Received: 5 March 2020 | Revised: 15 November 2020 | Accepted: 21 December 2020

DOI: 10.1002/rnc.5464

SPECIAL ISSUE ARTICLE

WILEY

Output-feedback Lyapunov redesign of uncertain systems with delayed measurements

Jing Xu[1,2] | **Leonid M. Fridman**[3] | **Emilia Fridman**[4] | **Yugang Niu**[2]

[1]Engineering Research Center of Intelligent Control for Underground Space, Ministry of Education, China University of Mining and Technology, Beijing, China

[2]Key Laboratory of Advanced Control and Optimization for Chemical Process, (East China University of Science and Technology), Ministry of Education, Shanghai, 200237, China

[3]Facultad de Ingeniería, UNAM, Av. Universidad 3000, Ciudad Universitaria, Mexico City, Mexico

[4]School of Electrical Engineering, Tel-Aviv University, Tel-Aviv, 69978, Israel

Correspondence

Jing Xu, Engineering Research Center of Intelligent Control for Underground Space, Ministry of Education, China University of Mining and Technology, Beijing, China.
Email: jingxu@ecust.edu.cn

Yugang Niu, Key Laboratory of Advanced Control and Optimization for Chemical Process, (East China University of Science and Technology), Ministry of Education, Shanghai 200237, China.
Email: acniuyg@ecust.edu.cn

Funding information

Higher Education Discipline Innovation Project, Grant/Award Number: B17017; Israel Science Foundation, Grant/Award Number: 673/19; National Natural Science Foundation of China, Grant/Award Numbers: 61803156, 62073139; Natural Science Foundation of Shanghai, Grant/Award Number: 18ZR1409300; Open Research Fund of Engineering Research Center of Intelligent Control for Underground Space, Ministry of Education, China University of Mining and Technology, Grant/Award Number: ICUS-2020-A05; Programa de Apoyo a Proyectos de Investigacion e Innovacion Tecnologica (PAPIIT-UNAM), Grant/Award Number: IN 115419

Abstract

This paper presents an output-feedback Lyapunov redesign for uncertain systems with the delayed measurements, which recasts the state estimation and robust control into a unified framework. Instead of the traditional observer/differentiator-based output-feedback design, a static state estimator is constructed by the Taylor expansion of delayed measurements with the integral remainders. Then, a sliding variable is constructed according to the nominal Lyapunov function. A Lyapunov redesign approach is used to keep the system trajectory in predefined vicinity of origin, even subject to approximation errors and exogenous disturbances. The maximum value of the allowable delays for the closed-loop stability is found via linear matrix inequalities. Finally, the effectiveness of the proposed method is verified in the magnetic suspension system.

KEYWORDS

Lyapunov redesign, output-feedback control, robustness, state estimation, time-delay

1 INTRODUCTION

The rejection of disturbances and parameter variations is an important task for uncertain systems.[1] Lyapunov redesign (LR) was first illustrated in References 2 and 3 for the stabilization of uncertain systems. LR was recast into a sliding mode control framework, and the trajectories of the closed-loop system approached the desired vicinity of the sliding manifold.[4] The sliding variable was designed based on Lyapunov function ensuring the asymptotic convergence of nominal closed-loop system. Further, an additional robustifying component was added, which was discontinuous on the manifold for compensating the matched uncertainties. However, the main disadvantage of LR is that the knowledge of all states of the system is needed.

The main approaches for state estimation were focused on the observer-based methodologies. Standard techniques of exact feedback linearization and LR were used in Reference 5, with the speed of the motor estimated via the high gain observer design. In Reference 6, the output feedback stabilization of an inertia wheel pendulum was studied, which extended state feedback control design to the output-feedback case by using a high gain observer. Regarding the expensive implementation of observers, it is desirable to develop static output-feedback LR controllers, which raises two subproblems: (1) *how to estimate the unmeasurable states without introducing an extra dynamical structure for observation?* and (2) *how to simultaneously attenuate the approximation errors and uncertainty?*

Unfortunately, the presence of measurement delays is very common situation in control systems, due to the imperfect sensors. The use of observers/differentiors introduced extra dynamical structures in the feedback loops, requiring extensive numerical computation. Therefore, the static output-feedback term may be preferable for the nominal output-feedback control. In this work, inspired by the recent results,[7-16] a static time-delay estimator of a simple implementation is inserted into the feedback loops, which estimates the unavailable states from analyzing the delayed outputs. Then, the delay-dependent robust control law composed of a nominal term and a switching term is designed based on the redesign. For the nominal design, the Lyapunov–Krasovskii functional is used to investigate the admissible measurement delays for the closed-loop stability of uncertain systems. Then, a new delayed sliding variable is proposed based on nominal Lyapunov Function derivative, and an additional switching control is used for the boundness of system trajectories around equilibrium in the presence of matched uncertainties. The main contributions of this paper are:

- The time-delay state estimator is constructed without preliminary knowledge of open-loop system structure or system parameters, which is of simple implementation and fast computation;
- The maximum measurement delay allowing the closed-loop stability of LR-based output feedback control is estimated such that the state estimation and the uncertainty compensation can be investigated in a unified design framework.

1.1 Notation

For a real symmetric matrix P, the notation $P \leq 0$ (respectively, $P < 0$) means that P is negative semi-definite (respectively, negative definite). The expression $A - B \geq 0$ means that $A - B$ is positive semi-definite. The superscripts "-1" and "T" stand for inverse and transpose, respectively. For a symmetric positive (negative) definite matrix A, $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ represent its maximum and the minimum eigenvalue, respectively. Define the signum function as $\mathrm{sign}(x) = [\mathrm{sign}(x_1), \ldots, \mathrm{sign}(x_n)]$. The notations col$\{\cdot\}$ and diag$\{\cdot\}$ represents the column and diagonal block-vectors, respectively. Define a symmetric matrix as $\mathrm{He}(M) = M + M^T$, and the symmetric elements of a symmetric matrix is represented by $\star$.

2 PRELIMINARIES AND PROBLEM FORMULATION

2.1 Preliminarity for Lyapunov redesign

Consider the following linear time-invariant system with delayed measurements:

$$\dot{x}_n(t) = A_1 x_1(t) + A_2 x_2(t) + \ldots + A_n x_n(t) + B_1(u(t) + \delta(t, x)), \tag{1}$$

$$y(t) = x_1(t-h), \tag{2}$$

where $x_i(t) = x_1^{(i-1)}(t) \in \mathbb{R}^k$ is the ith derivative of $x_1(t)$ $(i=2, \dots, n)$, $y(t) \in \mathbb{R}^k$ is the measured output, $u(t) \in \mathbb{R}^m$ is the control input, $\delta(t,x)$ is the lumped uncertainty due to model simplification and parameter uncertainty, and h is a known transmission delay. The occurrence of the delay h in the sensor channel is destroying, due to the inaccessibility of the current measurement $x_1(t)$.

To improve the controller performance, an additional buffer is introduced in the feedback loop for the storage of the delayed measurements:

$$\hat{x}_1(t,\hbar) = \mathrm{col}\{x_1(t-h), x_1(t-2h), \dots, x_1(t-nh)\}, \tag{3}$$

where $\hbar = \mathrm{col}\{h, 2h, \dots, nh\}$.

Remark 1. Model (1) represents the linearized nonlinear system of relative degree n with the matched state dependent uncertainty, which describes the local behavior of any Lipschitz system linearized in the vicinity of the equilibrium, see Reference 17.

Assumption 1. The matched uncertainty is bounded, that is,

$$\|\delta(t,x)\| \le \varrho, \tag{4}$$

where ϱ is a positive bound.

Let $x(t) = \mathrm{col}\{x_1(t), x_2(t), \dots, x_n(t)\}$, and the system dimension is then given as $\overline{n} = nk$. System (1) with a buffer (3) is further rewritten as

$$\begin{cases} \dot{x}(t) = Ax(t) + B(u(t) + \delta(t,x)) \\ y(t) = \hat{x}_1(t,\hbar), \end{cases} \tag{5}$$

with the initial condition given as $x(0) = x_0$, where $x(t) \in \Omega \subset \mathbb{R}^{\overline{n}}$ is system state vector (Ω is an neighborhood of the origin), and

$$A = \begin{bmatrix} 0 & I_k & 0 & \dots & 0 \\ 0 & 0 & I_k & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & I_k \\ A_1 & A_2 & A_3 & \dots & A_n \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ B_1 \end{bmatrix}.$$

For the region $-nh \le \phi < 0$ that is not defined for system (1), we define

$$x_1(-nh) = \dots = x_1(-(n-1)h) = \dots = x_1(-2h) = x_1(-h) = 0.$$

Remark 2. The region Ω characterizes where the linearized nonlinear system (5) holds, which is defined as $\|x(t)\| \le \psi$.[16] The value of ψ is estimated during the linearization in Reference 17, which is assumed to be known for model (5). Moreover, the estimation of ψ is helpful to establish that the redesigned controller attenuates the effect of the disturbance.

Definition 1. A solution $x(t)$ of system (5) is said to be globally uniformly ultimately bounded with ultimate bound γ, if there exists a positive constant γ independent of t_0 and for every arbitrarily large positive constant c, there is $T(c,\gamma) \ge 0$, independent of t_0, such that

$$\|x(t_0)\| \le c \Rightarrow \|x(t)\| \le \gamma, \ t \ge t_0 + T.$$

Since only the delayed measurements in $\hat{x}_1(t, \hbar)$ are accessible, it is necessary to consider the output-feedback LR for system (5). Generally (Reference 18), two types of methods are always used for the output-feedback control: (1) pure output-feedback control and (2) the use of observers/differentiators for state estimation. The first one uses the outputs for controlling reduced-order dynamics, which sacrifices certain control specification for simple control structure. The latter one introduces an additional dynamical structure for state estimation, which increases system dimension. As a third group of output-feedback control, static output-feedback control via the time-delay estimation has aroused heated research interests, see References 10-16. This new time-delay estimator is of simple static form and design flexibility, which avoids introducing extra dynamical structure for state estimation.

