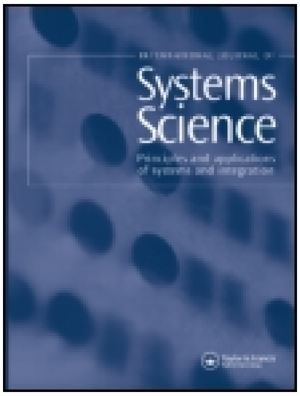

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Sliding mode control of systems with time-varying delays via descriptor approach

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Sliding mode control of systems with time-varying delays via descriptor approach

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A descriptor approach to stability and control of linear systems with time-varying delays, which is based on the Lyapunov–Krasovskii techniques, are combined with a recent result on the sliding mode control of such systems. The systems under consideration have norm-bounded uncertainties and uncertain bounded delays. The solution is given in terms of linear matrix inequalities and improves the previous results based on other Lyapunov techniques. A numerical example illustrates the advantages of the new method.

1. Introduction

During the last decade a rich literature has been dedicated to robust control of time-delay systems (e.g. Boyd *et al.* 1994, Dugard and Verriest 1998, Fridman 2001, 2002, Fridman and Shaked, 2002a, b, 2003, Fu *et al.* 1997, Gouaisbaut *et al.* 2002, Ivanescu 2000, Kolmanovskii and Richard 1999, Kolmanovskii *et al.* 1999, Kolmanovskii and Myshkis 1999, Li and de Souza 1997, Mahmoud 2000, Moon *et al.* 2001, Niculescu 2001 and references therein). Many existing results concern systems with unknown but constant delays. However, in some applications, such as networked control or teleoperated systems, the assumption of a constant delay is too restrictive; this can lead to bad performances or, even worse, to unstable behaviours.

This paper combines two previous results to obtain a more efficient sliding mode controller for uncertain systems with time-varying delays and norm-bounded uncertainties. Other results (Gouaisbaut *et al.* 2002) concern varying delays but may lead to strong conditions which reduce the dynamic performances.

The first of these results is the sliding mode design (Gouaisbaut *et al.* 2002), which copes with stabilization of systems with time-varying delays. The approach relies on the construction of a Lyapunov-Razumikhin function that allows fast variations of the delay but leads to some conservatism on the upper bound of the time-delay.

The second result given in Fridman (2001) concerns the construction of a new class of Lyapunov–Krasovskii functionals using a descriptor model transformation. Unlike previous transformations, the descriptor model leads to a system that is equivalent to the original one (from the point of view of stability) and requires bounding of fewer cross-terms. Furthermore, following this approach, stability criteria have been given in Fridman and Shaked (2003) for systems with time-varying delays without any assumption on their derivatives (which was the case with the usual Lyapunov–Krasovskii functionals).

The paper is organized as follows. Section 2 develops a Lyapunov–Krasovskii approach on a descriptor representation for an uncertain, linear, time-delay system. This provides a stability condition expressed in term of feasibility of a linear matrix inequality (LMI) (Boyd *et al.* 1994). Then the design of a stabilizing memoryless state feedback is derived. Section 3 deals with the design of a sliding mode controller. This is achieved through the resolution of a generalized eigenvalue problem that can be solved efficiently using semidefinite programming tools. Section 4 solves an illustrative example using the present approach and compares it with previous results.

Throughout, the superscript T stands for matrix transposition, $\mathcal{R}^n$ denotes the n dimensional Euclidean space, and $\mathcal{R}^{n\times m}$ is the set of all $n \times m$ real matrices. The notation $P > 0$ for $P \in \mathcal{R}^{n\times n}$ means that P is symmetric and positive definite. I_n represents the $n \times n$ identity matrix.

