INTERNATIONAL JOURNAL OF ROBUST AND NONLINEAR CONTROL
Int. J. Robust. Nonlinear Control 2012; **22**:420–436
Published online 4 February 2011 in Wiley Online Library (wileyonlinelibrary.com). DOI: 10.1002/rnc.1704

Networked-based stabilization via discontinuous Lyapunov functionals

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SUMMARY

This paper presents a new stability and L_2-gain analysis of linear Networked Control Systems (NCS). The new method is inspired by discontinuous Lyapunov functions that were introduced by Naghshtabrizi *et al.* (*Syst. Control Lett.* 2008; **57**:378–385; *Proceedings 26th American Control Conference*, New York, U.S.A., July 2007) in the framework of impulsive system representation. Most of the existing works on the stability of NCS (in the framework of time delay approach) are reduced to some Lyapunov-based analysis of systems with uncertain and bounded time-varying delays. This analysis via time-independent Lyapunov functionals does not take advantage of the sawtooth evolution of the delays induced by sample-and-hold. The latter drawback was removed by Fridman (*Automatica* 2010; **46**:421–427), where *time-dependent* Lyapunov functionals for sampled-data systems were introduced. This led to essentially less conservative results. The objective of the present paper is to extend the time-dependent Lyapunov functional approach to NCS, where variable sampling intervals, data packet dropouts, and *variable network-induced delays* are taken into account. The Lyapunov functionals in this paper depend on time and on the *upper bound of the network-induced* delay, and these functionals do not grow along the input update times. The new analysis is applied to the state-feedback and to a novel network-based static output-feedback H_∞ control problems. Numerical examples show that the novel discontinuous terms in Lyapunov functionals essentially improve the results. Copyright © 2011 John Wiley & Sons, Ltd.

Received 18 September 2009; Revised 10 December 2010; Accepted 16 December 2010

KEY WORDS: networked control systems; time-varying delay; Lyapunov–Krasovskii method

1. INTRODUCTION

Three main approaches have been used to the sampled-data control and later to the Networked Control Systems (NCS), where the plant is controlled via communication network. The first one is based on discrete-time models [1, 2]. This approach is not applicable to the performance analysis (e.g. to the exponential decay rate) of the resulting continuous-time closed-loop system. The second one is a *time delay approach*, where the system is modeled as a continuous-time system with a time-varying *sawtooth delay* in the control input [3, 4]. The time delay approach via *time-independent* Lyapunov–Krasovskii functionals or Lyapunov–Razumikhin functions leads to linear matrix inequalities (LMIs) [5] for analysis and design of linear uncertain NCS [6–10]. The third approach is based on the representation of the sampled-data system in the form of *impulsive model* [11, 12]. Recently, the impulsive model approach was extended to the case of uncertain sampling intervals [13] and to NCS [14]. In [13, 14] a discontinuous Lyapunov function method was introduced, which improved the existing Lyapunov-based results.

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For systems with time-varying delays, stability conditions via time-independent Lyapunov functionals guarantee also the stability of the corresponding systems with constant delay. However, it is well known (see examples in [15] and discussions on *quenching* in [16], as well as Example 1) that in many particular systems the upper bound on the sawtooth delay that preserves the stability may be higher than the corresponding bound for the constant delay. In the recent paper [17], *time-dependent Lyapunov functionals* have been introduced for the analysis of sampled-data systems in the framework of time delay approach. The introduced time-dependent terms of Lyapunov functionals lead to qualitatively new results, taking into account the sawtooth evolution of the delays induced by sample-and-hold. In some well-studied numerical examples, the results of [17] approach the analytical values of minimum L_2-gain and of maximum sampling interval, preserving the stability.

The objective of the present paper is to extend the *discontinuous Lyapunov functional method* (in the framework of time delay approach) *to network-based H_∞ control*, where data packet dropouts and *variable network-induced delays* are taken into account. Our Lyapunov functional depends on time and on the upper bound of the network-induced delay and it does not grow along the input update times. We apply our analysis results to state-feedback and to a novel *static output-feedback H_∞ control*. We note that the observer-based control via network is usually encountered with some waiting strategy and buffers [18]. The implementation of the network-based static output-feedback controller is simple. Sufficient conditions for the stabilization via the continuous static output-feedback can be found in the survey [19]. Similar to the sampled-data H_∞ control [20], we consider an H_∞ performance index that takes into account the updating rates of the measurement. This index is related to the energy of the measurement noise. Numerical examples show that the novel discontinuous terms in the Lyapunov functional essentially reduce the conservatism.

A conference version of the paper has been presented in [21].

Notation

Throughout the paper the superscript 'T' stands for matrix transposition, $\mathscr{R}^n$ denotes the n-dimensional Euclidean space with vector norm $\|\cdot\|$, $\mathscr{R}^{n\times m}$ is the set of all $n\times m$ real matrices, and the notation $P>0$, for $P\in\mathscr{R}^{n\times n}$ means that P is symmetric and positive definite. The symmetric elements of the symmetric matrix will be denoted by $*$. Given a positive number $\tau_M>0$, the space of functions $\phi:[-\tau_M,0]\to\mathscr{R}^n$, which are absolutely continuous on $[-\tau_M,0)$, have a finite $\lim_{\theta\to 0^-}\phi(\theta)$ and have square integrable first-order derivatives denoted by W with the norm

$$\|\phi\|_W=\max_{\theta\in[-\tau_M,0]}|\phi(\theta)|+\left[\int_{-\tau_M}^{0}|\dot{\phi}(s)|^2\,\mathrm{d}s\right]^{\frac{1}{2}}.$$

We also denote $x_t(\theta)=x(t+\theta)$, $\dot{x}_t(\theta)=\dot{x}(t+\theta)$, $(\theta\in[-\tau_M,0])$.

2. PROBLEM FORMULATION

Consider the system

$$\begin{aligned}\dot{x}(t)&=Ax(t)+B_2u(t)+B_1w(t),\\ z(t)&=C_1x(t)+D_{12}u(t),\end{aligned}\tag{1}$$

where $x(t)\in\mathscr{R}^n$ is the state vector, $w(t)\in\mathscr{R}^q$ is the disturbance, $u(t)\in\mathscr{R}^m$ is the control input, and $z(t)\in\mathscr{R}^r$ is the signal to be controlled or estimated, A, B_1, B_2, C_1, and D_{12} are system matrices with appropriate dimensions. We will consider exponential stabilization (for $w=0$) and H_∞ control of (1) via state feedback or static output feedback.

2.1. Static output-feedback control

Consider the static output-feedback control of NCS shown in Figure 1. The sampler is time-driven, whereas the controller and the Zero-Order Hold (ZOH) are event-driven (in the sense that the

DOI: 10.1002/rnc

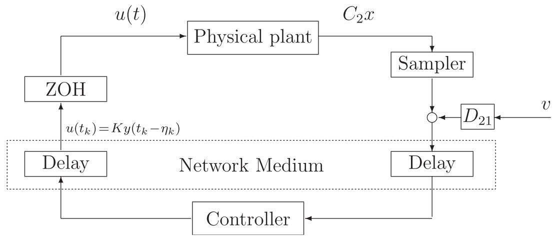

Figure 1. Networked static output-feedback control system.

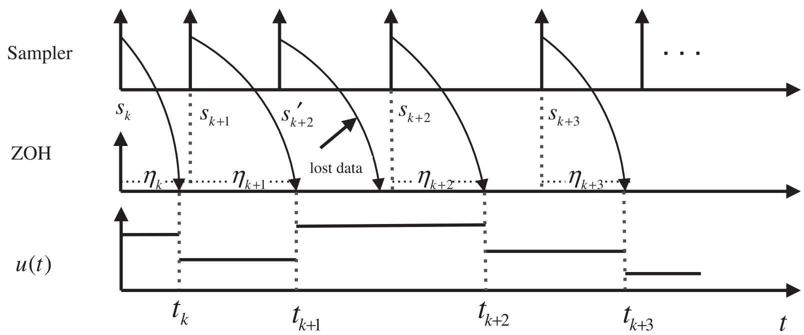

Figure 2. The timing diagram of the NCS (s'_k denotes the sampling instant that the measurement is lost).

controller and the ZOH update their outputs as soon as they receive a new sample). We assume that the measurement output $y(s_k)\in\mathscr{R}^p$ is available at discrete sampling instants

$$0=s_0<s_1<\cdots<s_k<\cdots,\quad \lim_{k\to\infty} s_k=\infty$$

and it may be corrupted by a measurement noise signal $v(s_k)$ (see Figure 1):

$$y(s_k)=C_2x(s_k)+D_{21}v(s_k). \tag{2}$$

We take into account data packet dropouts by allowing the sampling to be nonuniform. Thus, in our formulation $y(s_k)$, $k=0,1,2\dots$, correspond to the measurements that are not lost.

