t^*) \le MC^2 < C_1^2$. This contradicts the second relation of (4.38). Thus, (4.36) and consequently, (4.19) and (4.20) are true on $[0, h]$.

Step 3: We continue to find sufficient conditions in terms of LMIs to guarantee $\dot V(t) + 2\delta V(t) \le 0$ for all $t > h$.

Differentiating V along (4.1) and integrating by parts, we obtain (4.27) and (4.28). For $t > h$, (4.25)–(4.26) become

$$\begin{aligned} \dot V_{aug} &= -2P_1\beta \int_0^1 z_x^2(x,t)dx + 2P_1\lambda \int_0^1 z^2(x,t)dx \\ &\quad -2P_1\mu \int_0^1 z(x,t)z(x,t-h)dx \\ &\quad -2P_2 \int_0^1 z(x,t)z_x(x,t) \int_{t-h}^t z(x,s)dsdx \\ &\quad -2P_2\beta \int_0^1 z_x(x,t) \int_{t-h}^t z_x(x,s)dsdx \\ &\quad +2P_2 \int_0^1 z_{xx}(x,t) \int_{t-h}^t z_x(x,s)dsdx \\ &\quad +2P_2\lambda \int_0^1 z(x,t) \int_{t-h}^t z(x,s)dsdx \\ &\quad -2P_2\mu \int_0^1 z(x,t-h) \int_{t-h}^t z(x,s)dsdx \\ &\quad +2P_2 \int_0^1 z(x,t)[z(x,t) - z(x,t-h)]dx \\ &\quad +2P_3 \int_0^1 \int_{t-h}^t z(x,s)ds[z(x,t) - z(x,t-h)]dx, \end{aligned} \tag{4.39}$$

and

$$\begin{aligned} \dot V_P &= -2P\beta \int_0^1 z_{xx}^2(x,t)dx + 2P\lambda \int_0^1 z_x^2(x,t)dx \\ &\quad +2P \int_0^1 z_{xx}(x,t)z_x(x,t)z(x,t)dx \\ &\quad +2P\mu \int_0^1 z_{xx}(x,t)z(x,t-h)dx. \end{aligned} \tag{4.40}$$

By adding to $\dot V(t) + 2\delta V(t)$ the equality (4.29), and using (4.27)–(4.28), (4.39), (4.40), we obtain

$$\dot V(t) + 2\delta V(t) \le \int_0^1 \psi^\top(x,t)\Phi\psi(x,t)dx, \tag{4.41}$$

where $\psi(x,t)$ is given by (4.31).

From Step 1-Step 2, we obtain if $\|z_0\|_{H^1_{per}} < C$, then the feasibility of LMIs (4.5), (4.6), (4.14) imply that $V(h) < C_1^2$. Furthermore, if LMIs (4.3), (4.8) hold, then by using (4.4), (4.34) we obtain

$$\begin{aligned} \max_{0\le x\le 1} |z(x,h)|^2 &\le (1+\Gamma) \int_0^1 z^2(x,h)dx + \frac{1}{\Gamma} \int_0^1 z_x^2(x,h)dx \\ &\le \gamma \int_0^1 z^2(x,h)dx + P \int_0^1 z_x^2(x,h)dx \le V(h) < C_1^2. \end{aligned}$$

Matrix Φ given by (4.10) is affine in z. Hence, $\Phi \le 0$ for all $z \in (-C_1, C_1)$ if LMIs (4.7) are satisfied. Therefore, (4.7) guarantees $\dot V(t) + 2\delta V(t) \le 0$, which implies

$$V(t) \le e^{-2\delta(t-h)} V(h) \ \forall t \ge h. \tag{4.42}$$

Using (4.23) and (4.42), we obtain (4.15). □

Remark 4.3. Given any $\mu > \lambda$, the LMIs in Lemma 4.2 are always feasible for appropriate decision variables, large enough δ_1 and small enough h. Indeed, consider $\Xi = \{\Xi_{ij}\}$, $\Phi = \{\phi_{ij}\}$ and $\Lambda = \{\Lambda_{ij}\}$ given by (4.9), (4.10) and (4.11) respectively. Note that for any $P > 0$, LMI (4.8) holds with appropriate $\Gamma > 0$. We will show that strict inequalities (4.5)–(4.7) hold with $\delta = 0$ and $C_1 = 0$. Then LMIs (4.5)–(4.7) hold with small enough $\delta > 0$ and $C_1 > 0$. Choose $R = S = 1$, $\nu = 0$, $P_1 = 1$, $P_2 = -\mu$, $P_3 = \mu^2$ such that $\phi_{12} = -P_1\mu - P_2 = 0$ and $\phi_{25} = -P_2\mu - P_3 = 0$. Hence, $\Xi_{|z=0} < 0$ for large enough $\delta_1 > \lambda$. By applying Schur complement, we obtain

$$\begin{aligned} \Phi\Big|_{z=0} &= \begin{bmatrix} \phi_{11} & 0 & 0 & 0 & \mu(\mu-\lambda) & 0 \\ * & -Q & 0 & P\mu & 0 & 0 \\ * & * & \phi_{33} & 0 & 0 & \mu\beta \\ * & * & * & -2P\beta & 0 & -\mu \\ * & * & * & * & -\frac{1}{h} & 0 \\ * & * & * & * & * & -\frac{1}{h} \end{bmatrix} \\ &\le 0 \Longleftrightarrow \\ &\begin{bmatrix} \phi_{11} & 0 & 0 & \mu(\mu-\lambda) & 0 \\ * & \phi_{33} & 0 & 0 & \mu\beta \\ * & * & -2P\beta + (P\mu)^2 Q^{-1} & 0 & -\mu \\ * & * & * & -\frac{1}{h} & 0 \\ * & * & * & * & -\frac{1}{h} \end{bmatrix} \le 0, \end{aligned} \tag{4.43}$$

where $\phi_{11} = -2(\mu-\lambda) + h + Q$, $\phi_{33} = -2\beta + 2P\lambda + h$. Choose $Q = \mu - \lambda$. For small enough h, LMI (4.43) is feasible with appropriate (small enough) $P > 0$, whereas $\Lambda_{|z=0} < 0$ holds for large enough $\delta_1 > \lambda$.

5. Regional delayed stabilization of KdVB equation

5.1. Controller distributed on the whole domain

Our main objective is to provide sufficient conditions for regional stability of the closed-loop system (2.6) subject to (2.7) or (2.8) with b_j defined by (2.2). For this purpose we define a Lyapunov–Krasovskii functional of the form:

$$V_1(t) = V(t) + W \int_0^1 \int_{t-h}^t e^{-2\delta(t-s)} z_x^2(x,s)dsdx, \tag{5.1}$$

where $W > 0$ and $V(t)$ is given by (4.2).

We will employ the following inequalities:

Lemma 5.1 (*Wirtinger Inequality* Wang, 1994)**.** *For $a < b$, let $g \in H^1(a,b)$ be a scalar function with $g(a) = 0$ or $g(b) = 0$. Then*

$$\int_a^b g^2(x)dx \le \frac{4(b-a)^2}{\pi^2} \int_a^b \left[\frac{dg(x)}{dx}\right]^2 dx.$$

Moreover, if $g(a) = g(b) = 0$, then

$$\int_a^b g^2(x)dx \le \frac{(b-a)^2}{\pi^2} \int_a^b \left[\frac{dg(x)}{dx}\right]^2 dx.$$

Lemma 5.2 (*Poincaré Inequality* Payne & Weinberger, 1960)**.** *For $a < b$, let $g \in H^1(a,b)$ be a scalar function with $\int_a^b g(x)dx = 0$. Then*

$$\int_a^b g^2(x)dx \le \frac{(b-a)^2}{\pi^2} \int_a^b \left[\frac{dg(x)}{dx}\right]^2 dx.$$

Theorem 5.1. *Consider the system (2.6) subject to (2.7) or (2.8). Given positive scalars Δ, $\mu>\lambda$, h, δ, and positive tuning parameters $\delta_1>\lambda$, $C>0$ and $C_1>0$. Let there exist scalars $\gamma>0$, $\Gamma>0$, $P_1>0$, $P_2<0$, $P>0$, $Q>0$, $R>0$, $S>0$, $W>0$, $\alpha>0$, $P_3\in\mathbb{R}$ and $\nu\in\mathbb{R}$ such that the LMIs (4.3), (4.5), (4.8) and*

$$-We^{-2\delta h}+\alpha\le 0, \tag{5.2}$$

$$\Sigma\big|_{z=C_1}\le 0,\ \ \Sigma\big|_{z=-C_1}\le 0, \tag{5.3}$$

$$\Theta\big|_{z=C_1}\le 0,\ \ \Theta\big|_{z=-C_1}\le 0, \tag{5.4}$$

hold, where Ξ is given by (4.9),

$$\Sigma=\Lambda+\mathrm{diag}\{0,0,W,0,0,0\}, \tag{5.5}$$

and

$$\Theta=\left[\begin{array}{c:c}\Phi & *\\ \hdashline P_1\mu\ \ 0\ \ 0\ \ 0\ \ -P\mu\ \ P_2\mu & -\alpha\dfrac{\pi^2}{\Delta^2}\end{array}\right]+\mathrm{diag}\{0,0,W,0,0,0,0\}$$

with Φ defined by (4.10).

