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Event-Triggered H_∞ Control: a Switching Approach

Anton Selivanov and Emilia Fridman, *Senior Member, IEEE*

***Abstract*—Event-triggered approach to networked control systems is used to reduce the workload of the communication network. For the static output-feedback continuous event-trigger may generate an infinite number of sampling instants in finite time (Zeno phenomenon) what makes it inapplicable to the real-world systems. Periodic event-trigger avoids this behavior but does not use all the available information. In the present paper we aim to exploit the advantage of the continuous-time measurements and guarantee a positive lower bound on the inter-event times by introducing a switching approach for finding a waiting time in the event-triggered mechanism. Namely, our idea is to present the closed-loop system as a switching between the system under periodic sampling and the one under continuous event-trigger and take the maximum sampling preserving the stability as the waiting time. We extend this idea to the L_2-gain and ISS analysis of perturbed networked control systems with network-induced delays. By examples we demonstrate that the switching approach to event-triggered control can essentially reduce the amount of measurements to be sent through a communication network compared to the existing methods.**

I. Introduction

NETWORKED control systems (NCS), that are comprised of sensors, actuators, and controllers connected through a communication network, have been recently extensively studied by researchers from a variety of disciplines [2]–[5]. One of the main challenges in such systems is that only sampled in time measurements can be transmitted through a communication network. Namely, consider the system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t), \tag{1}$$

with a state $x \in \mathbb{R}^n$, input $u \in \mathbb{R}^m$, and output $y \in \mathbb{R}^l$. Assume that there exists $K \in \mathbb{R}^{m\times l}$ such that the control signal $u(t) = -Ky(t)$ stabilizes the system (1). In NCS the measurements can be transmitted to the controller only at discrete time instants

$$0 = s_0 < s_1 < s_2 < \dots, \quad \lim_{k\to\infty} s_k = \infty. \tag{2}$$

Therefore, the closed-loop system has the form

$$\dot{x}(t) = Ax(t) - BKCx(s_k), \quad t \in [s_k, s_{k+1}), \quad k \in \mathbb{N}_0, \tag{3}$$

where $\mathbb{N}_0$ is the set of nonnegative integers. There are different ways of obtaining the sequence of sampling instants s_k that preserve the stability. The simplest approach is *periodic sampling* where one chooses $s_k = kh$ with appropriate period h. Under periodic sampling the measurements are sent even when the output fluctuation is small and does not significantly change the control signal. To avoid these "redundant" packets one can use *continuous event-trigger* [6], where

$$\begin{aligned} s_{k+1} = \min\{t > s_k \,|\, (y(t) - y(s_k))^T\Omega(y(t) - y(s_k)) \\ \geq \varepsilon y^T(t)\Omega y(t)\} \end{aligned} \tag{4}$$

with a matrix $\Omega \geq 0$ and a scalar $\varepsilon > 0$. In case of a static *output*-feedback execution times s_k, implicitly defined by (4), can be such that $\lim_{k\to\infty} s_k < \infty$ [7]. That is, an infinite number of events is generated in finite time what makes (4) inapplicable to NCS. To avoid this Zeno phenomenon one can use *periodic event-trigger* [8]–[11] by choosing

$$\begin{aligned} s_{k+1} = \min\{s_k + ih \,|\, i \in \mathbb{N},\, (y(s_k + ih) - y(s_k))^T\Omega\times \\ (y(s_k + ih) - y(s_k)) > \varepsilon y^T(s_k + ih)\Omega y(s_k + ih)\}. \end{aligned} \tag{5}$$

This approach guarantees that the inter-event times are at least h and fits the case where the sensor measures only sampled in time outputs $y(ih)$.

However, when the continuous measurements are available one can use this additional information to improve the control algorithm. In [12]–[14] the following strategy of choosing the sampling instants has been considered:

$$s_{k+1} = \min\{t \geq s_k + T \,|\, \eta \geq 0\}, \tag{6}$$

where $T > 0$ is a constant waiting time and η is an event-trigger condition. In [13], [14] the value of T that preserves the stability was obtained by solving a scalar differential equation. For $\eta = |y(t) - y(t_k)| - C$ with a constant C some qualitative results concerning *practical* stability have been obtained in [12].

In this work we propose a new constructive and efficient method of finding an appropriate waiting time. Our idea is to present the closed-loop system as a switching between the system under periodic sampling and the one under continuous event-trigger and take the maximum sampling preserving the stability as a waiting time. We extend this idea to the systems with network-induced delays, external disturbances, and measurement noise (Section III). Differently from [8], [12]–[14] our method is applicable to uncertain linear systems and the waiting time is found from LMIs. Comparatively to periodic event-trigger of [9]–[11] our method leads to error separation between the system under periodic sampling and the one under continuous event-trigger that allows for larger sampling periods for the same values of the event-trigger parameter ε. The latter allows to reduce the amount of sent measurements as illustrated by examples brought from [7] and [15] (Section IV).

