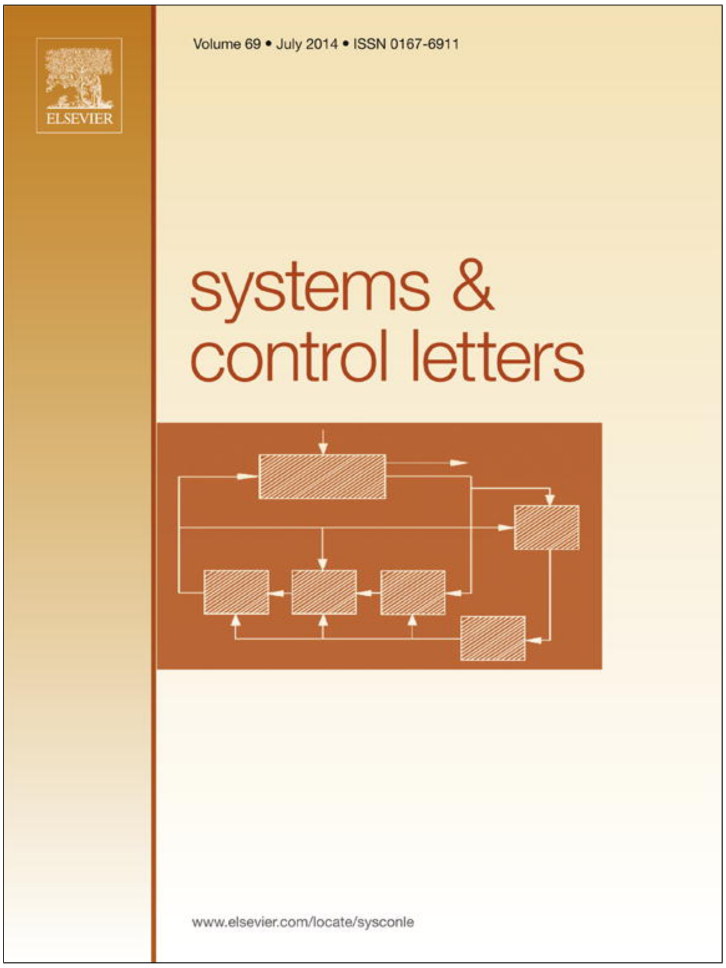
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A Round-Robin type protocol for distributed estimation with H_∞ consensus[☆,☆☆]

V. Ugrinovskii [a,*], E. Fridman [b]

[a] School of Engineering and Information Technology, UNSW Canberra at the Australian Defence Force Academy, Canberra, Australia

[b] School of Electrical Engineering, Tel-Aviv University, Israel

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ABSTRACT

The paper considers a distributed robust estimation problem over a network with directed topology involving continuous time observers. While measurements are available to the observers continuously, the nodes interact according to a Round-Robin rule, at discrete time instances. The results of the paper are sufficient conditions which guarantee a suboptimal H_∞ level of consensus between observers with sampled interconnections.

1. Introduction

The problem of distributed estimation is one of very active topics in the modern control theory and signal processing literature. Interest in this problem is motivated by a growing number of applications where a decision about the observed process must be made simultaneously by spatially distributed sensors, each taking partial measurements of the process.

When the process and measurements are subject to noise and disturbance, robustness aspects of the problem come into prominence. In the past several years, a number of results have been presented in the literature which develop the H_∞ control and estimation theory for distributed systems subject to uncertain perturbations; e.g., see [1–7]. In particular, methodologies of distributed sampled-data H_∞ filtering have been considered, e.g., in [8]. That reference emphasized several distinctive aspects of realistic sensor networks, among them coupling between sensor nodes through the information communicated between neighbouring sensor nodes and the sampled nature of that coupling, which is dictated by the digital communication technology. The latter feature of sampled data networks is an important consideration in any network design, because the amount of information that can be transmitted to/received at each node of the network is constrained, due to data rate limitations of digital communication channels.

In this paper, we address some of the challenges specific to Round-Robin type communication protocols. The Round-Robin protocol is a commonly used protocol for information transmission in networked control systems. It allows each node to communicate with its neighbours intermittently, during scheduled time slots and is known to lead to bandwidth savings. From a hybrid systems perspective this protocol has been studied in details in [9,10]. More recently, it has been considered in the context of time-delay systems in [11], where an analysis of exponential stability and L_2 properties of networked control systems with Round-Robin scheduling was presented using a delay switching system modelling. In this paper, we further develop this technique in the context of robust distributed estimation with intermittent communications between sensing nodes. The type of communication we consider is where the nodes broadcast their information at

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* Corresponding author. Tel.: +61 2 62688219; fax: +61 2 62688443.
E-mail addresses: v.ougrinovski@adfa.edu.au, v.ugrinovskii@gmail.com (V. Ugrinovskii), emilia@eng.tau.ac.il (E. Fridman).

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every scheduled time instant to all nodes in their vicinity, but they listen to only one node within their neighbourhood at a time, according to the Round-Robin rule. For instance, this can be achieved by encoding the transmitted information with a node-specific key. i.e., when node i receives signals from multiple sources, it extracts the information sent by node j by utilizing the key of that node, and continues doing this by rotating the keys.

The objective of this paper is to develop an algorithm for the synthesis of a Round-Robin type protocol for a network of distributed observers, to allow this network to track dynamics of a linear uncertain plant. Unlike many existing approaches to distributed estimation, the salient feature of our methodology is that individual estimators may not be able to track the plant, if they rely solely on their own measurements, because the plant may not be observable from the node's measurements. This issue has recently been emphasized in [5–7] which demonstrated that the consensus between sensors plays a crucial role in enabling individual sensor nodes to overcome the lack of detectability and successfully track the plant. Necessary conditions on the network to ensure the plant is detectable/observable by the network have recently been presented in [12,13].

The first contribution of this paper is a version of the protocol of [11] to be used with the distributed estimation schemes proposed [5–7]. We show that instead of continuously exchanging information (the type of networks considered in those references), the node observers can achieve the relative H_∞ consensus objective by exchanging information at certain sampling times, by polling one neighbour at a time. It is assumed that sampling times are known and agreed upon at each node. Technically, this would require all nodes to have their clocks synchronized, e.g., by means of a network time protocol [14].

Our second contribution demonstrates that the Round-Robin design of [11] can be applied to derive a network of non-switching observers. Of course, each observer periodically switches between input channels, but the observer gains remain unchanged. This is an important feature of our methodology to ensure its scalability. In a large network of distributed estimators, switching of observer gains typically leads to a combinatorially complex scheduling problem, and necessitates the development of additional tools to resolve this complexity; see [7]. We show that these issues are avoidable in observer networks of the type considered in this paper.

Our main result is a sufficient condition, expressed in the form of Linear Matrix Inequalities (LMIs), from which filter and interconnection gains for each node estimator can be computed, to ensure the network of sampled data observers using switching communications converges to the trajectory of the observed plant by achieving consensus between the filters at every node. Conditions for consensus of multi-agent systems using sampled data communications or systems communicating over switching graphs are well known in the literature [15,16]. This includes an observation that the sampling period has a significant effect on the system performance [17]. As the example presented in Section 4 illustrates, our conditions allow to investigate the effects of intermittent sampled communications on the system performance as well. At the same time, our result provides a guarantee of the network consensus performance in the presence of disturbances. Since the consensus between observers is essential for the network to be able to overcome observability/detectability limitations of individual observers [5,13], consensus performance is seen as an important design consideration which results in this paper address.