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2. Stabilization of linear systems with norm-bounded uncertainties by delayed feedback

This section considers the following uncertain linear system with a time-varying delay:

$$\begin{aligned}\dot{x}(t) &= (A_0 + H\Delta(t)E_0)x(t) + (A_1 + H\Delta(t)E_1)x(t-\tau(t)) \\ &\quad + (B_0 + H\Delta(t)E_2)u(t) + B_1 u(t-\tau(t)), \\ x(t) &= \phi(t), \ t \in [-h, 0], \end{aligned} \tag{1}$$

where $x(t) \in \mathcal{R}^n$ is the system state, $u(t) \in \mathcal{R}^m$ is the control input, h is an upper-bound on the time-delay function $(0 \le \tau(t) \le h, \quad \forall t \ge 0)$. The matrix $\Delta(t) \in \mathcal{R}^{p \times q}$ is a matrix of time-varying, uncertain parameters satisfying

$$\Delta^{\mathrm{T}}(t)\Delta(t) \le I_q \quad \forall\, t. \tag{2}$$

For simplicity, we consider only one delay, but the results here may be easily generalized to the case of multiple delays.

We seek a control law

$$u(t) = Kx(t) \tag{3}$$

that will asymptotically stabilize the system.

2.1. *Stability issue*

This subsection considers the following equation:

$$\dot{x}(t) = (\bar{A}_0 + H\Delta(t)\bar{E}_0)x(t) + (\bar{A}_1 + H\Delta(t)\bar{E}_1)x(t-\tau(t)). \tag{4}$$

Representing (1) in an equivalent descriptor form (Fridman 2001):

$$\begin{aligned}\dot{x}(t) &= y(t), \\ 0 &= -y(t) + (\bar{A}_{\mathrm{T}} + H\Delta\bar{E}_{\mathrm{T}})x(t) \\ &\quad - (\bar{A}_1 + H\Delta\bar{E}_1)\int_{t-\tau(t)}^{t} y(s)\mathrm{d}s\end{aligned}$$

or

$$E\dot{\bar{x}}(t) = \begin{bmatrix} 0 & I_n \\ \bar{A}_{\mathrm{T}} + H\Delta\bar{E}_{\mathrm{T}} & -I_n \end{bmatrix}\bar{x}(t) - \begin{bmatrix} 0 \\ \bar{A}_1 + H\Delta\bar{E}_1 \end{bmatrix}\int_{t-\tau(t)}^{t} y(s)\mathrm{d}s, \tag{5}$$

with

$$\bar{x}(t) = col\{x(t),\ y(t)\},\ E = \mathrm{diag}\{I_n, 0\},$$
$$\bar{A}_{\mathrm{T}} = \bar{A}_0 + \bar{A}_1, \quad \bar{E}_{\mathrm{T}} = \bar{E}_0 + \bar{E}_1,$$

the following Lyapunov–Krasovskii functional is applied:

$$V(t) = \bar{x}^{\mathrm{T}}(t)EP\bar{x}(t) + V_2(t), \tag{6}$$

where

$$\begin{aligned} &P = \begin{bmatrix} P_1 & 0 \\ P_2 & P_3 \end{bmatrix}, \quad P_1 > 0, \quad EP = P^{\mathrm{T}}E \ge 0, \\ &V_2(t) = \int_{-h}^{0}\int_{t+\theta}^{t} y^{\mathrm{T}}(s)[R + \delta_2\bar{E}_1^{\mathrm{T}}\bar{E}_1]y(s)\,\mathrm{d}s\,\mathrm{d}\theta. \end{aligned} \tag{7a–d}$$

The following result is obtained:

Lemma 1: *The system* (4) *is asymptotically stable if there exist* $n \times n$ *matrices* $0 < P_1,\ P_2,\ P_3,\ R > 0$ *and positive numbers* δ_1, δ_2 *that satisfy the following LMI*:

$$\Gamma = \begin{bmatrix} \Psi & hP^{\mathrm{T}}\begin{bmatrix} 0 \\ \bar{A}_1 \end{bmatrix} & P^{\mathrm{T}}\begin{bmatrix} 0 \\ H \end{bmatrix} & hP^{\mathrm{T}}\begin{bmatrix} 0 \\ H \end{bmatrix} \\ * & -hR & 0 & 0 \\ * & * & -\delta_1 I_p & 0 \\ * & * & * & -\delta_2 h I_p \end{bmatrix} < 0 \tag{8}$$