Denote

$$M_1=\max\left\{\left(P_1+2P_2h+P_3h^2+(\frac{Rh^2}{2}+Qh)\right)P_1^{-1},\ (P+\frac{Sh^2}{2}+Wh)P^{-1}\right\}+(e^{2\delta_1 h}-1). \tag{5.6}$$

If

$$M_1C^2<C_1^2, \tag{5.7}$$

then for any initial state $z_0\in H^3(0,1)\cap H^1_{per}(0,1)$ satisfying the compatible conditions (3.3) and $\|z_0\|_{H^1_{per}}<C$, system (2.6) subject to (2.7) or (2.8) possesses a unique (weak) solution in the sense that for any $T>0$, $z(x,t)\in C(0,T;H^1_{per}(0,1))$. Moreover, the solution of (2.6) subject to (2.7) or (2.8) satisfies

$$V_1(t)\le M_1e^{-2\delta(t-h)}\left[P_1\int_0^1 z_0^2(x)dx+P\int_0^1[z_0'(x)]^2dx\right] \tag{5.8}$$

for all $t\ge h$.

Proof. Consider the system (2.6) subject to (2.7) or (2.8). Since the solution to the system (2.6) subject to (2.7) or (2.8) does not depend on the values of $z(x,t)$ for $t<0$, we redefine the initial condition to be constant:

$$z(x,t)=z_0(x),\ \ t\le 0.$$

Next the substitution $V(t)\to V_1(t)$ in Lemma 4.2 leads to the following changes:

$$V(0)\to V_1(0),\ \ M\to M_1,\ \ V(h)\to V_1(h),$$

where

$$\begin{aligned}V_1(0)\ &\le\left(P_1+2P_2h+P_3h^2+(\frac{Rh^2}{2}+Qh)\right)\int_0^1 z_0^2(x)dx\\ &\quad+(P+\frac{Sh^2}{2}+Wh)\int_0^1[z_0'(x)]^2dx.\end{aligned} \tag{5.9}$$

Here V_1 and M_1 are given by (5.1) and (5.6) respectively.

By Lemma 4.2, the LMIs (4.3), (4.5) and (4.8) guarantee (4.19). We show next that the feasibility of the LMIs (5.2), (5.3) and (5.4) guarantee

$$\dot V_1(t)+2\delta V_1(t)-2\delta_1V_0(t)\le 0\ \forall t\in[0,h] \tag{5.10}$$

and

$$\dot V_1(t)+2\delta V_1(t)\le 0\ \forall t>h. \tag{5.11}$$

Then from (4.19), (5.10), it follows that if (5.7) holds, then for initial state satisfying $\|z_0\|_{H^1_{per}}<C$, the following inequality holds:

$$V_1(h)\ \ \le M_1\left[P_1\int_0^1 z_0^2(x)dx+P\int_0^1[z_0'(x)]^2dx\right]<C_1^2. \tag{5.12}$$

Differentiating V_1 along (2.6), we find by employing (4.30)

$$\begin{aligned}&\dot V_1(t)+2\delta V_1(t)-2\delta_1V_0(t)\\ &=\dot V(t)+2\delta V(t)-2\delta_1V_0(t)+W\int_0^1 z_x^2(x,t)dx\\ &-We^{-2\delta h}\int_0^1 z_x^2(x,t-h)dx\\ &\le\int_0^1\psi^\top(x,t)\Sigma\psi(x,t)dx\\ &-We^{-2\delta h}\int_0^1 z_x^2(x,t-h)dx,\end{aligned} \tag{5.13}$$

where $\psi(x,t)$ is defined by (4.31) and Σ is given by (5.5).

Due to the affinity of Σ in $z\in(-C_1,C_1)$, we conclude that (5.3) guarantees (5.10).

Note that (2.6) comparatively to (4.1) has an additional term $-\mu\sum_{j=1}^N b_j(x)f_j(x,t-h)$ for $t>h$. Then from (4.41) and the substitution $\mu z(x,t-h)\to\mu[z(x,t-h)-f_j(x,t-h)]$, we have

$$\begin{aligned}&\dot V_1(t)+2\delta V_1(t)\le\int_0^1\psi^T(x,t)\Phi\psi(x,t)dx\\ &+2P_1\mu\sum_{j=1}^N\int_{x_{j-1}}^{x_j}z(x,t)f_j(x,t-h)dx\\ &+2P_2\mu\sum_{j=1}^N\int_{x_{j-1}}^{x_j}\int_{t-h}^t z(x,s)dsf_j(x,t-h)dx\\ &-2P\mu\sum_{j=1}^N\int_{x_{j-1}}^{x_j}z_{xx}(x,t)f_j(x,t-h)dx\\ &+W\int_0^1 z_x^2(x,t)dx-We^{-2\delta h}\int_0^1 z_x^2(x,t-h)dx\\ &\forall t>h,\end{aligned} \tag{5.14}$$

where $\psi(x,t)$ is defined by (4.31) and Φ is given by (4.10).

For the case of the point measurements, $f_j(x,t-h)$ is given by (2.7). Wirtinger's inequality yields

$$\begin{aligned}&\int_{x_{j-1}}^{x_j}f_j^2(x,t-h)dx=\int_{x_{j-1}}^{x_j}\left[\int_{\bar x_j}^{x}z_\zeta(\zeta,t-h)d\zeta\right]^2dx\\ &=\int_{x_{j-1}}^{\bar x_j}\left[\int_{\bar x_j}^{x}z_\zeta(\zeta,t-h)d\zeta\right]^2dx\\ &+\int_{\bar x_j}^{x_j}\left[\int_{\bar x_j}^{x}z_\zeta(\zeta,t-h)d\zeta\right]^2dx\\ &\le\frac{\Delta^2}{\pi^2}\int_{x_{j-1}}^{x_j}z_x^2(x,t-h)dx\ \forall t>h.\end{aligned}$$

For the case of the averaged measurements, $f_j(x,t-h)$ is given by (2.8). Since $\int_{x_{j-1}}^{x_j} f_j(x,t-h)dx = 0$, and $\frac{d}{dx}f_j(x,t-h) = z_x(x,t-h)$, the Poincaré inequality (Lemma 5.2) leads to

$$\int_{x_{j-1}}^{x_j} f_j^2(x,t-h)dx \le \frac{\Delta^2}{\pi^2}\int_{x_{j-1}}^{x_j} z_x^2(x,t-h)dx \ \forall t > h.$$

Hence, in both cases the following inequality

$$\alpha\sum_{j=1}^N \int_{x_{j-1}}^{x_j}\left[z_x^2(x,t-h) - \frac{\pi^2}{\Delta^2}f_j^2(x,t-h)\right]dx \ge 0 \tag{5.15}$$

holds for some constant $\alpha > 0$.

Set $\eta = \mathrm{col}\{z(x,t), z(x,t-h), z_x(x,t), z_{xx}(x,t), \int_{t-h}^t z(x,s)ds, \int_{t-h}^t z_x(x,s)ds, f_j(x,t-h)\}$. We add (4.29) to (5.14) and find

$$\begin{aligned}
&\dot V_1(t) + 2\delta V_1(t)\\
&\le \dot V_1(t) + 2\delta V_1(t) + \alpha\sum_{j=1}^N\int_{x_{j-1}}^{x_j}\left[z_x^2(x,t-h)\right.\\
&\left.-\frac{\pi^2}{\Delta^2}f_j^2(x,t-h)\right]dx\\
&\le \sum_{j=1}^N\int_{x_{j-1}}^{x_j}\eta^\top\Theta\eta dx - (We^{-2\delta h}-\alpha)\int_0^1 z_x^2(x,t-h)dx.
\end{aligned}$$

Since $V(h) < C_1^2$, we obtain that if $-We^{-2\delta h}+\alpha \le 0$ and $\Theta \le 0$ hold for all $z \in (-C_1, C_1)$, then (5.11) holds. The matrix Θ is affine with respect to z. Therefore, $\Theta \le 0$ for all $z\in(-C_1,C_1)$ if the LMIs (5.4) are satisfied. Hence, the LMIs (5.2), (5.4) yield (5.11), which implies

$$V_1(t) \le e^{-2\delta(t-h)}V_1(h) \qquad \forall t \ge h. \tag{5.16}$$

Inequalities (5.16) and (5.12) imply (5.8) that completes the proof. □

Remark 5.1. Note that Σ and Θ are convex in h in the sense that if the conditions of Theorem 5.1 are satisfied for some $\bar h > 0$, then they are also satisfied with the same decision variables for all $h \in [0,\bar h]$. Hence, the feasibility of the LMIs in Theorem 5.1 guarantees the regional stability of (2.6) subject to (2.7) or (2.8) for all constant delays $h\in[0,\bar h]$. Hence, the delay in (2.1) may be uncertain, but upper-bounded by the known bound $\bar h$ meaning that there is robustness of the stability with respect to small enough delays.