However, the aforehand researches can hardly be applied to system (5), because the current measurement $x_1(t)$ is unavailable. Taking Definition 1 into account, the following problem is addressed:

2.1.1 | Static delayed output-feedback LR design problem

Instead of observers/differentiators, design a static time-delay estimator to reconstruct the system state of system (5) from a few past measurements, and design a delay-dependent output-feedback robustifying control law:

$$u(t, h, y) = u_n(t, h, y) + u_s(t, h, y), \tag{6}$$

such that it guarantees uniform ultimate boundedness of every system response $x(t)$ with an ultimate bound $\gamma = \gamma(\varpi)$, where ϖ consists of all tunable design parameters, $u_n(t, h, y)$ is used for stabilizing the nominal system (5) ($\delta(t, x) = 0$), and $u_s(t, h, y)$ is used in the nominal stability design by cancelling the effects of the matched system uncertainty $\delta(t, x)$.

2.2 | Useful preliminaries

Besides, the following three lemmas are necessary for the development of the main results in this work.

An effective method for investigating the stability of a linear system is the Lyapunov–Krasovskii stability method shown in Lemma 1, where the proposed Lyapunov–Krasovskii functional is a potential measure quantifying the deviation of the state $x(t)$ from the trivial solution 0. Denote by $W[-h, 0]$ the Banach space of absolutely continuous functions $\phi : [-h, 0] \to \mathbb{R}^n$ with $\dot{\phi} \in \mathcal{L}_2(-h, 0)$ (the space of square integrable functions) with the norm

$$\|\phi\|_W = \max_{s \in [-h,0]} \phi(s)| + \left[\int_{-h}^{0} |\dot{\phi}(s)|^2 ds\right]^{1/2}.$$

Lemma 1. *(Lyapunov–Krasovskii stability theorem[11,12]) Consider a retarded differential equation:*

$$\dot{x}(t) = f(t, x_t), \tag{7}$$

where $f : \mathbb{R} \times \mathbb{C}[-h, 0] \to \mathbb{R}^n$ maps $\mathbb{R}\times$ (bounded sets in $\mathbb{C}[-h, 0]$) into bounded sets of $\mathbb{R}^n$, and $x_t \triangleq x(t + \theta)$, $\theta \in [-h, 0]$. Suppose that $\mu_1, \mu_2, \mu_3 : \mathbb{R}_+ \to \mathbb{R}_+$ are continuous nondecreasing functions, $\mu_1(s)$ and $\mu_2(s)$ are positive for $s > 0$, and $\mu_1(0) = \mu_2(0) = 0$. The trivial solution of system (7) is uniformly stable if there exists a continuous functional $V : \mathbb{R} \times W[-h, 0] \times \mathcal{L}_2(-h, 0) \to \mathbb{R}^+$, which is positive definite:

$$\mu_1(\|x\|) \le V(t, x_t, \dot{x}_t) \le \mu_2(\|x_t\|_W), \tag{8}$$

such that its derivative along (7) is nonpositive in the sense that

$$\dot{V}(t, x_t, \dot{x}_t) \le -\mu_3(\|x\|). \tag{9}$$

If $\mu_3(s) > 0$ for $s > 0$, then system (7) is uniformly asymptotically stable. If, in addition, $\lim_{s\to\infty}\mu_3(s) = \infty$, then it is globally uniformly asymptotically stable.

Lemma 2. *(Jensen's inequality[13]) Define $G = \int_a^b f(\vartheta)x(\vartheta)\mathrm{d}\vartheta$, where $a \le b$, $f : [a,b] \to [0,\infty)$, $x(s) \in \mathbb{R}^n$, and the integration concerned is well defined. Then, for any $n \times n$ matrix $R > 0$, the following inequality holds:*

$$G^T R G \le \int_a^b f(v)\mathrm{d}v \int_a^b f(\vartheta)x^T(\vartheta)Rx(\vartheta)\mathrm{d}\vartheta. \tag{10}$$

Remark 3. The inequality (10) is used for handling the quadratic terms of remainder in Taylor expansion, which is still valid in the presence of uncertainty. It directly relates the quadratic form $x^T(\vartheta)Rx(\vartheta)$ inside the integral symbol with the quadratic form $(\int_a^b f(\vartheta)x(\vartheta)\mathrm{d}\vartheta)^T R(\int_a^b f(\vartheta)x(\vartheta)\mathrm{d}\vartheta)$.

Lemma 3. *(Reference 19) For any matrices $U, V \in \mathbb{R}^{n\times n}$ with $V > 0$, and any positive scalar ϵ, we have*

$$UV^{-1}U^T \ge \epsilon\mathrm{He}(U) - \epsilon^2 V.$$

3 | OUTPUT-FEEDBACK LYAPUNOV REDESIGN VIA TIME-DELAY ESTIMATION

3.1 | Time-delay-based estimation

The nominal system corresponding to system (5) with $\delta(t,x) = 0$ is linear time-invariant, which can be stabilized by a state-feedback control law. In classical LR, the following nominal state-feedback control law is designed:

$$u_n^*(t,x) = -Kx(t), \tag{11}$$

which is used as a basis for static output-feedback control design, where $K \in \mathbb{R}^{m\times\bar{n}}$ is the controller gain.

Due to the specific system structure in model (5), state $x(t)$ depends on the output and its derivatives. In this sense, the state feedback control also refers to the derivative-dependent feedback control.[10] In the output-feedback sense, the system state $x(t)$ requires to be estimated for the implementation of (11) by using the delayed measurements:

$$\hat{x}_1(t,\hbar) = \mathrm{col}\{x_1(t-h), x_1(t-2h), \dots, x_1(t-nh)\}.$$

Based on the Taylor series expansion with the integral (Lagrange) form of the remainder, the following relations are used:

$$\begin{cases} x_1(t-h) = x_1(t-h) - \theta_1 x_2(t-h) + \frac{\theta_1^2}{2!}x_3(t-h) + \dots + \frac{(-1)^{n-1}\theta_1^{n-1}}{(n-1)!}x_n(t-h) + r_1(t) \\ x_1(t-2h) = x_1(t-h) - \theta_2 x_2(t-h) + \frac{\theta_2^2}{2!}x_3(t-h) + \dots + \frac{(-1)^{n-1}\theta_2^{n-1}}{(n-1)!}x_n(t-h) + r_2(t) \\ x_1(t-3h) = x_1(t-h) - \theta_3 x_2(t-h) + \frac{\theta_3^2}{2!}x_3(t-h) + \dots + \frac{(-1)^{n-1}\theta_3^{n-1}}{(n-1)!}x_n(t-h) + r_3(t) \\ \qquad \dots \quad \dots \\ x_1(t-nh) = x_1(t-h) - \theta_n x_2(t-h) + \frac{\theta_n^2}{2!}x_3(t-h) + \dots + \frac{(-1)^{n-1}\theta_n^{n-1}}{(n-1)!}x_n(t-h) + r_n(t), \end{cases} \tag{12}$$

where

$$\theta_i = (i-1)h, \quad r_i(t) = \frac{(-1)^n}{(n-1)!}\int_{t-ih}^{t-h}(s-t+ih)^{n-1}x_{n+1}(s)\,\mathrm{d}s, \; i \in \overline{1,n}.$$

Apparently, $\theta_1 = 0$ and $r_1(t) = 0$.

Defining $\theta \triangleq \mathrm{col}\{\theta_1, \theta_2, \dots, \theta_n\} = \mathrm{col}\{0, h, \dots, (n-1)h\}$, it is easy to verify that (12) is equivalent to

$$\hat{x}_1(t,\hbar) = M(\theta)x(t-h) + r(t), \tag{13}$$

where

$$\begin{aligned}\hat{x}_1(t,\hbar) &= \mathrm{col}\{x_1(t-h), x_1(t-2h), \dots, x_1(t-nh)\}\\ r(t) &= \mathrm{col}\{r_1(t), r_2(t), \dots, r_n(t)\}\\ M(\theta) &= \begin{bmatrix} I_k & 0_k & 0_k & \dots & 0_k \\ I_k & -\theta_2 I_k & \frac{\theta_2^2}{2!} I_k & \dots & \frac{(-1)^{n-1}\theta_2^{n-1}}{(n-1)!} I_k \\ I_k & -\theta_3 I_k & \frac{\theta_3^2}{2!} I_k & \dots & \frac{(-1)^{n-1}\theta_3^{n-1}}{(n-1)!} I_k \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ I_k & -\theta_n I_k & \frac{\theta_n^2}{2!} I_k & \dots & \frac{(-1)^{n-1}\theta_n^{n-1}}{(n-1)!} I_k \end{bmatrix}.\end{aligned}$$

From Reference 10, the matrix $M(\theta)$ is invertible such that $M^{-1}(\theta)$ exists, and the remainder $r(t)$ characterizing the estimation errors satisfies that $r(t) = O(h^n)$.

Matrices A, B, and K are correspondingly partitioned as

$$\begin{aligned} A &= \left[\begin{array}{c|cccc} 0_k & I_k & 0_k & \dots & 0_k \\ 0_k & 0_k & I_k & \dots & 0_k \\ \vdots & \vdots & \vdots & & \vdots \\ 0_k & 0_k & 0_k & \dots & I_k \\ \hline A_1 & A_2 & A_3 & \dots & I_k \end{array}\right] \triangleq \begin{bmatrix} 0_{(\bar{n}-k)\times k} & I_{\bar{n}-k} \\ \bar{A}_1 & \bar{A}_2 \end{bmatrix} \\ B &= \begin{bmatrix} 0_{k\times(\bar{n}-k)} & B_1^T \end{bmatrix}^T, K = \left[\begin{array}{cccc|c} K_1 & K_2 & \dots & K_{n-1} & K_n \end{array}\right] \triangleq \begin{bmatrix} \bar{K}_1 & \bar{K}_2 \end{bmatrix}. \end{aligned} \tag{14}$$

From Leibniz–Newton formula, we have

$$x(t-h) = x(t) - \chi(t), \tag{15}$$

where $\chi(t) = \int_{t-h}^{t} \dot{x}(s)\mathrm{d}$. Substituting (15) into (13), we obtain:

$$x(t) = M^{-1}(\theta)\hat{x}_1(t,\hbar) - M^{-1}(\theta)r(t) + \chi(t), \tag{16}$$

which constructs the estimation of $x(t)$ as

$$x(t) \approx \hat{x}(t) \triangleq M^{-1}(\theta)\hat{x}_1(t,\hbar), \tag{17}$$

by ignoring the approximation errors $r(t)$ and $\chi(t)$. With Assumption 2, it is easy to verify that

$$\|\chi(t)\| \le \beta_\chi, \quad \beta_\chi = \alpha h.$$

Compared with the time-delay estimator in Reference 10, the bad effects of the delay h in sensor channels is the additional introduction of the error $\chi(t)$, which is $O(h)$.