II. A switching approach to event-trigger

Consider (1). Assume that there exists K such that $A - BKC$ is Hurwitz. For $C = I$ such K exists if (A, B) is

A. Selivanov (antonselivanov@gmail.com) and E. Fridman (emilia@eng.tau.ac.il) are with School of Electrical Engineering, Tel Aviv University, Israel.

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stabilizable. For the static output-feedback case such K exists if the transfer function $C(sI-A)^{-1}B$ is hyper-minimum-phase (has stable zeroes and positive leading coefficient of the numerator, see, e.g., [16]). Assume that the measurements are sent at time instants (2). The closed-loop system (3) can be rewritten in the form

$$\dot{x}(t)=(A-BKC)x(t)-BKe(t), \tag{7}$$

where $e(t)=y(s_k)-y(t)$ for $t\in[s_k,s_{k+1})$. In the case of periodic sampling $s_k=kh$, $e(t)$ is the error due to sampling that can be presented as [17]

$$\begin{aligned} e(t)&=-C\int_{t-\tau(t)}^{t}\dot{x}(s)\,ds,\\ \tau(t)&=t-kh,\quad t\in[kh,(k+1)h),\quad k=0,1,\dots \end{aligned} \tag{8}$$

In the case of continuous event-trigger, $e(t)$ is the error due to triggering that can be bounded using relation (4) [6].

Under periodic sampling (leading to (7), (8)) "redundant" packets can be sent while continuous event-trigger (that leads to (4), (7)) can cause Zeno phenomenon. To avoid the above drawbacks periodic event-trigger (5) can be used, where the closed-loop system can be written as

$$\dot{x}(t)=(A-BKC)x(t)+BKC\int_{t-\tau(t)}^{t}\dot{x}(s)\,ds-BKe(t) \tag{9}$$

with $\tau(t)=t-s_k-ih\le h$, $e(t)=y(s_k)-y(s_k+ih)$ for $t\in[s_k+ih,s_k+(i+1)h)$, $i\in\mathbb{N}_0$ such that $s_k+(i+1)h\le s_{k+1}$. As one can see, (9) contains both error due to sampling (the integral term) and the error due to triggering $e(t)$ what makes it more difficult to ensure the stability of (9) compared to (7) with only one error.

We propose an event-trigger that allows to *separate these errors* by considering the switching between periodic sampling and continuous event-trigger. Namely, after the measurement has been sent, the sensor waits for at least h seconds (that corresponds to T in (6)). During this time the system is described by (7), (8). Then the sensor begins to continuously check the event-trigger condition and sends the measurement when it is violated. During this time the system is described by (7) with $e(t)$ satisfying the event-trigger condition. This leads to the following choice of sampling:

$$\begin{aligned} s_{k+1}=\min\{s\ge s_k+h\,|\,(y(s)-y(s_k))^T\Omega(y(s)-y(s_k))\\ \ge\varepsilon y^T(s)\Omega y(s)\} \end{aligned} \tag{10}$$

with a matrix $\Omega\ge 0$ and scalars $\varepsilon\ge 0$, $h>0$, where the inter-event times are not less than h. The system (3), (10) can be presented as a switching between (7), (8) and (4), (7):

$$\begin{aligned} \dot{x}(t)=(A-BKC)x(t)+\chi(t)BKC\int_{t-\tau(t)}^{t}\dot{x}(s)\,ds\\ -(1-\chi(t))BKe(t), \end{aligned} \tag{11}$$

where

$$\begin{aligned} \chi(t)&=\begin{cases}1, & t\in[s_k,s_k+h),\\ 0, & t\in[s_k+h,s_{k+1}),\end{cases}\\ \tau(t)&=t-s_k\le h,\qquad t\in[s_k,s_k+h),\\ e(t)&=y(s_k)-y(t),\quad t\in[s_k+h,s_{k+1}). \end{aligned} \tag{12}$$

By using the functional $V=x^TPx+\chi(V_U+V_X)$, where V_U and V_X are defined in (13) and (26) of [18], the following stability conditions can be derived (see [1] for the proof).

Theorem 1: For given scalars $h>0$, $\varepsilon\ge 0$, $\delta>0$ let there exist $n\times n$ matrices $P>0$, $U>0$, X, X_1, P_2, P_3, Y_1, Y_2, Y_3 and $l\times l$ matrix $\Omega\ge 0$ such that

$$\Xi>0,\quad \Psi_0\le 0,\quad \Psi_1\le 0,\quad \Phi\le 0, \tag{13}$$

where

$$\Xi=\begin{bmatrix}P+h\frac{X+X^T}{2} & hX_1-hX\\ * & -hX_1-hX_1^T+h\frac{X+X^T}{2}\end{bmatrix},$$

$$\Phi=\begin{bmatrix}\Phi_{11} & \Phi_{12} & -P_2^TBK\\ * & -P_3^T-P_3 & -P_3^TBK\\ * & * & -\Omega\end{bmatrix},$$