Remark 5.2. Consider (2.6) subject to (2.7) or (2.8) without delay (i.e. with $h=0$). For the energy function $E(t) = \frac{1}{2}\int_0^1 z^2(x,t)dx$, by employing (5.15) with $h=0$ and applying S-procedure with $\alpha > 0$, the following inequalities hold for all $t \ge 0$

$$\begin{aligned}
&\dot E(t) + 2\delta E(t)\\
&\le \dot E(t) + 2\delta E(t) + \alpha\sum_{j=1}^N\int_{x_{j-1}}^{x_j}\left[z_x^2(x,t) - \frac{\pi^2}{\Delta^2}f_j^2(x,t)\right]dx\\
&\le -(\beta-\alpha)\int_0^1 z_x^2 dx + \sum_{j=1}^N\int_{x_{j-1}}^{x_j}[z \quad f_j]\Upsilon\begin{bmatrix} z\\ f_j\end{bmatrix}dx \le 0
\end{aligned}$$

if $\alpha < \beta$ and

$$\Upsilon \triangleq \begin{bmatrix} -\mu+\lambda+\delta & \frac{\mu}{2}\\ * & -\alpha\frac{\pi^2}{\Delta^2}\end{bmatrix} \le 0. \tag{5.17}$$

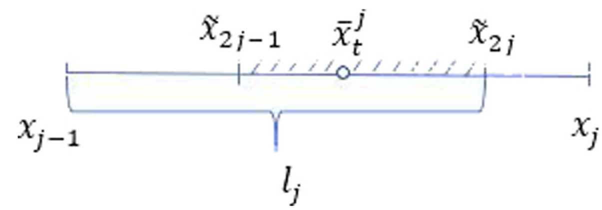


Fig. 1. Subdomain $[\tilde x_{2j-1}, \tilde x_{2j}]$, point $\bar x_t^j$ and l_j.

Therefore, given positive scalars Δ and $\mu > \lambda$, if there exist $\alpha > 0$ and $\delta > 0$ such that $\alpha < \beta$ and the LMI (5.17) holds, then the closed-loop system is *globally exponentially stable* in the L^2-sense: $E(t) \le e^{-2\delta t}E(0) \quad \forall t \ge 0$.

5.2. Controller distributed on subdomains

In this subsection, we are concerned with the case that the actuation does not cover the whole domain [0, 1] and the averaged measurements are measured over the parts of the subdomains. As in Wang and Wu (2014), let

$$0 \le \tilde x_1 < \tilde x_2 \le \tilde x_3 < \tilde x_4 \le \cdots \le \tilde x_{2N-1} < \tilde x_{2N} \le 1,$$

$$[\tilde x_{2j-1}, \tilde x_{2j}] \subset [x_{j-1}, x_j], \ j = 1,2,\ldots,N.$$

Define the shape functions

$$\begin{cases} b_j(x) = 1, \ x \in [\tilde x_{2j-1}, \tilde x_{2j}],\\ b_j(x) = 0, \ otherwise,\end{cases} \quad j = 1,\ldots,N. \tag{5.18}$$

Now we consider the system (2.1) under the averaged measurements

$$y_j(t) = \frac{\int_{\tilde x_{2j-1}}^{\tilde x_{2j}} z(\xi,t)d\xi}{\tilde\Delta_j}, \quad \tilde\Delta_j = \tilde x_{2j} - \tilde x_{2j-1} \le \tilde\Delta. \tag{5.19}$$

By applying the first mean value theorem, we obtain that there exists a point $\bar x_t^j \in [\tilde x_{2j-1}, \tilde x_{2j}]$ such that

$$y_j(t)\tilde\Delta_j = \int_{\tilde x_{2j-1}}^{\tilde x_{2j}} z(\xi,t)d\xi = z(\bar x_t^j, t)\tilde\Delta_j. \tag{5.20}$$

See Fig. 1, where

$$l_j = \max\{\tilde x_{2j} - x_{j-1}, x_j - \tilde x_{2j-1}\}. \tag{5.21}$$

By selecting the controller (2.5), we arrive at the closed-loop system

$$\begin{cases} z_t(x,t) + z(x,t)z_x(x,t) - \beta z_{xx}(x,t) - \lambda z(x,t) + z_{xxx}(x,t)\\ = -\mu\sum_{j=1}^N b_j(x)(1-\chi_{[0,h]}(t))\left[z(\bar x_t^j, t) - f_j(t)\right],\\ z(0,t) = z(1,t), \ z_x(0,t) = z_x(1,t),\\ z_{xx}(0,t) = z_{xx}(1,t),\end{cases} \tag{5.22}$$

where

$$f_j(t) \triangleq y_j(t) - y_j(t-h) = \frac{\int_{\tilde x_{2j-1}}^{\tilde x_{2j}}[z(\xi,t) - z(\xi,t-h)]d\xi}{\tilde\Delta_j}.$$

For the shape function (5.18), we arrive at the following result:

Theorem 5.2. *Consider the closed-loop system (5.22) under the shape function (5.18). Denote*

$$l \triangleq \max_j l_j, \qquad \bar{\Delta} \triangleq \min_j \frac{\tilde{\Delta}_j}{\Delta_j}. \tag{5.23}$$

Given positive scalars $\mu > \lambda$, h, δ, l, $\bar{\Delta}$ and positive tuning parameters $\delta_1 > \lambda$, $C > 0$ and $C_1 > 0$. Let there exist scalars $\gamma > 0$, $\Gamma > 0$, $P_1 > 0$, $P_2 < 0$, $P > 0$, $Q > 0$, $R > 0$, $S > 0$, $\alpha_i > 0\,(i = 0, 1, 2, 3)$, $P_3 \in \mathbb{R}$ and $\nu \in \mathbb{R}$ such that the LMIs (4.3), (4.5), (4.6), (4.8) and

$$\tilde{\Theta}|_{z=C_1} \le 0,\ \tilde{\Theta}|_{z=-C_1} \le 0 \tag{5.24}$$

hold, where

$$\tilde{\Theta} = \begin{bmatrix} \tilde{\theta}_{11} & \tilde{\theta}_{12} & 0 & \nu & \tilde{\theta}_{15} & 0 & \tilde{\theta}_{17} & 0 & 0 & 0 \\ * & \tilde{\theta}_{22} & 0 & 0 & -P_3 & 0 & 0 & 0 & 0 & 0 \\ * & * & \tilde{\theta}_{33} & Pz & -P_2 z & -P_2\beta & 0 & 0 & 0 & 0 \\ * & * & * & \tilde{\theta}_{44} & 0 & P_2 & 0 & 0 & 0 & 0 \\ * & * & * & * & \tilde{\theta}_{55} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & \tilde{\theta}_{66} & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & \tilde{\theta}_{77} & \tilde{\theta}_{78} & P\mu & -P_2\mu \\ * & * & * & * & * & * & * & \tilde{\theta}_{88} & -P\mu & P_2\mu \\ * & * & * & * & * & * & * & * & -\alpha_2 & 0 \\ * & * & * & * & * & * & * & * & * & -\alpha_1 \end{bmatrix},$$

$$\tilde{\theta}_{11} = 2P_2 + 2P_1\lambda + Rh + Q + 2\delta P_1 + \alpha_0 - \alpha_3\frac{\pi^2}{4l^2},$$

$$\tilde{\theta}_{12} = -P_2 - \alpha_0,$$

$$\tilde{\theta}_{15} = P_2\lambda + P_3 + 2\delta P_2,$$

$$\tilde{\theta}_{17} = \alpha_3\frac{\pi^2}{4l^2},$$

$$\tilde{\theta}_{22} = -Qe^{-2\delta h} + \alpha_0,$$

$$\tilde{\theta}_{33} = -2P_1\beta + 2P\lambda + 2\nu + Sh + 2\delta P + \alpha_3,$$

$$\tilde{\theta}_{44} = -2P\beta + \alpha_2,$$

$$\tilde{\theta}_{55} = -Re^{-2\delta h}\frac{1}{h} + 2\delta P_3 + \alpha_1,$$

$$\tilde{\theta}_{66} = -Se^{-2\delta h}\frac{1}{h},$$

$$\tilde{\theta}_{77} = -2P_1\mu\bar{\Delta} - \alpha_3\frac{\pi^2}{4l^2},$$

$$\tilde{\theta}_{78} = P_1\mu\bar{\Delta},$$

$$\tilde{\theta}_{88} = -\alpha_0\bar{\Delta}.$$

If (4.14) with M given by (4.13) is satisfied, then for any initial state $z_0 \in H^3(0,1) \cap H^1_{per}(0,1)$ satisfying the compatible conditions (3.3) and $\|z_0\|_{H^1_{per}} < C$, the solution of the closed-loop system (5.22) satisfies (4.15) for all $t \ge h$.