Remark 4. Since the time delay h is a known parameter, the time delay in (6) and (17) is selected to be same as h.

Remark 5. The nominal control (19) fully starts from $t \ge nh$, because the estimation of state $x(t)$ is accurate since $t \ge nh$.

It follows from Assumption 2 that

$$\|r(t)\| \le \beta_r, \ \beta_r = \sum_{i=1}^{n} \frac{\alpha}{n!}\theta_i^i. \tag{18}$$

Remark 6. In Reference 20, the Taylor expansion is directly performed to the nonlinear item for the state estimation, but the ellipsoid of accuracy can not be determined. In this work, the approximation errors are characterized as the remainder $r(t)$ for analyzing the closed-loop stability, which is more accurate.

3.2 | Nominal design

Inspired from (11) and (17), the static part of the robust controller (6) is taken as

$$u_n(t,h,y) = -KM^{-1}(\theta)\hat{x}_1(t,\hbar), \tag{19}$$

which coincides with the virtual control law (11) through replacing $x(t)$ by its estimation $\hat{x}(t)$ in (17).

By substituting (19) into (5) together with (17), the following nominal closed-loop system with $\delta(t,x) = 0$ is obtained:

$$\dot{x}(t) = Ax(t) - A_r(\theta)\hat{x}_1(t,\hbar), \tag{20}$$

which is equivalent to

$$\dot{x}(t) = A_s x(t) + A_d\chi(t) - A_r(\theta)r(t), \tag{21}$$

where $A_s = A - BK$, $A_d = BK$ and $A_r(\theta) = BKM^{-1}(\theta)$.

System (21) is rewritten in a more compact form:

$$\dot{x}(t) = \Gamma_1(\theta)\xi(t), \tag{22}$$

where $\xi(t) = \mathrm{col}\{x(t),\ \chi(t),\ r(t)\}$ and $\Gamma_1(\theta) = \begin{bmatrix} A_s & A_d & -A_r(\theta)\end{bmatrix}$.

Moreover, we have

$$x_{n+1}(t) = A_p x(t) + A_q\chi(t) - A_m(\theta)r(t), \tag{23}$$

where $x_{n+1}(t) \triangleq \dot{x}_n(t)$, and

$$\begin{cases} A_p = \bar{A}_n - B_1K, & \bar{A}_n = \begin{bmatrix} A_1 & A_2 & \dots & A_n\end{bmatrix} \\ A_q = B_1K, & A_m(\theta) = B_1KM^{-1}(\theta). \end{cases}$$

By letting $\Gamma_2(\theta) = \begin{bmatrix} A_p & A_q & -A_m(\theta)\end{bmatrix}$, Equation (23) becomes

$$x_{n+1}(t) = \Gamma_2(\theta)\xi(t). \tag{24}$$

The following theorem investigates the stabilization of the closed-loop system (22) in the presence of the estimation errors $r(t)$ and $\chi(t)$.

Theorem 1. *Given the time delay h, the matrix $Z \in \mathbb{R}^{\bar{n}\times k}$, and the positive scalar ϵ, the nominal system (5) with $\delta(t,x) = 0$ under the static output-feedback control law $u_n(t,h,y)$ in (19) is uniformly asymptotically stable, if there exists the symmetric matrices $P,\ Q_{11},\ Q_{22},\ Q_{33},\ S, \in \mathbb{R}^{\bar{n}\times\bar{n}}$, and $T_i \in \mathbb{R}^{k\times k}$, and the general matrices $Q_{12} \in \mathbb{R}^{\bar{n}\times\bar{n}}$, and $Q_{13},\ Q_{23},\ L_1,\ L_2 \in \mathbb{R}^{\bar{n}\times k}$, such that the following LMIs hold:*

$$\begin{bmatrix} \mathrm{He}(PA - L_1 - L_2) + Q_{11} & L_1 + L_2 + Q_{12} & \Lambda_{13} & A^TP^T - L_1^T - L_2^T & \Lambda_{15} \\ \star & -S + Q_{22} & Q_{23}M(\theta) & L_1^T + L_2^T & L_1^T + L_2^T \\ \star & \star & \Lambda_{33} & -(L_1^T + L_2^T) & -(L_1^T + L_2^T) \\ \star & \star & \star & -\epsilon\mathrm{He}(P) + \epsilon^2\overline{S}(h) & 0 \\ \star & \star & \star & \star & \Lambda_{55} \end{bmatrix} < 0, \tag{25}$$

$$P > 0, \quad S > 0, \quad T_i > 0, \quad Q > 0 \tag{26}$$

where $\overline{S}(h) = hS,\ \overline{T}(\theta) = \sum_{i=1}^{n}\theta_i^{2i}T_i,\ D = \begin{bmatrix} 0 & I_k\end{bmatrix}^T$, and

$$\Lambda_{13} = -(L_1 + L_2) + Q_{13}M(\theta), \quad \Lambda_{15} = \bar{A}_n^T D^T P_o^T - L_1^T - L_2^T$$
$$\Lambda_{33} = -M^T(\theta)TM(\theta) + M^T(\theta)Q_{33}M(\theta), \quad \Lambda_{55} = -\mathrm{He}(ZD^TP) + Z\overline{T}(\theta)Z^T,$$
$$T = \mathrm{diag}\{T_1, 4T_2, \dots, n^2T_n\}, \quad Q = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ \star & Q_{22} & Q_{23} \\ \star & \star & Q_{33} \end{bmatrix}.$$

Then, the controller gain $K \in \mathbb{R}^{m\times\bar{n}}$ can be computed from the following LMI:

$$\begin{bmatrix} -\mu I_{\bar{n}} & P_2B_1K - L_1 - L_2 \\ \star & -I_k \end{bmatrix} \leq 0, \tag{27}$$

where μ is a given positive scalar characterizing the accuracy of approximation $P_2B_1K \approx L_1 + L_2$, and $P_2 = PD$.

Proof. Choose the following Lyapunov–Krasovskii functional for the nominal design:

$$V(t, x_t, \dot{x}_t) = V_1(t, x_t) + V_2(t, x_t, \dot{x}_t) + V_3(t, x_t, \dot{x}_t), \tag{28}$$

where

$$\begin{aligned} V_1(t, x_t) &= x^T(t)Px(t) \\ V_2(t, x_t, \dot{x}_t) &= \sum_{i=1}^{n} \theta_i^i \int_{t-ih}^{t-h} (s - t + ih)^i x_{i+1}^T(s)T_ix_{i+1}(s)\,\mathrm{d}s \\ &\quad + \sum_{i=1}^{n} \theta_i^{2i} \int_{t-h}^{t} x_{i+1}^T(s)T_ix_{i+1}(s)\,\mathrm{d}s \\ V_3(t, x_t, \dot{x}_t) &= h\int_{t-h}^{t} (s - t + h)\dot{x}^T(s)S\dot{x}(s)\,\mathrm{d}s, \end{aligned}$$

with $P > 0$, $T_i > 0$, and $S > 0$, $i = \overline{1, n}$. The function $V_1(t)$ is of quadratic form, which satisfies

$$\lambda_{\min}(P)\|x\|^2 \leq V_1(t, x_t) \leq \lambda_{\max}(P)\|x\|^2. \tag{29}$$

Moreover, for fixed i, it can be seen that function $f(s) = (s - t + ih)^i$ is increasing over the time interval $s \in (t - ih, t - h)$ such that

$$0 \leq (s - t + ih)^i x_{i+1}^T(s)T_ix_{i+1}(s) \leq \theta_i^i x_{i+1}^T(s)T_ix_{i+1}(s) \leq \theta_i^i \lambda_{\max}(T_i)\|x_{i+1}(s)\|^2.$$

Then, we have

$$\begin{aligned} 0 \leq V_2(t, x_t, \dot{x}_t) &\leq \sum_{i=1}^{n} \theta_i^{2i} \lambda_{\max}(T_i) \int_{t-ih}^{t-h} \|x_{i+1}(s)\|^2\mathrm{d}s + \sum_{i=1}^{n} \theta_i^{2i} \lambda_{\max}(T_i) \int_{t-h}^{t} \|x_{i+1}(s)\|^2\mathrm{d}s \\ &\leq \sum_{i=1}^{n} \theta_i^{2i} \lambda_{\max}(T_i) \int_{t-ih}^{t} \|x_{i+1}(s)\|^2\mathrm{d}s. \end{aligned} \tag{30}$$

It follows from Assumption A1 that inequality (30) becomes

$$0 \leq V_2(t, x_t, \dot{x}_t) \quad \leq \alpha^2 \sum_{i=1}^{n} \lambda_{\max}(T_i)\theta_i^{2i}(\theta_i + h). \tag{31}$$

Similarly, by virtue of $0 \le s-t+h \le h$, we have

$$0 \le V_3(t, x_t, \dot{x}_t) \le h^2 \lambda_{\max}(S) \int_{t-h}^{t} \|\dot{x}(s)\|^2 \, \mathrm{d}s \le \alpha^2 h^3 \lambda_{\max}(S). \tag{32}$$

By combing (29), (31), and (32), we obtain:

$$\lambda_{\min}(P)\|x\|^2 \le V(t, x_t, \dot{x}_t) \quad \le \lambda_{\max}(P)\|x\|^2 + \alpha^2 \sum_{i=1}^{n} \lambda_{\max}(T_i)\theta_i^{2i}(\theta_i + h) + \alpha^2 h^3 \lambda_{\max}(S), \tag{33}$$

which indicates that the functional $V(t, x_t, \dot{x}_t)$ is bounded fitting the requirement in (8).

Corresponding to the partition in (14), matrix P is of the following form:

$$P = \begin{bmatrix} P_1 & P_2 \end{bmatrix},$$

where $P_1 \in \mathbb{R}^{\bar{n}\times(n-1)k}$, $P_2 \in \mathbb{R}^{\bar{n}\times k}$, and $P_2 = PD$. We can successively make the approximation errors to be small by properly designing S and T_i.