As in [5–7], our methodology relies on certain vector dissipativity properties of the large-scale system comprised of the observers' error dynamics. However, different from these references, to establish these vector dissipativity properties, we employ a novel class of generalized supply rates which reflects the sampled-data nature of interconnections between observers. The general idea behind introducing such generalized supply rates can be traced to [18] (also, see [6]), but our proposal here makes use of special properties of sampled signals. In the limit, when the maximum sampling period approaches zero, these generalized supply rates vanish, and one recovers the vector dissipativity properties of error dynamics established in [5]. Thus, the feasibility of the conditions proposed for systems with continuously operating interconnections can be used as a preliminary (but not conclusive) test for the conditions proposed here. Our numerical example illustrates this point very well, showing a negligible difference between the disturbance attenuation levels obtained using the benchmark algorithm of [6] and those obtained using our conditions under a small sampling rate. At the same time, the feasibility of our conditions makes explicit the dependence of the proposed algorithm on the sampling period. Being only sufficient conditions, our conditions are potentially conservative, but the fact that the techniques used in the derivation of these conditions showed substantial reduction of conservatism in similar problems [11] is encouraging.

The paper is organized as follows. The problem formulation, along with the graph theory preliminaries is presented in Section 2. The main results of the paper are given in Section 3. In Section 4, we discuss an illustrative example, and Section 5 concludes the paper.

Notation: throughout the paper, $\mathbf{R}^n$ denotes a real Euclidean n-dimensional vector space, with the norm $\|x\| \triangleq (x'x)^{1/2}$; here the symbol $'$ denotes the transpose of a matrix or a vector. $L_2[0,\infty)$ will denote the Lebesgue space of $\mathbf{R}^n$-valued vector-functions $z(\cdot)$, defined on the time interval $[0,\infty)$, with the norm $\|z\|_2 \triangleq (\int_0^\infty \|z(t)\|^2 dt)^{1/2}$ and the inner product $\int_0^\infty z_1(t)'z_2(t)dt$. $\otimes$ is the Kronecker product of matrices, $\mathbf{1}_n \in \mathbf{R}^n$ is the column-vector of ones. Also, $\det X$ is the determinant of X.

2. The problem formulation

2.1. Graph theory

Consider a filter network with N nodes and a directed graph topology $\mathscr{G} = (\mathscr{V}, \mathscr{E})$; $\mathscr{V} = \{1, 2, \ldots, N\}$, $\mathscr{E} \subset \mathscr{V} \times \mathscr{V}$ are the set of vertices and the set of edges, respectively. The notation (j, i) will denote the edge of the graph originating at node j and ending at node i. In accordance with a common convention [15], we consider graphs without self-loops, i.e., $(i, i) \notin \mathbf{E}$. However, each node is assumed to have complete information about its filter and measurements.

For each $i \in \mathscr{V}$, we denote $\mathscr{V}_i = \{j : (j, i) \in \mathscr{E}\}$ to be the *ordered* set of nodes supplying information to node i, i.e., the neighbourhood of i. Without loss of generality, suppose that the elements of $\mathscr{V}_i$ are ordered in the ascending order. The cardinality of $\mathscr{V}_i$, known as the in-degree of node i, is denoted by p_i; i.e., p_i is equal to the number of incoming edges for node i. Also, the out-degree of node i (i.e., the number of outgoing edges) is denoted by q_i.

Without loss of generality the graph $\mathscr{G}$ will be assumed to be weakly connected, that is, for every two nodes of $\mathscr{G}$ there is an undirected path between these two nodes. The rationale for this assumption is based on [5, Proposition 1]; according to that proposition H_∞ consensus optimization problems over disconnected graphs are reducible to the corresponding problems over individual weakly connected components.

Let $\mathscr{A} = [\mathbf{a}_{ij}]_{i,j=1}^N$ be the adjacency matrix of the digraph $\mathscr{G}$, i.e., $\mathbf{a}_{ij} = 1$ if $(j, i) \in \mathscr{E}$, otherwise $\mathbf{a}_{ij} = 0$. Also, let $\mathscr{L}$ be the $N \times N$ Laplacian matrix of the graph $\mathscr{G}$, $\mathscr{L} = \operatorname{diag}[p_1, \ldots, p_N] - \mathscr{A}$.

In the sequel, a shift permutation operator defined on elements of the set $\mathscr{V}_i$ will be used:

$$\Pi\{j_1, \ldots, j_{p_i-1}, j_{p_i}\} = \{j_{p_i}, j_1, \ldots, j_{p_i-1}\}. \quad (1)$$

Furthermore, $\Pi^k(\mathscr{V}_i)$ will denote the set obtained from $\mathscr{V}_i$ using k consecutive shift permutations (1). In regard to this set, the following notation will be used throughout the paper unless stated otherwise: for $\nu \in \{1, \dots, p_i\}, j_\nu$ is the ν-th element in the ordered set $\mathscr{V}_i$. Conversely, $\nu_j^{k,i} \in \{1, \dots, p_i\}$ is the index of element j in the permutation $\Pi^k(\mathscr{V}_i)$. We will omit the superscript ^{k,i} if this does not lead to ambiguity.

2.2. Distributed estimation with H_∞ consensus

Consider a plant described by the equation

$$\dot{x} = Ax + B_2 w(t). \tag{2}$$

Here $x \in \mathbf{R}^n$ is the state of the plant, and $w(t) \in \mathbf{R}^{m_w}$ is a disturbance. We also assume $w(t) \in L_2[0, \infty)$, so that the L_2-integrable solution of (2) with the initial condition $x(0) = x_0$ exists on any finite interval $[0, T]$ [19, p. 125]. Furthermore, it will be convenient to assume that the plant was at the state x_0 for all $t \leq 0$.

The distributed filtering problem under consideration is to estimate the state of the system (2) using a network of filters connected according to the graph $\mathscr{G}$. Each node takes measurements

$$y_i(t) = C_i x(t) + D_{2i} w(t) + \bar{D}_{2i} v_i(t); \tag{3}$$

$v_i(t) \in \mathbf{R}^{m_v}$ is a measurement disturbance. As seen from (3), the measurements are assumed to be taken continuously. Although in practice, measurements are usually taken at discrete time instances, we assume that the data rate of the sensors is high enough to allow for the continuous-time interpretation of the measurement signals y_i. This will enable us to focus exclusively on the effects due to sampling and intermittence of interconnections.

The measurements are processed by a network of observers connected over the graph $\mathscr{G}$. The key assumption in this paper is to allow the observers to make use of their local measurements continuously, however they can only interact with each other at discrete time instances t_k, $k = 0, 1 \dots$, with $t_0 = 0$. For simplicity, we assume that this schedule of updates is known to all participants in the network, and therefore all nodes exchange information at the same time instance t_k. However, at every time instance t_k only one neighbour in the set $\mathscr{V}_i$ is polled by each node i, according to the 'Round-Robin' rule. Formally, this leads us to define the following observer protocol: for $t \in [t_k, t_{k+1})$, $k = 0, 1, \dots$,

$$\begin{aligned}\dot{\hat{x}}_i &= A\hat{x}_i(t) + L_i(y_i(t) - C_i\hat{x}_i(t)) \\ &\quad + K_i \sum_{j \in \Pi^k(\mathscr{V}_i)} H_i(\hat{x}_j(t_{k-\nu_j^{k,i}+1}) - \hat{x}_i(t_{k-\nu_j^{k,i}+1})),\end{aligned} \tag{4}$$

where $\hat{x}_i(t)$ is the estimate of the plant state $x(t)$ calculated at node i, the matrices L_i, K_i are parameters of the filters to be determined, and H_i is a given matrix. All observers are initiated with zero initial condition, $\hat{x}_i(t) = 0$ for all $t \leq 0$ and all $i = 1, \dots, N$. In particular, this ensures that in (4), the terms sampled at times $t_{k-\nu_j^{k,i}+1} < 0$ are equal to zero.