Proof. Consider V_0 given by (4.18) and V given by (4.2). For $t \le h$, by Theorem 5.1, if LMIs (4.3), (4.5), (4.6) and (4.8) are feasible, then

$$\dot{V}_0(t) - 2\delta_1 V_0(t) \le 0,\ t \in [0, h],$$
$$\dot{V}(t) + 2\delta V(t) - 2\delta_1 V_0(t) \le 0,\ t \in [0, h].$$

For $t > h$, differentiating V along (5.22) and integrating by parts, we have (4.27) and (4.28). Hence,

$$\begin{aligned} \dot{V}_{aug}(t) &= -2P_1\beta\int_0^1 z_x^2(x,t)dx + 2P_1\lambda\int_0^1 z^2(x,t)dx \\ &-2P_1\mu\sum_{j=1}^N z^2(\bar{x}_t^j,t)\tilde{\Delta}_j \\ &+2P_1\mu\sum_{j=1}^N f_j(t)z(\bar{x}_t^j,t)\tilde{\Delta}_j \\ &-2P_2\int_0^1 z(x,t)z_x(x,t)\int_{t-h}^t z(x,s)dsdx \\ &-2P_2\beta\int_0^1 z_x(x,t)\int_{t-h}^t z_x(x,s)dsdx \\ &+2P_2\int_0^1 z_{xx}(x,t)\int_{t-h}^t z_x(x,s)dsdx \\ &+2P_2\lambda\int_0^1 z(x,t)\int_{t-h}^t z(x,s)dsdx \\ &-2P_2\mu\sum_{j=1}^N[z(\bar{x}_t^j,t) - f_j(t)]\int_{x_{j-1}}^{x_j} b_j(x) \\ &\times\int_{t-h}^t z(x,s)dsdx \\ &+2P_2\int_0^1 z(x,t)[z(x,t) - z(x,t-h)]dx \\ &+2P_3\int_0^1\int_{t-h}^t z(x,s)ds[z(x,t) - z(x,t-h)]dx, \end{aligned} \tag{5.25}$$

and

$$\begin{aligned} \dot{V}_P(t) &= -2P\beta\int_0^1 z_{xx}^2(x,t)dx + 2P\lambda\int_0^1 z_x^2(x,t)dx \\ &+2P\int_0^1 z_{xx}(x,t)z_x(x,t)z(x,t)dx \\ &+2P\mu\sum_{j=1}^N z(\bar{x}_t^j,t)\int_{x_{j-1}}^{x_j} b_j(x)z_{xx}(x,t)dx \\ &-2P\mu\sum_{j=1}^N f_j(t)\int_{x_{j-1}}^{x_j} b_j(x)z_{xx}(x,t)dx. \end{aligned} \tag{5.26}$$

The Cauchy–Schwarz inequality leads to

$$\begin{aligned} &\int_{x_{j-1}}^{x_j} f_j^2(t)dx = \Delta_j f_j^2(t) \\ &\le \frac{\Delta_j}{\tilde{\Delta}_j}\int_{\bar{x}_{2j-1}}^{\bar{x}_{2j}} [z(\xi,t) - z(\xi,t-h)]^2 d\xi \\ &\le \frac{\Delta_j}{\tilde{\Delta}_j}\int_{x_{j-1}}^{x_j} [z(\xi,t) - z(\xi,t-h)]^2 d\xi, \end{aligned}$$

$$\begin{aligned} &\int_{x_{j-1}}^{x_j}\left(b_j(x)\int_{t-h}^t z(x,s)ds\right)^2 dx \\ &\le \int_{x_{j-1}}^{x_j}\left(\int_{t-h}^t z(x,s)ds\right)^2 dx, \end{aligned}$$

and

$$\int_{x_{j-1}}^{x_j}\left(b_j(x)z_{xx}(x,t)\right)^2 dx \le \int_{x_{j-1}}^{x_j} z_{xx}^2(x,t)dx.$$

Hence, the following inequalities

$$\alpha_0\sum_{j=1}^N\int_{x_{j-1}}^{x_j}\left[[z(x,t) - z(x,t-h)]^2 - \frac{\tilde{\Delta}_j}{\Delta_j}f_j^2(t)\right]dx \ge 0,$$

$$\alpha_1\sum_{j=1}^N\int_{x_{j-1}}^{x_j}\left[\left(\int_{t-h}^t z(x,s)ds\right)^2 -\left(b_j(x)\int_{t-h}^t z(x,s)ds\right)^2\right]dx\geq 0,$$

$$\alpha_2\sum_{j=1}^N\int_{x_{j-1}}^{x_j}\left[z_{xx}^2(x,t)-\left(b_j(x)z_{xx}(x,t)\right)^2\right]dx\geq 0$$

hold for some $\alpha_i>0\,(i=0,1,2)$.

Wirtinger's inequality leads to

$$\begin{aligned}&\int_{x_{j-1}}^{x_j}[z(x,t)-z(\bar{x}_t^j,t)]^2dx\\&=\int_{x_{j-1}}^{\bar{x}_t^j}[z(x,t)-z(\bar{x}_t^j,t)]^2dx+\int_{\bar{x}_t^j}^{x_j}[z(x,t)-z(\bar{x}_t^j,t)]^2dx\\&\leq\frac{4(\bar{x}_t^j-x_{j-1})^2}{\pi^2}\int_{x_{j-1}}^{\bar{x}_t^j}z_x^2(x,t)dx+\frac{4(x_j-\bar{x}_t^j)^2}{\pi^2}\int_{\bar{x}_t^j}^{x_j}z_x^2(x,t)dx.\end{aligned}$$

Then from (5.21), we obtain

$$\int_{x_{j-1}}^{x_j}[z(x,t)-z(\bar{x}_t^j,t)]^2dx\leq\frac{4l^2}{\pi^2}\int_{x_{j-1}}^{x_j}z_x^2(x,t)dx. \tag{5.27}$$

Hence,

$$\alpha_3\sum_{j=1}^N\int_{x_{j-1}}^{x_j}\left[z_x^2(x,t)-\frac{\pi^2}{4l^2}[z(x,t)-z(\bar{x}_t^j,t)]^2\right]dx\geq 0$$

holds for some $\alpha_3>0$.

Therefore, for $t>h$, using (4.27), (4.28), (4.29), (5.25), (5.26) and applying the S-Procedure we obtain

$$\begin{aligned}&\dot{V}+2\delta V\\&\leq\dot{V}+2\delta V\\&+\alpha_0\sum_{j=1}^N\int_{x_{j-1}}^{x_j}\left[[z(x,t)-z(x,t-h)]^2-\frac{\tilde{\Delta}_j}{\Delta_j}f_j^2(t)\right]dx\\&+\alpha_1\sum_{j=1}^N\int_{x_{j-1}}^{x_j}\left[\left(\int_{t-h}^t z(x,s)ds\right)^2\right.\\&\left.-\left(b_j(x)\int_{t-h}^t z(x,s)ds\right)^2\right]dx\\&+\alpha_2\sum_{j=1}^N\int_{x_{j-1}}^{x_j}\left[z_{xx}^2(x,t)-\left(b_j(x)z_{xx}(x,t)\right)^2\right]dx\\&+\alpha_3\sum_{j=1}^N\int_{x_{j-1}}^{x_j}\left[z_x^2(x,t)-\frac{\pi^2}{4l^2}[z(x,t)-z(\bar{x}_t^j,t)]^2\right]dx\\&\leq\sum_{j=1}^N\int_{x_{j-1}}^{x_j}\tilde{\eta}^T\tilde{\Theta}\tilde{\eta}dx,\end{aligned} \tag{5.28}$$

where

$$\begin{aligned}\tilde{\eta}=\ &\mathrm{col}\{z(x,t),z(x,t-h),z_x(x,t),z_{xx}(x,t),\\&\int_{t-h}^t z(x,s)ds,\int_{t-h}^t z_x(x,s)ds,z(\bar{x}_t^j,t),f_j(t),\\&b_jz_{xx}(x,t),b_j\int_{t-h}^t z(x,s)ds\}.\end{aligned}$$

Therefore, the LMIs (5.24) and the inequality (4.14) yield (4.15). □

Remark 5.3. Consider next the system (2.1) without delay (i.e. $h=0$) under the shape functions (5.18), measurements (5.19) and the controller (2.5). In the non-delayed case, such a controller may globally exponentially stabilize the system in L^2 norm. Consider E given by Remark 5.2. Then from (5.25) we obtain

$$\begin{aligned}&\dot{E}(t)+2\delta E(t)\\&=-\beta\int_0^1 z_x^2(x,t)dx+(\lambda+\delta)\int_0^1 z^2(x,t)dx\\&-\mu\sum_{j=1}^N z^2(\bar{x}_t^j,t)\tilde{\Delta}_j.\end{aligned} \tag{5.29}$$

