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On reachable sets for linear systems with delay and bounded peak inputs[☆]

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Abstract

Linear systems with constant coefficients and time-varying delays are considered. We address the problem of finding an ellipsoid that bounds the set of the states in the Euclidean space that are reachable from the origin, in finite time, by inputs with peak value that is bounded by a prechosen positive scalar. The system may encounter uncertainties in the matrices of its state space model and in the delay length. The Lyapunov–Razumikhin approach is applied and a bounding ellipsoid is obtained by solving a set of linear matrix inequalities that depend on the upper-bound of the delay length.

Keywords: Time delays; Lyapunov–Razumikhin function; Ellipsoidal state bounding; Delay-dependent conditions; LMI

1. Introduction

Reachable set bounding was first considered in the late 1960s in the context of state estimation and it has later received a lot of attention in parameter estimation (see Durieui, Walter, & Polyak, 2001 and references therein). The bounding of the reachable set by an ellipsoid is an important issue even in robust control. It may be used for solving peak-to-peak minimization problem (Abegor, Nagpal, & Poolla, 1996) or to control problems with saturating actuators (Tarbouriech, Garcia, & Gomes da Silva, 2002; Hu, Lin, & Chen, 2002). In cases without time delay, a linear matrix inequality (LMI) solution to this problem is given in Boyd, El Ghaoui, Feron, and Balakrishnan, (1994) via Lyapunov function applying the S-procedure. The result obtained there stems from the work of Schweppe (1973).

In the present note we derive, for the first time, an ellipsoid bound on the reachable set of linear system with time delay. We adopt the method of Boyd et al. (1994) for treating systems with time delay. *Reachable sets* for such a system is, the set of all the states in the Euclidean space that are reachable from the origin, in finite time, by inputs with peak value that is bounded by some given positive scalar.

The methods which are most commonly used in the analysis and synthesis of time-delay systems are based on the Lyapunov–Krasovskii *functionals*, which are a natural generalization of the direct method of Lyapunov for ordinary differential equations (see Hale & Lunel, 1993; Kolmanovskii & Myshkis, 1999; Niculescu, 2001). On the other hand, functions are much simpler to use, and it is more natural to explore Lyapunov–Razumikhin *functions* when seeking sufficient, finite dimensional, conditions for the existence of ellipsoids that bound the set of the reachable states in the Euclidean space.

For systems with uncertain (but probably bounded) delay, LMI delay-independent and delay-dependent stability conditions have been derived by using either Lyapunov–Krasovskii functionals or Lyapunov–Razumikhin functions (see e.g. Boyd et al., 1994; Li & de Souza, 1997; Verriest & Niculescu, 1998; Kolmanovskii, Niculescu, & Richard, 1999; Niculescu, 2001 and references therein). Delay-dependent conditions via Lyapunov–Krasovskii functionals are based on different model transformations of the original system (Kolmanovskii & Myshkis, 1999; Niculescu, 2001; Fridman & Shaked, 2002). The corresponding conditions via Lyapunov–Razumikhin functions

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are derived by using the 'first-order model transformation' (see Niculescu, Dion, & Dugard, 1998; Niculescu, 2001).

In the present note, an ellipsoid is derived that contains the reachable set in the Euclidean space for a linear system with time-varying delay. A delay-dependent sufficient condition is obtained in terms of a LMI via the Razumikhin approach. The 'first-order' model transformation is applied, together with the 'parameterized' model transformation (see Niculescu, 2001), in order to derive this condition.

Notation. Throughout the paper the superscript 'T' stands for matrix transposition, $\mathscr{R}^n$ denotes the n-dimensional Euclidean space with vector norm $|\cdot|$, $\mathscr{R}^{n\times m}$ is the set of all $n\times m$ real matrices, and the notation $P>0$, for $P\in\mathscr{R}^{n\times n}$ means that P is symmetric and positive definite. The space of continuous functions $\phi:[-h,0]\to\mathscr{R}^n$ with the supremum norm $|\cdot|$ is denoted by $C_n[-h,0]$ and $x_t(\theta)\triangleq x(t+\theta)$ $(\theta\in[-h,0])$.