$$\Psi_0=\begin{bmatrix}\Psi_{11}-X_\delta & \Psi_{12}+h\frac{X+X^T}{2} & \Psi_{13}+X_{1\delta}\\ * & \Psi_{22}+hU & \Psi_{23}-h(X-X_1)\\ * & * & \Psi_{33}-X_{2\delta}|_{\tau=0}\end{bmatrix},$$

$$\Psi_1=\begin{bmatrix}\Psi_{11}-\frac{X+X^T}{2} & \Psi_{12} & \Psi_{13}+X-X_1 & hY_1^T\\ * & \Psi_{22} & \Psi_{23} & hY_2^T\\ * & * & \Psi_{33}-X_{2\delta}|_{\tau=h} & hY_3^T\\ * & * & * & -hUe^{-2\delta h}\end{bmatrix},$$

$$\begin{aligned}
\Phi_{11}&=P_2^T(A-BKC)+(A-BKC)^TP_2+\varepsilon C^T\Omega C+2\delta P,\\
\Phi_{12}&=P+(A-BKC)^TP_3-P_2^T,\\
\Psi_{11}&=A^TP_2+P_2^TA+2\delta P-Y_1-Y_1^T,\\
\Psi_{12}&=P-P_2^T+A^TP_3-Y_2,\\
\Psi_{13}&=Y_1^T-P_2^TBKC-Y_3,\\
\Psi_{22}&=-P_3-P_3^T,\\
\Psi_{23}&=Y_2^T-P_3^TBKC,\\
\Psi_{33}&=Y_3+Y_3^T,\\
X_\delta&=(1/2-\delta h)(X+X^T),\\
X_{1\delta}&=(1-2\delta h)(X-X_1),\\
X_{2\delta}&=(1/2-\delta(h-\tau))(X+X^T-2X_1-2X_1^T).
\end{aligned}$$

Then the system (3) under the event-trigger (10) is exponentially stable with a decay rate δ.

Remark 1: Using the functional of Theorem 1 with $\chi=1$ the following result is obtained [1]:

For given scalars $h>0$, $\varepsilon\ge 0$, $\delta>0$ let there exist $n\times n$ matrices $P>0$, $U>0$, X, X_1, P_2, P_3, Y_1, Y_2, Y_3 and $l\times l$ matrix $\Omega\ge 0$ such that

$$\begin{aligned} &\Xi>0,\\ &\left[\begin{array}{c|c}\Psi_i & \begin{matrix}-P_2^TBK\\ -P_3^TBK\\ 0\end{matrix}\\ \hline * & -\Omega\end{array}\right]+\varepsilon[I_n\,0]^TC^T\Omega C[I_n\,0]\le 0,\ i=0,1. \end{aligned} \tag{14}$$

Then the system (3) under periodic event-trigger (5) is exponentially stable with a decay rate δ.

Remark 2: The feasibility of (14) implies the feasibility of (13). Therefore, the stability of (3) under (10) can be guaranteed for not smaller h and ε than under (5). Examples in Section IV show that these values under (10) are essentially larger what allows to reduce the amount of sent measurements. Note that for *the same* h, ε, and Ω the amount of sent measurements under periodic event-trigger (5) is deliberately

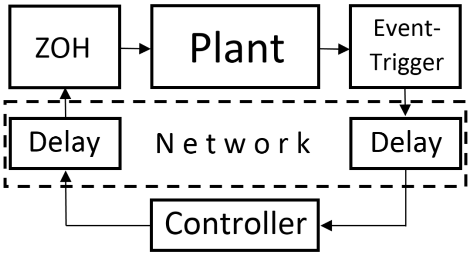

Fig. 1. A system with network-induced delays

less than under (10). Indeed, if the measurement is sent at s_k and the event-trigger rule is satisfied at s_k+h, according to (5) the sensor will wait till at least s_k+2h before sending the next measurement, while according to (10) the next measurement can be sent before s_k+2h.

III. Event-trigger under network-induced delays and disturbances

Consider the system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_1 w(t) + B_2 u(t),\\ z(t) &= C_1 x(t) + D_1 u(t),\\ y(t) &= C_2 x(t) + D_2 v(t) \end{aligned} \tag{15}$$

with a state $x \in \mathbb{R}^n$, input $u \in \mathbb{R}^m$, controlled output $z \in \mathbb{R}^{n_z}$, measurements $y \in \mathbb{R}^l$, and disturbances $w \in \mathbb{R}^{n_w}$, $v \in \mathbb{R}^{n_v}$. Denote by $\eta_k \le \eta_M$ the overall network-induced delay from the sensor to the actuator that affects the transmitted measurement $y(s_k)$ (see Fig. 1). Here s_k is a sampling instant on the sensor side. We assume that η_k are such that the ZOH updating times $t_k = s_k + \eta_k$ satisfy

$$t_k = s_k + \eta_k \le s_{k+1} + \eta_{k+1} = t_{k+1}, \quad k \in \mathbb{N}_0. \tag{16}$$

Then the system (15) with $u(t) = Ky(s_k)$ for $t \in [t_k, t_{k+1})$ has the form