Differentiating the first term of (28) with respect to time t yields:

$$\frac{\mathrm{d}V_1}{\mathrm{d}t} = x^T(t)\mathrm{He}(PA_s)x(t) + 2x^T(t)PA_d\chi(t) - 2x^T(t)PA_r(\theta)r(t). \tag{34}$$

The derivatives of $V_2(t, x_t, \dot{x}_t)$ and $V_3(t, x_t, \dot{x}_t)$ along the trajectory (21) are, respectively, given by

$$\frac{\mathrm{d}V_2}{\mathrm{d}t} = \sum_{i=1}^{n} \{\theta_i^{2i} x_{i+1}^T(t) T_i x_{i+1}(t) - i\theta_i^i \int_{t-ih}^{t-h} (s-t+ih)^{i-1} x_{i+1}^T(s) T_i x_{i+1}(s) \, \mathrm{d}s\}, \text{ and} \tag{35}$$

$$\frac{\mathrm{d}V_3}{\mathrm{d}t} = h\dot{x}^T(t)S\dot{x}(t) - h\int_{t-h}^{t} \dot{x}^T(s)S\dot{x}(s) \, \mathrm{d}s. \tag{36}$$

It follows from Lemma 2 that the following inequalities hold:

$$\begin{cases} -h\int_{t-h}^{t} \dot{x}^T(s)S\dot{x}(s) \, \mathrm{d}s \le -\chi^T(t)S\chi(t) \\ -\int_{t-ih}^{t-h} (s-t+ih)^{i-1} \mathrm{d}s \int_{t-ih}^{t-h} (s-t+ih)^{i-1} x_{i+1}^T(s) T_i x_{i+1}(s) \, \mathrm{d}s \le -r_i^T(t) T_i r_i(t), \end{cases}$$

which is equivalent to

$$\begin{cases} -h\int_{t-h}^{t} \dot{x}^T(s)S\dot{x}(s) \, \mathrm{d}s \le -\chi^T(t)S\chi(t) \\ -\frac{\theta_i^i}{i}\int_{t-ih}^{t-h} (s-t+ih)^{i-1} x_{i+1}^T(s) T_i x_{i+1}(s) \, \mathrm{d}s \le -r_i^T(t) T_i r_i(t). \end{cases} \tag{37}$$

With (37), we have

$$\begin{aligned} \frac{\mathrm{d}V_2}{\mathrm{d}t} &\le \sum_{i=1}^{n} \theta_i^{2i} x_{i+1}^T(t) T_i x_{i+1}(t) - r^T(t) T r(t) \\ \frac{\mathrm{d}V_3}{\mathrm{d}t} &\le h\dot{x}^T(t)S\dot{x}(t) - \chi^T(t)S\chi(t), \end{aligned} \tag{38}$$

where $T = \mathrm{diag}\{T_1,\ 4T_2,\ \dots,\ n^2T_n\}$.

Adding (34) and (38) results in

$$\begin{aligned} \frac{\mathrm{d}V}{\mathrm{d}t} \le\, & x^T(t)\mathrm{He}(PA_s)x(t) + 2x^T(t)PA_d\chi(t) \\ & - 2x^T PA_r(\theta)r(t) - r^T(t)Tr(t) - \chi^T(t)S\chi(t) \\ & + \sum_{i=1}^{n} \theta_i^{2i} x_{i+1}^T(t) T_i x_{i+1}(t) + h\dot{x}^T(t)S\dot{x}(t). \end{aligned}$$

To satisfy (9), we select $\gamma_3(t, x_t) = \xi^T(t)Q\xi(t)$ by containing both $x(t)$ and $x_t(\vartheta)$. From (22) and (23), we obtain:

$$\frac{\mathrm{d}V}{\mathrm{d}t} \le \xi^T(t)(\Theta_1 + \Theta_2 + Q)\xi(t) \le 0, \tag{39}$$

where

$$\Theta_1 = \begin{bmatrix} \mathrm{He}(PA_s) & PA_d & -PA_r(\theta) \\ \star & -S & 0 \\ \star & \star & -T \end{bmatrix}, \quad Q = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ \star & Q_{22} & Q_{23} \\ \star & \star & Q_{33} \end{bmatrix}$$

$$\Theta_2 = \Gamma_1^T \overline{S}(h)\Gamma_1 + \Gamma_2^T \overline{T}(h)\Gamma_2, \quad \overline{S}(h) = hS, \quad \overline{T}(\theta) = \sum_{i=1}^{n} \theta_i^{2i} T_i.$$

Noting that $\mu_1(t) > 0,\ \mu_2(t) > 0,\ \mu_3(t) > 0$ for $t > 0$, it follows from Lemma 1 that the closed-loop system (20) is uniformly asymptotically stable, if conditions (33) and (39) hold.

Inequality (39) is further rewritten as

$$\Theta_1 + \begin{bmatrix} A_s^T \\ A_d^T \\ -A_r^T(\theta) \end{bmatrix} \overline{S}(h) \begin{bmatrix} A_s & A_d & -A_r(\theta) \end{bmatrix} + \begin{bmatrix} A_p^T \\ A_q^T \\ -A_m^T(\theta) \end{bmatrix} \overline{T}(\theta) \begin{bmatrix} A_p & A_q & -A_m(\theta) \end{bmatrix} + Q < 0. \tag{40}$$

Applying the Schur Complement Lemma to (40) yields the following bilinear matrix inequality:

$$\begin{bmatrix} \mathrm{He}(PA_s) + Q_{11} & PA_d + Q_{12} & -PA_r(\theta) + Q_{13} & A_s^T & A_p^T \\ \star & -S + Q_{22} & Q_{23} & A_d^T & A_q^T \\ \star & \star & -T + Q_{33} & -A_r^T(\theta) & -A_m^T(\theta) \\ \star & \star & \star & -\overline{S}^{-1}(h) & 0 \\ \star & \star & \star & \star & -\overline{T}^{-1}(\theta) \end{bmatrix} < 0, \tag{41}$$

with the nonlinear terms in

$$\begin{aligned} PA_s &= PA - PBK = PA - P_2B_1K \\ PA_d &= PBK = P_2B_1K \\ PA_r(\theta) &= PBKM^{-1}(\theta) = P_2B_1KM^{-1}(\theta). \end{aligned}$$

Define the transformation matrix as

$$\mathcal{T} = \mathrm{diag}\{I, I, M(\theta), P^T, P_2^T\}.$$

Premultiplying and postmultiplying (41) by $\mathcal{T}^T$ and $\mathcal{T}$, respectively, yields:

$$\begin{bmatrix} \mathrm{He}(PA_s) + Q_{11} & PA_d + Q_{12} & -PA_d + Q_{13}M(\theta) & A_s^T P^T & A_p^T P_2^T \\ \star & -S + Q_{22} & Q_{23}M(\theta) & A_d^T P^T & A_q^T P_2^T \\ \star & \star & \Lambda_{33} & -A_r^T(\theta)P^T & -A_m^T(\theta)P_2^T \\ \star & \star & \star & -P\overline{S}^{-1}(h)P^T & 0 \\ \star & \star & \star & \star & -P_2\overline{T}^{-1}(\theta)P_2^T \end{bmatrix} < 0, \tag{42}$$

where $\Lambda_{33} = -M^T(\theta)TM(\theta) + M^T(\theta)TQ_{33}M(\theta)$, and

$$A_p^T P_2^T = \bar{A}_n^T P_2^T - K^T B_1^T P_2^T, \quad A_q^T P_2^T = K^T B_1^T P_2^T, \quad A_m^T(\theta) P_2^T = M^{-T}(\theta) K^T B_1^T P_2^T.$$

The coupling of the Lyapunov matrix P with system matrices A_p, A_q and A_m makes (42) a bilinear matrix inequality, which can hardly be solved by the LMI toolbox in Matlab. To confront this issue, the following approximation is constructed:

$$P_2 B_1 K = L_1 + L_2, \tag{43}$$

where $L_1 \in \mathbb{R}^{\overline{n} \times \overline{n}}$, $L_2 \in \mathbb{R}^{\overline{n} \times \overline{n}}$, which can be guaranteed by

$$\|P_2 B_1 K - L_1 - L_2\| \le \mu, \tag{44}$$

where μ is a positive scalar characterizing the approximation accuracy. The equivalent form of inequality (44) is given as (27).

It is readily verified from Lemma 3 that

$$\begin{cases} -P\overline{S}^{-1}(h)P^T \le -\epsilon \mathrm{He}(P) + \epsilon^2 \overline{S}(h) \\ -(ZT(\theta) - P_2)T^{-1}(\theta)(T(\theta)Z^T - P_2^T) \le 0, \end{cases} \tag{45}$$

where $\epsilon > 0$ is prescribed, and Z is a given matrix. By virtue of (43) and (45), the sufficient condition for the establishment of (42) is given as (27). ■

After the nominal design, the nominal closed-loop system (21) with $\delta(t, x) = 0$ is stable. Thus, we assume the following:

Assumption 2. State $\dot{x}(t)$ is bounded over the time interval $[t - nh,\ t]$:

$$\|\dot{x}(s)\| \le \alpha, \text{ for all } s \in [t - nh, t], \tag{46}$$

where the estimated parameter α relates with the nominal design.

Remark 7. The right-hand side of (5), that is, $\dot{x}(t)$, represents the source of the dynamical behaviors of system (5). Condition $\dot{x}(t) = 0$ holds, when system states reach their equilibrium.

Remark 8. Condition (46) implies a standard assumption in control systems:

$$\|x_i(t)\| \le \alpha,\ i = \overline{2, n+1}. \tag{47}$$

With the sufficient large control law, the state trajectories of a nonlinear system are assumed to evolve in a compact set where its linearized model (1) holds.