From now on, we will omit the time variable when a signal is considered at time t, and will write, for example, $\hat{x}_i$ for $\hat{x}_i(t)$.

The last term in (4) reflects the desire of each node observer to update its estimate of the plant using feedback from the neighbours in its neighbourhood, according to the consensus estimation paradigm [20,5]. However, unlike these references, under the proposed protocol, only one neighbour is polled at each time t_k to provide a 'neighbour feedback', and this sample is stored and used by the observer until time t_{k+p_i}. The feature of the proposed Round-Robin type protocol is to poll the neighbours one at a time, in a cyclic manner. Formally, this can be described by first applying the shift permutation operator Π to the neighbourhood set at every time instance t_k, and then selecting the first element from the resulting permutation $\Pi^k(\mathscr{V}_i)$ for feedback.

Let $e_i = x - \hat{x}_i$ be the local estimation error at node i. This error satisfies the equation:

$$\begin{aligned}\dot{e}_i &= (A - L_iC_i)e_i + (B - L_iD_i)\xi_i \\ &\quad + K_iH_i \sum_{j \in \Pi^k(\mathscr{V}_i)} (e_j(t_{k-\nu_j^{k,i}+1}) - e_i(t_{k-\nu_j^{k,i}+1})).\end{aligned} \tag{5}$$

Here we used the notation ξ_i to represent the perturbation vector $[w' \ v_i']'$, and the matrices B, D_i are defined as follows: $B = [B_2 \ 0]$, $D_i = [D_{2i} \ \bar{D}_{2i}]$. Since the plant was at the state $x(t) = x_0$ for all $t \leq 0$, the initial conditions for (5) are $e_i(t) = x_0 \ \forall t \leq 0$.

Since the error dynamics (5) are governed by L_2 integrable disturbance signals ξ_i, we can only expect the node observers to converge in L_2 sense. To quantify the transient consensus performance of the observer network (4) under disturbances, consider the cost of disagreement between the observers caused by a particular vector of disturbance signals $\xi(\cdot) = [\xi_1(\cdot)' \dots \xi_N(\cdot)']'$,

$$\begin{aligned}J(\xi) &= \frac{1}{N}\int_0^\infty \sum_{i=1}^N \sum_{j \in \Pi^k(\mathscr{V}_i)} \|\hat{x}_j(t) - \hat{x}_i(t)\|^2 dt \\ &= \frac{1}{N}\int_0^\infty \sum_{i=1}^N \sum_{j \in \Pi^k(\mathscr{V}_i)} \|e_j(t) - e_i(t)\|^2 dt,\end{aligned} \tag{6}$$

where k is a time-dependent index, $k = 0, 1, \dots$, defined so that for every $t \in [0, \infty)$, $t_k \leq t < t_{k+1}$. The functional (6) was originally introduced in [5] as a measure of consensus performance of a corresponding continuous-time observer network. It is worth noting that for each t, $\sum_{j \in \Pi^k(\mathscr{V}_i)} \|\hat{x}_j(t) - \hat{x}_i(t)\|^2$ is independent of the order in which node i polls its neighbours, so that

$$\sum_{j \in \Pi^k(\mathscr{V}_i)} \|\hat{x}_j(t) - \hat{x}_i(t)\|^2 = \sum_{j \in \mathscr{V}_i} \|\hat{x}_j(t) - \hat{x}_i(t)\|^2.$$

Therefore, the inner summation in (6) can be replaced with summation over the neighbourhood set $\mathscr{V}_i$. This observation leads to the same expression for $J(\xi)$ as in the case of continuous-time networks [5],

$$J(\xi) = \frac{1}{N}\int_0^\infty \sum_{i=1}^N \left[(p_i + q_i)\|e_i(s)\|^2 - 2e_i' \sum_{j \in \mathscr{V}_i} e_j(s)\right] ds. \tag{7}$$

The following distributed estimation problem is a version of the distributed H_∞ consensus-based estimation problem originally introduced in [5,6], modified to include the Round-Robin type protocol (4).

Definition 1. The distributed estimation problem under consideration is to determine a collection of observer gains L_i and interconnection coupling gains K_i, $i = 1, \dots, N$, for the filters (4) which ensure that the following conditions are satisfied:

(i) in the absence of uncertainty, the interconnection of unperturbed systems (5) must be exponentially stable;

(ii) the filter must ensure a specified level of transient consensus performance, as follows:

$$\sup_{x_0, \xi \neq 0} \frac{J(\xi)}{\|x_0\|_P^2 + \frac{1}{N}\|\xi\|_2^2} \leq \gamma^2. \tag{8}$$

Here, $\|x_0\|_P^2 = x_0'Px_0$, $P = P' > 0$ is a matrix to be determined, and $\gamma > 0$ is a given constant.

In [5], the quantity on the left-hand-side of (8) was referred to as the mean-square L_2 disagreement gain of the distributed observer.

Note that unlike [5,6], here we aim to achieve internal stability and H_∞ performance of the observer using a different communication protocol, which involves sampling of observer inputs according to the Round-Robin rule.

3. The main results

Our approach to solving the problem in Definition 1 will follow the methodology for the analysis of stability and L_2-gain for networked control systems proposed in [11]. In this paper, this methodology is further extended to derive *synthesis conditions* for a network of observers. The methodology in [11] makes use of the time-delay approach to sampled-data control started in [21]. In [11] the closed-loop system under consideration is presented as a switched system with multiple and ordered time-varying delays.

As can be seen from (5), if the observer at node i polls a channel at time t_{k-p_i+1}, the next time the same channel will be polled at time t_{k+1}. The longest time between polls of the same channel at node i constitutes the maximum delay in communication between node i and its neighbours, which will be denoted by τ_i:

$$\tau_i = \max_k (t_{k+1} - t_{k-p_i+1}).$$

The largest communication delay in the network is then $\tau = \max_i \tau_i$. It is easy to see from these definitions that $\tau = \max_k(t_{k+1} - t_{k-\bar{p}+1})$, where $\bar{p} = \max_i p_i$.

Consider the following Lyapunov–Krasovskii candidate for the system (5):

$$V_i(e_i) = e_i' Y_i^{-1} e_i + \int_{t-\tau_i}^{t} e^{-2\alpha_i(t-s)} e_i(s)' S_i e_i(s) ds + \tau_i \int_{t-\tau_i}^{t} e^{-2\alpha_i(t-s)} \dot{e}_i(s)'(\tau_i + s - t) R_i \dot{e}_i(s) ds, \quad (9)$$

where $Y_i = Y_i' > 0$, $R_i = R_i' \geq 0$, $S_i = S_i' \geq 0$ and $\alpha_i \geq 0$, $i = 1, \ldots, N$, are matrices and constants to be determined. $V_i(e_i)$ is a standard Lyapunov–Krasovskii functional used in the literature on exponential stability of systems with time-varying delays; e.g., see [11].

Given a matrix $W_i = W_i > 0$, define

$$\mathscr{W}_i(u, z) = \frac{\pi^2}{4}(u - z)' W_i (u - z).$$

Theorem 1. *Suppose there exist gains K_i, L_i, matrices $W_i = W_i' > 0$, and constants $\alpha_i > 0$, $0 < \pi_i < 2\alpha_i q_i^{-1}$, $i = 1, \ldots, N$, such that the following vector dissipation inequality holds for all $i = 1, \ldots, N$: For $t \in [t_k, t_{k+1})$,*

$$\dot{V}_i(e_i) + 2\alpha_i V_i(e_i) - \sum_{j \in \mathscr{V}_i} \pi_j V_j(e_j) + \left(\sum_{j: i \in \mathscr{V}_j} \tau_j^2 \right) \dot{e}_i' W_i \dot{e}_i - \sum_{j \in \mathscr{V}_i} \mathscr{W}_j(e_j, e_j(t_{k-\nu_j^{k,i}+1})) + \frac{1}{\gamma^2}(p_i + q_i)\|e_i\|^2 - \frac{2}{\gamma^2} e_i' \sum_{j \in \mathscr{V}_i} e_j - \|\xi_i\|^2 \leq 0, \quad (10)$$

where $\nu_j^{k,i}$ is the index of j in the ordered permutation set $\Pi^k(\mathscr{V}_i)$. Then the system (5) satisfies conditions (i) *and* (ii) *in Definition 1.*

The proof of this theorem and other statements are given in the Appendix.