By using (5.27), the following inequality holds for all $t\geq 0$

$$\begin{aligned}&\dot{E}(t)+2\delta E(t)\\&\leq-\beta\frac{\pi^2}{4l^2}\sum_{j=1}^N\int_{x_{j-1}}^{x_j}[z(x,t)-z(\bar{x}_t^j,t)]^2dx\\&+(\lambda+\delta)\int_0^1 z^2(x,t)dx-\mu\sum_{j=1}^N z^2(\bar{x}_t^j,t)\tilde{\Delta}_j\\&\leq\sum_{j=1}^N\int_{x_{j-1}}^{x_j}\left[z(x)\quad z(\bar{x}_t^j)\right]\tilde{\Upsilon}\begin{bmatrix}z(x)\\z(\bar{x}_t^j)\end{bmatrix}dx\leq 0.\end{aligned} \tag{5.30}$$

if

$$\tilde{\Upsilon}\triangleq\begin{bmatrix}-\beta\frac{\pi^2}{4l^2}+\lambda+\delta & \beta\frac{\pi^2}{4l^2}\\ * & -\beta\frac{\pi^2}{4l^2}-\mu\bar{\Delta}\end{bmatrix}\leq 0. \tag{5.31}$$

Therefore, given positive scalars $l<\sqrt{\frac{\beta\pi^2}{4\lambda}}$ and $\bar{\Delta}$ (c.f. (5.23)), if there exists $\delta>0$ such that the LMI (5.31) hold, then the closed-loop system is globally exponentially stable in the L^2-sense: $E(t)\leq e^{-2\delta t}E(0),\ \forall t\geq 0$.

6. Simulation examples

Example 1. Consider the system (2.1) under controller distributed on the whole domain with parameters $\beta=0.5$ and $\lambda=15$. Fig. 2 demonstrates the time evolution of the L^2-norm $\|z(\cdot,t)\|_{L^2}^2$ for the open-loop system initialized by $z(x,0)=0.0025\sin(2\pi x)$, $0\leq x\leq 1$. It is seen that the open-loop system is unstable.

For the control law (2.5) with the point measurements or averaged measurement, by using Yalmip we verify LMI conditions of Theorem 5.1 with $\mu=20$, $\Delta=0.1$, $\delta=1$, $\delta_1=20$, $C=0.044$, $C_1=0.05$. We find that the closed-loop system (2.6) subject to (2.7) or (2.8) preserves the exponential stability with a decay rate $\delta=1$ for $h\leq 0.0019$ for any initial values satisfying $\|z_0\|_{H^1_{per}}<0.044$. In order to obtain the estimate on the set of initial conditions (inside the domain of attraction) as large as possible, we minimize P_1+P. The feasible solutions of the LMIs are given as follows: $P_1=387.3590$, $P=5.5792$.

Next a finite difference method is applied to compute the state of the closed-loop system (2.6) subject to (2.7) to illustrate the effect of the proposed feedback control law (2.5) with the point measurements. We choose the same values of parameters and the initial condition $z_0(x)=0.0025\sin(2\pi x)$, $0\leq x\leq 1$. Hence,

$$\|z_0\|_{H^1_{per}}^2=387.3590\|z_0\|^2+5.5792\|z_0'\|^2<0.044^2.$$

The steps of space and time are taken as 0.05 and 10^{-7}, respectively. Simulation of solutions under the controller

$$u_j(t)=\begin{cases}-20z(\bar{x}_j,t), & t>0,\\ 0, & t\leq 0\end{cases}$$

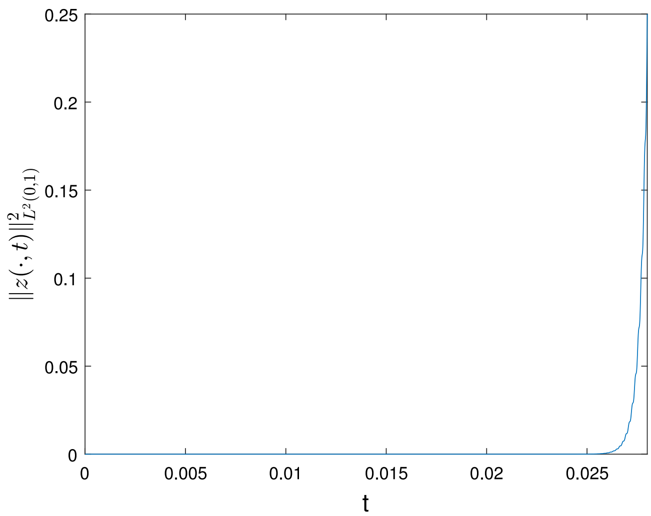


Fig. 2. Open-loop system (without control input) with $\lambda = 15$.

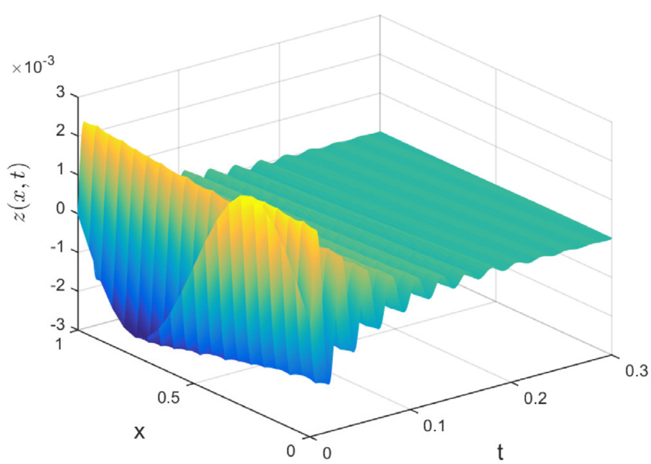


Fig. 3. State $z(x,t)$ with $h = 0.0019$ and $x_j - x_{j-1} = 0.1$ under the point measurements.

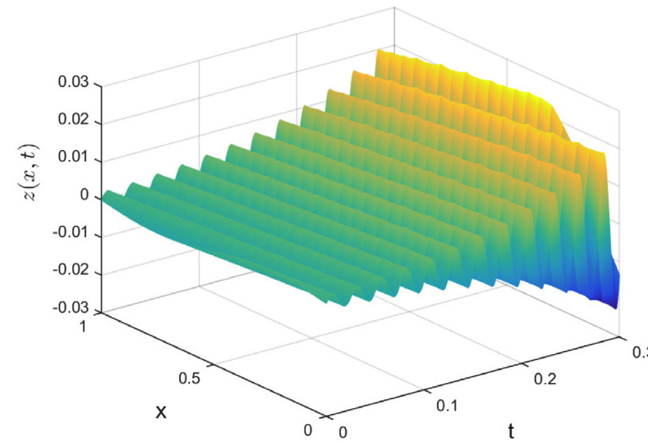


Fig. 4. State $z(x,t)$ with $h = 0.01$ and $x_j - x_{j-1} = 0.1$ under the point measurements.

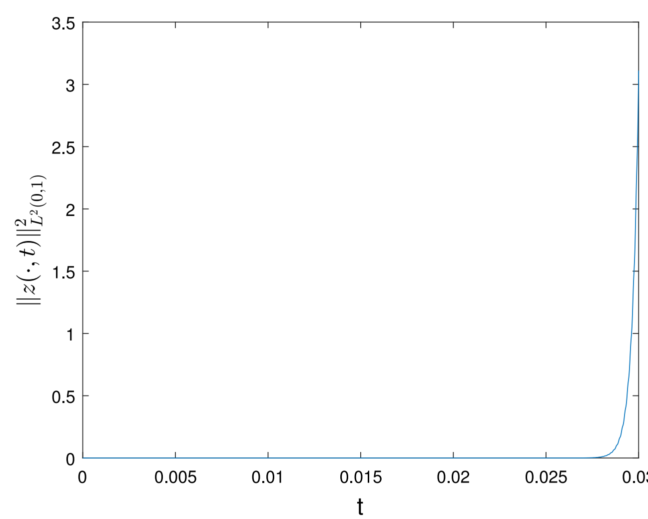


Fig. 5. Open-loop system (without control input) with $\lambda = 0.2$.

with $x_j - x_{j-1} = \Delta_j = \Delta = 0.1$, $\bar{x}_j = \frac{x_{j-1}+x_j}{2}$, $j = 1, \ldots, 10$, where the spatial domain is divided into ten sub-domains, shows that the closed-loop system is exponentially stable (see Fig. 3). Enlarging the value of h until 0.01, we find that the solution starting from the same initial condition is unbounded (see Fig. 4). The simulations of the solutions confirm the theoretical results. For the case of the averaged measurements, the simulation results are similar.

Example 2. Consider now the system (2.1) under the controller distributed on one subdomain. Here we choose parameters $\beta = 0.5$ and $\lambda = 0.2$. Fig. 5 shows that the corresponding open-loop system is unstable with the initial function $z_0(x) = 0.0025\sin(2\pi x)$, $0 \le x \le 1$.