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_1 w(t) + B_2 K[C_2 x(t_k - \eta_k) + D_2 v(t_k - \eta_k)],\\ z(t) &= C_1 x(t) + D_1 K[C_2 x(t_k - \eta_k) + D_2 v(t_k - \eta_k)]. \end{aligned} \tag{17}$$

Similar to Section II we would like to present the resulting closed-loop system (10), (17) as a system with periodic sampling for $t \in [t_k, t_k+h)$ (i.e. $t \in [s_k+\eta_k, s_k+\eta_k+h)$) and as a system with continuous event-trigger for $t \in [t_k+h, t_{k+1})$. If $t_k + h = s_k + \eta_k + h > s_{k+1} + \eta_{k+1} = t_{k+1}$ (what may happen due to the communication delay η_k) no switching occurs. Therefore, the system (10), (17) can be presented as

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_1 w(t) + \chi(t) B_2 K[C_2 x(t - \tau(t)) + D_2 v(t - \tau(t))]\\ &\quad + (1 - \chi(t)) B_2 K[C_2 x(t - \bar\eta(t)) + D_2 v(t - \bar\eta(t)) + e(t)],\\ z(t) &= C_1 x(t) + \chi(t) D_1 K[C_2 x(t - \tau(t)) + D_2 v(t - \tau(t))]\\ &\quad + (1 - \chi(t)) D_1 K[C_2 x(t - \bar\eta(t)) + D_2 v(t - \bar\eta(t)) + e(t)], \end{aligned} \tag{18}$$

where

$$\begin{aligned} \chi(t) &= \begin{cases} 1, & t \in [t_k, \min\{t_k + h, t_{k+1}\}),\\ 0, & t \in [\min\{t_k + h, t_{k+1}\}, t_{k+1}), \end{cases}\\ \tau(t) &= t - s_k, \qquad\qquad t \in [t_k, \min\{t_k + h, t_{k+1}\}),\\ e(t) &= y(s_k) - y(t - \bar\eta(t)),\ t \in [\min\{t_k + h, t_{k+1}\}, t_{k+1}). \end{aligned}$$

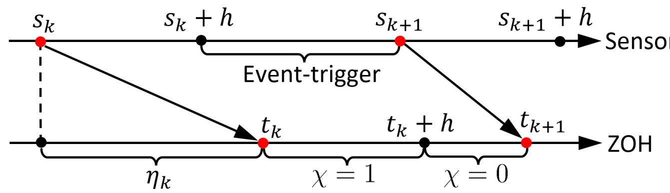

Fig. 2. Switching between the subsystems of (18)

Here $\tau(t) \le h + \eta_M \triangleq \tau_M$ and $\bar\eta(t) \in [0, \eta_M]$ is a "fictitious" delay to be defined hereafter.

Consider the case where $t_k + h < t_{k+1}$ (see Fig. 2). To use the event-trigger condition we would like to choose such $\bar\eta(t)$ that (10) implies

$$\begin{aligned} 0 \le \varepsilon[C_2 x(t - \bar\eta(t)) + D_2 v(t - \bar\eta(t))]^T \Omega \times\\ [C_2 x(t - \bar\eta(t)) + D_2 v(t - \bar\eta(t))] - e^T(t) \Omega e(t) \end{aligned} \tag{19}$$

for $t \in [t_k + h, t_{k+1})$. Relation (19) is true if $t - \bar\eta(t) \in [s_k + h, s_{k+1})$ for $t \in [t_k + h, t_{k+1})$. Therefore, the simplest choice of $\bar\eta(t)$ is a linear function with $\bar\eta(t_k + h) = \eta_k$ and $\bar\eta(t_{k+1}) = \eta_{k+1}$, i.e. for $t \in [\min\{t_k + h, t_{k+1}\}, t_{k+1})$

$$\bar\eta(t) = \frac{t_{k+1} - t}{t_{k+1} - t_k - h}\eta_k + \frac{t - t_k - h}{t_{k+1} - t_k - h}\eta_{k+1}.$$

Though for both $\chi(t) = 0$ and $\chi(t) = 1$ the system (18) includes time-delays, the upper bound η_M for $\bar\eta(t)$ is smaller than τ_M since $\tau(t)$ includes the delay due to sampling.

We say that the system (10), (17) is *internally exponentially stable* if it is exponentially stable with $w(t) \equiv 0$, $v(t) \equiv 0$. Let us extend the definition of $\tau(t)$ by setting $\tau(t) = \bar\eta(t)$ for $t \in [\min\{t_k + h, t_{k+1}\}, t_{k+1})$. We say that the system (10), (17) has an L_2-gain (H_∞ gain) less than γ if for the zero initial condition $x(0) = 0$ and all $w, v \in L_2[0, \infty)$ such that $w^T(t)w(t) + v^T(t-\tau(t))v(t-\tau(t)) \not\equiv 0$ the following relation holds on the trajectories of (10), (17):

$$\begin{aligned} J = \int_0^\infty \Big\{ z^T(t) z(t) - \gamma^2 [w^T(t) w(t)\\ + v^T(t - \tau(t)) v(t - \tau(t))] \Big\} dt < 0. \end{aligned} \tag{20}$$