Remark 1. Let $\mathscr{V}_i = \{j_1, \ldots, j_{p_i}\}$, and define

$$\mathscr{S}_i(e_i, \dot{e}_i, e_{j_1}, \ldots, e_{j_{p_i}}, \xi_i) = \left(\sum_{j: i \in \mathscr{V}_j} \tau_j^2 \right) \dot{e}_i' W_i \dot{e}_i - \sum_{j \in \mathscr{V}_i} \mathscr{W}_j(e_j, e_j(t_{k-\nu_j^{k,i}+1})) + \frac{1}{\gamma^2}(p_i + q_i)\|e_i\|^2 - \frac{2}{\gamma^2} e_i' \sum_{j \in \mathscr{V}_i} e_j - \|\xi_i\|^2.$$

Then, inequality (10) can be written in the standard form of a vector dissipation inequality [5],

$$\dot{V}_i(e_i) + 2\alpha_i V_i(e_i) - \sum_{j \in \mathscr{V}_i} \pi_j V_j(e_j) \leq -\mathscr{S}_i(e_i, \dot{e}_i, e_{j_1}, \ldots, e_{j_{p_i}}, \xi_i).$$

This prompts for an interpretation of $V(e) = [V_1(e_1), \ldots, V_N(e_N)]'$ and $[\mathscr{S}_1, \ldots, \mathscr{S}_N]'$ as, respectively, a vector storage function and a vector supply rate for the large scale system comprised of the error dynamics subsystems (5) [22,5]. Strictly speaking, in our case such an interpretation is somewhat artificial, since for example, the derivative signal $\dot{e}_i$ is not an output of 'subsystem' i, and is not used for feedback by any of the neighbours this node. Nonetheless, in the proof of Theorem 1 the functions $\mathscr{S}_i$ will play a role analogous to that played by generalized supply rates in [18,5,6].

In what follows we derive a sufficient condition for the dissipation inequality (10) to hold. We begin with a technical lemma which essentially restates the corresponding lemma of [23] in the form convenient for the subsequent use in the paper. Consider a vector $\delta = [\delta_0', \ldots, \delta_{p_i}']'$, $\delta_\nu \in \mathbf{R}^n$. Also, for given $n \times n$ matrices $R_i = R_i' \geq 0$ and G_i, define

$$\Psi_i = \begin{bmatrix} R_i & \frac{1}{2}(G_i + G_i') & \cdots & \frac{1}{2}(G_i + G_i') \\ \frac{1}{2}(G_i + G_i') & R_i & \cdots & \frac{1}{2}(G_i + G_i') \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2}(G_i + G_i') & \frac{1}{2}(G_i + G_i') & \cdots & R_i \end{bmatrix}.$$

Lemma 1. *Suppose the matrices $R_i = R_i' \geq 0$ and G_i are such that*

$$\begin{bmatrix} R_i & G_i \\ G_i' & R_i \end{bmatrix} \geq 0. \quad (11)$$

Then

$$\tau_i \left[\frac{1}{t - t_k} \delta_0' R_i \delta_0 + \sum_{\nu=1}^{p_i - 1} \frac{1}{t_{k-\nu+1} - t_{k-\nu}} \delta_\nu' R_i \delta_\nu + \frac{1}{t_{k-p_i+1} - t + \tau_i} \delta_{p_i}' R_i \delta_{p_i} \right] \geq \delta' \Psi_i \delta.$$

Let $\mathbf{e}_i = [e_i(t_k)' \ldots e_i(t_{k-p_i+2})' \, e_i(t_{k-p_i+1})']'$, $\bar{\mathbf{e}}_i = [e_i' \, \mathbf{e}_i' \, e_i(t - \tau_i)']'$, and $T_i \in R^{(p_i+1)n \times (p_i+2)n}$ be the following matrix

$$T_i = \begin{bmatrix} 1 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 & -1 \end{bmatrix} \otimes I.$$

Define $\bar{\Psi}_i = e^{-2\alpha_i \tau_i} T_i' \Psi_i T_i$ and partition this matrix in accordance with the partition of $\bar{\mathbf{e}}_i$:

$$\bar{\Psi}_i = e^{-2\alpha_i \tau_i} T_i' \Psi_i T_i = \begin{bmatrix} \bar{\Psi}_{i,11} & \bar{\Psi}_{i,12} & \bar{\Psi}_{i,13} \\ \bar{\Psi}_{i,12}' & \bar{\Psi}_{i,22} & \bar{\Psi}_{i,23} \\ \bar{\Psi}_{i,13}' & \bar{\Psi}_{i,23}' & \bar{\Psi}_{i,33} \end{bmatrix}.$$

Also, let us introduce the correspondingly partitioned matrix

$$\tilde{\Psi}_i = \begin{bmatrix} \tilde{\Psi}_{i,11} & \tilde{\Psi}_{i,12} & \tilde{\Psi}_{i,13} \\ \tilde{\Psi}_{i,12}' & \tilde{\Psi}_{i,22} & \tilde{\Psi}_{i,23} \\ \tilde{\Psi}_{i,13}' & \tilde{\Psi}_{i,23}' & \tilde{\Psi}_{i,33} \end{bmatrix}, \quad (12)$$

where we let $\tilde{\Psi}_{i,11} = \bar{\Psi}_{i,11} - 2\alpha_i Y_i^{-1} - S_i$, $\tilde{\Psi}_{i,33} = \bar{\Psi}_{i,33} + e^{-2\alpha_i \tau_i} S_i$, and $\tilde{\Psi}_{i,\mu\nu} = \bar{\Psi}_{i,\mu\nu}$ for all other elements of $\tilde{\Psi}_i$. Then the following statement holds.

Lemma 2. *Under the conditions of Lemma 1,*

$$\dot{V}_i \leq -2\alpha_i V_i(e_i) + 2e_i' Y_i^{-1}\dot{e}_i + \tau_i^2 \dot{e}_i(t)' R_i \dot{e}_i(t) - \bar{\mathbf{e}}_i' \tilde{\Psi}_i \bar{\mathbf{e}}_i. \quad (13)$$

Furthermore, since $R_j, S_j \geq 0$, for every $j \in \mathscr{V}_i$, we have

$$-\pi_j V_j(e_j) \leq -\pi_j e_j' Y_j^{-1} e_j. \quad (14)$$

This leads to the following statement.