where

$$\Psi = \Psi_0 + \begin{bmatrix} \delta_1\bar{E}_{\mathrm{T}}^{\mathrm{T}}\bar{E}_{\mathrm{T}} & 0 \\ 0 & h(R + \delta_2\bar{E}_1^{\mathrm{T}}\bar{E}_1) \end{bmatrix},$$

$$\Psi_0 = P^{\mathrm{T}}\begin{bmatrix} 0 & I_n \\ \bar{A}_{\mathrm{T}} & -I_n \end{bmatrix} + \begin{bmatrix} 0 & I_n \\ \bar{A}_{\mathrm{T}} & -I_n \end{bmatrix}^{\mathrm{T}} P,$$

and $*$ *denotes symmetrical entries.*

Proof: Note that

$$\bar{x}^{\mathrm{T}}(t)EP\bar{x}(t) = x^{\mathrm{T}}(t)P_1x(t)$$

and, hence, differentiating the first term of (6) with respect to t gives:

$$\frac{\mathrm{d}}{\mathrm{d}t}\{\bar{x}^{\mathrm{T}}(t)EP\bar{x}(t)\} = 2x^{\mathrm{T}}(t)P_1\dot{x}(t) = 2\bar{x}^{\mathrm{T}}(t)P^{\mathrm{T}}\begin{bmatrix} \dot{x}(t) \\ 0 \end{bmatrix}. \tag{9}$$

Replacing $\begin{bmatrix} \dot{x}(t) \\ 0 \end{bmatrix}$ by the right side of (5) we obtain:

$$\begin{aligned}\frac{\mathrm{d}V(t)}{\mathrm{d}t} &= \bar{x}^{\mathrm{T}}(t)\Psi_0\bar{x}(t) + \eta_0 + \eta_1 + \eta_2 \\ &\quad + hy^{\mathrm{T}}(t)\big[R + \delta_2\bar{E}_1^{\mathrm{T}}\bar{E}_1\big]y(t) \\ &\quad - \int_{t-h}^{t} y^{\mathrm{T}}(s)\big[R + \delta_2\bar{E}_1^{\mathrm{T}}\bar{E}_1\big]y(s)\,\mathrm{d}s, \qquad (10)\end{aligned}$$

where

$$\eta_0(t) \triangleq -2\int_{t-\tau(t)}^{t} \bar{x}^{\mathrm{T}}(t)P^{\mathrm{T}}\begin{bmatrix} 0 \\ \bar{A}_1 \end{bmatrix} y(s)\,\mathrm{d}s,$$

$$\eta_1(t) \triangleq 2\bar{x}^{\mathrm{T}}(t)P^{\mathrm{T}}\begin{bmatrix} 0 \\ H \end{bmatrix}\Delta(\bar{E}_0 + \bar{E}_1)x(t),$$

$$\eta_2(t) \triangleq -2\int_{t-\tau(t)}^{t} \bar{x}^{\mathrm{T}}(t)P^{\mathrm{T}}\begin{bmatrix} 0 \\ H \end{bmatrix}\Delta\bar{E}_1 y(s)\,\mathrm{d}s.$$

Applying the standard bounding

$$a^{\mathrm{T}}b \le a^{\mathrm{T}}Ra + b^{\mathrm{T}}R^{-1}b, \quad \forall a,b \in \mathcal{R}^n, \forall R \in \mathcal{R}^{n\times n} : R > 0,$$

and using the fact that $\tau(t) \le h$, we have

$$\begin{aligned}\eta_0(t) &\le \tau\bar{x}^{\mathrm{T}}(t)P^{\mathrm{T}}\begin{bmatrix} 0 \\ \bar{A}_1 \end{bmatrix}R^{-1}[0\ \ \bar{A}_1^{\mathrm{T}}]P\bar{x}(t) + \int_{t-\tau(t)}^{t} y^{\mathrm{T}}(s)Ry(s)\mathrm{d}s \\ &\le h\bar{x}^{\mathrm{T}}(t)P^{\mathrm{T}}\begin{bmatrix} 0 \\ \bar{A}_1 \end{bmatrix}R^{-1}[0\ \ \bar{A}_1^{\mathrm{T}}]P\bar{x}(t) + \int_{t-h}^{t} y^{\mathrm{T}}(s)Ry(s)\mathrm{d}s. \qquad (11)\end{aligned}$$