Theorem 2: For given $\gamma > 0$, $h > 0$, $\eta_M \ge 0$, $\varepsilon \ge 0$, $\delta > 0$ let there exist $n \times n$ matrices $P > 0$, $S_0 \ge 0$, $S_1 \ge 0$, $R_0 \ge 0$, $R_1 \ge 0$, G_1, G_0 and $l \times l$ matrix $\Omega \ge 0$ such that

$$\Psi \le 0, \quad \Phi \le 0, \quad \begin{bmatrix} R_0 & G_0\\ G_0^T & R_0 \end{bmatrix} \ge 0, \quad \begin{bmatrix} R_1 & G_1\\ G_1^T & R_1 \end{bmatrix} \ge 0, \tag{21}$$

where $\Psi = \{\Psi_{ij}\}$ and $\Phi = \{\Phi_{ij}\}$ are symmetric matrices composed from the matrices

$$\begin{aligned} &\Psi_{11} = \Phi_{11} = A^T P + PA + 2\delta P + S_0 - e^{-2\delta\eta_M} R_0 + C_1^T C_1,\\ &\Psi_{12} = e^{-2\delta\eta_M} R_0,\\ &\Psi_{14} = PB_2KC_2 + C_1^T D_1 K C_2,\\ &\Psi_{15} = \Phi_{16} = PB_1,\\ &\Psi_{16} = \Phi_{17} = PB_2KD_2 + C_1^T D_1 K D_2,\\ &\Psi_{17} = \Phi_{18} = A^T H,\\ &\Psi_{23} = e^{-2\delta\tau_M} G_1,\\ &\Psi_{24} = e^{-2\delta\tau_M}(R_1 - G_1),\\ &\Psi_{22} = \Phi_{22} = e^{-2\delta\eta_M}(S_1 - S_0 - R_0) - e^{-2\delta\tau_M} R_1,\\ &\Psi_{33} = \Phi_{33} = -e^{-2\delta\tau_M}(R_1 + S_1), \end{aligned}$$

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$$
\begin{aligned}
\Psi_{34} &= e^{-2\delta\tau_M}(R_1 - G_1^T),\\
\Psi_{44} &= e^{-2\delta\tau_M}(G_1 + G_1^T - 2R_1) + (D_1KC_2)^T D_1KC_2,\\
\Psi_{46} &= (D_1KC_2)^T D_1KD_2,\\
\Psi_{47} &= \Phi_{48} = (B_2KC_2)^T H,\\
\Psi_{55} &= \Phi_{66} = -\gamma^2 I,\\
\Psi_{57} &= \Phi_{68} = B_1^T H,\\
\Psi_{77} &= \Phi_{88} = -H,\\
\Psi_{66} &= (D_1KD_2)^T D_1KD_2 - \gamma^2 I,\\
\Psi_{67} &= \Phi_{78} = (B_2KD_2)^T H,\\
\Phi_{12} &= e^{-2\delta\eta_M} G_0,\\
\Phi_{23} &= e^{-2\delta\tau_M} R_1,\\
\Phi_{24} &= e^{-2\delta\eta_M}(R_0 - G_0^T),\\
\Phi_{14} &= PB_2KC_2 + e^{-2\delta\eta_M}(R_0 - G_0) + C_1^T D_1KC_2,\\
\Phi_{15} &= PB_2K + C_1^T D_1K,\\
\Phi_{45} &= (D_1KC_2)^T D_1K,\\
\Phi_{44} &= e^{-2\delta\eta_M}(G_0 + G_0^T - 2R_0) + (D_1KC_2)^T D_1KC_2 + \varepsilon C_2^T \Omega C_2,\\
\Phi_{47} &= (D_1KC_2)^T D_1KD_2 + \varepsilon C_2^T \Omega D_2,\\
\Phi_{58} &= (B_2K)^T H,\\
\Phi_{55} &= (D_1K)^T D_1K - \Omega,\\
\Phi_{57} &= (D_1K)^T D_1KD_2,\\
\Phi_{77} &= (D_1KD_2)^T D_1KD_2 + \varepsilon D_2^T \Omega D_2 - \gamma^2 I,\\
H &= \eta_M^2 R_0 + h^2 R_1,
\end{aligned}
$$

$\tau_M = h + \eta_M$, other blocks are zero matrices. Then the system (17) under the event-trigger (10) is internally exponentially stable with a decay rate δ and has L_2-gain less than γ.

Proof: See Appendix.

Corollary 1: If (21) are valid with $C_1 = 0$, $D_1 = 0$ then the system (18) under the event-trigger (10) is Input-to-State Stable with respect to $\bar{w}(t) = \mathrm{col}\{w(t), v(t - \tau(t))\}$.