Denote by t_k the updating instant time of the ZOH, and suppose that the updating signal at the instant t_k has experienced a signal transmission delay η_k. The timing diagram of the considered NCS with both delay and packet dropout is shown in Figure 2, where $s_k=t_k-\eta_k$ denotes the sampling time of the data that has not been lost. Following [14], we allow the delays η_k to grow larger than $s_{k+1}-s_k$, provided that the sequence of input update times t_k remains strictly increasing. This means that if an old sample gets to the destination after the most recent one, it should be dropped.

The static output-feedback controller has a form $u(t_k)=Ky(t_k-\eta_k)$, where K is the controller gain. Thus, considering the behavior of the ZOH, we have

$$u(t)=Ky(t_k-\eta_k),\quad t_k\leqslant t<t_{k+1},\ k=0,1,2,\dots \tag{3}$$

with t_{k+1} being the next updating instant time of the ZOH after t_k.

As in [7–9, 14], we assume that

$$t_{k+1}-t_k+\eta_k\leqslant\tau_M,\quad 0\leqslant\eta_k\leqslant\eta_M,\ k=0,1,2,\dots \tag{4}$$

where η_M is a known upper bound on the network-induced delays η_k and τ_M denotes the maximum time span between the time $s_k=t_k-\eta_k$ at which the state is sampled, and the time t_{k+1} at which next update arrives at the ZOH. As a Corollary from the main result, we will formulate sufficient conditions for stabilization of NCS with constant delay $\eta_k\equiv\eta_M$.

DOI: 10.1002/rnc

Remark 1
The assumption (4) is equivalent to

$$s_{k+1}-s_k+\eta_{k+1}\leqslant\tau_M,\quad 0\leqslant\eta_{k+1}\leqslant\eta_M,\ \ k=0,1,2,\dots \tag{5}$$

The latter implies that $s_{k+1}-s_k\leqslant\tau_M$, i.e. the sampling intervals and the numbers of successive packet dropouts are uniformly bounded.

Remark 2
Consider now a more general situation, where the older sample can get the destination later than the most recent one and where the older data packet is not discarded. In this more general case our results will remain true provided that τ_M satisfying (4) can be found and that $\lim_{k\to\infty} t_k=\infty$.

Defining

$$\tau(t)=t-t_k+\eta_k,\quad t_k\leqslant t<t_{k+1}, \tag{6}$$

we obtain the following closed-loop system (1), (3):

$$\begin{aligned}\dot{x}(t)&=Ax(t)+A_1x(t-\tau(t))+A_2v(t-\tau(t))+B_1w(t),\\ z(t)&=C_1x(t)+D_1x(t-\tau(t))+D_2v(t-\tau(t)),\end{aligned} \tag{7}$$

where

$$\begin{aligned}A_1&=B_2KC_2,\quad A_2=B_2KD_{21},\\ D_1&=D_{12}KC_2,\quad D_2=D_{12}KD_{21}.\end{aligned} \tag{8}$$

Under (4) and (6), we have $0\leqslant\tau(t)\leqslant t_{k+1}-t_k+\eta_k\leqslant\tau_M$ and $\dot{\tau}(t)=1$ for $t\neq t_k$.

Denote $\bar{v}(t)=v(t-\tau(t))$ $(t\geqslant t_0)$. Then (7) has two disturbances $\bar{v}\in L_2[t_0,\infty)$ and $w\in L_2[t_0,\infty)$, where

$$\|\bar{v}\|^2_{L_2}=\int_{t_0}^{\infty} v^{\mathrm{T}}(t-\tau(t))v(t-\tau(t))\mathrm{d}t=\sum_{k=0}^{\infty}(t_{k+1}-t_k)v^{\mathrm{T}}(t_k-\eta_k)v(t_k-\eta_k). \tag{9}$$

For a given scalar $\gamma>0$, we thus define the following performance index [20]:

$$\begin{aligned}J&=\|z\|^2_{L_2}-\gamma^2(\|\bar{v}\|^2_{L_2}+\|w\|^2_{L_2})\\ &=\int_{t_0}^{\infty}[z^{\mathrm{T}}(s)z(s)-\gamma^2w^{\mathrm{T}}(s)w(s)]\mathrm{d}s-\gamma^2\sum_{k=0}^{\infty}(t_{k+1}-t_k)v^{\mathrm{T}}(t_k-\eta_k)v(t_k-\eta_k).\end{aligned} \tag{10}$$

Our objective is to find a controller of (3) that internally exponentially stabilizes the system and that leads to L_2-gain of (7) less than γ. The latter means that along (7) $J<0$ for the zero initial function and for all non-zero $w\in L_2$, $v\in l_2$, and for all allowable sampling intervals, data packet dropouts, and network-induced delays, satisfying (4).

We note that *the last term* of the performance index J *takes into account the updating rates of the measurement* and is thus related to the *energy of the measurement noise* [20]. For the sampled-data control under uniform sampling intervals, a conventional performance index has a form [12, 22]:

$$J_{\mathrm{samp}}=\int_{t_0}^{\infty}[z^{\mathrm{T}}(s)z(s)-\gamma^2w^{\mathrm{T}}(s)w(s)]\mathrm{d}s-\gamma^2\sum_{k=0}^{\infty}v^{\mathrm{T}}(t_k)v(t_k). \tag{11}$$

Index J_{samp} has a little physical sense for NCS since it does not take the updating rates into account.

DOI: 10.1002/rnc

2.2. State-feedback control

For the state-feedback case, we consider the static output-feedback formulation, where $C_2 = I$, $D_{21} = 0$, and $v(t_k - \eta_k) \equiv 0$. Thus, the resulting state-feedback controller has a form

$$u(t) = Kx(t_k - \eta_k), \quad t_k \leqslant t < t_{k+1} \tag{12}$$

and leads to the following closed-loop system (1), (12):

$$\begin{aligned} \dot{x}(t) &= Ax(t) + A_1 x(t - \tau(t)) + B_1 w(t), \\ z(t) &= C_1 x(t) + D_1 x(t - \tau(t)), \end{aligned} \tag{13}$$

where $\tau(t)$ is defined by (6) and where

$$A_1 = B_2 K, \quad D_1 = D_{12} K. \tag{14}$$

The corresponding performance index has a form:

$$J_1 = J_{|v(t_k - \eta_k) = 0} = \int_{t_0}^{\infty} [z^{\mathrm{T}}(s)z(s) - \gamma^2 w^{\mathrm{T}}(s)w(s)]\mathrm{d}s. \tag{15}$$

Our objective is to find a state feedback (12) that internally exponentially stabilizes the system and that leads to $J_1 < 0$ for the zero initial function and for all non-zero $w \in L_2$, and for all allowable sampling intervals, data packet dropouts, and network-induced delays, satisfying (4). For the sake of brevity, in the remainder of this paper the notation τ stands for the time-varying delay $\tau(t)$.

3. MAIN RESULTS

3.1. Exponential stability and L_2-gain analysis

In this section, we analyze the closed-loop systems (7) (and its particular case (13), where $C_2 = I$ and $v(t_k - \eta_k) \equiv 0$). Exponential stability of (7) with $w = v = 0$, i.e. of

$$\dot{x}(t) = Ax(t) + A_1 x(t - \tau), \quad \tau = t - t_k + \eta_k, \quad t_k \leqslant t < t_{k+1}, \tag{16}$$

as well as the L_2-gain analysis of (7) will be based on the following.

Lemma 1

Let there exist positive numbers α, β, δ, and a functional $V : \mathscr{R} \times W \times L_2[-\tau_M, 0] \to [t_0, \infty)$ such that

$$\beta|\phi(0)|^2 \leqslant V(t, \phi, \dot{\phi}) \leqslant \delta \|\phi\|_W^2. \tag{17}$$

Let the function $\bar{V}(t) = V(t, x_t, \dot{x}_t)$ be continuous from the right for $x(t)$ satisfying (7), absolutely continuous for $t \neq t_k$ and satisfies

$$\lim_{t \to t_k^-} \bar{V}(t) \geqslant \bar{V}(t_k). \tag{18}$$

(i) If along (16)

$$\dot{\bar{V}}(t) + 2\alpha \bar{V}(t) \leqslant 0 \quad \text{almost for all } t, \tag{19}$$

then $\bar{V}(t) \leqslant \mathrm{e}^{-2\alpha t} \bar{V}(t_0)$, i.e. $|x(t)|^2 \leqslant \mathrm{e}^{-2\alpha t} \frac{\delta}{\beta} \|x_{t_0}\|_W^2$ for $x_{t_0} \in W$ and thus (16) is exponentially stable with the decay rate α.