2. Problem formulation

Consider the following linear system with delay:

$$\dot{x}(t)=A_0x(t)+A_1x(t-\tau(t))+Bw(t), \tag{1a}$$

$$x(t)\equiv 0,\quad t\in[-h,0], \tag{1b}$$

where $x(t)\in\mathscr{R}^n$ is the system state, $w(t)\in\mathscr{R}^p$ is the constrained input, and A_0, A_1 and B are constant matrices of appropriate dimensions. For simplicity, we consider the case of a single delay. The results are easily generalized to the case of multiple delays. We assume that τ satisfies

$$0\leqslant\tau(t)\leqslant h\quad\forall t\geqslant 0 \tag{2}$$

and that

$$w^{\mathrm{T}}(t)w(t)\leqslant\bar{w}\quad\forall t\geqslant 0, \tag{3}$$

where $\bar{w}$ is a given positive scalar.

We denote the set of the reachable states with w that satisfies (3) by

$$\mathscr{R}_x\triangleq\{x(t)\in\mathscr{R}^n\,|\,x(t),\ w(t)\text{ satisfy (1) and (3)},\ t\geqslant 0\}. \tag{4}$$

We will bound $\mathscr{R}_x$ by an ellipsoid of the form

$$\mathscr{E}=\{\xi\,|\,\xi^{\mathrm{T}}P\xi\leqslant 1,\ \xi\in\mathscr{R}^n\}, \tag{5}$$

where $0<P$.

For the case without delay, a LMI condition for an ellipsoid that bounds the reachable set has been derived in Boyd et al. (1994). In the present note, an ellipsoid of the form (5) is derived that contains the reachable set $\mathscr{R}_x$ for the time-delay system (1). A delay-dependent sufficient condition for the existence of such an ellipsoid is obtained in terms of a LMI via the Razumikhin approach. A method for reducing the size of this ellipsoid is introduced.

3. The bounding ellipsoid

We apply the following 'first-order' model transformation, together with the 'parameterized' transformation (Niculescu, 2001):

$$\begin{aligned}\dot{x}(t)=&(A_0+F)x(t)+(A_1-F)x(t-\tau)\\&-F\int_{t-\tau}^{t}[A_0x(s)+A_1x(s-\tau)+Bw(s)]\,\mathrm{d}s+Bw(t),\\x(t)\equiv&\ 0,\quad t\in[-2h,0].\end{aligned} \tag{6}$$

We suppose that $w(t)=0$ and $x(t)=0$ for $t<0$. Since $x_0=0$, if $x(t)$ satisfies (1) for $t\geqslant 0$, then it satisfies also (6) for the same values of t (and not only for $t\geqslant h$ as in the case of the nonzero initial condition). The latter model transformation will lead to a delay-independent condition if one takes $F=0$. Taking $F=A_1$ a delay-dependent condition will be obtained which corresponds to the 'first-order' model transformation (Niculescu, 2001). Note that (6) is not equivalent to (1), having a double delay and additional dynamics. However, all the solutions of (1) satisfy (6).

Clearly if the function

$$V(\xi)=\xi^{\mathrm{T}}P\xi \tag{7}$$

satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}V(x(t))\leqslant 0\quad\forall x(t),\quad w(t)\text{ satisfying (6), (3)}$$
$$\text{and } V(x(t))\geqslant 1,\ t>0, \tag{8}$$

then the ellipsoid $\mathscr{E}$ given by (5) contains all the solutions of (6) starting from 0 with $w(t)$ satisfying (3) (and thus the reachable set $\mathscr{R}_x$ of (1)).

Since the expression for $(\mathrm{d}/\mathrm{d}t)V(x(t))$ depends on $x(t+\theta)$, $\theta\in[-2h,0]$ (and not just on $x(t)$), it is difficult to satisfy (8) for all $x(t+\theta)$. In fact, if a solution of (1) begins inside the ellipsoid $\mathscr{E}$, and is to leave this ellipsoid at some time t, then

$$x^{\mathrm{T}}(t+\theta)Px(t+\theta)\leqslant x^{\mathrm{T}}(t)Px(t),\quad\forall\theta\in[-2h,0]. \tag{9}$$

Consequently, (8) need only be satisfied if (9) is true. This is the basic idea of Razumikhin approach (see Razumikhin, 1960; Hale & Lunel, 1993).