Similarly

$$\eta_1 \le \delta_1^{-1}\bar{x}^{\mathrm{T}}(t)P^{\mathrm{T}}\begin{bmatrix} 0 \\ H \end{bmatrix}[0\ \ H^{\mathrm{T}}]P\bar{x}(t) + \delta_1 x^{\mathrm{T}}(t)\bar{E}_{\mathrm{T}}^{\mathrm{T}}\bar{E}_{\mathrm{T}}x(t),$$

$$\begin{aligned}\eta_2 &\le h\delta_2^{-1}\bar{x}^{\mathrm{T}}(t)P^{\mathrm{T}}\begin{bmatrix} 0 \\ H \end{bmatrix}[0\ \ H^{\mathrm{T}}]P\bar{x}(t) \\ &\quad + \delta_2\int_{t-h}^{t} y^{\mathrm{T}}(s)\bar{E}_1^{\mathrm{T}}\bar{E}_1 y(s)\,\mathrm{d}s.\end{aligned}$$

Substituting the right sides of the latter inequalities into (10), we obtain

$$\frac{\mathrm{d}V(t)}{\mathrm{d}t} \le \bar{x}^{\mathrm{T}}(t)\bar{\Gamma}\bar{x}(t) \qquad (12)$$

where

$$\begin{aligned}\bar{\Gamma} &= \Psi + hP^{\mathrm{T}}\begin{bmatrix} 0 \\ \bar{A}_1 \end{bmatrix}R^{-1}[0\ \ \bar{A}_1^{\mathrm{T}}]P \\ &\quad + (\delta_1^{-1} + h\delta_2^{-1}) \times P^{\mathrm{T}}\begin{bmatrix} 0 \\ H \end{bmatrix}[0\ \ H^{\mathrm{T}}]P.\end{aligned}$$

Therefore, LMI (8) yields by Schur complements that $\bar{\Gamma} < 0$ and hence $\dot{V} < 0$, while $V \ge 0$, and thus (4) is asymptotically stable (Kolmanovskii and Myshkis 1999, Fridman 2002). □

2.2. *State-feedback stabilization*

The results of Lemma 1 can also be used to verify the stability of the closed-loop obtained by applying (3) to the system (1) if we set in (8)

$$\bar{A}_i = A_i + B_iK,\ i = 0,1, \quad \bar{E}_0 = E_0 + E_2K \qquad (13)$$

and verify that the resulting LMI is feasible. The problem with (8) is that it is linear in its variables only when the state-feedback gain K is given. To find K, we apply again the Schur formula to $\bar{\Gamma}$, the Ψ term being expanded. We thus obtain the following matrix inequality:

$$\begin{bmatrix} \Psi_0 & hP^{\mathrm{T}}\begin{bmatrix} 0 \\ \bar{A}_1R^{-1} \end{bmatrix} & \begin{bmatrix} 0 \\ hI_n \end{bmatrix} & \begin{bmatrix} \bar{E}_{\mathrm{T}}^{\mathrm{T}} \\ 0 \end{bmatrix} & h\begin{bmatrix} 0 \\ \bar{E}_1^{\mathrm{T}} \end{bmatrix} & \delta_1^{-1}P^{\mathrm{T}}\begin{bmatrix} 0 \\ H \end{bmatrix} & \delta_2^{-1}hP^{\mathrm{T}}\begin{bmatrix} 0 \\ H \end{bmatrix} \\ * & -hR^{-1} & 0 & 0 & 0 & 0 & 0 \\ * & * & -hR^{-1} & 0 & 0 & 0 & 0 \\ * & * & * & -\delta_1^{-1}I_q & 0 & 0 & 0 \\ * & * & * & * & -\delta_2^{-1}hI_q & 0 & 0 \\ * & * & * & * & * & -\delta_1^{-1}I_p & 0 \\ * & * & * & * & * & * & -\delta_2^{-1}hI_p \end{bmatrix} < 0. \qquad (14)$$