(ii) For a given $\gamma > 0$, if along (7)

$$\dot{\bar{V}}(t) + z^{\mathrm{T}}(t)z(t) - \gamma^2 w^{\mathrm{T}}(t)w(t) - \gamma^2 v^{\mathrm{T}}(t_k - \eta_k)v(t_k - \eta_k) < 0 \quad \forall w \neq 0, v \neq 0, t_k \leqslant t \leqslant t_{k+1}, \tag{20}$$

then the performance index (10) achieves $J < 0$ for all non-zero $w \in L_2$, $v \in l_2$ and for the zero initial function.

Proof
For (i) see [17].

(ii) Given $N>>1$, we integrate the first inequality (20) from t_0 till t_N. We have

$$\bar{V}(t_N)-\bar{V}(t_{N-1})+\bar{V}(t_{N-1}^-)-\bar{V}(t_{N-2})\cdots+\bar{V}(t_1^-)-\bar{V}(t_0)$$
$$+\int_{t_0}^{t_N}[z^{\mathrm{T}}(t)z(t)-\gamma^2 w^{\mathrm{T}}(t)w(t)]\mathrm{d}t-\gamma^2\sum_{k=0}^{N-1}(t_{k+1}-t_k)v^{\mathrm{T}}(t_k-\eta_k)v(t_k-\eta_k)<0.$$

As $\bar{V}(t_N)\geqslant 0$, $\bar{V}(t_{k-1}^-)-\bar{V}(t_{k-1})\geqslant 0$ for $k=2,\ldots,N$ and $\bar{V}(t_0)=0$, we find

$$\int_{t_0}^{t_N}[z^{\mathrm{T}}(t)z(t)-\gamma^2 w^{\mathrm{T}}(t)w(t)]\mathrm{d}t-\gamma^2\sum_{k=0}^{N-1}(t_{k+1}-t_k)v^{\mathrm{T}}(t_k-\eta_k)v(t_k-\eta_k)<0.$$

Thus, for $N\to\infty$ we arrive to $J<0$. □

A standard time-independent functional for delay-dependent stability of (16) with fast varying delay $\tau\in[0,\tau_M]$ has a form (see [23–25])

$$V_0(x_t,\dot{x}_t)=x^{\mathrm{T}}(t)Px(t)+\int_{t-\tau_M}^{t}\mathrm{e}^{2\alpha(s-t)}x^{\mathrm{T}}(s)Sx(s)\mathrm{d}s$$
$$+\frac{1}{\tau_M}\int_{-\tau_M}^{0}\int_{t+\theta}^{t}\mathrm{e}^{2\alpha(s-t)}\dot{x}^{\mathrm{T}}(s)R\dot{x}(s)\mathrm{d}s\,\mathrm{d}\theta,\quad P>0,\; S>0,\; R>0, \tag{21}$$

where $\alpha>0$ corresponds to exponential stability with the decay rate $\alpha>0$. In the existing papers [4–9] in the framework of input delay approach, time-independent Lyapunov functionals are usually involved.

For the case of $\eta_k\equiv 0$ (when there are no network-induced delays) and $\tau=t-t_k$, the following time-dependent functional has been introduced in [17]:

$$V_s(t,x_t,\dot{x}_t)=\bar{V}_s(t)=x^{\mathrm{T}}(t)Px(t)+\sum_{i=1}^{2}V_{is}(t,x_t,\dot{x}_t),$$

where the discontinuous terms V_{1s} and V_{2s} have the form

$$\begin{aligned}V_{1s}(t,x_t,\dot{x}_t)&=\frac{\tau_M-\tau}{\tau_M}[x(t)-x(t-\tau)]^{\mathrm{T}}X[x(t)-x(t-\tau)],\quad X>0,\\ V_{2s}(t,x_t,\dot{x}_t)&=\frac{\tau_M-\tau}{\tau_M}\int_{t-\tau}^{t}\mathrm{e}^{2\alpha(s-t)}\dot{x}^{\mathrm{T}}(s)U\dot{x}(s)\mathrm{d}s,\quad U>0,\; \alpha>0.\end{aligned} \tag{22}$$

For $\eta_k\equiv 0$, V_{1s} and V_{2s} do not increase along the jumps, since these terms are nonnegative before the jumps at t_k and become zero just after the jumps (because $t_{|t=t_k}=(t-\tau)_{|t=t_k}$). Thus, $\bar{V}_s(t)$ does not increase along the jumps and the condition $\lim_{t\to t_k^-}\bar{V}_s(t)\geqslant\bar{V}_s(t_k)$ holds.

In the case of $\eta_k\neq 0$ and $\tau=t-t_k+\eta_k$, the discontinuous terms (22) cannot be used, because $t_{|t=t_k}\neq(t-\tau)_{|t=t_k}=t_k-\eta_k$. One can modify V_{1s} and V_{2s} as follows:

$$\begin{aligned}\tilde{V}_{1s}(t,x_t,\dot{x}_t)&=(t_{k+1}-t)[x(t)-x(t_k)]^{\mathrm{T}}X[x(t)-x(t_k)],\quad X>0,\\ \tilde{V}_{2s}(t,x_t,\dot{x}_t)&=(t_{k+1}-t)\int_{t_k}^{t}\mathrm{e}^{2\alpha(s-t)}\dot{x}^{\mathrm{T}}(s)U\dot{x}(s)\mathrm{d}s,\quad U>0,\;\; \alpha>0\end{aligned} \tag{23}$$

and use the bounds $0\leqslant\eta_k\leqslant\eta_M$ and $t_{k+1}-t_k=t_{k+1}-s_k-\eta_k\leqslant\tau_M$. However, this leads to the overall bound $\tau_M+\eta_M$ on the delay τ, which is greater than τ_M, because

$$\tau=t-t_k+\eta_k\leqslant t_{k+1}-t_k+\eta_k\leqslant\tau_M+\eta_M. \tag{24}$$

DOI: 10.1002/rnc

Such an extension can be efficient only for $\eta_M \to 0$, whereas for bigger η_M it can lead to more conservative results than the standard results for the uncertain time-varying delays $\tau \in [0, \tau_M]$.

In the case of $\eta_k \neq 0$, the upper bound τ_M, that preserves the stability, is between the corresponding upper bounds on the (arbitrary) fast varying delay and on the sampling in the absence of network-induced delays (with $\eta_k \equiv 0$). As the biggest bound is the one for the case of $\eta_k \equiv 0$, we construct the discontinuous terms of Lyapunov functional that correspond to the 'worst case', where we have the maximum network-induced delay η_M. Defining

$$\tau_1 = \max\{0, \ \tau - \eta_M\} = \max\{0, \ t - t_k - \eta_M + \eta_k\}, \quad t_k \leqslant t < t_{k+1}, \tag{25}$$

we note that $\tau_{1|t=t_k} = 0$ and that $\tau_1 \leqslant \tau_M - \eta_M$. We consider the functional of the form

$$V(t, x_t, \dot{x}_t) = \bar{V}(t) = V_0(x_t, \dot{x}_t) + \sum_{i=1}^{2} V_i(t, x_t, \dot{x}_t), \tag{26}$$

where V_0 is defined by (21) and

$$\begin{aligned} V_1(t, x_t, \dot{x}_t) &= \frac{\tau_M - \tau}{\tau_M - \eta_M}[x(t) - x(t-\tau_1)]^{\mathrm{T}} X [x(t) - x(t-\tau_1)], \quad X > 0, \\ V_2(t, x_t, \dot{x}_t) &= \frac{\tau_M - \tau}{\tau_M - \eta_M} \int_{t-\tau_1}^{t} \mathrm{e}^{2\alpha(s-t)} \dot{x}^{\mathrm{T}}(s) U \dot{x}(s) \mathrm{d}s, \quad U > 0, \ \alpha > 0. \end{aligned} \tag{27}$$

Along the input update times $t = t_k$, V_1 and V_2 do not increase since these terms are nonnegative before t_k and become zero just after t_k (because $t_{|t=t_k} = (t - \tau_1)_{|t=t_k}$). Thus, $\bar{V}$ does not increase along the input update times and the condition $\lim_{t \to t_k^-} \bar{V}(t) \geqslant \bar{V}(t_k)$ holds.