Lemma 1. *Assume that* (8) *holds if* (9) *is satisfied. Then* $\mathscr{R}_x\subset\mathscr{E}$.

Proof. Denoting

$$\bar{V}(x_t)=\sup_{-2h\leqslant\theta\leqslant 0}V(x(t+\theta)),$$

there exists $\theta_0\in[-2h,0]$ such that $\bar{V}(x_t)=V(x(t+\theta_0))$ and either $\theta_0=0$ or $\theta_0<0$ and $V(x(t+\theta))<V(x(t+\theta_0))$ if $\theta_0<\theta\leqslant 0$. We first prove that for all x_t (and not just for those satisfying (9)) $\dot{\bar{V}}(x_t)\leqslant 0$ if $w^{\mathrm{T}}w\leqslant\bar{\omega}$ and

$V(x(t)) \geqslant 1$, where $\dot{\bar{V}}$ is defined as an upper right derivative as follows:

$$\dot{\bar{V}}(x_t) = \lim_{r\to 0^+} \sup \frac{1}{r}[\bar{V}(x_{t+r}) - \bar{V}(x_t)].$$

Given that $w(t)$ satisfies (3), we consider the case where $V(x(t)) \geqslant 1$. If $\theta_0 < 0$, then for a sufficiently small scalar $r > 0$, $\bar{V}(x_{t+r}) = \bar{V}(x_t)$, and thus $\dot{\bar{V}}(x_t) = 0$. If $\theta_0 = 0$, then $\dot{\bar{V}}(x_t) \leqslant 0$ by (8). Therefore,

$$\dot{\bar{V}}(x_t) \leqslant 0 \quad \forall x(t),\ w(t) \text{ satisfying (6), (3)}$$
$$\text{if } V(x(t)) \geqslant 1,\ t > 0. \tag{10}$$

We show next that it is impossible for $V(x(t))$ to be greater than 1. Assuming that $V(x(t_0)) > 1$ for some $t_0 > 0$ then, since $V(x(0)) = V(0) = 0$, there exist t_1 and t_2 such that $V(x(t_1)) = 1$, $V(x(t)) < 1$ for $t < t_1$ and $V(x(t)) > 1$ for $t_1 \leqslant t \leqslant t_2$. Hence, $\bar{V}(x_{t_1}) = 1$. On the other hand, from (10) we find that

$$1 = \bar{V}(x_{t_1}) \geqslant \bar{V}(x_t) \geqslant V(x(t)) \quad \text{for } t_1 \leqslant t \leqslant t_2.$$

We obtained a contradiction and thus $V(x(t)) \leqslant 1$ for all $t \geqslant 0$. □

Using the S-procedure (see e.g. Boyd et al., 1994), the requirements of (8) are satisfied if there exist positive scalars λ_1 and λ_2 such that

$$\Delta(x(t)) \triangleq \frac{d}{dt}V(x(t)) + \lambda_1(V(x(t)) - 1) + \lambda_2(\bar{w} - w^T(t)w(t)) \leqslant 0. \tag{11}$$

It readily follows from (11) and (6) that

$$\Delta(x(t)) = \begin{bmatrix} x(t) \\ w(t) \end{bmatrix}^T \times \begin{bmatrix} P(A_0+F) + (A_0^T + F^T)P + \lambda_1 P & PB \\ * & -\lambda_2 I \end{bmatrix} \times \begin{bmatrix} x(t) \\ w(t) \end{bmatrix} + \mu_0 + \mu_1 + \mu_2 + \mu_3 + \lambda_2\bar{w} - \lambda_1, \tag{12a}$$