Proof: If $\bar{w}^T(t)\bar{w}(t)$ is bounded by Δ^2 then (28) (see Appendix) with $C_1 = 0$, $D_1 = 0$ transforms to $\dot{V} \le -2\delta V + \gamma^2\Delta^2$. This implies the assertion of the corollary. ∎

Remark 3: The system (17) under periodic event-trigger (5) can be presented in the form (18) with $\chi = 0$ and $\bar{\eta}(t) \le \tau_M$. By modifying the proof of Theorem 2 one can obtain the stability conditions using the functional (23) with arbitrary chosen "delay partitioning" parameter $\eta_M \in (0, \tau_M)$ [19], [20].

Remark 4: Though there are no universal methods of finding optimal event-trigger parameters h and ε, for practical use, one can find the maximum h_* that ensures stability of a system under periodic sampling (by using Theorem 1 or 2 with $\varepsilon = 0$) and calculate the maximum $\varepsilon > 0$ for some $h < h_*$

Remark 5: The proposed approach can be easily extended to cope with packet dropouts with bounded amount of consecutive packet losses. Consider the unreliable network with the maximum number of consecutive packet losses d^{sc} (from the sensor to the controller) and d^{ca} (from the controller to the actuator). To cope with this issue we set the sensor to send the measurement $y(s_k)$ $d^{sc}+1$ times at time instants $s_k + ih_d/d^{sc}$, where $i = 0, \ldots, d^{sc}$, $h_d > 0$. The same strategy is applied to the data sent from the controller. Denote by r_k^{sc} and r_k^{ca} network delays that correspond to the first successfully sent packets. Then the closed-loop system is given by (17) with

$$\eta_k = (d_k^{sc}/d^{sc} + d_k^{ca}/d^{ca})h_d + r_k^{sc} + r_k^{ca} \le \eta_M,$$

where d_k^{sc} and d_k^{ca} are the actual amounts of consecutive packets that were lost. If $r_k^{sc} + r_k^{ca} < \eta_M$ one can choose $h_d > 0$ such that $\eta_k \le \eta_M$ and apply the results of this section. This approach can be improved by introducing the acknowledgement signal of successful reception as suggested in [21]. Such improvement is a possible direction of the future work.

Remark 6: Differently from periodic event-trigger approach considered in [8] our method is applicable to linear systems with polytopic-type uncertainties, since LMIs of Theorems 1 and 2 are affine in A, B, B_1, and B_2.

TABLE I
AVERAGE AMOUNTS OF SENT MEASUREMENTS (SM)

	ε	h	SM
Periodic sampling	—	1.173	18
Event-trigger (5)	4.6×10^{-3}	1.115	17.47
Event-trigger (5)	0.555	0.344	24.8
Switching approach (10)	0.555	0.899	11.13

IV. NUMERICAL EXAMPLES

Example 1 [7]. Consider the system (3) with

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad K = 3. \quad (22)$$

As it has been shown in [7] for this system an accumulation of events occurs under continuous event-trigger (4). In what follows we compare three approaches of choosing the sampling instants s_k: periodic sampling with $s_k = kh$, periodic event-trigger (5), and switching event-trigger (10).

For $\varepsilon = 0$ (10) transforms into periodic sampling, therefore, Theorem 1 can be used to obtain the maximum period h. Under periodic sampling the amount of sent measurements is $\left[\frac{T_f}{h}\right] + 1$, where T_f is the time of simulation and $[\cdot]$ is the integer part of a given number. To obtain the amount of sent measurements for s_k given by (5) (or (10)), for each $\varepsilon = i \times 10^{-4}$ $(i = 0, 1, \ldots, 10^4)$ we find the maximum h that satisfies Remark 1 (or Theorem 1) and for each pair of (ε, h) we perform numerical simulations for several initial conditions given by $(x_1(0), x_2(0)) = (10\cos(2\pi k/30), 10\sin(2\pi k/30))$ with $k = 1, \ldots, 30$. Then we choose the pair (ε, h) that ensures the minimum average amount of sent measurements. The obtained average amount of sent measurements for $\delta = 0.24$ and $T_f = 20$ are presented in Table I. As one can see periodic event-trigger (5) does not give any significant improvement compared to periodic sampling, while the switching event-trigger (10) allows to reduce the network workload by almost 40%. Note that for the same value of ε the value of h obtained for switching event-trigger (10) is more than 2.5 times larger than the one for periodic event-trigger (5).

Example 2 [15]. Consider an inverted pendulum on a cart described by (3) with

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 10/3 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0.1 \\ 0 \\ -1/30 \end{bmatrix}, \quad C = I.$$

For $K = -[2, 12, 378, 210]$ Theorem 1 gives $h = 0.242$, $\varepsilon = 0.35$. According to the numerical simulations, performed

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TABLE II
AMOUNTS OF SENT MEASUREMENTS (SM) WITH TIME-DELAYS AND DISTURBANCES

	ε	h	SM
Periodic sampling	—	0.091	330
Event-trigger (5)	0.033	0.036	195
Event-trigger (10)	0.044	0.065	173

for T_f and $x(0)$ from [15], the average release period under switching event-trigger (10) is 0.5769, which is larger than 0.5131 obtained for the same system in [10] (where the average release period is larger than in [15], [22]–[24]).