For the control law (2.5) with the shape function (5.18) and the averaged measurement (5.19), by using Yalmip we verify LMI conditions of Theorem 5.2 with $\mu = 2$, $l = 0.8$, $\bar{\Delta} = 0.4$, $\delta = 0.14$, $\delta_1 = 20$, $C = 0.044$, $C_1 = 0.05$. We find that the closed-loop system (5.22) preserves the exponential stability with a decay rate $\delta = 0.14$ for $h \le 0.00638$ for any initial values satisfying $\|z_0\|_{H^1_{per}} < 0.044$. We proceed further with the numerical simulations of the solutions of the closed-loop system (5.22) under the controller with $\mu = 2$, $\tilde{x}_1 = 0.2$, $\tilde{x}_2 = 0.6$. Let $x_0 = 0$, $x_1 = 1$. Hence, $[\tilde{x}_1, \tilde{x}_2] \subset [x_0, x_1]$, $l = 0.8$, $\bar{\Delta} = 0.4$. In this case, the control law is given by

$$u_1(t) = -2\int_{0.2}^{0.6} z(x,t)dx/0.4 = -5\int_{0.2}^{0.6} z(x,t)dx, \ \ t \ge 0. \qquad (6.1)$$

The simulations show that the state of KdVB equation converges to zero (see Fig. 6). Simulations of the solutions confirm the theoretical results that follow from LMIs.

Next we consider the control law distributed on the whole domain:

$$u_1(t) = -2\int_0^1 z(x,t)dx, \ \ t \ge 0. \qquad (6.2)$$

Using Theorem 5.1 with $\mu = 2$, $\Delta = 1$, $\delta_1 = 20$, $C = 0.044$, $C_1 = 0.05$, we find that the LMI conditions with the same $h = 0.00638$ leads to a larger decay rate $\delta = 0.8$.

7. Concluding remarks

In this paper, distributed control of the KdVB equation in the presence of uncertain and bounded constant delay was studied under the spatially distributed (either point or averaged) measurements. By constructing a novel augmented Lyapunov function, sufficient conditions were derived such that the regional stability

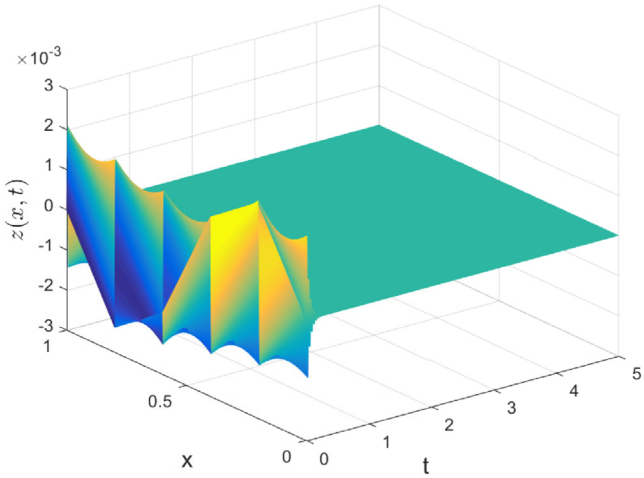


Fig. 6. State $z(x,t)$ with $[\tilde{x}_1, \tilde{x}_2] = [0.2, 0.6]$ under the averaged measurements.

of the closed-loop system is guaranteed. Numerical example illustrated the efficiency of the proposed design method.

One of the directions for the future research is extension of the obtained results to the observer-based boundary control of nonlinear PDEs.

Appendix. Proof of Proposition 3.1

The proof is based on the Galerkin approximation method. By arguments of Theorem 2.2 of Larkin (2004) we establish the well-posedness of weak solution of (2.6) subject to (2.7) or (2.8).

Define $H^3_{per}(0,1) = \{g \in H^3(0,1) \cap H^1_{per}(0,1) : g'(0) = g'(1),\ g''(0) = g''(1)\}$ equipped with the norm of $H^3(0,1)$. Given $T > 0$. Suppose that $\{\phi_n\}_1^\infty$ is an orthonormal basis for $H^3_{per}(0,1)$. For any $N \in \mathbb{Z}^+$, define a finite-dimensional subspace of $H^1_{per}(0,1)$ by $V_N = \text{span}\{\phi_1, \phi_2, \ldots, \phi_N\}$. A Galerkin approximation solution to (2.6) is constructed as follows:

$$z^N(x,t) = \sum_{n=1}^{N} g_n^N(t)\phi_n(x), \tag{A.1}$$

which satisfies

$$\begin{cases} \langle z_t^N(\cdot,t),\phi\rangle + \langle z^N(\cdot,t)z_x^N(\cdot,t),\phi\rangle + \beta\langle z_x^N(\cdot,t),\phi_x\rangle \\ -\lambda\langle z^N(\cdot,t),\phi\rangle - \langle z_{xx}^N(\cdot,t),\phi_x\rangle + \langle z_{xxx}^N,\phi\rangle \\ = -\mu\sum_{j=1}^{N}\int_{\Omega_j}(1-\chi_{[0,h]}(t)) \\ \qquad\qquad \times[z^N(x,t-h) - f_j^N(x,t-h)] \\ \qquad\qquad \times\phi(x)dx \qquad \forall\,\phi \in V_N, \\ g_n^N(0) = \langle z_0,\phi_n\rangle,\ n = 1,\ldots,N. \end{cases} \tag{A.2}$$

Set $X(t) = (g_1^N(t),\ldots,g_n^N(t))^T$. From system (A.2), it follows that $X(t)$ satisfies a nonlinear ODE system:

$$\dot{X}(t) = AX(t) + F(X(t)),$$

where

$$A = -\beta(\{\langle\phi_i',\phi_j'\rangle\}_{i,j=1}^N)^T + \lambda I_N + (\{\langle\phi_i'',\phi_j'\rangle\}_{i,j=1}^N)^T,$$
$$F(X(t)) = (\langle X^T(\{\phi_i\phi_j'\}_{i,j=1}^N)X,\phi_k\rangle_{k=1}^N)^T$$
$$-\mu\sum_{j=1}^{N}\int_{\Omega_j}(1-\chi_{[0,h]}(t))[X(t-h) - (\langle f_j^N(t-h),\phi_k\rangle_{k=1}^N)^T].$$

Here $\{\langle\phi_i',\phi_j'\rangle\}_{i,j=1}^N$ and $\{\langle\phi_i'',\phi_j'\rangle\}_{i,j=1}^N$ are the matrices with the corresponding entries. Note that the nonlinearity $F(X)$ is locally Lipschitz continuous in X. Hence, the existence of functions $g_1^N(t),\ldots,g_N^N(t)$ on some interval $[0,t_N)$ is ensured by the local Lipschitz condition. To extend these functions to any $t_N = T < \infty$ and to pass to the limit as $N \to \infty$, we need a-priori estimates. The proof is split into several lemmas.

Lemma A.1. *For any $t \in [0,h]$, the following inequality holds:*

$$\max_{t\in[0,h]}\sup_N\left[\|z^N(\cdot,t)\|_{L^2}^2 + \int_0^t \|z_x^N(\cdot,s)\|_{L^2}^2 ds\right] < \infty. \tag{A.3}$$

Proof. For $t \in [0,h]$, substituting $\phi = z^N(x,t)$ into (A.2), we have

$$\frac{1}{2}\frac{d}{dt}\|z^N(\cdot,t)\|_{L^2}^2 + \beta\|z_x^N(\cdot,t)\|_{L^2}^2 = \lambda\|z^N(\cdot,t)\|_{L^2}^2. \tag{A.4}$$

Note that $\langle z^N z_x^N, z^N\rangle$ and $\langle z_{xx}^N, z_x^N\rangle$ vanish after integration by parts. Then, by Gronwall inequality

$$\|z^N(\cdot,t)\|_{L^2}^2 + \int_0^t \|z_x^N(\cdot,s)\|_{L^2}^2 ds \le K\|z_0\|_{L^2}^2\ \forall t \in [0,h], \tag{A.5}$$

where K is a constant independent on N. □

Lemma A.2. *For any $t \in [0,h]$ and $z_0 \in H^3_{per}(0,1)$, the following inequality holds:*

$$\sup_N \|z_t^N(\cdot,0)\|_{L^2} < \infty. \tag{A.6}$$

Proof. Set $t = 0$ in (A.2). Then

$$\begin{aligned} &\langle z_t^N(\cdot,0),\phi\rangle + \langle z^N(\cdot,0)z_x^N(\cdot,0),\phi\rangle - \beta\langle z_{xx}^N(\cdot,0),\phi\rangle \\ &-\lambda\langle z^N(\cdot,0),\phi\rangle + \langle z_{xxx}^N(\cdot,0),\phi\rangle = 0. \end{aligned} \tag{A.7}$$

Substituting $\phi = z_t^N(x,0)$ into (A.7) we obtain

$$\begin{aligned} \|z_t^N(\cdot,0)\|_{L^2}^2 = &-\langle z^N(\cdot,0)z_x^N(\cdot,0) - \beta z_{xx}^N(\cdot,0) - \lambda z^N(\cdot,0) \\ &+ z_{xxx}^N(\cdot,0), z_t^N(\cdot,0)\rangle. \end{aligned} \tag{A.8}$$

From (A.8) the Minkowski and Cauchy–Schwarz inequalities lead to

$$\begin{aligned} \|z_t^N(\cdot,0)\|_{L^2} &\le \|z^N(\cdot,0)z_x^N(\cdot,0)\|_{L^2} + \beta\|z_{xx}^N(\cdot,0)\|_{L^2} \\ &\quad + \lambda\|z^N(\cdot,0)\|_{L^2} + \|z_{xxx}^N(\cdot,0)\|_{L^2} \\ &\le K_1\|z^N(\cdot,0)\|_{H^3} + \|z^N(\cdot,0)\|_{L^\infty}\|z_x{}^N(\cdot,0)\|_{L^2} \end{aligned} \tag{A.9}$$

for some constant $K_1 > 0$.