where

$$\mu_0 = -2x^T(t)P(A_1 - F)x(t-\tau) \leqslant \gamma_0^{-1}x^T(t)P(A_1-F)P^{-1}(A_1-F)^T Px(t) + \gamma_0 x^T(t)Px(t), \tag{12b}$$

$$\mu_1 = -2x^T(t)PF\int_{t-\tau}^{t} A_0 x(s)\,ds \leqslant \tau\gamma_1^{-1}x^T(t)PFA_0P^{-1}A_0^TF^TPx(t) + \tau\gamma_1 x^T(t)Px(t), \tag{12c}$$

$$\mu_2 = -2x^T(t)PFA_1\int_{t-\tau}^{t} x(s-\tau)\,ds \leqslant \tau\gamma_2^{-1}x^T(t)PFA_1P^{-1}A_1^TF^TPx(t) + \tau\gamma_2 x^T(t)Px(t), \tag{12d}$$

$$\mu_3 = -2x^T(t)PF\int_{t-\tau}^{t} Bw(s)\,ds \leqslant \tau\gamma_3^{-1}x^T(t)PFBB^TF^TPx(t) + \tau\gamma_3\bar{w} \tag{12e}$$

and where γ_i, $i = 0, \ldots, 3$ are positive scalars. Requiring that

$$(\lambda_2 + \gamma_3 h)\bar{w} - \lambda_1 \leqslant 0 \tag{13}$$

we obtain that $\Delta \leqslant 0$ for all τ that satisfy (2) if the following inequality holds:

$$\begin{bmatrix} \Psi & PB & P(A_1-F) & hPFA_0 & hPFA_1 & hPFB \\ * & -\lambda_2 I & 0 & 0 & 0 & 0 \\ * & * & -\gamma_0 P & 0 & 0 & 0 \\ * & * & * & -h\gamma_1 P & 0 & 0 \\ * & * & * & * & -h\gamma_2 P & 0 \\ * & * & * & * & * & -h\gamma_3 I \end{bmatrix} \leqslant 0, \tag{14a}$$

where

$$\Psi = P(A_0+F) + (A_0^T + F^T)P + (\lambda_1 + \gamma_0 + h\gamma_1 + h\gamma_2)P. \tag{14b}$$

Denoting $W \triangleq PF$ and noticing that the smaller λ_1 is in (14) the less restrictive the inequality becomes, we obtain the following result.

Theorem 1. *Consider system (1) with a delay that satisfies (2). The reachable set of the system states achieved by the input that satisfies (3) is bounded by the prescribed ellipsoid $\mathscr{E}$ of (5) if for some positive scalars γ_i, $i = 0, \ldots, 3$ and λ there exists $W \in \mathscr{R}^{n\times n}$ that satisfy the following LMI:*

$$\begin{bmatrix} \bar{\Psi} & PB & PA_1 - W & hWA_0 & hWA_1 & hWB \\ * & -\lambda I & 0 & 0 & 0 & 0 \\ * & * & -\gamma_0 P & 0 & 0 & 0 \\ * & * & * & -h\gamma_1 P & 0 & 0 \\ * & * & * & * & -h\gamma_2 P & 0 \\ * & * & * & * & * & -h\gamma_3 I \end{bmatrix} \leqslant 0, \tag{15a}$$

where

$$\bar{\Psi} = W + W^T + PA_0 + A_0^TP + (\lambda\bar{w} + \gamma_0 + \gamma_3 h\bar{w} + h\gamma_1 + h\gamma_2)P. \tag{15b}$$

Remark 1. The LMI (15) guarantees also the internal stability of (1), where $w=0$, but not necessarily the asymptotic stability of the system.

Remark 2. Inequality (15) is linear in the decision variables only when the ellipsoid $\mathscr{E}$ is given. If the latter is not the case and one consider the matrix $0<P$ as an additional decision variable, (15) is no longer linear in all its variable. One can, however, regard then the positive scalars γ_i, $i=0,\ldots,3$ and λ, as tuning parameters. For each combination of these parameters the inequality is linear in P and W.