3.3 | Delay-dependent Lyapunov redesign

Inspired from that the ideal state-feedback sliding variable is given as $\sigma^*(t, x) = 2B^T P x(t)$, the following output-feedback sliding variable is constructed by replacing $x(t)$ with its estimation $\hat{x}(t)$:

$$\hat{\sigma}(t, x_t, \hbar) = 2B^T P \hat{x}(t) = 2B^T P M^{-1}(\theta) \hat{x}_1(t, \hbar),$$

which is equivalent to

$$\begin{aligned} \hat{\sigma}(t, x_t, \hbar) &= 2B^T P x(t) + 2B^T P M^{-1}(\theta) r(t) - 2B^T P \chi(t) \\ &= \sigma^*(t, x) + 2B^T P M^{-1}(\theta) r(t) - 2B^T P \chi(t). \end{aligned}$$

It is easy to verify that both the sliding surfaces $\sigma^*(t, x) = 0$ and $\hat{\sigma}(t, x_t, \hbar) = 0$ cross the equilibrium $x(t) = 0$, when the system uncertainty $\delta(t, x)$ is fully compensated. However, due to the estimation errors $\chi(t)$ and $r(t)$, there is always discrepancy between the ideal state-feedback sliding motion $\sigma^*(t, x)$ and its estimation $\hat{\sigma}(t, x_t, \hbar)$. Only for small h, the sliding motion $\hat{\sigma}(t, x_t, \hbar)$ captures the dominant dynamical behaviors of $\sigma^*(t, x)$.

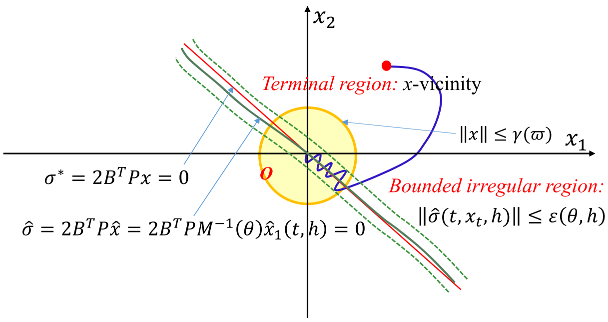

FIGURE 1 Attractivity of the ideal state-feedback sliding manifold [Colour figure can be viewed at wileyonlinelibrary.com]

As shown in Figure 1, the system states can approach the vicinity of the output-feedback sliding surface $\hat{\sigma}(t,x_t,\hbar)=0$ in a finite time, that is, $\|\hat{\sigma}(t,x_t,\hbar)\| \le \varepsilon(\theta,\alpha)$, where the vicinity size $\varepsilon(\theta,\alpha)$ is fixed for given h and α. Once the system states reach the vicinity $\{x|\ \|\hat{\sigma}(t,x_t,\hbar)\| \le \varepsilon(\theta,\alpha)\}$, they will slide along the surface $\hat{\sigma}(t,x_t,\hbar)=0$ toward the terminal region around the small vicinity of $x=0$, that is, $\|x(t)\| \le \gamma(\varpi)$.

Then, the switching control law to cancel the effects of the uncertainty $\delta(t,x)$ is given by

$$u_s(t,h,y) = \begin{cases} -\varphi_1 \mathrm{sign}(\hat{\sigma}(t,x_t,\hbar)) - \frac{\varphi_2}{\|\hat{\sigma}(t,x_t,\hbar)\|}\mathrm{sign}(\hat{\sigma}(t,x_t,\hbar)), & \text{if } \hat{\sigma}(t,x_t,\hbar) \ge \varepsilon(\theta,\alpha), \\ -\varphi_1 \mathrm{sign}(\hat{\sigma}(t,x_t,\hbar)), & \text{otherwise.} \end{cases} \tag{48}$$

which yields the following overall control law:

$$\begin{aligned} u(t,h,y) &= u_n(t,h,y) + u_s(t,h,y) \\ &= \begin{cases} -KM^{-1}(\theta)\hat{x}_1(t,\hbar) - \varphi_1 \mathrm{sign}(\hat{\sigma}(t,x_t,\hbar)) - \frac{\varphi_2}{\|\hat{\sigma}(t,x_t,\hbar)\|}\mathrm{sign}(\hat{\sigma}(t,x_t,\hbar)), & \text{if } \hat{\sigma}(t,x_t,\hbar) \ge \varepsilon(\theta,\alpha) \\ -KM^{-1}(\theta)\hat{x}_1(t,\hbar) - \varphi_1 \mathrm{sign}(\hat{\sigma}(t,x_t,\hbar)), & \text{otherwise.} \end{cases} \end{aligned} \tag{49}$$

The existence and uniqueness of solutions for the closed-loop system (5) under the SMC control law (49) can still be guaranteed by means of the method of steps, see, for example, References 21,22.

Theorem 2. *For given h and α, the matrices P, S and T_i ($i=\overline{1,n}$), Q obtained from Theorem 1, the trajectories of the closed-loop system (5) under the control law in (48) converge to the neighborhood of the sliding surface $\|\hat{\sigma}(t,x_t,\hbar)\| \le \varepsilon(\theta,\alpha)$, if the delayed switching gains φ_1 and φ_2 satisfying*

$$\begin{cases} \varphi_1 \ge \varrho \\ \varphi_2 \ge \frac{\varepsilon(\theta,\alpha)\cdot(\varphi_1+\varrho)\|\hat{\sigma}(t,x_t,\hbar)\| + \varepsilon(\theta,\alpha)\cdot\lambda_{\max}(\Pi(\theta,h))\varrho^2\|\hat{\sigma}(t,x_t,\hbar)\|}{\|\hat{\sigma}(t,x_t,\hbar)\| - \varepsilon(\theta,\alpha)}, \end{cases} \tag{50}$$

where $\theta = \mathrm{col}\{\theta_1,\ \theta_2,\ \dots,\ \theta_n\},\ \theta_i = (i-1)h$, and

$$\Pi(\theta,h) = hS + \sum_{i=1}^{n} (\theta_i)^{2i} T_i,\ \ \varepsilon(\theta,\alpha) = 2(\beta_r\|M^{-T}(\theta)PB\| + \beta_\chi\|PB\|).$$

Moreover, the solution of the closed-loop system of (5) under the control law in (49) has the predefined ultimate bound:

$$\{x|\ \|x(t)\| \le \gamma(\varpi)\},\ \ \gamma(\varpi) \ge \varepsilon(\theta,\alpha),$$

where $\varpi = \{\theta, \varphi_1, \varphi_2\}$, and

$$\begin{aligned} \gamma(\varpi) &= \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)\lambda_{\min}(Q)}} \cdot \sqrt{d(\varpi)} \\ d(\varpi) &= (\varphi_2 - \varepsilon(\theta,\alpha)\cdot(\varphi_1+\varrho) - \lambda_{\max}(\Pi(\theta,h))\varrho^2) + q\varepsilon(\theta,\alpha)\cdot(\varphi_1-\varrho) \\ &\quad + \lambda_{\min}(Q)\beta_r^2 + \lambda_{\min}(Q)\beta_\chi^2. \end{aligned}$$

Proof. When taking the term $\delta(t,x)$ into consideration, we have

$$\frac{\mathrm{d}V}{\mathrm{d}t} \leq g(t,x_t,\dot{x}_t) + 2x^T(t)PB(u_s(t,h,y) + \delta(t,x)) + \delta^T(t,x)\Pi(\theta,h)\delta(t,x) \tag{51}$$

where $\Pi(\theta,h) = hS + \sum_{i=1}^{n}(\theta_i)^{2i}T_i$, and

$$\begin{aligned} g(t,x_t,\dot{x}_t) = {} & x^T(t)\mathrm{He}(PA_s)x(t) + 2x^T(t)PA_d\chi(t) - 2x^T(t)PA_r(\theta)r(t) \\ & + \sum_{i=1}^{n}\{(\theta_i)^{2i}x_{i+1}^T(t)T_ix_{i+1}(t) - i(\theta_i)^i\int_{t-ih}^{t-h}(s-t+ih)^{i-1}x_{i+1}^T(s)T_ix_{i+1}(s)\,\mathrm{d}s\} \\ & + h\dot{x}^T(t)S\dot{x}(t) - h\int_{t-h}^{t}\dot{x}^T(s)S\dot{x}(s)\,\mathrm{d}s. \end{aligned}$$

The item $g(t,x_t,\dot{x}_t)$ in (51) comes from the nominal design in Theorem 1, which is obtained by adding (34),(35),(36). By virtue of (22) and (23), $g(t,x_t,\dot{x}_t)$ is converted into

$$\begin{aligned} g(t,x_t,\dot{x}_t) = {} & x^T(t)\mathrm{He}(PA_s)x(t) + 2x^T(t)PA_d\chi(t) - 2x^T(t)PA_r(\theta)r(t) \\ & + \xi^T(t)\Gamma_2^T(\theta)\overline{T}(\theta)\Gamma_2(\theta)\xi(t) - \sum_{i=1}^{n}i(\theta_i)^i\int_{t-ih}^{t-h}(s-t+ih)^{i-1}x_{i+1}^T(s)T_ix_{i+1}(s)\,\mathrm{d}s \\ & + h\xi^T(t)\Gamma_1^T(\theta)S\Gamma_1(\theta)\xi(t) - h\int_{t-h}^{t}\dot{x}^T(s)S\dot{x}(s)\,\mathrm{d}s, \end{aligned} \tag{52}$$

where $\overline{T}(\theta) = \sum_{i=1}^{n}(\theta_i)^{2i}T_i$. It follows from Lemma 2 and Remark 3 that inequality (38) still holds, even in the presence of uncertainty. Thus, (52) represents the derivative of $V(t,x_t,\dot{x}_t)$ with $\delta(t,x) = 0$, which has been investigated in the nominal design.

It follows thereby from (39) that the inequality below holds:

$$g(t,x_t,\dot{x}_t) \leq -\xi^T(t)Q\xi(t) \leq -\lambda_{\min}(Q)\|\xi\|^2,$$

where the locations of eigenvalues of the matrix Q can be used for estimating the stability margin.