By using the discontinuous Lyapunov functional (26), we obtain the following sufficient conditions:

Theorem 1

(i) Given $\alpha > 0$, let there exist $n \times n$-matrices $P > 0$, $R > 0$, $U > 0$, $X > 0$, $S > 0$, T_{11}, P_{2i}, P_{3i}, T_{2i}, M_{2i}, Y_{ij}, and $Z_{ij} (i, j = 1, 2)$ such that the following four LMIs:

$$\Psi_{11} = \begin{bmatrix} \Phi_{11} & \Phi_{12} & \tau_M Z_{11}^{\mathrm{T}} & Z_{11}^{\mathrm{T}} & \Phi_{16} \\ * & \Phi_{22} & \tau_M Z_{12}^{\mathrm{T}} & Z_{12}^{\mathrm{T}} & \Phi_{26} \\ * & * & -R\mathrm{e}^{-2\alpha\tau_M} & 0 & 0 \\ * & * & * & -S\mathrm{e}^{-2\alpha\tau_M} & 0 \\ * & * & * & * & T_{11} + T_{11}^{\mathrm{T}} \end{bmatrix} < 0, \tag{28}$$

$$\Psi_{12} = \begin{bmatrix} \Phi_{11} & \Phi_{12} & \eta_M Y_{11}^{\mathrm{T}} & (\tau_M - \eta_M) Z_{11}^{\mathrm{T}} & Z_{11}^{\mathrm{T}} & \Phi_{16} \\ * & \Phi_{22} & \eta_M Y_{12}^{\mathrm{T}} & (\tau_M - \eta_M) Z_{12}^{\mathrm{T}} & Z_{12}^{\mathrm{T}} & \Phi_{26} \\ * & * & -\frac{\eta_M}{\tau_M} R\mathrm{e}^{-2\alpha\tau_M} & 0 & 0 & \eta_M T_{11} \\ * & * & * & -\frac{\tau_M - \eta_M}{\tau_M} R\mathrm{e}^{-2\alpha\tau_M} & 0 & 0 \\ * & * & * & * & -S\mathrm{e}^{-2\alpha\tau_M} & 0 \\ * & * & * & * & * & T_{11} + T_{11}^{\mathrm{T}} \end{bmatrix} < 0, \tag{29}$$

$$
\Psi_{21}=\begin{bmatrix}
\Omega_{11}-\frac{1-2\alpha(\tau_M-\eta_M)}{\tau_M-\eta_M}X & \Omega_{12}+X & Z_{21}^{\mathrm{T}} & (\tau_M-\eta_M)Z_{21}^{\mathrm{T}} & \eta_M M_{21}^{\mathrm{T}} & \Omega_{17}+\frac{1-2\alpha(\tau_M-\eta_M)}{\tau_M-\eta_M}X & \Omega_{18}\\
* & \Omega_{22}+U & Z_{22}^{\mathrm{T}} & (\tau_M-\eta_M)Z_{22}^{\mathrm{T}} & \eta_M M_{22}^{\mathrm{T}} & \Omega_{27}-X & \Omega_{28}\\
* & * & -S\mathrm{e}^{-2\alpha\tau_M} & 0 & 0 & 0 & T_{22}\\
* & * & * & \Omega_{55}|_{\tau=\eta_M} & 0 & 0 & (\tau_M-\eta_M)T_{22}\\
* & * & * & * & \Omega_{66} & 0 & 0\\
* & * & * & * & * & \Omega_{77}-\frac{1-2\alpha(\tau_M-\eta_M)}{\tau_M-\eta_M}X & 0\\
* & * & * & * & * & * & -T_{22}-T_{22}^{\mathrm{T}}
\end{bmatrix}<0, \tag{30}
$$

$$
\Psi_{22}=\begin{bmatrix}
\Omega_{11}-X & \Omega_{12} & Z_{21}^{\mathrm{T}} & (\tau_M-\eta_M)Y_{21}^{\mathrm{T}} & \eta_M M_{21}^{\mathrm{T}} & \Omega_{17}+\frac{1}{\tau_M-\eta_M}X & \Omega_{18}\\
* & \Omega_{22} & Z_{22}^{\mathrm{T}} & (\tau_M-\eta_M)Y_{22}^{\mathrm{T}} & \eta_M M_{22}^{\mathrm{T}} & \Omega_{27} & \Omega_{28}\\
* & * & -S\mathrm{e}^{-2\alpha\tau_M} & 0 & 0 & 0 & T_{22}\\
* & * & * & \Omega_{44}|_{\tau=\tau_M} & 0 & (\tau_M-\eta_M)T_{21} & 0\\
* & * & * & * & \Omega_{66} & 0 & 0\\
* & * & * & * & * & \Omega_{77}-\frac{1}{\tau_M-\eta_M}X & 0\\
* & * & * & * & * & * & -T_{22}-T_{22}^{\mathrm{T}}
\end{bmatrix}<0 \tag{31}
$$

are feasible, where

$$
\begin{aligned}
&\Phi_{11}=A^{\mathrm{T}}P_{21}+P_{21}^{\mathrm{T}}A+2\alpha P+S-Y_{11}-Y_{11}^{\mathrm{T}},\quad \Phi_{12}=P-P_{21}^{\mathrm{T}}+A^{\mathrm{T}}P_{31}-Y_{12},\\
&\Phi_{22}=-P_{31}-P_{31}^{\mathrm{T}}+R,\quad \Phi_{16}=Y_{11}^{\mathrm{T}}-Z_{11}^{\mathrm{T}}+P_{21}^{\mathrm{T}}A_1-T_{11},\quad \Phi_{26}=Y_{12}^{\mathrm{T}}-Z_{12}^{\mathrm{T}}+P_{31}^{\mathrm{T}}A_1,\\
&\Omega_{11}=A^{\mathrm{T}}P_{22}+P_{22}^{\mathrm{T}}A+2\alpha P+S-Y_{21}-Y_{21}^{\mathrm{T}},\quad \Omega_{12}=P-P_{22}^{\mathrm{T}}+A^{\mathrm{T}}P_{32}-Y_{22},\\
&\Omega_{17}=Y_{21}^{\mathrm{T}}-M_{21}^{\mathrm{T}}-T_{21},\quad \Omega_{18}=M_{21}^{\mathrm{T}}-Z_{21}^{\mathrm{T}}+P_{22}^{\mathrm{T}}A_1,\quad \Omega_{22}=-P_{32}-P_{32}^{\mathrm{T}}+R,\\
&\Omega_{27}=Y_{22}^{\mathrm{T}}-M_{22}^{\mathrm{T}},\quad \Omega_{28}=M_{22}^{\mathrm{T}}-Z_{22}^{\mathrm{T}}+P_{32}^{\mathrm{T}}A_1,\\
&\Omega_{44}=-(\tau-\eta_M)\left[\frac{1}{\tau_M}\mathrm{e}^{-2\alpha\tau_M}R+\frac{1}{\tau_M-\eta_M}\mathrm{e}^{-2\alpha(\tau_M-\eta_M)}U\right],\\
&\Omega_{55}=-\frac{\tau_M-\tau}{\tau_M}R\mathrm{e}^{-2\alpha\tau_M},\quad \Omega_{66}=-\frac{\eta_M}{\tau_M}R\mathrm{e}^{-2\alpha\tau_M},\quad \Omega_{77}=T_{21}+T_{21}^{\mathrm{T}}.
\end{aligned} \tag{32}
$$

Then system (16) is exponentially stable with the decay rate α for all delays (6) satisfying (4). If the above LMIs hold with $\alpha=0$, then they are feasible for a small enough $\alpha_0>0$, i.e. (16) is exponentially stable with the decay rate α_0.

(ii) Given $\gamma>0$, if the following LMIs:

$$
\left[\begin{array}{c|ccc}
 & P_{2i}^{\mathrm{T}}A_2 & P_{2i}^{\mathrm{T}}B_1 & C_1^{\mathrm{T}}\\
 & P_{3i}^{\mathrm{T}}A_2 & P_{3i}^{\mathrm{T}}B_1 & 0\\
\Psi_{ij}|_{\alpha=0} & 0 & 0 & 0\\
 & 0 & 0 & D_1^{\mathrm{T}}\\
\hline
* & -\gamma^2 I & 0 & D_2^{\mathrm{T}}\\
* & * & -\gamma^2 I & 0\\
* & * & * & -I
\end{array}\right]<0,\quad i,j=1,2 \tag{33}
$$

DOI: 10.1002/rnc

with notations given in (32) are feasible. Then (7) is internally exponentially stable and has L_2-gain less than γ.

Proof

(i) In order to apply V of (26), we consider two cases: $\tau\in[0,\eta_M)$ and $\tau\in[\eta_M,\tau_M]$.