Since inequality (15) is affine in the system matrices the criterion of Lemma 1 can be applied to the case where these matrices are uncertain. In this case we denote

$$\Omega=[A_0\ \ A_1\ \ B]$$

and assume that $\Omega\in\mathscr{C}o\{\Omega_j,\ j=1,\ldots,N\}$, namely,

$$\Omega=\sum_{j=1}^{N} f_j\Omega_j \quad \text{for some } 0\leqslant f_j\leqslant 1,\ \sum_{j=1}^{N} f_j=1, \tag{16}$$

where the N vertices of the polytope are described by

$$\Omega_j=[A_0^{(j)}\ \ A_1^{(j)}\ \ B^{(j)}].$$

We obtain the following.

Corollary 1. *Consider system* (1) *where the delay satisfies* (2) *and the parameters of the system reside in the given polytope* Ω. *The reachable set of the system states achieved by the input that satisfies* (3) *is bounded by the prescribed ellipsoid* $\mathscr{E}$ *of* (5), *over the entire polytope, if there exist positive scalars* γ_i, $i=0,\ldots,3$ *and* λ *and* $W\in\mathscr{R}^{n\times n}$ *that satisfy the following set of LMIs*:

$$\begin{bmatrix} \bar{\Psi}_j & PB^{(j)} & PA_1^{(j)}-W & hWA_0^{(j)} & hWA_1^{(j)} & hWB^{(j)} \\ * & -\lambda I & 0 & 0 & 0 & 0 \\ * & * & -\gamma_0 P & 0 & 0 & 0 \\ * & * & * & -h\gamma_1 P & 0 & 0 \\ * & * & * & * & -h\gamma_2 P & 0 \\ * & * & * & * & * & -h\gamma_3 I \end{bmatrix} \leqslant 0, \quad j=1,\ldots,N, \tag{17a}$$

where

$$\bar{\Psi}_j=W+W^{\mathrm{T}}+PA_0^{(j)}+A_0^{(j)\mathrm{T}}P + (\gamma_0+\lambda\bar{w}+\gamma_3 h\bar{w}+h\gamma_1+h\gamma_2)P. \tag{17b}$$

Remark 3. The solution for (17), or (15), if it exists, need not be unique. One may then seek the 'smallest' possible ellipsoid. For this purpose, one may consider P as a decision variable and add the additional requirement that

$$\delta I\leqslant P. \tag{18}$$

The problem will then become one of solving (17) and (18), following the method of Remark 3, while maximizing δ. This is achieved by adding the following LMI to those solved in Corollary 1:

$$\begin{bmatrix} \bar{\delta} I_n & I_n \\ I_n & P \end{bmatrix} \geqslant 0, \tag{19}$$

where $\bar{\delta}=\delta^{-1}$ is minimized.

It is also noted that since (17a) is convex in h, a bound on the reachable set that is found by Corollary 1 for a given delay $\bar{h}$ will also be a bound on all the reachable sets that are obtained via Corollary 1 for $h<\bar{h}$. It is thus expected that $\bar{\delta}$ will be an increasing function of h.

The result of Corollary 1 leads to the following delay-independent criterion.

Corollary 2. *The reachable set of the states of* (1) *achieved by the input that satisfies* (3) *is bounded by the prescribed ellipsoid* $\mathscr{E}$ *of* (5), *over the entire polytope, for all positive delays, if there exist positive scalars* γ *and* λ *that satisfy the following set of LMIs*:

$$\begin{bmatrix} \hat{\Psi}_j & PB^{(j)} & PA_1^{(j)} \\ * & -\lambda I & 0 \\ * & * & -\gamma P \end{bmatrix} \leqslant 0, \quad j=1,\ldots,N, \tag{20a}$$

where

$$\hat{\Psi}_j=PA_0^{(j)}+A_0^{(j)\mathrm{T}}P+(\gamma+\lambda\bar{w})P. \tag{20b}$$

Remark 4. The results of Theorem 1 and its corollaries apply the tuning parameters λ and γ_i, $i=0,\ldots,3$. The question arises, how to find the optimal combination of these parameters. One way to address the tuning issue is to choose for a cost function the parameter $t_{\min}$ that is obtained while solving the feasibility problem using Matlab's LMI toolbox (Gahinet, Nemirovski, Laub, & Chilali, 1995). This scalar parameter is positive in cases where the combination of the tuning parameters is one that does not allow a feasible solution to the set of LMIs considered. Applying a numerical optimization algorithm, such as the program *fminsearch* in the optimization toolbox of Matlab (Coleman, Branch, & Grace, 1999), to the above cost function, a locally convergent solution to the problem is obtained. If the resulting minimum value of the cost function is negative, the tuning parameters that solve the problem are found.