The uncertainty $\delta(t,x)$ appears on the right-hand side of (51) at the same time, when the delayed switching control $u_s(t,h,y)$ occurs. From (48), it is shown that

$$\begin{aligned} \frac{\mathrm{d}V}{\mathrm{d}t} \leq {} & -\lambda_{\min}(Q)\|\xi\|^2 + (\sigma^*(t,x))^T\cdot[-\varphi_1\mathrm{sign}(\hat{\sigma}(t,x_t,\hbar)) - \frac{\varphi_2}{\|\hat{\sigma}(t,x_t,\hbar)\|}\mathrm{sign}(\hat{\sigma}(t,x_t,\hbar)) + \delta(t,x)] \\ & + \delta^T(t,x,u)\Pi(\theta,h)\delta(t,x) \\ \leq {} & -\lambda_{\min}(Q)\|\xi\|^2 + (\hat{\sigma}(t,x_t,\hbar))^T\cdot[-\varphi_1\mathrm{sign}(\hat{\sigma}(t,x_t,\hbar)) - \frac{\varphi_2}{\|\hat{\sigma}(t,x_t,\hbar)\|}\mathrm{sign}(\hat{\sigma}(t,x_t,\hbar)) + \delta(t,x)] \\ & - 2r^T(t)M^{-T}(\theta)PB\cdot[-\varphi_1\mathrm{sign}(\hat{\sigma}(t,x_t,\hbar)) - \frac{\varphi_2}{\|\hat{\sigma}(t,x_t,\hbar)\|}\mathrm{sign}(\hat{\sigma}(t,x_t,\hbar)) + \delta(t,x)] \\ & + 2\chi^T(t)PB\cdot[-\varphi_1\mathrm{sign}(\hat{\sigma}(t,x_t,\hbar)) - \frac{\varphi_2}{\|\hat{\sigma}(t,x_t,\hbar)\|}\mathrm{sign}(\hat{\sigma}(t,x_t,\hbar)) + \delta(t,x)] \\ & + \delta^T(t,x,u)\Pi(\theta,h)\delta(t,x). \end{aligned} \tag{53}$$

The following relation holds

$$(\hat{\sigma}(t,x_t,\hbar))^T\cdot\mathrm{sign}(\hat{\sigma}(t,x_t,\hbar)) = |\hat{\sigma}(t,x_t,\hbar)|, \quad \|\hat{\sigma}(t,x_t,\hbar)\| \leq |\hat{\sigma}(t,x_t,\hbar)| \tag{54}$$

which yields an equivalent form of (53):

$$\begin{aligned} \frac{\mathrm{d}V}{\mathrm{d}t} \leq {} & -\lambda_{\min}(Q)\|\xi\|^2 - \varphi_1|\hat{\sigma}(t,x_t,\hbar)| + (\hat{\sigma}(t,x_t,\hbar))^T\cdot\delta(t,x) \\ & - 2r^T(t)M^{-T}(\theta)PB\cdot[-\varphi_1\mathrm{sign}(\hat{\sigma}(t,x_t,\hbar)) - \frac{\varphi_2}{\|\hat{\sigma}(t,x_t,\hbar)\|}\mathrm{sign}(\hat{\sigma}(t,x_t,\hbar)) + \delta(t,x)] \end{aligned}$$

$$
\begin{aligned}
&+2\chi^T(t)PB\cdot[-\varphi_1\mathrm{sign}(\hat{\sigma}(t,x_t,\hbar))-\frac{\varphi_2}{\|\hat{\sigma}(t,x_t,\hbar)\|}\mathrm{sign}(\hat{\sigma}(t,x_t,\hbar))+\delta(t,x)]\\
&-\varphi_2\frac{|\hat{\sigma}(t,x_t,\hbar)|}{\|\hat{\sigma}(t,x_t,\hbar)\|}+\delta^T(t,x)\Pi(\theta,h)\delta(t,x)\\
&\leq-\lambda_{\min}(Q)\|\xi\|^2-\varphi_1|\hat{\sigma}(t,x_t,\hbar)|+\varrho\|\hat{\sigma}(t,x_t,\hbar)\|\\
&+\varepsilon(\theta,\alpha)\cdot[\varphi_1+\frac{\varphi_2}{\|\hat{\sigma}(t,x_t,\hbar)\|}+\varrho]-\varphi_2+\lambda_{\max}(\Pi(\theta,h))\varrho^2\\
&\leq-\lambda_{\min}(Q)\|\xi\|^2-|\hat{\sigma}(t,x_t,\hbar)|\cdot(\varphi_1-\varrho)\\
&-\left(1-\frac{\varepsilon(\theta,\alpha)}{\|\hat{\sigma}(t,x_t,\hbar)\|}\right)\varphi_2+\varepsilon(\theta,\alpha)\cdot(\varphi_1+\varrho)+\lambda_{\max}(\Pi(\theta,h))\varrho^2
\end{aligned}\tag{55}
$$

where $\varepsilon(\theta,\alpha)=2(\beta_r\|M^{-T}(\theta)PB\|+\beta_\chi\|PB\|)$.

1. Vicinity of sliding surface $\|\hat{\sigma}(t,x_t,\hbar)\|=0$.

If $1-\frac{\varepsilon(\theta,\alpha)}{\|\hat{\sigma}(t,x_t,\hbar)\|}\geq 0$, or equivalently $\|\hat{\sigma}(t,x_t,\hbar)\|\geq\varepsilon(\theta,\alpha)$, system trajectories are attracted to approach the sliding surface $\hat{\sigma}(t,x_t,\hbar)=0$ under the switching gains φ_1 and φ_2:

$$
\begin{cases}\varphi_1\geq\varrho\\ \varphi_2\geq\frac{\varepsilon(\theta,\alpha)\cdot(\varphi_1+\varrho)+\lambda_{\max}(\Pi(\theta,h))\varrho^2}{1-\frac{\varepsilon(\theta,\alpha)}{\|\hat{\sigma}(t,x_t,\hbar)\|}}.\end{cases}
$$

Otherwise, the finite-time reachability property is destroying, because the constraint $1-\frac{\varepsilon(\theta,\alpha)}{\|\hat{\sigma}(t,x_t,\hbar)\|}\geq 0$ hardly holds. To sum up, the system trajectories can reach the vicinity of the sliding surface $\hat{\sigma}(t,x_t,\hbar)=0$ under the switching gains satisfying (50).

2. Estimation of the boundary of x-vicinity.

Considering (55) together with $1-\frac{\varepsilon(\theta,\alpha)}{\|\hat{\sigma}(t,x_t,\hbar)\|}\geq 0$, we have

$$
\begin{aligned}
\frac{\mathrm{d}V}{\mathrm{d}t}&\leq-\lambda_{\min}(Q)\|\xi\|^2-|\hat{\sigma}(t,x_t,\hbar)|\cdot(\varphi_1-\varrho)\\
&-\left(1-\frac{\varepsilon(\theta,\alpha)}{\|\hat{\sigma}(t,x_t,\hbar)\|}\right)\varphi_2+\varepsilon(\theta,\alpha)\cdot(\varphi_1+\varrho)+\lambda_{\max}(\Pi(\theta,h))\varrho^2\\
&\leq-\lambda_{\min}(Q)\|x\|^2-\lambda_{\min}(Q)\beta_r^2-\lambda_{\min}(Q)\beta_\chi^2-\varepsilon(\theta,\alpha)\cdot(\varphi_1-\varrho)\\
&-(\varphi_2-\varepsilon(\theta,\alpha)\cdot(\varphi_1+\varrho)-\lambda_{\max}(\Pi(\theta,h))\varrho^2)\\
&\leq 0.
\end{aligned}\tag{56}
$$

For the case $1-\frac{\varepsilon(\theta,\alpha)}{\|\hat{\sigma}(t,x_t,\hbar)\|}<0$, the attractivity of the sliding surface fails such that only the convergence of system states to the vicinity of the equilibrium is ensured.

It thereby follows from (56) that $\|x(t)\|\leq c(\varpi)$, where $\varpi=\{\theta,\varphi_1,\varphi_2\}$, and

$$
\begin{aligned}
c(\varpi)&=\sqrt{\frac{d(\varpi)}{\lambda_{\min}(Q)}}\\
d(\varpi)&=(\varphi_2-\varepsilon(\theta,\alpha)\cdot(\varphi_1+\varrho)-\lambda_{\max}(\Pi(\theta,h))\varrho^2)+\varepsilon(\theta,\alpha)\cdot(\varphi_1-\varrho)\\
&\quad+\lambda_{\min}(Q)\beta_r^2+\lambda_{\min}(Q)\beta_\chi^2.
\end{aligned}
$$

Apparently, another constraint on $\|x(t)\|$ is given by

$$
\lambda_{\min}(P)\|x(t)\|^2\leq V_1(t,x_t)\leq\lambda_{\max}(P)\|x(t)\|^2.\tag{57}
$$

By combing (57) with $\|x(t)\|\leq c(\varpi)$, we obtain:

$$
\|x(t)\|\leq\gamma(\varpi),
$$

where $\gamma(\varpi)=\sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)\lambda_{\min}(Q)}}\cdot\sqrt{d(\varpi)}$. This completes the proof. ■

TABLE 1 System parameters

Physical notations	l	r	J	A_{gap}	R	g
Value	0.15 m	0.01 m	$0.0967\ \text{kg} \cdot \text{m}^2$	$101.69\ \text{mm}^2$	$1.6\ \Omega$	$9.8\ \text{m} \cdot \text{s}^{-2}$

3.3.1 | Design steps

The model (5) is derived based on the linearization of any Lipschitz nonlinear system around the equilibrium. Then, the matrices $A_1, A_2, \dots, A_n$ and B_1 are obtained, where the domain for model (5) is defined as $\|x(t)\| \le \psi$. Here, ψ and ϱ are known scalars. With Assumptions A1, we will follow

(1) For given h, if the solutions L_1, L_2, P, Q, S, T_i, to the LMIs (25),(26),(27) exists;
(2) Compute the static controller gain K by investigating the LMI (27);
(3) Select the value α satisfying that $\alpha \ge \|A_s\|\psi + \|A_d\|\beta_\chi + \|A_r(\theta)\|\beta_r$;
(4) With P, Q, S, T_i, h, α obtained in Step (1)–(3), the switching gains φ_1 and φ_2 are chosen satisfying the constraint (57);
(5) Formulate the overall control law in (49) by using P, K, h, φ_1 and φ_2.

4 | SIMULATION EXAMPLE: AN ACTIVE MAGNETIC BEARING SYSTEM WITH VOLTAGE CONTROL

In this section, the effectiveness of the output-feedback LR is verified in an active magnetic bearing system with voltage control. The nonlinear dynamic model of an active magnetic bearing system is represented as

$$\begin{cases} \dot{\phi}(t) = (U - iR) \\ F(t) = cp_1^2(t)\phi^2(t), \quad c = 0.75/\mu_0 A_{\text{gap}}, \quad p_1(t) = 1/(ay(t) + b) \\ J\ddot{\theta}(t) = lF(t) - Mgr, \end{cases} \tag{58}$$

where J is the beam polar moment of inertia, A_{gap} is the pole face area, M is the mass of beam, g is the gravitational acceleration, l is the distance from pivot to actuator center, r is the displacement of the center of mass to the pivot, U is the voltage supplied across the coils, i is the current flowing through the coils, and R is the electrical resistance. Here, a and b are constants determined from the fit in the experiments. From Reference 23, the system parameters are shown in Table 1.