Consider the system (17) with the same A, $B_1^T = C_1 = [1,1,1,1]$, $B_2 = B$, $C_2 = I$, $D_1 = 0.1$, $D_2 = [0,0,0,0]$, $K = [2.9129, 10.4357, 287.9029, 160.3271]$. For $\gamma = 200$, $\eta_M = 0.1$ Theorem 2 (with $\delta = 0$) gives $h = 0.117$, $\varepsilon = 0.13$. From the numerical simulations, performed for T_f and $w(t)$ from [10], we obtained an average release period 0.3488, which is larger than 0.3098 obtained for the same system in [10] (where the average release period is larger than the one obtained in [15] for a different controller gain).

For $\gamma = 100$ in a manner similar to Example 1 we obtained the amount of sent measurements presented in Table II. As one can see both event-triggers reduce the network workload and switching event-trigger (10) allows to reduce the amount of sent measurements by more than 11% compared to periodic event-trigger (5).

V. CONCLUSION

We proposed a new approach to event-triggered control under the continuous-time measurements that guarantees a positive lower bound for inter-event times and can significantly reduce the workload of the network. Our idea is based on a switching between periodic sampling and continuous event-trigger. We extended this approach to the L_2-gain and ISS analyses of perturbed NCS with network-induced delays. Our results are applicable to linear systems with polytopic-type uncertainties. The presented method can be extended to nonlinear NCSs that may be a topic for the future research.

APPENDIX
PROOF OF THEOREM 2

The system (10), (17) is rewritten as (18). Similar to [19] we consider Lyapunov functional

$$V = V_P + V_{S_0} + V_{S_1} + V_{R_0} + V_{R_1}, \tag{23}$$

where $x_t(\theta) = x(t+\theta)$ for $\theta \in [-h, 0]$, $V_P(x_t) = x^T(t)Px(t)$,

$$\begin{aligned}
V_{S_0}(t,x_t) &= \int_{t-\eta_M}^{t} e^{2\delta(s-t)} x^T(s) S_0 x(s)\, ds,\\
V_{R_0}(t,x_t) &= \eta_M \int_{-\eta_M}^{0} \int_{t+\theta}^{t} e^{2\delta(s-t)} \dot{x}^T(s) R_0 \dot{x}(s)\, ds\, d\theta,\\
V_{S_1}(t,x_t) &= \int_{t-\tau_M}^{t-\eta_M} e^{2\delta(s-t)} x^T(s) S_1 x(s)\, ds,\\
V_{R_1}(t,x_t) &= h \int_{-\tau_M}^{-\eta_M} \int_{t+\theta}^{t} e^{2\delta(s-t)} \dot{x}^T(s) R_1 \dot{x}(s)\, ds\, d\theta.
\end{aligned}$$

By differentiating these functionals we obtain

$$\begin{aligned}
\dot{V}_{S_0} &= -2\delta V_{S_0} + x^T(t) S_0 x(t)\\
&\qquad - e^{-2\delta\eta_M} x^T(t-\eta_M) S_0 x(t-\eta_M),\\
\dot{V}_{S_1} &= -2\delta V_{S_1} + e^{-2\delta\eta_M} x^T(t-\eta_M) S_1 x(t-\eta_M)\\
&\qquad - e^{-2\delta\tau_M} x^T(t-\tau_M) S_1 x(t-\tau_M),\\
\dot{V}_{R_0} &= -2\delta V_{R_0} + \eta_M^2 \dot{x}^T(t) R_0 \dot{x}(t)\\
&\qquad - \eta_M \int_{t-\eta_M}^{t} e^{2\delta(s-t)} \dot{x}^T(s) R_0 \dot{x}(s)\, ds,\\
\dot{V}_{R_1} &= -2\delta V_{R_1} + h^2 \dot{x}^T(t) R_1 \dot{x}(t)\\
&\qquad - h \int_{t-\tau_M}^{t-\eta_M} e^{2\delta(s-t)} \dot{x}^T(s) R_1 \dot{x}(s)\, ds.
\end{aligned} \tag{24}$$

A. System (18) *with* $\chi(t) = 0$, $\bar{\eta}(t) \in [0, \eta_M]$. We have

$$\begin{aligned}
\dot{V}_P = 2x^T(t)P[&Ax(t) + B_1 w(t) + B_2 K C_2 x(t-\bar{\eta}(t))\\
&+ B_2 K D_2 v(t-\bar{\eta}(t)) + B_2 K e(t)].
\end{aligned} \tag{25}$$