4. Example

We consider system (1) where

$$A_0 = \begin{bmatrix} -2 & 0 \\ 0 & -0.9+\rho \end{bmatrix}, \quad A_1 = \begin{bmatrix} -1 & 0 \\ -1 & -1+0.5\rho \end{bmatrix},$$

$$B = \begin{bmatrix} -0.5 \\ 1 \end{bmatrix}, \quad \text{and} \quad \bar{w} = 1$$

and where ρ is a scalar parameter that satisfies $|\rho| < 0.2$ and $h = 0.7$. This system has been considered by many authors in the past. In Fridman and Shaked (2002) the H_∞-norm of the system was considered for $\rho = 0$. The uncertainty polytope in this case possesses $N = 2$ vertices. A solution for the three LMIs in (17) and (19) was obtained for the parameters: $\lambda = 0.25$; $\gamma_0 = 0.2$, $\gamma_1 = 1$, $\gamma_2 = 1$, and $\gamma_3 = 0.17$. The P and F obtained were

$$P = \begin{bmatrix} 0.7855 & 0.0164 \\ 0.0164 & 0.0511 \end{bmatrix} \quad \text{and}$$

$$F = -\begin{bmatrix} 0.6635 & 0.0274 \\ 0.4983 & 0.9024 \end{bmatrix}.$$

The corresponding value of $\bar{\delta}$ is 19.7052. The maximum axis length of the resulting ellipsoid is 4.4391 and the smallest axis length is 1.128.

The reachable set that corresponds to the system is depicted in Figs. 1–4. The size of the bounding ellipsoid depends on the delay bound h of (2). It increases with h. For example, for $h = 0.75$ a minimum value of $\bar{\delta} = 65.42$ is obtained (using $\lambda = 0.14$, $\gamma_0 = \gamma_3 = 0.11$, $\gamma_1 = \gamma_2 = 1.1$).

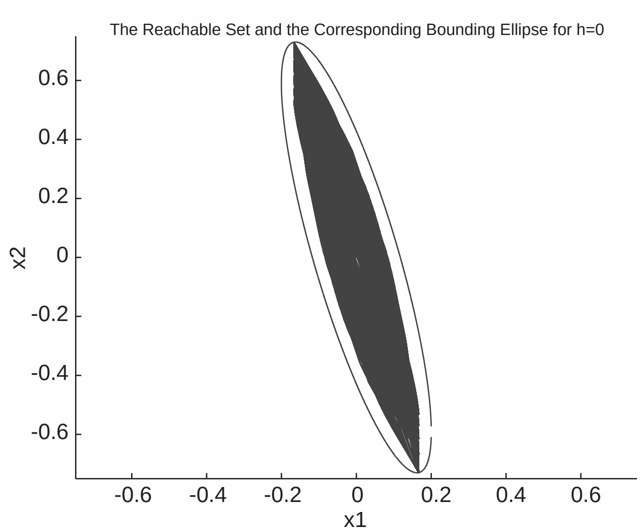

Fig. 1. The reachability set and the bounding ellipse for $h = 0$.