Lemma 3.

$$-\sum_{j\in\mathscr{V}_i}\pi_j V_j(e_j) - \sum_{j\in\mathscr{V}_i}\mathscr{W}_j(e_j, e_j(t_{k-\nu_j^{k,i}+1})) \leq -\begin{bmatrix}\mathbf{e}_{i,t}\\ \mathbf{e}_{i,s}\end{bmatrix}'\begin{bmatrix}\bar{\Phi}_{i,11} & \bar{\Phi}_{i,12}\\ \bar{\Phi}_{i,21} & \bar{\Phi}_{i,22}\end{bmatrix}\begin{bmatrix}\mathbf{e}_{i,t}\\ \mathbf{e}_{i,s}\end{bmatrix}, \quad (15)$$

where

$$\bar{\Phi}_{i,11} = \begin{bmatrix}\pi_{j_1}Y_{j_1}^{-1} + \frac{\pi^2}{4}W_{j_1} & \cdots & 0\\ \vdots & \ddots & \vdots \\ 0 & \cdots & \pi_{j_{p_i}}Y_{j_{p_i}}^{-1} + \frac{\pi^2}{4}W_{j_{p_i}}\end{bmatrix},$$

$$\bar{\Phi}_{i,22} = \begin{bmatrix}\frac{\pi^2}{4}W_{j_1} & 0 & \cdots & 0\\ 0 & \frac{\pi^2}{4}W_{j_2} & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \frac{\pi^2}{4}W_{j_{p_i}}\end{bmatrix},$$

$$\bar{\Phi}_{i,12} = \bar{\Phi}_{i,21} = -\bar{\Phi}_{i,22}.$$

Next, we apply the descriptor method [24] in order to derive LMIs for the design of observers' gains. For $t \in [t_k, t_{k+1})$, consider the neighbourhood set $\mathscr{V}_i$ and its corresponding permutation $\Pi^k(\mathscr{V}_i)$. Recall that for every $j \in \mathscr{V}_i$, $\nu_j^{k,i} \in \{1, \ldots, p_i\}$ is the index of node j in the ordered set $\Pi^k(\mathscr{V}_i)$. According to this notation, on the interval $[t_k, t_{k+1})$ the observer at node i utilizes the sample $\hat{x}_j(t_{k-\nu_j^{k,i}+1})$, and the corresponding error equation is driven by $e_j(t_{k-\nu_j^{k,i}+1})$. Let us define vectors

$$\mathbf{e}_{i,t} = [e_{j_1}(t)' \ldots e_{j_{p_i-1}}(t)' \, e_{j_{p_i}}(t)']',$$
$$\mathbf{e}_{i,s} = [e_{j_1}(t_{k-\nu_{j_1}^{k,i}+1})' \ldots e_{j_{p_i}}(t_{k-\nu_{j_{p_i}}^{k,i}+1})']',$$

which consist of the current and sampled error interconnection inputs, respectively, ordered in accordance with the ordering of the set $\mathscr{V}_i$. Note that for arbitrary compatible matrices X_i, Z_i and Q_i,

$$(X_i e_i + Z_i\dot{e}_i + (\mathbf{1}_{p_i}' \otimes Q_i)\mathbf{e}_{i,s})' \times \Big((A - L_iC_i)e_i + (\mathbf{1}_{p_i}' \otimes K_iH_i)\mathbf{e}_{i,s} - (\mathbf{1}_{p_i}' \otimes K_iH_i)\mathbf{e}_i + (B - L_iD_i)\xi_i - \dot{e}_i\Big) = 0. \quad (16)$$

From (13) and (16) it follows that

$$\begin{aligned}\dot{V}_i + 2\alpha_i V_i(e_i) \leq\ & 2e_i' Y_i^{-1}\dot{e}_i + \tau_i^2 \dot{e}_i' R_i \dot{e}_i - \bar{\mathbf{e}}_i'\tilde{\Psi}_i\bar{\mathbf{e}}_i \\ & + (X_i e_i + Z_i\dot{e}_i + (\mathbf{1}_{p_i}' \otimes Q_i)\mathbf{e}_{i,s})' \\ & \times \Big((A - L_iC_i)e_i + (\mathbf{1}_{p_i}' \otimes K_iH_i)\mathbf{e}_{i,s} \\ & - (\mathbf{1}_{p_i}' \otimes K_iH_i)\mathbf{e}_i + (B - L_iD_i)\xi_i - \dot{e}_i\Big).\end{aligned}$$

Along with condition (15) established in Lemma 3, this leads to the conclusion that

$$\begin{aligned}&\dot{V}_i(e_i) + 2\alpha_i V_i - \sum_{j\in\mathscr{V}_i}\pi_j V_j(e_j) + \left(\sum_{j:\, i\in\mathscr{V}_j}\tau_j^2\right)\dot{e}_i' W_i \dot{e}_i \\ &- \sum_{j\in\mathscr{V}_i}\mathscr{W}_j(e_j, e_j(t_{k-\nu_j^{k,i}+1})) + \frac{1}{\gamma^2}(p_i + q_i)\|e_i\|^2 \\ &- \frac{2}{\gamma^2}e_i'\sum_{j\in\mathscr{V}_i}e_j - \|\xi_i\|^2 \leq \eta_i' \Xi_i \eta_i. \end{aligned} \quad (17)$$

In the above inequality, η_i is the vector $\eta_i = [\dot{e}_i' \, e_i' \, \mathbf{e}_i' \, e_i(t - \tau_i)' \, \mathbf{e}_{i,t}' \, \mathbf{e}_{i,s}' \, \xi']'$, and Ξ_i is the matrix partitioned as follows

$$\Xi_i = \begin{bmatrix}\Xi_{aa} & \Xi_{ab} & \Xi_{ac} & 0 & 0 & \Xi_{af} & \Xi_{ag}\\ \star & \Xi_{bb} & \Xi_{bc} & -\tilde{\Psi}_{i,13} & \Xi_{be} & \Xi_{bf} & \Xi_{bg}\\ \star & \star & -\tilde{\Psi}_{i,22} & -\tilde{\Psi}_{i,23} & 0 & \Xi_{cf} & 0\\ \star & \star & \star & -\tilde{\Psi}_{i,33} & 0 & 0 & 0\\ \star & \star & \star & \star & -\bar{\Phi}_{i,11} & -\bar{\Phi}_{i,12} & 0\\ \star & \star & \star & \star & \star & \Xi_{ff} & \Xi_{fg}\\ \star & \star & \star & \star & \star & \star & -I\end{bmatrix}, \quad (18)$$

$$\Xi_{aa} = \tau_i^2 R_i + \left(\sum_{j:i\in\mathscr{V}_j}\tau_j^2\right)W_i - Z_i - Z_i',$$
$$\Xi_{ab} = Y_i^{-1} - X_i + Z_i'(A - L_iC_i),$$
$$\Xi_{ac} = -Z_i'(\mathbf{1}_{p_i}' \otimes K_iH_i), \qquad \Xi_{af} = \mathbf{1}_{p_i}' \otimes (-Q_i + Z_i'K_iH_i),$$
$$\Xi_{ag} = Z_i'(B - L_iD_i),$$
$$\Xi_{bb} = \frac{(p_i + q_i)}{\gamma^2}I - \tilde{\Psi}_{i,11} + X_i'(A - L_iC_i) + (A - L_iC_i)'X_i,$$
$$\Xi_{bc} = -\tilde{\Psi}_{i,12} - (\mathbf{1}_{p_i}' \otimes X_i'K_iH_i), \qquad \Xi_{be} = -\frac{1}{\gamma^2}(\mathbf{1}_{p_i}' \otimes I),$$
$$\Xi_{bf} = \mathbf{1}_{p_i}' \otimes (X_i'K_iH_i + (A - L_iC_i)'Q_i),$$
$$\Xi_{bg} = X_i'(B - L_iD_i), \qquad \Xi_{cf} = -\mathbf{1}_{p_i}\mathbf{1}_{p_i}' \otimes (H_i'K_i'Q_i),$$
$$\Xi_{ff} = \mathbf{1}_{p_i}\mathbf{1}_{p_i}' \otimes (Q_i'K_iH_i + H_i'K_i'Q_i) - \bar{\Phi}_{i,22},$$
$$\Xi_{fg} = \mathbf{1}_{p_i} \otimes Q_i'(B - L_iD_i).$$

It is worth noting that the matrix Ξ_i does not depend on k. Hence the dissipation inequality follows from the condition $\Xi_i < 0$ at any time t. By combining this conclusion with Theorem 1, we arrive at the following statement.