Consider the inverse of P. It is obvious from the requirement $P_1 > 0$, and the fact that in (8) $-(P_3 + P_3^{\mathrm{T}})$ must be negative definite, that P is non-singular. Defining

$$P^{-1} = Q = \begin{bmatrix} Q_1 & 0 \\ Q_2 & Q_3 \end{bmatrix} \quad \text{and} \quad M = \mathrm{diag}\{Q,\ I_{2(n+p+q)}\} \tag{15a, b}$$

we multiply (14) by M^{T} and M, on the left and on the right, respectively. Choosing

$$R^{-1} = Q_1\varepsilon,$$

where ε is a positive number, and introducing $\bar{\delta}_1 = \delta_1^{-1}$ and $\bar{\delta}_2 = \delta_2^{-1}$, we obtain the LMI

$$\begin{bmatrix} \Phi & h\begin{bmatrix} 0 \\ \bar{A}_1 Q_1 \varepsilon \end{bmatrix} & Q^{\mathrm{T}}\begin{bmatrix} 0 \\ hI \end{bmatrix} & Q^{\mathrm{T}}\begin{bmatrix} \bar{E}_{\mathrm{T}}^{\mathrm{T}} \\ 0 \end{bmatrix} & hQ^{\mathrm{T}}\begin{bmatrix} 0 \\ \bar{E}_1^{\mathrm{T}} \end{bmatrix} & \bar{\delta}_1\begin{bmatrix} 0 \\ H \end{bmatrix} & h\bar{\delta}_2\begin{bmatrix} 0 \\ H \end{bmatrix} \\ * & -hQ_1\varepsilon & 0 & 0 & 0 & 0 & 0 \\ * & * & -hQ_1\varepsilon & 0 & 0 & 0 & 0 \\ * & * & * & -\bar{\delta}_1 I_q & 0 & 0 & 0 \\ * & * & * & * & -h\bar{\delta}_2 I_q & 0 & 0 \\ * & * & * & * & * & -\bar{\delta}_1 I_p & 0 \\ * & * & * & * & * & * & -\bar{\delta}_2 h I_p \end{bmatrix} < 0, \tag{16}$$

where

$$\Phi = \begin{bmatrix} 0 & I_n \\ \bar{A}_{\mathrm{T}} & -I_n \end{bmatrix} Q + Q^{\mathrm{T}} \begin{bmatrix} 0 & I_n \\ \bar{A}_{\mathrm{T}} & -I_n \end{bmatrix}^{\mathrm{T}}.$$

Substituting (13) into (16) and denoting $Y = KQ_1$, $B_{\mathrm{T}} = B_0 + B_1$, we obtain the following.

Theorem 1: *The control law of* (3) *asymptotically stabilizes* (1) *if, for some positive number* ε, *there exist scalars* $\bar{\delta}_1 > 0$, $\bar{\delta}_2 > 0$ *and matrices* $0 < Q_1$, Q_2, Q_3, $\in \mathcal{R}^{n\times n}$ $Y \in \mathcal{R}^{m\times n}$ *that satisfy the following LMI*:

$$\begin{bmatrix} Q_2 + Q_2^{\mathrm{T}} & Q_1 A_{\mathrm{T}}^{\mathrm{T}} + Y^{\mathrm{T}} B_{\mathrm{T}}^{\mathrm{T}} - Q_2^{\mathrm{T}} + Q_3 & 0 & hQ_2^{\mathrm{T}} & Q_1 E_{\mathrm{T}}^{\mathrm{T}} + Y^{\mathrm{T}} E_2^{\mathrm{T}} & hQ_2^{\mathrm{T}} E_1^{\mathrm{T}} & 0 & 0 \\ * & -Q_3 - Q_3^{\mathrm{T}} & h\varepsilon(A_1 Q_1 + B_1 Y) & hQ_3^{\mathrm{T}} & 0 & hQ_3^{\mathrm{T}} E_1^{\mathrm{T}} & \bar{\delta}_1 H & h\bar{\delta}_2 H \\ * & * & -h\varepsilon Q_1 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -hQ_1\varepsilon & 0 & 0 & 0 & 0 \\ * & * & * & * & -\bar{\delta}_1 I_q & 0 & 0 & 0 \\ * & * & * & * & * & -h\bar{\delta}_2 I_q & 0 & 0 \\ * & * & * & * & * & * & -\bar{\delta}_1 I_p & 0 \\ * & * & * & * & * & * & * & -\bar{\delta}_2 h I_p \end{bmatrix} < 0 \tag{17}$$

The state-feedback gain is then given by

$$K = YQ_1^{-1}. \tag{18}$$

3. Sliding mode controller

This section focuses on time-delay systems that can be represented, possibly, after a change of state coordinates and input, in the following regular form (Gouaisbaut *et al.* 2002, Perruquetti and Barbot 2002):

$$\begin{cases} \dfrac{\mathrm{d}z_1(t)}{\mathrm{d}t} = (A_{11} + H\Delta(t)E_0)z_1(t) + (A_{d11} + H\Delta(t)E_1) \\ \qquad \times z_1(t-\tau(t)) + (A_{12} + H\Delta(t)E_2)z_2(t) \\ \qquad + A_{d12}z_2(t-\tau(t)) \\ \dfrac{\mathrm{d}z_2(t)}{\mathrm{d}t} = \displaystyle\sum_{i=1}^{2}(A_{2i}z_i(t) + A_{d2i}z_i(t-\tau)) + Du(t) + f(t, z_t), \\ z(t) = \phi(t) \text{ for } t \in [-h, 0] \end{cases} \tag{19}$$

where $z(t) = (z_1, z_2)^{\mathrm{T}}$, $z_1 \in \mathcal{R}^{n-m}$, $z_2 \in \mathcal{R}^m$, A_{ij}, A_{dij}, $i = 1, 2$, $j = 1, 2$, E_k, $k = 0, 1, 2, H$ are constant matrices of appropriate dimensions, D is a regular $m \times m$ matrix, the matrix $\Delta(t)$ is a time-varying matrix of uncertain parameters, $u \in \mathcal{R}^m$ is the input vector, τ is a time-varying delay satisfying $0 \le \tau(t) \le h$, $\forall t \ge 0$, $z_t(\theta)$ is the function associated with z and defined on $[-h, 0]$ by $z_t(\theta) = z(t+\theta)$, ϕ is the initial piecewise continuous function defined on $[-h, 0]$.

We will assume that:

(A1) $(A_{11} + A_{d11}, A_{12} + A_{d12})$ is controllable.

(A2) f is Lipschitz continuous and satisfies the inequality

$$\| f(t, z_t)\| < F_M(t, z_t), \quad \forall t \ge 0,$$

where $F_M(t, z_t)$ is a continuous functional assumed to be known a priori,

(A3) $\Delta(t)$ is a time-varying matrix of uncertain parameters satisfying $\Delta^{\mathrm{T}}(t)\Delta(t) \le I \;\; \forall\, t$.

Consider the following switching function:

$$s(z) = z_2 - K z_1, \tag{20}$$

with $K \in \mathcal{R}^{m\times(n-m)}$. Let Ω, Θ be the linear functions defined by

$$\begin{aligned} \Omega(z(t)) &= \sum_{i=1}^{2}(A_{2i} - KA_{1i})z_i(t), \\ \Theta(z(t)) &= E_0 z_1(t) + E_2 z_2(t) \end{aligned} \tag{21}$$

and let D_M be the following functional:

$$\begin{aligned} D_M(z_t) = {}&(\|A_{d21} - KA_{d11}\| + \|KH\|\|E_1\|) \sup_{-h\le\theta\le 0} \|z_1(t+\theta)\| \\ &+ \|A_{d22} - KA_{d12}\| \sup_{-h\le\theta\le 0}\|z_2(t+\theta)\|. \end{aligned} \tag{22}$$

Following Gouaisbaut *et al.* (2002) and using the results of Section 2, we designed a sliding mode controller that will stabilize system (19) under less conservative assumptions on the delay law.