The derivatives of F and $p_1(t)$ with respect to time t are, respectively, given as

$$\begin{aligned} \dot{F}(t) &= 2cp_1^2(t)\phi(t)\dot{\phi}(t) + 2c\phi^2(t)p_1(t)\dot{p}_1(t) \\ \dot{p}_1(t) &= -p_1^2\dot{y}(t). \end{aligned} \tag{59}$$

With (60), differentiating $\ddot{\theta}(t)$ along the state trajectory of (58) yields

$$\dddot{\theta}(t) = \frac{2lcp_1(t)}{JN}\sqrt{\frac{F}{c}}(U - iR) - 2lFp_1(t)a\dot{y}(t). \tag{60}$$

The control input voltage can be specified as

$$U = \frac{JN}{2lcp_1\sqrt{F/c}}V(t) + iR + \frac{NFa\dot{y}(t)}{c\sqrt{F/c}}$$

where V is a virtual control input to be designed based on the delayed LR, and $\delta(t, x)$ is the lumped disturbance in the actuator channel. Then, the nonlinear system (58) is transformed into the following triple integrator model:

$$\dddot{\theta}(t) = V(t) + \delta(t, x), \tag{61}$$

which can be rewritten as (5), with

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

In the simulation setup, only $\theta(t-h)$ is measurable in system (61). The disturbance is chosen as $\delta(t, x) = 2\sin(50t)$, and the matching condition is given as

$$\|\delta(t, x)\| \leq \varrho, \quad \varrho = 2. \tag{62}$$

In light of Theorem 1 with $h = 0.25$, the following feasible solutions are obtained:

$$P = \begin{bmatrix} 0.0173 & 0.0942 & 3.2808 \\ \star & 27.9793 & 60.4991 \\ \star & \star & 73.3582 \end{bmatrix}, \quad L_1 = \begin{bmatrix} 45.5522 & -144.4984 & 27.9799 \\ \star & 1.8128 & -1.9019 \\ \star & \star & 3.8832 \end{bmatrix}$$

$$L_2 = \begin{bmatrix} 12.5346 & 143.1508 & -27.9598 \\ \star & -1.6853 & 1.8992 \\ \star & \star & -3.8827 \end{bmatrix}.$$

Then, the static output-feedback controller gain is obtained as

$$K = \begin{bmatrix} 0.003 & 0.0600 & 0.5420 \end{bmatrix},$$

and system (58) is stabilized for given $h = 0.25$. Hence, $\hbar = \{h, 2h, 3h\}$. The sliding manifold is chosen as $\hat{\sigma}(t, \hbar) = 0$, and

$$\hat{\sigma}(t, \hbar) = P_b M^{-1}(h) \hat{x}_1(t, \hbar).$$

where $\hat{x}_1(t, \hbar) = \text{col}\{x_1(t-h), x_1(t-2h), x_1(t-3h)\}$, and

$$P_b = \begin{bmatrix} 6.5616 & 120.9982 & 146.7164 \end{bmatrix}$$

$$M(h) = \begin{bmatrix} 1 & 0 & 0 \\ 1 & -h & 0.5h^2 \\ 1 & -2h & 2h^2 \end{bmatrix}.$$

Then, the switching gain in (49) is specified as constant for attenuating the bounded disturbance $\delta(t, x)$:

$$\varphi_1 = 2, \quad \varphi_2 = 2.5.$$

To this end, the output-feedback delayed controller via LR is formulated as

$$u(t) = -\begin{bmatrix} 0.003 & 0.0600 & 0.5420 \end{bmatrix} \hat{x}_1(t, \hbar) - 2\text{sign}(\hat{\sigma}(t, \hbar)) - \frac{2}{\|\hat{\sigma}(t, \hbar)\|}\text{sign}(\hat{\sigma}(t, \hbar)). \tag{63}$$

For comparison, two different types of state-feedback controller including the delayed controller and the delayed controller via LR are, respectively, formulated as

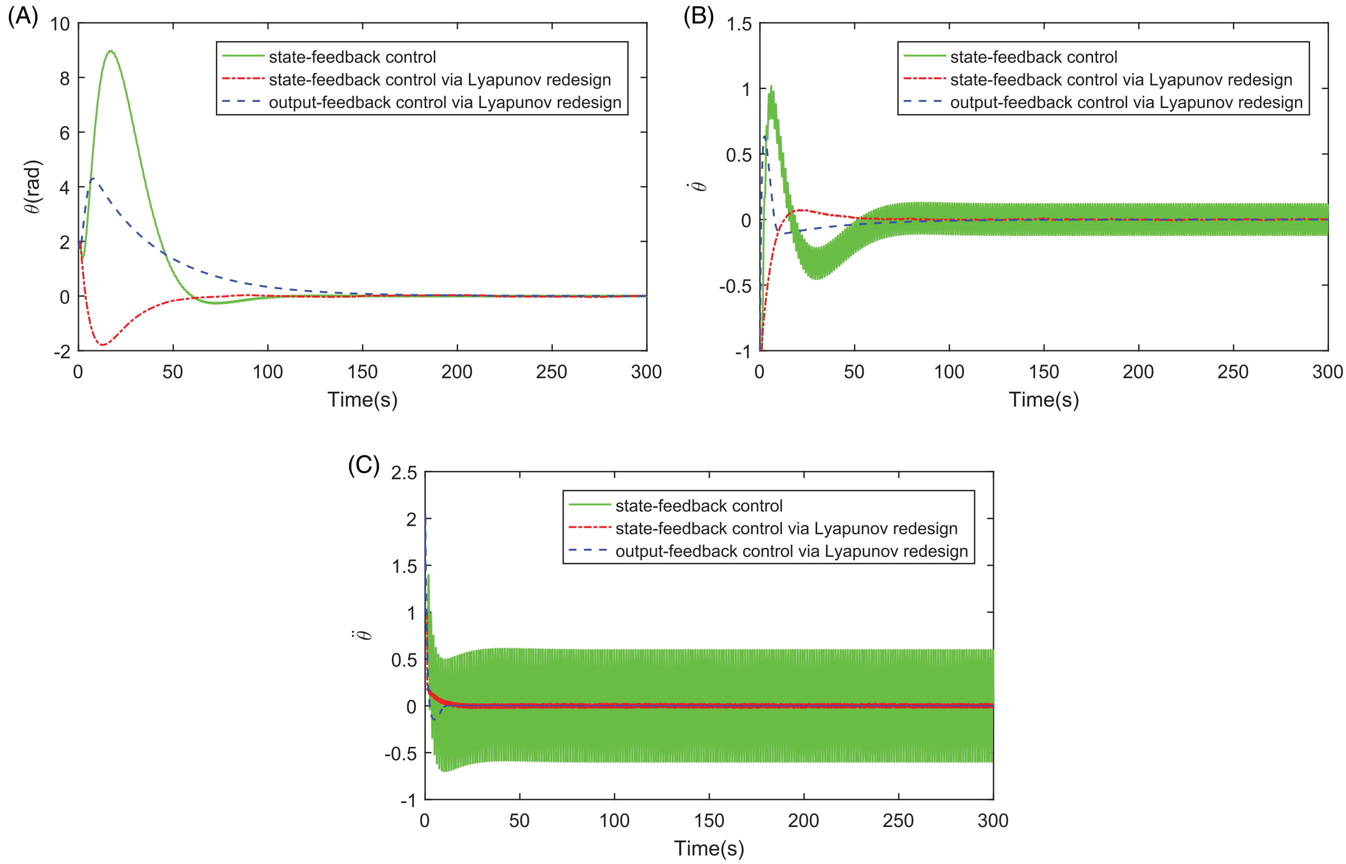

FIGURE 2 Simulated and system responses to the distance input $\delta(t,u,x) = 2\sin(t)$. (A) The state trajectory of $\theta(t)$; (B) The state trajectory of $\dot{\theta}(t)$; (C) The state trajectory of $\ddot{\theta}(t)$ [Colour figure can be viewed at wileyonlinelibrary.com]

$$u_s(t) = -\begin{bmatrix} 1.1002 & -2.1 & 1 \end{bmatrix} x(t)$$
$$u_{slr}(t) = -\begin{bmatrix} 0.003 & 0.0600 & 0.5420 \end{bmatrix} x(t) - 1.2\frac{\sigma^*(t,x)}{\|\sigma^*(t,x)\|}$$

where $\sigma^*(t,x) = K_{sb}x(t)$, and

$$K_{sb} = \begin{bmatrix} 8.6618 & 150.9982 & 546.7244 \end{bmatrix}.$$

Figure 2 reveals the states of system (58) under the state-feedback control law $u_s(t)$, the robust state-feedback control law $u_{slr}(t)$ via LR and the proposed output-feedback robust control law $u(t)$ designed in this work, respectively. The green lines in Figure 2 reveals the effectiveness of state-feedback control $u_s(t)$ based on the nominal design, which has worse disturbance attenuation capability than the robust law $u_{slr}(t)$ and $u(t)$ in (63). It can be seen that the output-feedback controller (63) introduces the stabilizing delays h, $2h$ and $3h$ in the closed loop for attenuating the oscillations in the system trajectories, which can achieve similar robust performance to $u_s(t)$ with full state information. It has been shown in Figure 3 that the sliding motion $\hat{\sigma}(t,\hbar)$ converges to the vicinity of the sliding surface $\hat{\sigma}(t,\hbar) = 0$, and eventually converges to the vicinity of origin, that is, $\{x| \|x\| \leq 0.28\}$ in the presence of matched uncertainty. Moreover, from the average theory, the delayed sliding motion $\hat{\sigma}(t,\hbar)$ has better filtering property of the state trajectories, when comparing with $\sigma^*(t,x)$.