Theorem 2. *Suppose there exist matrices $Y_i = Y_i' > 0$, X_i, Z_i, Q_i, $W_i = W_i' \geq 0$, $S_i = S_i' \geq 0$, $R_i = R_i' \geq 0$, G_i, constants $\alpha_i > 0$, $0 \leq \pi_i < 2\alpha_i q_i^{-1}$, and gain matrices K_i, L_i, $i = 1, \ldots, N$, which satisfy the LMI (11) and*

$$\Xi_i < 0. \quad (19)$$

Then the corresponding observer network (4) solves the problem posed in Definition 1. The matrix P in condition (8) corresponding to this solution is $P = \frac{1}{N}\sum_{i=1}^{N}(Y_i^{-1} + S_i\frac{1-e^{-2\alpha_i\tau_i}}{2\alpha_i})$.

Theorem 2 serves as the basis for the derivation of the main result of this paper, given below in Theorem 3, which is a sufficient condition for the synthesis of distributed observer networks of the form (4). Consider the following matrix

$$\bar{\Xi}_i = \begin{bmatrix}\bar{\Xi}_{aa} & \bar{\Xi}_{ab} & \bar{\Xi}_{ac} & 0 & 0 & \bar{\Xi}_{af} & \bar{\Xi}_{ag}\\ \star & \bar{\Xi}_{bb} & \bar{\Xi}_{bc} & -\tilde{\Psi}_{i,13} & \bar{\Xi}_{be} & \bar{\Xi}_{bf} & \bar{\Xi}_{bg}\\ \star & \star & -\tilde{\Psi}_{i,22} & -\tilde{\Psi}_{i,23} & 0 & \bar{\Xi}_{cf} & 0\\ \star & \star & \star & -\tilde{\Psi}_{i,33} & 0 & 0 & 0\\ \star & \star & \star & \star & -\bar{\Phi}_{i,11} & -\bar{\Phi}_{i,12} & 0\\ \star & \star & \star & \star & \star & \bar{\Xi}_{ff} & \bar{\Xi}_{fg}\\ \star & \star & \star & \star & \star & \star & -I\end{bmatrix}, \quad (20)$$

$$\bar{\Xi}_{aa} = \tau_i^2 R_i + \left(\sum_{j:i\in\mathcal{V}_j} \tau_j^2\right) W_i - \epsilon_i X_i - \epsilon_i X_i',$$
$$\bar{\Xi}_{ab} = Y_i^{-1} - X_i + \epsilon_i (X_i'A - U_iC_i),$$
$$\bar{\Xi}_{ac} = -\epsilon_i (\mathbf{1}'_{p_i} \otimes F_iH_i),$$
$$\bar{\Xi}_{af} = \mathbf{1}'_{p_i} \otimes (-\bar{\epsilon}_i X_i + \epsilon_i F_i H_i), \qquad \bar{\Xi}_{ag} = \epsilon_i (X_i'B - U_iD_i),$$
$$\bar{\Xi}_{bb} = \frac{(p_i+q_i)}{\gamma^2} I - \tilde{\Psi}_{i,11} + X_i'A - U_iC_i + A'X_i - C_i'U_i',$$
$$\bar{\Xi}_{bc} = -\tilde{\Psi}_{i,12} - \mathbf{1}'_{p_i} \otimes (F_iH_i), \qquad \bar{\Xi}_{be} = -\frac{1}{\gamma^2} (\mathbf{1}'_{p_i} \otimes I),$$
$$\bar{\Xi}_{bf} = \mathbf{1}'_{p_i} \otimes (F_iH_i + \bar{\epsilon}_i A X_i - \bar{\epsilon}_i C_i' U_i'),$$
$$\bar{\Xi}_{bg} = X_i'B - U_iD_i, \qquad \bar{\Xi}_{cf} = -\mathbf{1}_{p_i}\mathbf{1}'_{p_i} \otimes (\bar{\epsilon}_i H_i' F_i'),$$
$$\bar{\Xi}_{ff} = \bar{\epsilon}_i \mathbf{1}_{p_i}\mathbf{1}'_{p_i} \otimes (F_iH_i + H_i'F_i') - \bar{\Phi}_{i,22},$$
$$\bar{\Xi}_{fg} = \bar{\epsilon}_i \mathbf{1}_{p_i} \otimes (X_i'B - U_iD_i).$$

Theorem 3. *Suppose that there exist matrices $Y_i = Y_i' > 0$, X_i, $\det X_i \neq 0$, F_i, U_i, $S_i = S_i' \geq 0$, $R_i = R_i' \geq 0$, $W_i = W_i' \geq 0$, G_i, and constants $\alpha_i > 0$, $0 \leq \pi_i < 2\alpha_i q_i^{-1}$, $\epsilon_i > 0$, $\bar{\epsilon}_i > 0$, $i = 1, \ldots, N$, which satisfy the LMI (11) and*

$$\bar{\Xi}_i < 0. \tag{21}$$

Then the network of observers (4) with

$$K_i = (X_i')^{-1} F_i, \qquad L_i = (X_i')^{-1} U_i, \tag{22}$$

solves the distributed estimation problem posed in Definition 1. The matrix P in condition (8) corresponding to this solution is $P = \frac{1}{N}\sum_{i=1}^N (Y_i^{-1} + S_i \frac{1-e^{-2\alpha_i\tau_i}}{2\alpha_i})$.

Proof. Similar to [25], we observe that LMI (19) follows from (21), when we let $Z_i = \epsilon_i X_i$, $Q_i = \bar{\epsilon}_i X_i$, and take K_i, L_i to be matrices defined in (22). Then the claim of the theorem follows from Theorem 2. □

Remark 2. The proposed LMI conditions involve 'free' variables X_i, Z_i and Q_i. These variables are to reduce the conservatism of the proposed LMI conditions. At the same time they add to the number of unknowns to be used by the LMI solver. In a high-dimensional problem where this causes an excessive computational burden, additional constraints on these variables can be introduced to reduce the number of variables used by the solver, at the expense of a more conservative design; e.g., X_i can be assumed to be diagonal.

4. Example

Consider a plant of the form (2), with $A = \begin{bmatrix} -3.2 & 10 & 0 \\ 1 & -1 & 1 \\ 0 & -14.87 & 0 \end{bmatrix}$, $B_2 = \begin{bmatrix} -0.1246 \\ -0.4461 \\ 0.3350 \end{bmatrix}$. This plant was used in the example in [7]. The nominal part of the plant describes one of the regimes of the so-called Chua electronic circuit.

To estimate this plant, we will use the 3-node observer network shown in Fig. 1.

The measurement matrices are

$$C_1 = [0.0032\ -0.0047\ 0.0010],$$
$$C_2 = [-0.8986\ 0.1312\ -1.9703],$$
$$C_3 = [1\,0\,0], \quad \text{and} \quad D_{2i} = 0, \qquad \bar{D}_{2i} = 0.025.$$

With these parameters, the pairs (A, C_1) and (A, C_2) are not detectable, while (A, C_3) is observable. Since observer at node 3 does not receive information from other observers, it acts as a conventional continuous-time H_∞ filter, while the observers at nodes 1 and 2 utilize sampled data inputs they receive from their neighbours. This allows them to overcome difficulties due to unstable unobservable modes of A.