To compensate $x(t-\bar{\eta}(t))$ we apply Jensen's inequality [25] and Park's theorem [26] to obtain

$$\begin{aligned}
&- \eta_M \int_{t-\eta_M}^{t} e^{2\delta(s-t)} \dot{x}^T(s) R_0 \dot{x}(s)\, ds \le -e^{-2\delta\eta_M} \times\\
&\begin{bmatrix} x(t) - x(t-\bar{\eta}(t)) \\ x(t-\bar{\eta}(t)) - x(t-\eta_M) \end{bmatrix}^T \begin{bmatrix} R_0 & G_0 \\ G_0^T & R_0 \end{bmatrix} \begin{bmatrix} x(t) - x(t-\bar{\eta}(t)) \\ x(t-\bar{\eta}(t)) - x(t-\eta_M) \end{bmatrix},
\end{aligned} \tag{26}$$

$$\begin{aligned}
&- h \int_{t-\tau_M}^{t-\eta_M} e^{2\delta(s-t)} \dot{x}^T(s) R_1 \dot{x}(s) ds \le -e^{-2\delta\tau_M} \times\\
&[x(t-\eta_M) - x(t-\tau_M)]^T R_1 [x(t-\eta_M) - x(t-\tau_M)].
\end{aligned} \tag{27}$$

By summing up (19), (24), (25) in view of (26) and (27) and substituting z from (18) we obtain

$$\begin{aligned}
\dot{V} + 2\delta V + z^T z - \gamma^2 [w^T w + v^T(t-\bar{\eta}(t)) v(t-\bar{\eta}(t))]\\
\le \varphi^T(t) \Phi' \varphi(t) + \dot{x}^T(t) H \dot{x}(t),
\end{aligned}$$

where $\varphi(t) = \mathrm{col}\{x(t), x(t-\eta_M), x(t-\tau_M), x(t-\bar{\eta}(t)), e(t), w(t), v(t-\bar{\eta}(t))\}$ and the matrix Φ' is obtained from Φ by deleting the last block-column and the last block-row. Substituting expression for $\dot{x}$ and applying Schur complement formula we find that $\Phi \le 0$ guarantees that

$$\dot{V} + 2\delta V + z^T z - \gamma^2 [w^T w + v^T(t-\tau(t)) v(t-\tau(t))] \le 0. \tag{28}$$

B. System (18) *with* $\chi = 1$, $\tau(t) \in (\eta_M, \tau_M]$. For $\tau(t) \in [0, \eta_M]$ the system (18) with $\chi = 1$ is described by (18) with $\chi = 0$ and $e(t) = 0$ satisfying (19). That is, $\Phi \le 0$ guarantees (28) for (18) with $\chi = 1$, $\tau(t) \in [0, \eta_M]$. Therefore, we study the system (18) for $\chi = 1$, $\tau(t) \in (\eta_M, \tau_M]$. We have

$$\begin{aligned}
\dot{V}_P = 2x^T(t)P[&Ax(t) + B_1 w(t) + B_2 K C_2 x(t-\tau(t))\\
&+ B_2 K D_2 v(t-\tau(t))].
\end{aligned} \tag{29}$$

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To compensate $x(t-\tau(t))$ with $\tau(t)\in(\eta_M,\tau_M]$ we apply Jensen's inequality and Park's theorem to obtain

$$
\begin{aligned}
&-\eta_M\int_{t-\eta_M}^{t} e^{2\delta(s-t)}\dot{x}^T(s)R_0\dot{x}(s)\,ds\\
&\quad\le -e^{-2\delta\eta_M}[x(t)-x(t-\eta_M)]^T R_0[x(t)-x(t-\eta_M)],
\end{aligned}
\tag{30}
$$

$$
\begin{aligned}
&-h\int_{t-\tau_M}^{t-\eta_M} e^{2\delta(s-t)}\dot{x}^T(s)R_1\dot{x}(s)ds\le -e^{-2\delta\tau_M}\times\\
&\begin{bmatrix}x(t-\eta_M)-x(t-\tau(t))\\ x(t-\tau(t))-x(t-\tau_M)\end{bmatrix}^T\begin{bmatrix}R_1 & G_1\\ G_1^T & R_1\end{bmatrix}\begin{bmatrix}x(t-\eta_M)-x(t-\tau(t))\\ x(t-\tau(t))-x(t-\tau_M)\end{bmatrix}.
\end{aligned}
\tag{31}
$$

By summing up (24) and (29) in view of (30) and (31) and substituting z from (18) we obtain

$$
\begin{aligned}
\dot{V}+2\delta V+z^Tz-\gamma^2[w^Tw+v^T(t-\tau(t))v(t-\tau(t))]\\
\le\psi^T(t)\Psi'\psi(t)+\dot{x}^T(t)H\dot{x}(t),
\end{aligned}
$$

where $\psi(t)=\mathrm{col}\{x(t),x(t-\eta_M),x(t-\tau_M),x(t-\tau(t)),$ $w(t),v(t-\tau(t))\}$ and the matrix Ψ' is obtained from Ψ by deleting the last block-column and the last block-row. Substituting expression for $\dot{x}$ and applying Schur complement formula we find that $\Psi\le 0$ guarantees (28) for (18) with $\chi=1$.

Thus, (28) is true for the switched system (18). For $w\equiv 0$, $v\equiv 0$ (28) implies $\dot{V}\le -2\delta V$. Therefore, the system (18) is internally exponentially stable with the decay rate δ. By integrating (28) from 0 to ∞ with $x(0)=0$ we obtain (20).

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