Theorem 2: *Assume A1–A3. If, for some positive number ε, there exist positive numbers $\bar\delta_1$, $\bar\delta_2$ and matrices $0 < Q_1$, Q_2, $Q_3 \in \mathcal{R}^{(n-m)\times(n-m)}$, $Y \in \mathcal{R}^{m\times(n-m)}$ that satisfy the following LMI:*

$$\begin{bmatrix} Q_2 + Q_2^{\mathrm{T}} & X_{12} & 0 & hQ_2^{\mathrm{T}} & Q_1E_{\mathrm{T}}^{\mathrm{T}} + Y^{\mathrm{T}}E_2^{\mathrm{T}} & hQ_2^{\mathrm{T}}E_1^{\mathrm{T}} & 0 & 0 \\ * & -Q_3 - Q_3^{\mathrm{T}} & h\varepsilon(A_{d11}Q_1 + A_{d12}Y) & hQ_3^{\mathrm{T}} & 0 & hQ_3^{\mathrm{T}}E_1^{\mathrm{T}} & \bar\delta_1 H & h\bar\delta_2 H \\ * & * & -h\varepsilon Q_1 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -h\varepsilon Q_1 & 0 & 0 & 0 & 0 \\ * & * & * & * & -\bar\delta_1 I & 0 & 0 & 0 \\ * & * & * & * & * & -h\bar\delta_2 I & 0 & 0 \\ * & * & * & * & * & * & -\bar\delta_1 I & 0 \\ * & * & * & * & * & * & * & -\bar\delta_2 hI \end{bmatrix} < 0, \tag{23}$$

where

$$X_{12} = Q_1(A_{11}^{\mathrm{T}} + A_{d11}^{\mathrm{T}}) + Y^{\mathrm{T}}(A_{12}^{\mathrm{T}} + A_{d12}^{\mathrm{T}}) - Q_2^{\mathrm{T}} + Q_3,$$

then the sliding mode control law

$$\begin{aligned} u(t) = -D^{-1}\Big[&\Omega(z(t)) + (F_M(t, z_t) + D_M(z_t) \\ &+ \|KH\|\|\Theta(z(t))\| + M)\frac{s(z(t))}{\|s(z(t))\|}\Big], \end{aligned} \tag{24}$$

where $K = YQ_1^{-1}$, $M > 0$ and s, Ω, Θ, D_M are defined in (20–22), *asymptotically stabilizes system* (19) *for any delay function $\tau(t) \le h$.*

Proof: The proof is divided into two parts. The first is dedicated to the proof of the existence of an ideal sliding motion on the surface $s(z) = 0$; the second is dedicated to the proof of the stability of the reduced system.

Attractivity of the manifold:

Consider the Lyapunov–Krasovskii functional:

$$V(t) = s^{\mathrm{T}}(z(t))s(z(t)) = \|s(z(t))\|^2. \tag{25}$$

Differentiating (25) on the trajectories of the closed-loop system gives:

$$\dot{V}(t) = 2s^{\mathrm{T}}(t)(\Omega(z(t)) + \sum_{i=1}^{2}[A_{d2i} - KA_{d1i}]z_i(t-\tau) + Du(t) + f(t,z_t) - KH\Delta(t)[\Theta(z(t)) + E_1 z_1(t-\tau(t))]).$$

Using the expression of the control law (24), we get

$$\dot{V}(t) = 2s^{\mathrm{T}}(t)\Bigg(\sum_{i=1}^{2}(A_{d2i} - KA_{d1i})z_i(t-\tau) + f(t,z_t) - KH\Delta(t)[\Theta(z(t)) + E_1 z_1(t-\tau(t))] - [F_M(t,z_t) + D_M(z_t) + \|KH\|\,\|\Theta(z(t))\| + M]\frac{s}{\|s\|}\Bigg),$$

then we derive that:

$$\dot{V} \le -2M\|s(z(t))\| = -2MV(t)^{1/2}.$$

This last inequality is known to prove the finite-time convergence of the system (19) into the surface $s = 0$ (Perruquetti and Barbot 2002).