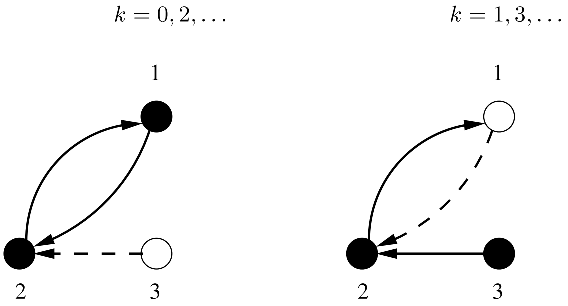

Fig. 1. An example 3-node network. The filled circles and solid lines represent nodes and links which are 'active' during the time interval $[t_k, t_{k+1})$, when k takes one of the values shown above the figure.

For simplicity we assume a constant sampling period of Δ, so that $t_k = k\Delta$. Then $\tau_1 = \Delta$, $\tau_2 = 2\Delta$ and $\tau_3 = 0$. We now apply Theorem 3 to compute observer and interconnection gains for this system. To this end, we solved the LMIs (21) numerically, with $\alpha_i = 0.1$, $\pi_i = \frac{2\alpha_i}{1+q_i}$, and $\bar{\epsilon}_i = 0$; that is, $Q_i = 0$ in this example. In fact, instead of solving the feasibility problem, we solved the optimization problem in which we sought to minimize γ^2 subject to the LMI constraints (11) and (21).

First, we compared the performance of our method with the performance guaranteed for estimators employing continuous-time interconnections by the method in [6]. To this end, we set the sampling rate to a high value by letting $\Delta = 0.0001$. With $\epsilon_i = 0.01$, we obtained the suboptimal γ^2 to be equal 0.2274, which is approximately equal to the level of H_∞ disagreement guaranteed for the comparison distributed estimator of [6], $\gamma^2 = 0.2299$. A slight discrepancy between the two values is likely due to numerical errors and/or conservative selection of parameters. Remarkably, both algorithms assign a high gain to the H_∞ filter at node 3 ($L_3 = 10^3 \times [0.2385\ 0.4724\ 3.9685]'$ using Theorem 3 versus $L_3 = 10^3 \times [0.0819\ 0.1707\ 1.5540]'$ using the method from [6]).

Next, we set the sampling rate to a larger value. After some experimenting with the tuning parameters ϵ_i, we chose $\epsilon_i = 0.1$. With $\Delta = 0.1$, Theorem 3 was found to guarantee the level of H_∞ disagreement $\gamma^2 = 0.5537$, and the gain L_3 reduced substantially, to the value $L_3 = [17.9083\ 13.1006\ -19.6797]'$. This gain is comparable with that obtained for the estimator of [6] with this value of γ^2. For $\Delta = 0.2$, the guaranteed level of H_∞ disagreement increased substantially, to the value of $\gamma^2 = 39.6506$. Further increasing the sampling period to $\Delta = 0.22$ resulted in a prohibitively large $\gamma^2 = 896.9248$.

5. Conclusions

The paper has presented a sufficient LMI condition for the design of a Round-Robin type interconnection protocol for networks of distributed observers. We have shown that the proposed protocol allows one to use sampled-data communications between the observers in the network, and does not require a combinatorial gain scheduling. As a result, the node observers are shown to be capable of achieving the H_∞ consensus objective introduced in [5–7]. As our example demonstrates, the proposed Round-Robin protocol achieves this objective at the expense of moderately deteriorated performance.

Appendix

A.1. Proof of Theorem 1

Define the vector function $V(e) = [V_1(e_1), \dots, V_N(e_N)]'$ and the matrix $M = [M_{ij}]$, $M_{ii} = -2\alpha_i$, $M_{ij} = \pi_j$ if $j \in \mathscr{V}_i$, and $M_{ij} = 0$ if $j \notin \mathscr{V}_i$ and $j \neq i$. It follows from (10) that for $t \in [t_k, t_{k+1})$,

$$\begin{aligned}\mathbf{1}_N'(\dot V - MV) &+ \frac{1}{\gamma^2}\sum_{i=1}^N\sum_{j\in\mathscr{V}_i}\|e_i - e_j\|^2 \\ &\le \sum_{i=1}^N\|\xi_i\|^2 - \sum_{i=1}^N\left(\sum_{j:i\in\mathscr{V}_j}\tau_j^2\right)\dot e_i' W_i \dot e_i \\ &+ \sum_{i=1}^N\sum_{j\in\mathscr{V}_i}\mathscr{W}_j(e_j, e_j(t_{k-\nu_j^{k,i}+1})).\end{aligned} \tag{23}$$

By changing the order of summation in the second term, we further obtain

$$\sum_{i=1}^N\left(\sum_{j:i\in\mathscr{V}_j}\tau_j^2\right)\dot e_i' W_i\dot e_i = \sum_{i=1}^N\sum_{j\in\mathscr{V}_i}\tau_i^2\dot e_j' W_j \dot e_j.$$

Hence, on the time interval $t \in [t_k, t_{k+1})$,

$$\begin{aligned}\mathbf{1}_N'(\dot V - MV) &+ \frac{1}{\gamma^2}\sum_{i=1}^N\sum_{j\in\mathscr{V}_i}\|e_i - e_j\|^2 \\ &\le \sum_{i=1}^N\left[\|\xi_i\|^2 - \sum_{j\in\mathscr{V}_i}\left[\tau_i^2\dot e_j' W_j\dot e_j - \mathscr{W}_j(e_j, e_j(t_{k-\nu_j^{k,i}+1}))\right]\right].\end{aligned} \tag{24}$$

Let $T > 0$ be a time instant, $T \in [t_{\bar k}, t_{\bar k+1})$. Let us fix i and $j \in \mathscr{V}_i$, and consider the partition of the interval $[0, T]$ into subintervals

$$\begin{aligned}[0, T] &= [0, t_{\bar k - \bar d p_i - \bar\nu + 1}) \cup [t_{\bar k - \bar\nu+1}, T) \\ &\cup \left(\bigcup_{d=1}^{\bar d}[t_{\bar k - dp_i - \bar\nu+1}, t_{\bar k-(d-1)p_i-\bar\nu+1})\right),\end{aligned} \tag{25}$$

where $\bar\nu = \nu_j^{\bar k,i}$ is the index of j in the permutation $\Pi^{\bar k}(\mathscr{V}_i)$, and $\bar d$ is the largest integer number such that $\bar d \le \frac{\bar k - \bar\nu+1}{p_i}$. Note that $0 \le \bar k - \bar d p_i - \bar\nu + 1$, and $t_{\bar k-\bar\nu+1} \le T < t_{\bar k - \bar\nu + p_i + 1}$. The significance of this partition is that on each interval $[t_{\bar k - dp_i - \bar\nu+1}, t_{\bar k-(d-1)p_i-\bar\nu+1})$, the observer at node i makes use of the sample $\hat x_j(t_{\bar k - dp_i-\bar\nu+1})$. Therefore the input e_j into the error dynamics equation (5) at node i holds the constant value $e_j(t_{\bar k - dp_i - \bar\nu+1})$ over this interval of time, with $e_j(t_l) = x_0 = \text{const}$ for $0 \le t_l < \bar k - \bar d p_i - \bar\nu + 1$. That is, for $t \in [t_{\bar k-dp_i-\bar\nu+1}, t_{\bar k-(d-1)p_i-\bar\nu+1})$,

$$\mathscr{W}_j(e_j, e_j(t_{k-\nu_j^{k,i}+1})) = \mathscr{W}_j(e_j, e_j(t_{\bar k - dp_i-\bar\nu+1}))$$

where $k = k(t)$ is determined from the condition $t_k \le t < t_{k+1}$, and $\nu_j^{k,i} = \nu_j^{k(t),i}$ is determined accordingly, as an index of j in the permutation $\Pi^{k(t)}(\mathscr{V}_i)$.