Case 1: $\tau\in[0,\eta_M)$, where $\tau_1=0$ and thus $V_1=V_2=0$. Differentiating $\bar{V}$ along (16), we find

$$[\dot{\bar{V}}(t)+2\alpha\bar{V}(t)]|_{\tau<\eta_M}\leqslant 2x^{\mathrm{T}}(t)P\dot{x}(t)+\dot{x}^{\mathrm{T}}(t)R\dot{x}(t)-\frac{1}{\tau_M}\mathrm{e}^{-2\alpha\tau_M}\int_{t-\tau_M}^{t}\dot{x}^{\mathrm{T}}(s)R\dot{x}(s)\mathrm{d}s$$
$$+2\alpha[x^{\mathrm{T}}(t)Px(t)]+x^{\mathrm{T}}(t)Sx(t)-x^{\mathrm{T}}(t-\tau_M)\mathrm{e}^{-2\alpha\tau_M}Sx(t-\tau_M). \quad (34)$$

Following [24], we employ the representation

$$-\int_{t-\tau_M}^{t}\dot{x}^{\mathrm{T}}(s)R\dot{x}(s)\mathrm{d}s=-\int_{t-\tau_M}^{t-\tau}\dot{x}^{\mathrm{T}}(s)R\dot{x}(s)\mathrm{d}s-\int_{t-\tau}^{t}\dot{x}^{\mathrm{T}}(s)R\dot{x}(s)\mathrm{d}s. \quad (35)$$

We apply the Jensen's inequality [26]

$$\begin{aligned}\int_{t-\tau}^{t}\dot{x}^{\mathrm{T}}(s)R\dot{x}(s)\mathrm{d}s &\geqslant \frac{1}{\tau}\int_{t-\tau}^{t}\dot{x}^{\mathrm{T}}(s)\mathrm{d}s\,R\int_{t-\tau}^{t}\dot{x}(s)\mathrm{d}s,\\ \int_{t-\tau_M}^{t-\tau}\dot{x}^{\mathrm{T}}(s)R\dot{x}(s)\mathrm{d}s &\geqslant \frac{1}{\tau_M-\tau}\int_{t-\tau_M}^{t-\tau}\dot{x}^{\mathrm{T}}(s)\mathrm{d}s\,R\int_{t-\tau_M}^{t-\tau}\dot{x}(s)\mathrm{d}s.\end{aligned} \quad (36)$$

Here for $\tau=0$ we understand by

$$\frac{1}{\tau}\int_{t-\tau}^{t}\dot{x}(s)\mathrm{d}s=\lim_{\tau\to 0}\frac{1}{\tau}\int_{t-\tau}^{t}\dot{x}(s)\mathrm{d}s=\dot{x}(t).$$

For $\tau_M-\tau=0$ the vector $1/(\tau_M-\tau)\int_{t-\tau_M}^{t-\tau}\dot{x}(s)\mathrm{d}s$ is defined similarly as $\dot{x}(t-\tau_M)$. Then, denoting

$$v_{11}=\frac{1}{\tau}\int_{t-\tau}^{t}\dot{x}(s)\mathrm{d}s,\quad v_{12}=\frac{1}{\tau_M-\tau}\int_{t-\tau_M}^{t-\tau}\dot{x}(s)\mathrm{d}s, \quad (37)$$

we obtain

$$[\dot{\bar{V}}(t)+2\alpha\bar{V}(t)]|_{\tau<\eta_M}\leqslant 2x^{\mathrm{T}}(t)P\dot{x}(t)+\dot{x}^{\mathrm{T}}(t)R\dot{x}(t)-\mathrm{e}^{-2\alpha\tau_M}\frac{\tau}{\tau_M}v_{11}^{\mathrm{T}}Rv_{11}$$
$$-\mathrm{e}^{-2\alpha\tau_M}\frac{\tau_M-\tau}{\tau_M}v_{12}^{\mathrm{T}}Rv_{12}+2\alpha[x^{\mathrm{T}}(t)Px(t)]+x^{\mathrm{T}}(t)Sx(t)-\mathrm{e}^{-2\alpha\tau_M}x^{\mathrm{T}}(t-\tau_M)Sx(t-\tau_M). \quad (38)$$

We further insert free-weighting $n\times n$-matrices by adding the following expressions to $\dot{\bar{V}}$ [24]:

$$\begin{aligned}0&=2[x^{\mathrm{T}}(t)Y_{11}^{\mathrm{T}}+\dot{x}^{\mathrm{T}}(t)Y_{12}^{\mathrm{T}}+x^{\mathrm{T}}(t-\tau)T_{11}^{\mathrm{T}}][-x(t)+x(t-\tau)+\tau v_{11}],\\ 0&=2[x^{\mathrm{T}}(t)Z_{11}^{\mathrm{T}}+\dot{x}^{\mathrm{T}}(t)Z_{12}^{\mathrm{T}}][-x(t-\tau)+x(t-\tau_M)+(\tau_M-\tau)v_{12}].\end{aligned} \quad (39)$$

We use also the descriptor method [23], where the right-hand side of the expression

$$0=2[x^{\mathrm{T}}(t)P_{2j}^{\mathrm{T}}+\dot{x}^{\mathrm{T}}(t)P_{3j}^{\mathrm{T}}][Ax(t)+A_1x(t-\tau)-\dot{x}(t)], \quad (40)$$

with some $n\times n$-matrices $P_{2j}, P_{3j} (j=1)$ is added into the right-hand side of (38). Setting

$$\eta_1(t)=\mathrm{col}\{x(t),\dot{x}(t),v_{11},v_{12},x(t-\tau_M),x(t-\tau)\},$$

we obtain that

$$[\dot{\bar{V}}(t)+2\alpha\bar{V}(t)]|_{\tau<\eta_M}\leqslant\eta_1^{\mathrm{T}}(t)\Psi|_{\tau<\eta_M}\eta_1(t), \quad (41)$$

where

$$\Psi|_{\tau<\eta_M} = \begin{bmatrix} \Phi_{11} & \Phi_{12} & \tau Y_{11}^{\mathrm{T}} & (\tau_M-\tau)Z_{11}^{\mathrm{T}} & Z_{11}^{\mathrm{T}} & \Phi_{16} \\ * & \Phi_{22} & \tau Y_{12}^{\mathrm{T}} & (\tau_M-\tau)Z_{12}^{\mathrm{T}} & Z_{12}^{\mathrm{T}} & \Phi_{26} \\ * & * & -\frac{\tau}{\tau_M} R\mathrm{e}^{-2\alpha\tau_M} & 0 & 0 & \tau T_{11} \\ * & * & * & -\frac{\tau_M-\tau}{\tau_M} R\mathrm{e}^{-2\alpha\tau_M} & 0 & 0 \\ * & * & * & * & -S\mathrm{e}^{-2\alpha\tau_M} & 0 \\ * & * & * & * & * & T_{11}+T_{11}^{\mathrm{T}} \end{bmatrix} \quad (42)$$

with notations are given in (32).

Denoting $\eta_{12}(t)=\mathrm{col}\{x(t), \dot{x}(t), v_{12}, x(t-\tau_M), x(t-\tau)\}$, we note that the two LMIs $\Psi_{11}<0$ and $\Psi_{12}<0$ imply $\Psi|_{\tau<\eta_M}<0$ because

$$\frac{\eta_M-\tau}{\eta_M}\eta_{12}^{\mathrm{T}}(t)\Psi_{11}\eta_{12}(t)+\frac{\tau}{\eta_M}\eta_1^{\mathrm{T}}(t)\Psi_{12}\eta_1(t)=\eta_1^{\mathrm{T}}(t)\Psi_{|\tau<\eta_M}\eta_1(t)<0.$$

This means that if $\Psi_{11}<0$ and $\Psi_{12}<0$, then $[\dot{\bar{V}}(t)+2\alpha\bar{V}(t)]|_{\tau<\eta_M}<0$.

Case 2: $\tau\in[\eta_M, \tau_M]$ and $\tau_1=\tau-\eta_M$. Taking into account that

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t-\tau_1)=(1-\dot{\tau}_1)\dot{x}(t-\tau_1)=0,$$

we have

$$\frac{\mathrm{d}}{\mathrm{d}t}V_1=-\frac{1}{\tau_M-\eta_M}[x(t)-x(t-\tau_1)]^{\mathrm{T}}X[x(t)-x(t-\tau_1)]+2\frac{\tau_M-\tau}{\tau_M-\eta_M}\dot{x}^{\mathrm{T}}(t)X[x(t)-x(t-\tau_1)],$$

$$\frac{\mathrm{d}}{\mathrm{d}t}V_2=-\frac{1}{\tau_M-\eta_M}\int_{t-\tau_1}^{t}\mathrm{e}^{2\alpha(s-t)}\dot{x}^{\mathrm{T}}(s)U\dot{x}(s)\mathrm{d}s+\frac{\tau_M-\tau}{\tau_M-\eta_M}\dot{x}^{\mathrm{T}}(t)U\dot{x}(t)-2\alpha V_2.$$

After differentiating $\bar{V}$ along (16), we employ the representation

$$-\int_{t-\tau_M}^{t}\dot{x}^{\mathrm{T}}(s)R\dot{x}(s)\mathrm{d}s=-\int_{t-\tau_M}^{t-\tau}\dot{x}^{\mathrm{T}}(s)R\dot{x}(s)\mathrm{d}s-\int_{t-\tau}^{t-\tau_1}\dot{x}^{\mathrm{T}}(s)R\dot{x}(s)\mathrm{d}s$$
$$-\int_{t-\tau_1}^{t}\dot{x}^{\mathrm{T}}(s)R\dot{x}(s)\mathrm{d}s. \quad (43)$$