By using Lemma 4.1, we obtain

$$\|z^N(\cdot,0)\|_{L^\infty}^2 \le (1+\Gamma)\,\|z^N(\cdot,0)\|_{L^2}^2 + \frac{1}{\Gamma}\|z_x^N(\cdot,0)\|_{L^2}^2 \quad \forall\Gamma > 0. \tag{A.10}$$

Thus, (A.6) follows from (A.9) and (A.10). □

Lemma A.3. *For any $t \in [0,h]$, the following inequality holds:*

$$\max_{t\in[0,h]}\sup_N\left[\|z_x^N(\cdot,t)\|_{L^2}^2 + \int_0^t \|z_{xx}^N(\cdot,s)\|_{L^2}^2 ds\right] < \infty. \tag{A.11}$$

Proof. For $t \in [0,h]$, substituting $\phi = z_{xx}^N(x,t)$ into (A.2) and integrating by parts, we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|z_x^N(\cdot,t)\|_{L^2}^2 &= \langle z^N(\cdot,t)z_x^N(\cdot,t), z_{xx}^N(\cdot,t)\rangle \\ &\quad - \beta\|z_{xx}^N(\cdot,t)\|_{L^2}^2 + \lambda\|z_x^N(\cdot,t)\|_{L^2}^2. \end{aligned} \tag{A.12}$$

Note that

$$\langle z^N(\cdot,t)z_x^N(\cdot,t), z_{xx}^N(\cdot,t)\rangle$$
$$\le \|z^N(\cdot,t)\|_{L^2}\|z_x^N(\cdot,t)\|_{L^\infty}\|z_{xx}^N(\cdot,t)\|_{L^2}.$$

From Lemma A.1, it follows that

$$\langle z^N(\cdot,t)z_x^N(\cdot,t), z_{xx}^N(\cdot,t)\rangle \le K_2\|z_x^N(\cdot,t)\|_{L^\infty}\|z_{xx}^N(\cdot,t)\|_{L^2}$$

holds for some constant $K_2>0$.

From Lemma 4.1, it follows that

$$\|z_x^N(\cdot,t)\|_{L^\infty}^2 \le (1+\Gamma)\|z_x^N(\cdot,t)\|_{L^2}^2 + \frac{1}{\Gamma}\|z_{xx}^N(\cdot,t)\|_{L^2}^2.$$

The latter inequality, together with Young inequality, yields

$$\begin{aligned}&\|z_x^N(\cdot,t)\|_{L^\infty}\|z_{xx}^N(\cdot,t)\|_{L^2}\\ &\le \left(\sqrt{1+\Gamma}\|z_x^N(\cdot,t)\|_{L^2} + \frac{1}{\sqrt{\Gamma}}\|z_{xx}^N(\cdot,t)\|_{L^2}\right)\|z_{xx}^N(\cdot,t)\|_{L^2}\\ &\le \frac{1+\Gamma}{\epsilon}\|z_x^N(\cdot,t)\|_{L^2}^2 + (\epsilon+\frac{1}{\sqrt{\Gamma}})\|z_{xx}^N(\cdot,t)\|_{L^2}^2\ \forall\epsilon>0.\end{aligned}$$

Choose $\epsilon>0$ and $\Gamma>0$ such that $K_2(\epsilon+\frac{1}{\sqrt{\Gamma}})<\beta$. Then, using the estimate (A.12), and exploiting the Gronwall lemma, we obtain

$$\|z_x^N(\cdot,t)\|_{L^2}^2 + \int_0^t \|z_{xx}^N(\cdot,s)\|_{L^2}^2 ds \le K_3\|z_x^N(\cdot,0)\|_{L^2}^2,$$

where K_3 is a constant independent on N. □

Lemma A.4. *For any $t\in[0,h]$, the following inequality holds:*

$$\max_{t\in[0,h]}\sup_N\left[\|z_t^N(\cdot,t)\|_{L^2}^2 + \int_0^t \|z_{xt}^N(\cdot,s)\|_{L^2}^2 ds\right] < \infty. \tag{A.13}$$

Proof. For $t\in[0,h]$, differentiating the first equation of (A.2) with respect to t, we have

$$\begin{aligned}&\langle z_{tt}^N(\cdot,t),\phi\rangle + \langle z_t^N(\cdot,t)z_x^N(\cdot,t),\phi\rangle + \langle z^N(\cdot,t)z_{xt}^N(\cdot,t),\phi\rangle\\ &+\beta\langle z_{xt}^N(\cdot,t),\phi_x\rangle - \lambda\langle z_t^N(\cdot,t),\phi\rangle - \langle z_{xxt}^N(\cdot,t),\phi_x\rangle = 0.\end{aligned} \tag{A.14}$$

Substituting $\phi = z_t^N(x,t)$ into (A.14) we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|z_t^N(\cdot,t)\|_{L^2}^2 &= \langle z^N(\cdot,t)z_{xt}^N(\cdot,t), z_t^N(\cdot,t)\rangle\\ &\quad - \beta\|z_{xt}^N(\cdot,t)\|_{L^2}^2 + \lambda\|z_t^N(\cdot,t)\|_{L^2}^2.\end{aligned} \tag{A.15}$$

Using Lemma 4.1, we obtain

$$\|z^N(\cdot,t)\|_{L^\infty} \le \sqrt{1+\Gamma}\|z^N(\cdot,t)\|_{L^2} + \frac{1}{\sqrt{\Gamma}}\|z_x^N(\cdot,t)\|_{L^2}\quad \forall\Gamma>0.$$

Hence,

$$\begin{aligned}&\langle z^N(\cdot,t)z_{xt}^N(\cdot,t), z_t^N(\cdot,t)\rangle\\ &\le \|z^N(\cdot,t)\|_{L^\infty}\|z_{xt}^N(\cdot,t)\|_{L^2}\|z_t^N(\cdot,t)\|_{L^2}\\ &\le \left(\sqrt{1+\Gamma}\|z^N(\cdot,t)\|_{L^2} + \frac{1}{\sqrt{\Gamma}}\|z_x^N(\cdot,t)\|_{L^2}\right)\|z_{xt}^N(\cdot,t)\|_{L^2}\\ &\times\|z_t^N(\cdot,t)\|_{L^2}.\end{aligned} \tag{A.16}$$

From Lemmas A.1 and A.3, together with the latter inequality and Young's inequality we obtain

$$\begin{aligned}&\langle z^N(\cdot,t)z_{xt}^N(\cdot,t), z_t^N(\cdot,t)\rangle\\ &\le K_3\|z_{xt}^N(\cdot,t)\|_{L^2}\|z_t^N(\cdot,t)\|_{L^2}\\ &\le K_3[\epsilon_1\|z_{xt}^N(\cdot,t)\|_{L^2}^2 + \frac{1}{\epsilon_1}\|z_t^N(\cdot,t)\|_{L^2}^2]\ \forall\epsilon_1>0\end{aligned} \tag{A.17}$$

for some constant $K_3>0$.