In Figure 4, different delays are used to show the sensitivity of system trajectories to the changes in h. For fixed K_o and φ_1, φ_2, the asymptotical stability can be ensured for the delays below their maximum value $h^* = 0.25$. Figure 5 shows that the use of $h = 0.04$ will be more efficient for smoothing the fluctuations in state $\ddot{\theta}(t)$ than $h = 0.02$, and thus the better chattering attenuation is achieved with larger h.

To sum up, the delayed output-feedback controller has the following specific features: (1) the time-delay estimator is of static form, which avoids introducing extra dynamical structure for state estimation and (2) the use of delayed

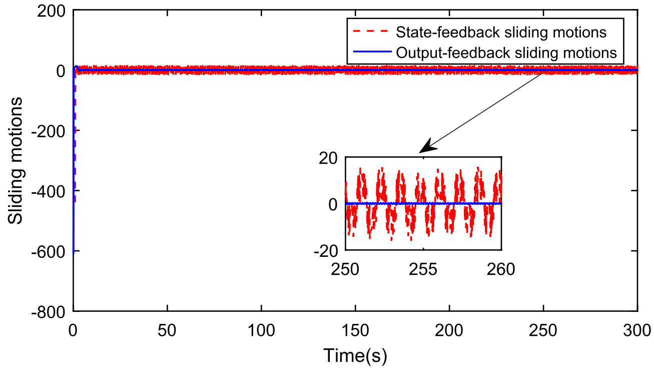

FIGURE 3 Sliding motions [Colour figure can be viewed at wileyonlinelibrary.com]

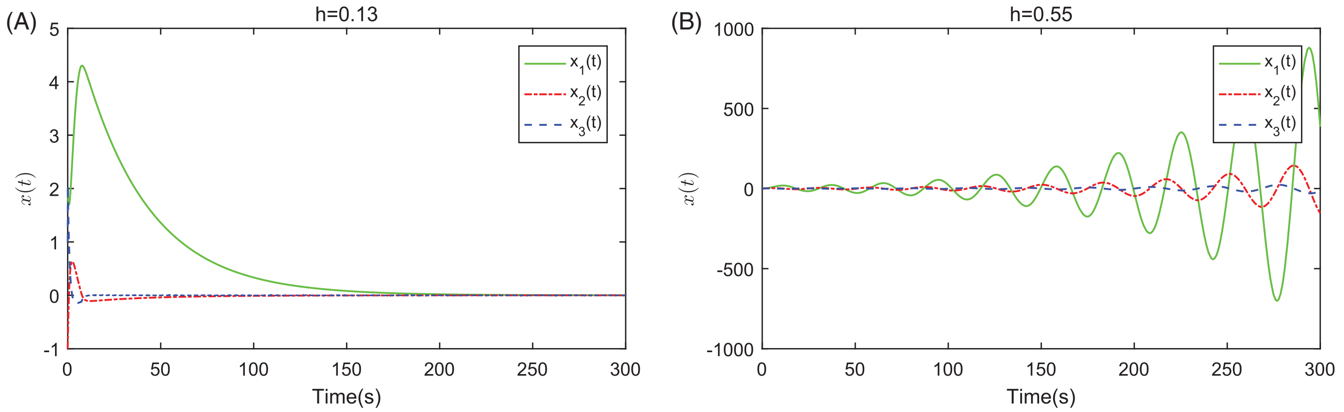

FIGURE 4 Simulation results for different delays. (A) $h = 0.13$; (B) $h = 0.55$ [Colour figure can be viewed at wileyonlinelibrary.com]

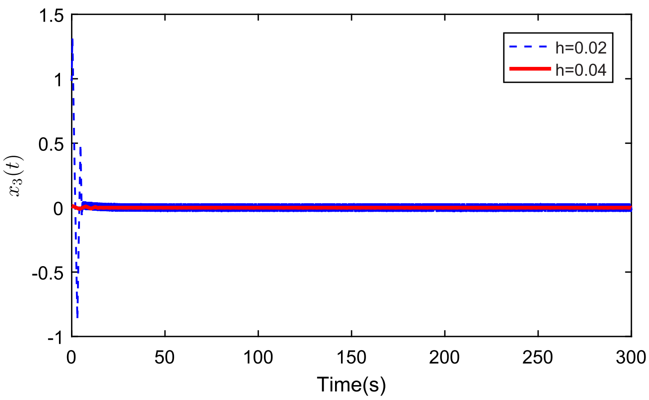

FIGURE 5 The performance of system trajectories under different time delays. [Colour figure can be viewed at wileyonlinelibrary.com]

output-feedback control via LR can reduce the chattering effects in system trajectories, when compared with the state-feedback case. Moreover, it is of design flexibility for the proper choice of h. Generally, there is a trade-off for appropriately selecting delays in the controller design: the bigger value of h results in a better reduction of chattering, and the smaller value of h leads to a better state estimation accuracy.

5 | CONCLUSION

A new approach to output feedback LR is proposed for uncertain system with measurement delays. The output-feedback robust controller has specific useful properties comparing with pure static/obeserver-based output-feedback controller, which is of simple structure and less real-time computation cost. The upper bound of measurement delay keeping a stability of system is estimated. The efficiency and merits of the proposed design procedure are validated through the simulations of behavior an active magnetic bearing system driven by static output feedback control based on delayed measurements.

ACKNOWLEDGEMENTS

This work is funded by the National Natural Science Foundation of China (61803156, 62073139), the Natural Science Foundation of Shanghai (18ZR1409300), the 111 Project (B17017), the Israel Science Foundation (grant no. 673/19), the Open Research Fund of Engineering Research Center of Intelligent Control for Underground Space, Ministry of Education, China University of Mining and Technology (ICUS-2020-A05), and Programa de Apoyo a Proyectos de Investigacion e Innovacion Tecnologica (PAPIIT-UNAM) under Grant IN 115419, and in part by the Fondo de Colaboraci on del II-FI UNAM under Project IISGBAS-100-2015.

DATA AVAILABILITY STATEMENT

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

ORCID

Jing Xu https://orcid.org/0000-0002-0647-9767
Leonid M. Fridman https://orcid.org/0000-0003-0208-3615
Yugang Niu https://orcid.org/0000-0002-3010-1879

REFERENCES

1. Khalil HK. *Nonlinear Systems*. Upper Saddle River, NJ: Prentice Hall; 2002.
2. Gutman S. Uncertain dynamical systems a Lyapunov min-max approach. *IEEE Trans Autom Control*. 1979;24(3):437-443.
3. Leitmann G. Guaranteed asymptotic stability for some linear-systems with bounded uncertainties. *J Dyn Syst Measur Control*. 1979;101(3):212-216.
4. Su W, Drakunov SV, Özgüner Ü. Constructing discontinuity surfaces for variable structure systems: a Lyapunov approach. *Automatica*. 1996;32(6):925-928.
5. Arreola RB. Output feedback nonlinear control for a linear motor in suspension mode. *Automatica*. 2004;40(12):2153-2160.
6. Khalid N, Memon YM. Output feedback stabilization of an inertia wheel pendulum using sliding mode control. Paper presented at: Proceedings of UKACC International Conference on Control; 2014. doi:https://doi.org/10.1109/control.2014.6915132.
7. Oliveira TR, Estrada A, Fridman LM. Global and exact HOSM differentiator with dynamic gains for output-feedback sliding mode control. *Automatica*. 2017;81:156-163.
8. Seuret A, Edwards C, Spurgeon SK, Fridman E. Static output feedback sliding mode control design via an artificial stabilizing delay. *IEEE Trans Autom Control*. 2009;54(2):256-265.
9. Fridman E, Shaikhet L. Delay-induced stability of vector second-order systems via simple Lyapunov functionals. *Automatica*. 2016;74:288-296.
10. Fridman E, Shaikhet L. Stabilization by using artificial delays: an LMI approach. *Automatica*. 2017;81:429-437.
11. Fridman E. *Introduction to Time-Delay Systems: Analysis and Control*. Basel, Switzerland: Birkhäuser; 2014.
12. Fridman E. Tutorial on Lyapunov-based methods for time-delay systems. *Europ J Control*. 2014;20:271-283.
13. Solomon O, Fridman E. New stability conditions for systems with distributed delays. *Automatica*. 2013;49:3467-3475.
14. Selivanov A, Fridman E. An improved time-delay implementation of derivative-dependent feedback. *Automatica*. 2018;98:269-276.
15. Xu J, Niu Y, Fridman E, Fridman LM. Finite frequency H_∞ control of singularly perturbed Euler-Lagrange systems: an artificial delay approach. *Int J Robust Nonlinear Control*. 2019;29(2):353-374.
16. Xu J, Fridman E, Fridman LM, Niu Y. Static sliding mode control of systems with arbitrary relative degree by using artificial delays. *IEEE Trans Autom Control*. 2020;65(12):5464-5471.
17. Ferreira de Loza A, Fridman L, Aguilar LT, Iriarte R. High-order sliding-mode observer-based input-output linearization. *Int J Robust Nonlinear Control*. 2019;29(10):3183-3199.
18. Yan X, Spurgeon S, Edwards C. Static output feedback sliding mode control for time-varying delay systems with time-delayed nonlinear disturbances. *Int J Robust Nonlinear control*. 2010;20:777-788.
19. Wei G, Wang Z, Shen B. Probability-dependent gain-scheduled control for discrete stochastic delayed systems with randomly occurring nonlinearities. *Int J Robust Nonlinear Control*. 2013;23(7):815-826.
20. Monticelli A. Electric power system state estimation. *Proc IEEE*. 2000;88(2):262-282.
21. Sanchez T, Polyakov A, Fridman E, Hetel L. A switching controller for a class of MIMO bilinear systems with time delay. *IEEE Trans Autom Control*. 2020;65(5):2250-2256.

22. Sanchez T, Polyakov A, Richard JP. A sliding mode controller for a model of flow separation in boundary layers. *Int J Robust Nonlinear Control*. 2020;30(3):1181-1202.
23. Lindlau JD, Knospe CR. Feedback linearization of an active magnetic bearing with voltage control. *IEEE Trans Control Syst Technol*. 2002;10(1):21-31.

How to cite this article: Xu J, Fridman LM, Fridman E, Niu Y. Output-feedback Lyapunov redesign of uncertain systems with delayed measurements. *Int J Robust Nonlinear Control*. 2021;31:3747–3766. https://doi.org/10.1002/rnc.5464