Then, similar to (36), applying Jensen's inequality and denoting

$$v_{21}=\frac{1}{\tau-\eta_M}\int_{t-\tau_1}^{t}\dot{x}(s)\mathrm{d}s, \quad v_{22}=\frac{1}{\tau_M-\tau}\int_{t-\tau_M}^{t-\tau}\dot{x}(s)\mathrm{d}s, \quad v_{23}=\frac{1}{\eta_M}\int_{t-\tau}^{t-\tau_1}\dot{x}(s)\mathrm{d}s, \quad (44)$$

we obtain

$$[\dot{\bar{V}}(t)+2\alpha\bar{V}(t)]|_{\tau\geqslant\eta_M}\leqslant 2x^{\mathrm{T}}(t)P\dot{x}(t)+\dot{x}^{\mathrm{T}}(t)\left[R+\frac{\tau_M-\tau}{\tau_M-\eta_M}U\right]\dot{x}(t)$$
$$-(\tau-\eta_M)v_{21}^{\mathrm{T}}\left[\frac{1}{\tau_M}\mathrm{e}^{-2\alpha\tau_M}R+\frac{1}{\tau_M-\eta_M}\mathrm{e}^{-2\alpha(\tau_M-\eta_M)}U\right]v_{21}$$
$$-\mathrm{e}^{-2\alpha\tau_M}\frac{\tau_M-\tau}{\tau_M}v_{22}^{\mathrm{T}}Rv_{22}-\mathrm{e}^{-2\alpha\tau_M}\frac{\eta_M}{\tau_M}v_{23}^{\mathrm{T}}Rv_{23}+x^{\mathrm{T}}(t)Sx(t)$$

DOI: 10.1002/rnc

$$
\begin{aligned}
&-x^{\mathrm{T}}(t-\tau_M)\mathrm{e}^{-2\alpha\tau_M}Sx(t-\tau_M)-\frac{1}{\tau_M-\eta_M}[x(t)-x(t-\tau_1)]^{\mathrm{T}}X[x(t)\\
&-x(t-\tau_1)]+2\frac{\tau_M-\tau}{\tau_M-\eta_M}\dot{x}^{\mathrm{T}}(t)X[x(t)-x(t-\tau_1)]\\
&+2\alpha\left[x^{\mathrm{T}}(t)Px(t)+\frac{\tau_M-\tau}{\tau_M-\eta_M}[x(t)-x(t-\tau_1)]^{\mathrm{T}}X[x(t)-x(t-\tau_1)]\right]. \qquad (45)
\end{aligned}
$$

We also insert free-weighting $n\times n$-matrices by adding the following expressions:

$$
\begin{aligned}
&0=2[x^{\mathrm{T}}(t)Y_{21}^{\mathrm{T}}+\dot{x}^{\mathrm{T}}(t)Y_{22}^{\mathrm{T}}+x^{\mathrm{T}}(t-\tau_1)T_{21}^{\mathrm{T}}][-x(t)+x(t-\tau_1)+(\tau-\eta_M)v_{21}],\\
&0=2[x^{\mathrm{T}}(t)Z_{21}^{\mathrm{T}}+\dot{x}^{\mathrm{T}}(t)Z_{22}^{\mathrm{T}}+x^{\mathrm{T}}(t-\tau)T_{22}^{\mathrm{T}}][-x(t-\tau)+x(t-\tau_M)+(\tau_M-\tau)v_{22}], \qquad (46)\\
&0=2[x^{\mathrm{T}}(t)M_{21}^{\mathrm{T}}+\dot{x}^{\mathrm{T}}(t)M_{22}^{\mathrm{T}}][-x(t-\tau_1)+x(t-\tau)+\eta_M v_{23}]
\end{aligned}
$$

and (40) with $j=2$ to the right-hand side of (45). Setting $\eta_2(t)=\mathrm{col}\{x(t),\dot{x}(t), x(t-\tau_M), v_{21}, v_{22}, v_{23}, x(t-\tau_1), x(t-\tau)\}$, we finally obtain that

$$
[\dot{\bar{V}}(t)+2\alpha\bar{V}(t)]|_{\tau\geqslant\eta_M}\leqslant\eta_2^{\mathrm{T}}(t)\Psi|_{\tau\geqslant\eta_M}\eta_2(t), \qquad (47)
$$

where

$$
\Psi|_{\tau\geqslant\eta_M}
=\begin{bmatrix}
\Omega_{11}-\frac{1-2\alpha(\tau_M-\tau)}{\tau_M-\eta_M}X & \Omega_{12}+\frac{\tau_M-\tau}{\tau_M-\eta_M}X & Z_{21}^{\mathrm{T}} & (\tau-\eta_M)Y_{21}^{\mathrm{T}} & (\tau_M-\tau)Z_{21}^{\mathrm{T}} & \eta_M M_{21}^{\mathrm{T}} & \Omega_{17}+\frac{1-2\alpha(\tau_M-\tau)}{\tau_M-\eta_M}X & \Omega_{18}\\
* & \Omega_{22}+\frac{\tau_M-\tau}{\tau_M-\eta_M}U & Z_{22}^{\mathrm{T}} & (\tau-\eta_M)Y_{22}^{\mathrm{T}} & (\tau_M-\tau)Z_{22}^{\mathrm{T}} & \eta_M M_{22}^{\mathrm{T}} & \Omega_{27}-\frac{\tau_M-\tau}{\tau_M-\eta_M}X & \Omega_{28}\\
* & * & -S\mathrm{e}^{-2\alpha\tau_M} & 0 & 0 & 0 & 0 & T_{22}\\
* & * & * & \Omega_{44} & 0 & 0 & (\tau-\eta_M)T_{21} & 0\\
* & * & * & * & \Omega_{55} & 0 & 0 & (\tau_M-\tau)T_{22}\\
* & * & * & * & * & \Omega_{66} & 0 & 0\\
* & * & * & * & * & * & \Omega_{77}-\frac{1-2\alpha(\tau_M-\tau)}{\tau_M-\eta_M}X & 0\\
* & * & * & * & * & * & * & -T_{22}-T_{22}^{\mathrm{T}}
\end{bmatrix} \qquad (48)
$$

with notations given in (32).

Similar to the case 1, we note that $\Psi_{21}<0$ and $\Psi_{22}<0$ imply $\Psi|_{\tau\geqslant\eta_M}<0$. So $[\dot{\bar{V}}(t)+2\alpha\bar{V}(t)]|_{\tau\geqslant\eta_M}<0$. Therefore, along (16) we have

$$
\dot{V}(t,x_t,\dot{x}_t)+2\alpha V(t,x_t,\dot{x}_t)\leqslant\max\{\eta_1^{\mathrm{T}}(t)\Psi|_{\tau<\eta_M}\eta_1(t),\eta_2^{\mathrm{T}}(t)\Psi|_{\tau\geqslant\eta_M}\eta_2(t)\}<0 \qquad (49)
$$

and the proof of (i) is completed.

(ii) Consider Lyapunov functional of (26) with $\alpha=0$. By using arguments similar to the proof of (i), we find that (20) holds if LMIs (33) are feasible. □

When the network-induced delay is constant, i.e. $\eta_k\equiv\eta_M$, we have only one case of $\tau\in[\eta_M,\tau_M]$. So the following result is obtained from Theorem 1:

Corollary 1

(i) Consider (7) with $\eta_k\equiv\eta_M$. Given $\alpha>0$, let there exist $n\times n$-matrices $P>0, R>0, U>0, X>0, S>0, P_{22}, P_{32}, T_{2i}, M_{2i}, Y_{2i}$, and $Z_{2i}(i=1,2)$ such that two LMIs $\Psi_{2i}<0(i=1,2)$ with notations given in (32) are feasible. Then (16) is exponentially stable with the decay rate α. If LMIs $\Psi_{2i}<0(i=1,2)$ hold with $\alpha=0$, then (16) is exponentially stable with a small enough decay rate.

(ii) Given $\gamma>0$, if two LMIs (33), where $i=2$, $j=1,2$ are feasible, then (7) is internally exponentially stable and the cost function (10) achieves $J<0$ for all non-zero $w\in L_2$, $v\in l_2$ and for the zero initial condition.