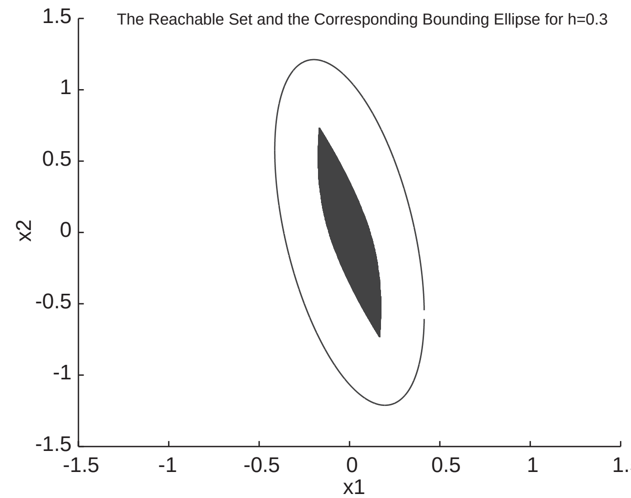

Fig. 2. The reachability set and the bounding ellipse for $h = 0.3$.

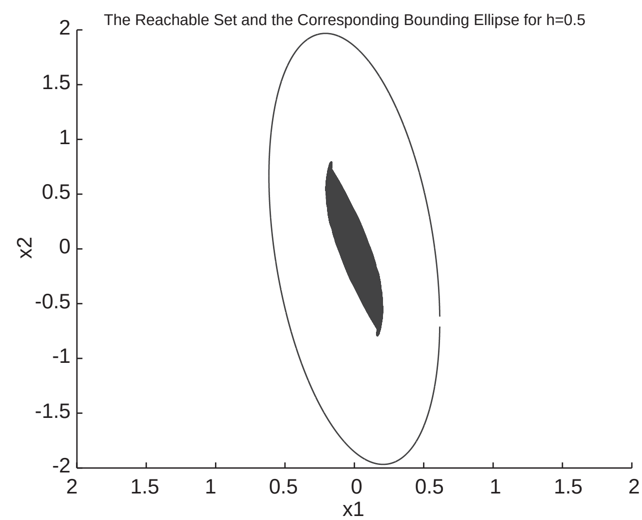

Fig. 3. The reachability set and the bounding ellipse for $h = 0.5$.

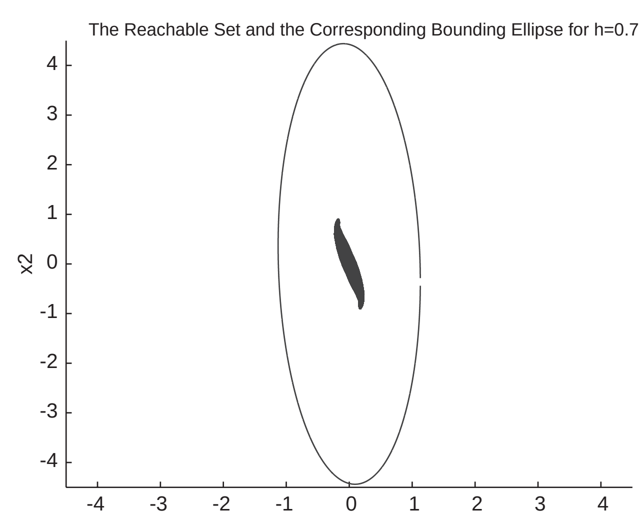

Fig. 4. The reachability set and the bounding ellipse for $h = 0.7$.

5. Conclusions

The problem of finding an ellipsoid that bounds the set of the reachable states (in the Euclidean space) for a linear system with time-varying delay is considered in the case of inputs with bounded peaks. A solution to the problem is derived by applying Lyapunov–Razumikhin functions and the S-procedure. Delay-dependent conditions for the reachable set to reside in a given ellipsoid are obtained. These conditions are based on the 'first-order' and the 'parameterized' model transformations. They are expressed in terms of inequalities that are affine in the system matrices. The latter fact allows the consideration of polytopic uncertainty in the parameters of these matrices.

The solution obtained can be used to verify whether a given ellipsoid bounds the set of the reachable states, or to find ellipsoids that bound this set. In the former problem, the solution is obtained by solving a single LMI, in the case with no uncertainty, or a set of LMIs that correspond to the vertices of the uncertainty polytope in the case with uncertainty. In the second problem, in order to obtain linear inequalities, the scalar decision parameters in the inequalities obtained should be used as tuning parameters.

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