Stability of the reduced system:

On the sliding manifold $s(z) = 0$, the system is driven by the following reduced system:

$$\frac{\mathrm{d}z_1(t)}{\mathrm{d}t} = (A_{11} + A_{12}K + H\Delta(t)(E_0 + E_2K))z_1(t) + (A_{d11} + A_{d12}K + H\Delta(t)E_1)z_1(t-\tau(t)) \qquad (26)$$

According to Theorem 1, this system is asymptotically stable for any delay law $\tau(t) \le h$ if, for some positive number ε, there exist positive numbers $\bar{\delta}_1$, $\bar{\delta}_2$ and matrices $0 < Q_1, Q_2, Q_3, Y \in \mathcal{R}^{m\times(n-m)}$ that satisfy the LMI (24). □

Remark 1: Note that the explicit knowledge of the time-dependance of the delay is not required in the expression of the control law $u(t)$, all is needed is the knowledge of an upper bound h.

4. Example

We demonstrate the applicability of the above theory by solving the example from (Gouaisbaut *et al.* 2002) for a system without uncertainty. Consider system

$$\dot{x}(t) = Ax(t) + A_d x(t-\tau) + B[u(t) + f(x,t)], \qquad (27)$$

with a time-varying delay, where

$$A = \begin{bmatrix} 2 & 0 \\ 1.75 & 0.25 \end{bmatrix}, \quad A_d = \begin{bmatrix} -1 & 0 \\ -0.1 & -0.25 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \qquad (28)$$

By an appropriate change of variables, this system is equivalent to:

$$\dot{z}(t) = \tilde{A}z(t) + \tilde{A}_d z(t-\tau) + \tilde{B}[u(t) + f(x,t)],$$

where

$$\tilde{A} = \begin{bmatrix} 0.25 & 0 \\ 1.75 & 2 \end{bmatrix}, \quad \tilde{A}_d = \begin{bmatrix} -0.9 & -0.65 \\ -0.1 & -0.35 \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \qquad (29)$$

As the pair $(\tilde{A}_{11}, \tilde{A}_{12})$ is not controllable, the system cannot be stabilized independently of the delay.

For this system, previous published works give the following results:

- In the case of a constant delay and $f = 0$, the system may be stabilized using a linear memoryless controller $u(t) = Kx(t)$ for the following maximum values of h: $h = 0.51$ by Li and de Souza (1997), $h = 0.984$ by Fu *et al.* (1997) and $h = 1.46$ by Ivanescu (2000). By sliding mode control for the case of constant delay and $f \ne 0$ the maximum value found for h is 1.65.
- Applying Theorem 2 in the case of a time-varying delay and $f \ne 0$, the corresponding value of $h = 3.999$ is achieved.

This is summarized in table 1.

5. Conclusions

The problem of finding a sliding mode controller that asymptotically stabilizes a system with time-varying delay and norm-bounded uncertainty has been solved. A delay-dependent solution has been derived using a special Lyapunov–Krasovskii functional. The result is based on a sufficient condition and it thus entails an overdesign. This overdesign is considerably reduced due to the fact that the method is based on the descriptor representation. As a byproduct, for the first time on the basis of the descriptor model transformation, the solution to the stabilization problem by the feedback,

Table 1. Comparison of results for example (26–27).

	Delay upper bound	Type of delay
Theorem 2	3.999	time-varying
Gouaisbaut *et al.* (2002)	1.650	constant
Ivanescu (2000)	1.460	constant
Fu *et al.* (1997)	0.984	constant
Li and de Souza (1997)	0.510	constant

which depends on both non-delayed and delayed state, is solved. Finally, a numerical example shows the effectiveness of the combined method: sliding mode and descriptor representation.

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