Remark 3
We note that the LMIs of Theorem 1 and Corollary 1 are affine in the system matrices. Therefore, in the case of polytopic type uncertainty with A, B_1, B_2, C_1, and D_{12} from the uncertain time-varying polytope

$$\Omega=\sum_{j=1}^{M} f_j(t)\Omega_j,\quad 0\leqslant f_j(t)\leqslant 1,\ \sum_{j=1}^{M} f_j(t)=1,$$
$$\Omega_j=[A^{(j)}\ \ B_1^{(j)}\ \ B_2^{(j)}\ \ C_1^{(j)}\ \ D_{12}^{(j)}], \tag{50}$$

one have to solve these LMIs simultaneously for all the M vertices Ω_j, applying the same decision variables.

Remark 4
The method of this paper can be extended to the case of $0<\eta_m\leqslant\eta_k\leqslant\eta_M$ by modifying the functional (26) as follows:

$$V_{\eta_m}(t,x_t,\dot{x}_t)=\bar{V}_{\eta_m}(t)=V_{\eta_m}(x_t,\dot{x}_t)+\sum_{i=1}^{2} V_i(t,x_t,\dot{x}_t),$$

where V_1, V_2 are defined by (27) and

$$\begin{aligned}V_{\eta_m}(x_t,\dot{x}_t)=x^{\mathrm{T}}(t)Px(t)+\int_{t-\eta_m}^{t} \mathrm{e}^{2\alpha(s-t)}x^{\mathrm{T}}(s)S_0x(s)\mathrm{d}s+\int_{t-\tau_M}^{t-\eta_m} \mathrm{e}^{2\alpha(s-t)}x^{\mathrm{T}}(s)S_1x(s)\mathrm{d}s\\ +\eta_m\int_{-\eta_m}^{0}\int_{t+\theta}^{t} \mathrm{e}^{2\alpha(s-t)}\dot{x}^{\mathrm{T}}(s)R_0\dot{x}(s)\mathrm{d}s\mathrm{d}\theta+\frac{1}{\tau_M-\eta_m}\int_{-\tau_M}^{-\eta_m}\int_{t+\theta}^{t} \mathrm{e}^{2\alpha(s-t)}\dot{x}^{\mathrm{T}}(s)\\ \times R_1\dot{x}(s)\mathrm{d}s\,\mathrm{d}\theta,\end{aligned}$$

where $P>0$, $S_i>0$, $R_i>0$ $(i=0,1)$. The latter functional leads to analysis in the following two cases: (1) $\eta_m+\eta_M\leqslant\tau_M$ (e.g. if $\eta_k\leqslant s_{k+1}-s_k$) and (2) $\eta_m+\eta_M>\tau_M$. In the first case one can consider three intervals for τ:

$$\eta_m+\eta_M\leqslant\tau_M\Longrightarrow\tau\in[\eta_m,\eta_M),\ \ \tau\in[\eta_M,\eta_m+\eta_M) \text{ and } \tau\in[\eta_m+\eta_M,\tau_M],$$

which will result in six LMIs (instead of four for $\eta_m=0$). The second case leads to consideration of two intervals for τ:

$$\eta_m+\eta_M>\tau_M\Longrightarrow\tau\in[\eta_m,\eta_M)\quad\text{and}\quad\tau\in[\eta_M,\tau_M]$$

and to four LMIs. It can be seen that such an extension essentially complicates the conditions and will not be given in this paper.

3.2. Application to network-based design

We apply the results of the previous section to *design* problems.

3.2.1. State-feedback design. In order to find the unknown gain K that exponentially stabilizes (13) with notations (14) and leads to $J_1<0$, we apply matrix inequalities of Theorem 1 to (13). This leads to nonlinear matrix inequalities because of the terms $P_2^{\mathrm{T}}B_2K$, $P_3^{\mathrm{T}}B_2K$. Following [20], we assume that $P_{3i}=\varepsilon P_{21}(i=1,2)$, where ε is a scalar. We arrive to

Corollary 2
Given $\alpha>0$, let there exist $n\times n$-matrices $\bar{P}>0$, $\bar{R}>0$, $\bar{U}>0$, $\bar{X}>0$, $\bar{S}>0$, Q, $\bar{T}_{11}$, $\bar{T}_{2i}$, $\bar{M}_{2i}$, $\bar{Y}_{ij}$, and $\bar{Z}_{ij}(i,j=1,2)$, an $n_u\times n$-matrix L and a tuning parameter $\varepsilon>0$ such that four LMIs (28)–(31) are

DOI: 10.1002/rnc

feasible, where P, R, U, X, S and T, M, Y, Z with subindexes are taken with bars and where

$$\begin{aligned}
\Phi_{11} &= Q^{\mathrm{T}}A^{\mathrm{T}}+AQ+2\alpha\bar{P}+\bar{S}-\bar{Y}_{11}-\bar{Y}_{11}^{\mathrm{T}}, \quad \Phi_{12}=\bar{P}-Q+\varepsilon Q^{\mathrm{T}}A^{\mathrm{T}}-\bar{Y}_{12},\\
\Phi_{22} &= -\varepsilon(Q+Q^{\mathrm{T}})+\bar{R}, \quad \Phi_{16}=\bar{Y}_{11}^{\mathrm{T}}-\bar{Z}_{11}^{\mathrm{T}}+B_2L-\bar{T}_{11}, \quad \Phi_{26}=\bar{Y}_{12}^{\mathrm{T}}-\bar{Z}_{12}^{\mathrm{T}}+\varepsilon B_2L,\\
\Omega_{11} &= Q^{\mathrm{T}}A^{\mathrm{T}}+AQ+2\alpha\bar{P}+\bar{S}-\bar{Y}_{21}-\bar{Y}_{21}^{\mathrm{T}}, \quad \Omega_{12}=\bar{P}-Q+\varepsilon Q^{\mathrm{T}}A^{\mathrm{T}}-\bar{Y}_{22},\\
\Omega_{18} &= \bar{M}_{21}^{\mathrm{T}}-\bar{Z}_{21}^{\mathrm{T}}+B_2L, \quad \Omega_{22}=-\varepsilon(Q+Q^{\mathrm{T}})+\bar{R}, \quad \Omega_{28}=\bar{M}_{22}^{\mathrm{T}}-\bar{Z}_{22}^{\mathrm{T}}+\varepsilon B_2L.
\end{aligned} \tag{51}$$

Then the stabilizing gain is given by $K=LQ^{-1}$. In the above conditions if only two LMIs $\Psi_{2,j}<0$, $j=1,2$, are feasible, then the results are valid for (13) with constant delay $\eta_k\equiv\eta_M$.

Proof
Consider (42), (48) with notations (14). Assuming $P_{3i}=\varepsilon P_{21}(i=1,2)$, where ε is a scalar, we denote $Q=P_{21}^{-1}$, $\bar{P}=Q^{\mathrm{T}}PQ$, $\bar{R}=Q^{\mathrm{T}}RQ$, $\bar{U}=Q^{\mathrm{T}}UQ$, $\bar{S}=Q^{\mathrm{T}}SQ$, $\bar{X}=Q^{\mathrm{T}}XQ$, $\bar{T}_{11}=Q^{\mathrm{T}}T_{11}Q$, $\bar{T}_{2i}=Q^{\mathrm{T}}T_{2i}Q$, $\bar{M}_{2i}=Q^{\mathrm{T}}M_{2i}Q$, $\bar{Y}_{ij}=Q^{\mathrm{T}}Y_{ij}Q$, $\bar{Z}_{ij}=Q^{\mathrm{T}}Z_{ij}Q$ $(i,j=1,2)$, and $L=KQ$. Multiplication of (42), (48) by diag$\{Q^{\mathrm{T}},\ldots,Q^{\mathrm{T}}\}$ and diag$\{Q,\ldots,Q\}$, from the right and the left, completes the proof. □

For the state-feedback H_∞ control, the resulting LMIs have the following form:

$$\left[\begin{array}{c|cc}
 & B_1 & C_1^{\mathrm{T}}\\
 & \varepsilon B_1 & 0\\
\Psi_{ij}|_{\alpha=0} & 0 & 0\\
 & 0 & L^{\mathrm{T}}D_{12}^{\mathrm{T}}\\
\hline
* & -\gamma^2 I & 0\\
* & * & -I
\end{array}\right]<0, \quad i,j=1,2. \tag{52}$$

Remark 5
The results of Corollary 2 apply the tuning scalar parameter ε. One way to address the tuning issue is to apply a numerical optimization algorithm, such as the program fminsearch in the optimization toolbox of Matlab.

3.2.2. Static output-feedback H_∞ control. It is well known that static output-feedback stabilization is a non-convex problem. We suggest here some solution to this problem (which may be conservative). Assume that B_2 is of full rank. Then there exists a mapping $x\mapsto\tilde{T}x$ with non-singular $n\times n$-matrix $\tilde{T}$, such that B_2 has the following partitioned form $B_2^{\mathrm{T}}=[0\ \ B^{\mathrm{T}}]$, where $B\in\mathscr{R}^{m\times m}$ is non-singular. Hence, without loss of generality, we take B_2 in the above form.