Choose $\epsilon_1>0$ such that $K_3\epsilon_1<\beta$. Then from (A.15), (A.16) and (A.17), we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|z_t^N(\cdot,t)\|_{L^2}^2 &\le -(\beta-K_3\epsilon)\|z_{xt}^N(\cdot,t)\|_{L^2}\\ &\quad+(\lambda+\frac{K_3}{\epsilon})\|z_t^N(\cdot,t)\|_{L^2}^2\end{aligned}$$

Application of the Gronwall inequality and Lemma A.2 yield (A.13). □

Proof of Proposition 3.1 (*Continuation*). Given $T\in[0,h]$. From Lemmas A.1–A.4 and Corollary 4.19 of Robinson (2001), by arguments of Larkin (2004) (see p.178) we can extract a subsequence N_k, which is still denoted by N, such that

$$\begin{cases} z^N \rightharpoonup z \text{ in } L^\infty(0,T;H^1_{per}(0,1)) \text{ weak star,}^1\\ z_t^N \rightharpoonup z_t \text{ in } L^\infty(0,T;L^2(0,1)) \text{ weak star,}\\ z_t^N \rightharpoonup z_t \text{ in } L^2(0,T;H^1_{per}(0,1)) \text{ weak.}^2\end{cases} \tag{A.18}$$

Since for any $\psi\in C_c^\infty(0,T)$ and $\phi\in H^1_{per}(0,1)$,

$$\begin{aligned}&\int_0^T\langle z_t^N(\cdot,t),\phi\rangle\psi(t)dt + \int_0^T\langle z^N(\cdot,t)z_x^N(\cdot,t),\phi\rangle\psi(t)dt\\ &+\beta\int_0^T\langle z_x^N(\cdot,t),\phi_x\rangle\psi(t)dt - \lambda\int_0^T\langle z^N(\cdot,t),\phi\rangle\psi(t)dt\\ &-\int_0^T\langle z_{xx}^N(\cdot,t),\phi_x\rangle\psi(t)dt = 0,\end{aligned} \tag{A.19}$$

by (A.18), passing to the limit as $N\to\infty$ in (A.19), we obtain

$$\langle z_t,\phi\rangle + \langle zz_x,\phi\rangle + \beta\langle z_x,\phi_x\rangle - \lambda\langle z,\phi\rangle - \langle z_{xx},\phi_x\rangle = 0,\quad t\in[0,T]\text{ a.e.}$$

Therefore, there exists a weak solution to the system (2.6) subject to (2.7) or (2.8) for all $t\in[0,h]$. □

We apply the same arguments step-by-step for $[h,2h]$, $[2h,3h]$, ... Let $T>0$. Following this procedure, we obtain that there exists a weak solution of system (2.6) subject to (2.7) or (2.8) for all $t\in[0,T]$.

References

Azouani, A., & Titi, E. S. (2014). Feedback control of nonlinear dissipative systems by finite determining parameters - A reaction–diffusion Paradigm. *Evolution Equations and Control Theory*, *3*, 579–594.

Balogh, A., & Krstic, M. (2000). Boundary control of the Korteweg–de Vries-Burgers equation: further results on stabilization and well-posedness, with numerical demonstration. *IEEE Transactions on Automatic Control*, *45*, 1739–1745.

Brezis, H. (2011). *Functional analysis, sobolev spaces and partial differential equations*. New York: Springer-Verlag.

Cerpa, E., & Coron, J.-M. (2013). Rapid stabilization for a Korteweg–de Vries equation from the left Dirichlet boundary condition. *IEEE Transactions on Automatic Control*, *58*, 1688–1695.

Demiray, H. (2004). A travelling wave solution to the KdV-Burgers equation. *Applied Mathematics and Computation*, *154*, 665–670.

Feng, B. F., & Kawahara, T. (2000). Stationary travelling-wave solutions of an unstable KdV-Burgers equation. *Physica D: Nonlinear Phenomena*, *137*, 228–236.

Fridman, E. (2001). New Lyapunov-Krasovskii functionals for stability of linear retarded and neutral type systems. *Systems & Control Letters*, *43*, 309–319.

Fridman, E. (2014). *Introduction to time-delay systems: Analysis and control*. Basel: Birkhäuser.

Fridman, E., & Bar Am, N. (2013). Sampled-data distributed H_∞ control of transport reaction systems. *SIAM Journal on Control and Optimization*, *51*, 1500–1527.

Fridman, E., & Blighovsky, A. (2012). Robust sampled-data control of a class of semilinear parabolic systems. *Automatica*, *48*, 826–836.

Fridman, E., & Orlov, Y. (2009). Exponential stability of linear distributed parameter systems with time-varying delays. *Automatica*, *45*, 194–201.

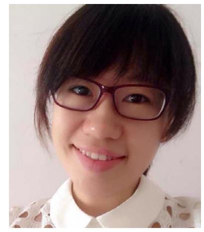
Gu, K., Kharitonov, V. L., & Chen, J. (2003). *Stability of time-delay systems*. Boston: Birkhäuser.

Hashimoto, T., & Krstic, M. (2016). Stabilization of reaction diffusion equations with state delay using boundary control input. *IEEE Transactions on Automatic Control*, *61*, 4041–4046.

[1] Definition: Let X be a Banach space. A sequence $f_n\in X^*$ converges weakly-∗ ("weakly star") to f, written $f_n \overset{*}{\rightharpoonup} f$, if $f_n(x)\to f(x)$ for every $x\in X$.

[2] Definition: Let X be a Banach space. A sequence $x_n\in X$ converges weakly to x (in X), written $x_n\rightharpoonup x$ in X, if $f_n(x)\to f(x)$ for every $f\in X^*$.

Kalantarov, V. K., & Titi, E. S. (2017). Global stabilization of the Navier–Stokes-Voight and the damped nonlinear wave equations by finite number of feedback controllers. arXiv:1706.00162.
Kang, W., & Fridman, E. (2017). Boundary control of delayed ODE-Heat cascade under actuator saturation. *Automatica*, *83*, 252–261.
Kang, W., & Fridman, E. (2018). Distributed delayed stabilization of Korteweg–de Vries-Burgers equation under sampled in space measurements. In *23rd international symposlum on mathematical theory of networks and systems* (pp. 66–71).
Krstic, M. (2009). *Delay compensation for nonlinear, adaptive, and PDE systems*. Birkhäuser.
Larkin, N. A. (2004). Korteweg–de Vries and Kuramoto-Sivashinsky equations in bounded domains. *Journal of Mathematical Analysis and Applications*, *297*, 169–185.
Liu, K., & Fridman, E. (2014). Delay-dependent methods and the first delay interval. *Systems & Control Letters*, *64*, 57–63.
Liu, W.-J., & Krstic, M. (2002). Global boundary stabilization of the Korteweg–de Vries-Burgers equation. *Journal on Computational and Applied Mathematics*, *21*, 315–354.
Lunasin, E., & Titi, E. S. (2017). Finite determining parameters feedback control for distributed nonlinear dissipative systems - a computational study. *Evolution Equations and Control Theory.*, *6*, 535–557.
Marx, S., & Cerpa, E. (2018). Output feedback stabilization of the Korteweg–de Vries equation. *Automatica*, *87*, 210–217.
Payne, L., & Weinberger, H. (1960). An optimal Poincaré inequality for convex domains. *Archive for Rational Mechanics and Analysis*, *5*, 286–292.
Robinson, J. C. (2001). *Infinite-dimensional dynamical systems: an introduction to dissipative parabolic PDEs and the theory of global attractors*. Cambridge University Press.
Selivanov, A., & Fridman, E. (2017). Sampled-data relay control of diffusion PDEs. *Automatica*, *82*, 59–68.
Seuret, A., & Gouaisbaut, F. (2013). Wirtinger-based integral inequality: Application to time-delay systems. *Automatica*, *49*, 2860–2866.
Wang, T. (1994). Stability in abstract functional-differential equations. II. Applications. *Journal of Mathematical Analysis and Applications*, *186*, 835–861.
Wang, J. W., & Wu, H. N. (2014). Lyapunov-based design of locally collocated controllers for semi-linear parabolic PDE systems. *Journal of the Franklin Institute*, *351*, 429–441.



Wen Kang received her B.S. degree from Wuhan University in 2009, and the Ph.D. degree from Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing, China, in 2014, both in Mathematics. From 2014 to 2017 she was a Lecturer at Harbin Institute of Technology. From November 2015 to November 2017 She was a Postdoctoral Researcher at the School of Electrical Engineering, Tel Aviv University, Israel. Currently she has a position at School of Automation and Electrical Engineering, University of Science and Technology Beijing. Her research interests include distributed parameter systems, time-delay systems, sliding mode control.

Emilia Fridman received the M.Sc. degree from Kuibyshev State University, USSR, in 1981 and the Ph.D. degree from Voronezh State University, USSR, in 1986, all in Mathematics. From 1986 to 1992 she was an Assistant and Associate Professor in the Department of Mathematics at Kuibyshev Institute of Railway Engineers, USSR. Since 1993 she has been at Tel Aviv University, where she is currently Professor of Electrical Engineering-Systems. Since 2018, she has been the incumbent for Chana and Heinrich Manderman Chair on System Control at Tel Aviv University. She is IEEE Fellow since 2019. She has held visiting positions at the Weierstrass Institute for Applied Analysis and Stochastics in Berlin (Germany), INRIA in Rocquencourt (France), Ecole Centrale de Lille (France), Valenciennes University (France), Leicester University (UK), Kent University (UK), CINVESTAV (Mexico), Zhejiang University (China), St. Petersburg IPM (Russia), Melbourne University (Australia), Supelec (France), KTH (Sweden).

Her research interests include time-delay systems, networked control systems, distributed parameter systems, robust control, singular perturbations and nonlinear control. She has published more than 150 articles in international scientific journals. She is the author of the monograph "Introduction to Time-Delay Systems: Analysis and Control" (Birkhauser, 2014). In 2014 she was Nominated as a Highly Cited Researcher by Thomson ISI. She serves/served as Associate Editor in Automatica, SIAM Journal on Control and Optimization and IMA Journal of Mathematical Control and Information. She is currently a member of the IFAC Council.