Corollary 3
Given $\gamma>0$ and tuning scalar parameters $\varepsilon_i(i=2,3)$ and a constant matrix $G\in\mathscr{R}^{m\times(n-m)}$, let there exist $n\times n$-matrices $P>0$, $R>0$, $U>0$, $X>0$, $S>0$, T_{11}, T_{2i}, M_{2i}, Y_{ij}, $Z_{ij}(i,j=1,2)$ and matrices $K\in\mathscr{R}^{m\times p}$, $G_{k1}\in\mathscr{R}^{(n-m)\times(n-m)}$, $G_{k2}\in\mathscr{R}^{(n-m)\times m}(k=2,3)$, such that four LMIs (33), where the slack variables P_{2i}, P_{3i}, $i=1,2$ are chosen of the following form:

$$P_{2i}=\begin{bmatrix} G_{21} & G_{22}\\ G & \varepsilon_2 I_m\end{bmatrix}, \quad P_{3i}=\begin{bmatrix} G_{31} & G_{32}\\ G & \varepsilon_3 I_m\end{bmatrix}, \quad i=1,2 \tag{53}$$

with notations given in (8), (32), are feasible. Then (7) is internally exponentially stable and the cost function (10) achieves $J<0$ for all non-zero $w\in L_2$, $v\in l_2$ and for the zero initial condition. If in the above conditions only two LMIs, corresponding to $i=2$, $j=1,2$, are feasible, then the results are valid for (7) with constant delay $\eta_k\equiv\eta_M$.

DOI: 10.1002/rnc

Proof
Taking into account (53), we have

$$P_{ji}^{\mathrm{T}} B_2 K C_2 = \mathrm{col}\{G^{\mathrm{T}} BKC_2, \varepsilon_j BKC_2\}, \quad i=1,2, \ j=2,3.$$

Substitution of (8) and (53) into matrix inequalities of Theorem 1 and Corollary 1 completes the proof. □

The result for the static output-feedback exponential stabilization with a given decay rate can be formulated similarly.

4. EXAMPLES

Example 1
Exponential stability and L_2-gain analysis.

Consider the system from [1, 9]:

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} 0 & 1 \\ 0 & -0.1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} u(t) + \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix} w(t), \\ z(t) &= [0 \ \ 1]x(t) + 0.1u(t), \end{aligned} \tag{54}$$

where $u(t) = -[3.75 \ \ 11.5]x(t_k - \eta_k)$, $t_k \leqslant t < t_{k+1}$. The closed-loop system with $w=0$ and with constant delay τ

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & -0.1 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ -0.375 & -1.15 \end{bmatrix} x(t-\tau) \tag{55}$$

is asymptotically stable for $\tau \leqslant 1.16$ and becomes unstable for $\tau > 1.17$. The latter means that all the existing methods via time-independent Lyapunov functionals cannot guarantee the stability of (55) for the sampling intervals that may be greater than 1.17. When there is no network-induced delay, i.e. $\eta_k \equiv 0$, the resulting τ_M determines an upper bound on the variable sampling intervals $t_{k+1} - t_k$. The results (obtained by various methods in the literature and by Theorem 1 with $\alpha = 0$) for the admissible upper bounds on the sampling intervals, which preserve the stability, are listed in Table I. From Table I, we can see that the result by Theorem 1 almost coincides with the result of [17] and is close to the exact bound 1.72 for the constant sampling.

For the values of η_M given in Table II, by applying various methods in the literature and by Theorem 1 with $\alpha = 0$, we obtain the maximum values of τ_M that preserve the stability (see Table II). The LMIs of Theorem 1 with $X > 0$ and $U > 0$ (the non-zero X and U correspond to the discontinuous terms of Lyapunov functional) lead to less conservative results than the same LMIs, where $U = 0$ or $X = 0$ (see Table II). We note that in this example the results of Corollary 1 for the constant $\eta_k \equiv \eta_M$ coincide with the results of Theorem 1 for the variable $0 \leqslant \eta_k \leqslant \eta_M$. Choosing next $\tau_M = 1.1137$, by applying Theorem 1, we obtain the maximum value of the decay rate α given in Table III for different bounds η_M.

Consider next the static output-feedback controller $u(t) = -0.1122y(t_k - \eta_k)$, $t_k \leqslant t < t_{k+1}$, where

$$y(t_k - \eta_k) = [1 \ \ 0]x(t_k - \eta_k) + 0.2v(t_k - \eta_k), \tag{56}$$

and where $\eta_M = 0.1$, $\tau_M = 1.5$. Applying LMIs of Theorem 1 (with the zero and with the non-zero X and U), we find that the resulting closed-loop system has an L_2-gain less than $\gamma = 1.27$

Table I. Maximum upper bound on the variable sampling.

Method	[25]	[13]	[27]	[28]	[17]	Th 1
τ_M	1.04	1.11	1.36	1.36	1.69	1.68

DOI: 10.1002/rnc

Table II. Maximum value of τ_M for different η_M.

τ_M/η_M	0	0.2	0.4	0.6	0.8
[25]	1.04	1.04	1.04	1.04	1.04
[14]	1.11	1.01	0.95	0.90	0.88
Th 1 ($U=0$)	1.28	1.22	1.17	1.13	1.09
Th 1 ($X=0$)	1.61	1.17	1.10	1.08	1.07
Th 1	1.68	1.26	1.18	1.14	1.10

Table III. Maximum value of α for different η_M.

η_M	0	0.01	0.1	0.2
$\alpha(U>0, X>0)$	0.26	0.21	0.12	0.07
$\alpha(U>0, X=0)$	0.14	0.12	0.05	0.01

(for $X=U=0$) and less than $\gamma=1.17$ (for $X>0$ and $U>0$). Hence, the discontinuous terms of Lyapunov functional improve the performance (the exponential decay rate and the L_2-gain).

Example 2
State-feedback stabilization.

Consider a two axis example of a three-axis milling machine tool from [2]:

$$\dot{x}(t)=\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -18.18 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -17.86 \end{bmatrix} x(t)+\begin{bmatrix} 0 & 0 \\ 515.38 & 0 \\ 0 & 0 \\ 0 & 517.07 \end{bmatrix} u(t), \tag{57}$$

where the *constant network-induced delays* $\eta_k \equiv \eta_M = 50$ ms were considered, the sampling period was chosen to be 100 ms and it was assumed at most two successive data packet dropouts. Under the above assumptions the resulting value of τ_M equals 350 ms. Under some additional assumptions on the distribution of packet dropouts, a state feedback controller has been found in [2] that exponentially stabilizes the system with the decay rate $\alpha=1.0735$.

Without additional assumptions on the packet dropouts, we apply Corollary 2 with $\varepsilon=0.4$ and we find that the state feedback with the gain

$$K=\begin{bmatrix} -0.0775 & -0.0043 & 0 & 0 \\ 0 & 0 & -0.0759 & -0.0042 \end{bmatrix}$$

stabilizes the system with a greater decay rate $\alpha=1.2408$ for *variable network-induced delays* $0 \leqslant \eta_k \leqslant 50$ ms.

Example 3
Static output-feedback H_∞ control.

We consider (54) with the measurement given by (56). It is assumed that the network-induced delay η_k satisfies $0 \leqslant \eta_k \leqslant \eta_M = 0.1$ and that $0 \leqslant t_{k+1} - s_k \leqslant \tau_M = 1.5$. Choosing $\varepsilon_2=\varepsilon_3=10$, $G=0.5$ and applying Corollary 3, we obtain a minimum performance level of $\gamma=1.51$. The corresponding static output feedback control law is $u(t)=-0.1122y(t)$. As we have seen in Example 1, the above controller, in fact, leads to a smaller performance level of $\gamma=1.17$ (which follows from the application of Theorem 1 to the resulting closed-loop system). The latter improvement of γ illustrates the conservatism of the design method.

DOI: 10.1002/rnc

5. CONCLUSIONS

A piecewise-continuous in time Lyapunov functional method has been presented for analysis and design of linear networked control systems, where variable sampling intervals, data packet dropouts, and variable network-induced delays are taken into account. This method has been developed in the framework of time delay approach. The presented results depend on the upper bound η_M of the network-induced delays. The new analysis has been applied to the state-feedback and to a novel static output-feedback H_∞ control. Different from the observer-based control, the static one is easy for implementation.

The presented method essentially reduces the conservatism. It gives insight for new constructions of Lyapunov functionals for systems with time-varying delays. The method can be applied to different networked control *design* problems.

ACKNOWLEDGEMENTS

This work was partially supported by Israel Science Foundation (grant no. 754/10) and by China Scholarship Council.

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DOI: 10.1002/rnc