It follows from the above discussion that

$$\begin{aligned}\int_0^T &\left(\tau_i^2\dot e_j' W_j\dot e_j - \mathscr{W}_j(e_j, e_j(t_{k-\nu_j^{k,i}+1}))\right)dt \\ &= \int_0^{t_{\bar k-\bar d p_i-\bar\nu+1}}\left(\tau_i^2\dot e_j'(t)W_j\dot e_j(t) - \mathscr{W}_j(e_j, e_j(0))\right)dt \\ &+ \sum_{d=1}^{\bar d}\int_{t_{\bar k-dp_i-\bar\nu+1}}^{t_{\bar k-(d-1)p_i-\bar\nu+1}}\left(\tau_i^2\dot e_j' W_j\dot e_j - \mathscr{W}_j(e_j, e_j(t_{\bar k-dp_i-\bar\nu+1}))\right)dt \\ &+ \int_{t_{\bar k-\bar\nu+1}}^T\left(\tau_i^2\dot e_j'(t)W_j\dot e_j(t) - \mathscr{W}_j(e_j, e_j(t_{\bar k-\bar\nu+1}))\right)dt.\end{aligned} \tag{26}$$

Using the Wirtinger's inequality [27], it follows that

$$\begin{aligned}\int_{t_{\bar k-dp_i-\bar\nu+1}}^{t_{\bar k-(d-1)p_i-\bar\nu+1}}&\left(\tau_i^2\dot e_j'(t)W_j\dot e_j(t) - \mathscr{W}_j(e_j, e_j(t_{\bar k-dp_i-\bar\nu+1}))\right)dt \\ &\ge (t_{\bar k-(d-1)p_i-\bar\nu+1} - t_{\bar k-dp_i-\bar\nu+1})^2 \\ &\quad\times\int_{t_{\bar k-dp_i-\bar\nu+1}}^{t_{\bar k-(d-1)p_i-\bar\nu+1}}\left(\dot e_j'(t)W_j\dot e_j(t) - \mathscr{W}_j(e_j, e_j(t_{\bar k-dp_i-\bar\nu+1}))\right)dt \\ &\ge 0.\end{aligned} \tag{27}$$

Similarly,

$$\begin{aligned}&\int_0^{t_{\bar k-\bar d p_i-\bar\nu+1}}\left(\tau_i^2\dot e_j'(t)W_j\dot e_j(t) - \mathscr{W}_j(e_j, e_j(0))\right)dt \ge 0, \\ &\int_{t_{\bar k-\bar\nu+1}}^T\left(\tau_i^2\dot e_j'(t)W_j\dot e_j(t) - \mathscr{W}_j(e_j, e_j(t_{\bar k-\bar\nu+1}))\right)dt \ge 0.\end{aligned} \tag{28}$$

Therefore, we conclude from (26)–(28) that

$$\int_0^T\left(\tau_i^2\dot e_j' W_j\dot e_j - \mathscr{W}_j(e_j, e_j(t_{k-\nu_j^{k,i}+1}))\right)dt \ge 0.$$

Hence, it follows from (24) that

$$\begin{aligned}\int_0^T\left(\mathbf{1}_N'(\dot V - MV)\right)dt &+ \frac{1}{\gamma^2}\sum_{i=1}^N\sum_{j\in\mathscr{V}_i}\int_0^T\|e_i - e_j\|^2 dt \\ &\le \sum_{i=1}^N\int_0^T\|\xi_i\|^2 dt.\end{aligned} \tag{29}$$

The statement of the theorem then follows from (29) using the same argument as that used in the proof of Theorem 1 in [5].

A.2. Proof of Lemma 2

Consider $\dot V_i$:

$$\begin{aligned}\dot V_i &= 2e_i'Y_i^{-1}\dot e_i + e_i'S_ie_i - e^{-2\alpha_i\tau_i}e_i'(t-\tau_i)S_ie_i(t-\tau_i) \\ &+ \tau_i\int_{t-\tau_i}^t\left[\dot e_i(t)'R_i\dot e_i(t) - e^{2\alpha_i(t-s)}\dot e_i(s)'R_i\dot e_i(s)\right]ds \\ &- 2\alpha_i\int_{t-\tau_i}^t e^{-2\alpha_i(t-s)}e_i(s)'S_ie_i(s)ds \\ &- 2\alpha_i\tau_i\int_{t-\tau_i}^t e^{-2\alpha_i(t-s)}\dot e_i(s)'(\tau + s - t)R_i\dot e_i(s)ds.\end{aligned}$$

Since $e^{2\alpha_i(t-s)} \ge e^{-2\alpha_i\tau_i}$ for $s \in [t-\tau_i, t]$, then

$$\begin{aligned}\dot V_i &\le -2\alpha_iV_i(e_i) + 2e_i'Y_i^{-1}\dot e_i + e_i'(2\alpha_iY_i^{-1} + S_i)e_i \\ &+ \tau_i^2\dot e_i(t)'R_i\dot e_i(t) - e^{-2\alpha_i\tau_i}e_i'(t-\tau_i)S_ie_i(t-\tau_i) \\ &- \tau_ie^{-2\alpha_i\tau_i}\left[\int_{t_k}^t\dot e_i(s)'R_i\dot e_i(s)ds + \sum_{\nu=1}^{p_i-1}\int_{t_{k-\nu}}^{t_{k-\nu+1}}\dot e_i(s)'R_i\dot e_i(s)ds\right. \\ &\left.+ \int_{t-\tau_i}^{t_{k-p_i+1}}\dot e_i(s)'R_i\dot e_i(s)ds\right].\end{aligned}$$

By Jensen's inequality,

$$\begin{aligned}\dot V_i &\le -2\alpha_iV_i(e_i) + 2e_i'Y_i^{-1}\dot e_i + e_i'(2\alpha_iY_i^{-1} + S_i)e_i \\ &+ \tau_i^2\dot e_i(t)'R_i\dot e_i(t) - e^{-2\alpha_i\tau_i}e_i'(t-\tau_i)S_ie_i(t-\tau_i) \\ &- \tau_ie^{-2\alpha_i\tau_i}\left[\frac{1}{t-t_k}(e_i - e_i(t-t_k))'R_i(e_i - e_i(t-t_k))\right. \\ &+ \sum_{\nu=1}^{p_i-1}\frac{1}{t_{k-\nu+1} - t_{k-\nu}}(e_i(t_{k-\nu+1}) - e(t_{k-\nu}))'\end{aligned}$$

$$\times R_i(e_i(t_{k-\nu+1}) - e(t_{k-\nu})) + \frac{1}{t_{k-p_i+1} - t + \tau_i}(e_i(t_{k-p_i+1}) - e(t-\tau_i))' \times R_i(e_i(t_{k-p_i+1}) - e(t-\tau_i))\Bigg].$$

Let $\delta = [\delta_0', \dots, \delta_{p_i}']'$, where

$$\delta_\nu = e_i(t_{k-\nu+1}) - e(t_{k-\nu}), \quad \nu = 1, \dots, p_i - 1,$$
$$\delta_0 = e_i(t) - e(t_k), \qquad \delta_{p_i} = e_i(t_{k-p_i+1}) - e(t-\tau_i).$$

Then, $\delta = T_i\bar{\mathbf{e}}_i$. Also, $\delta'\Psi_i\delta = \bar{\mathbf{e}}_i'T_i'\Psi_iT_i\bar{\mathbf{e}}_i$.

Using Lemma 1, we conclude that

$$\dot{V}_i \le -2\alpha_iV_i(e_i) + 2e_i'Y_i^{-1}\dot{e}_i + e_i'(2\alpha_iY_i^{-1} + S_i)e_i - e^{-2\alpha_i\tau_i}e_i'(t-\tau_i)S_ie_i(t-\tau_i) + \tau_i^2\dot{e}_i(t)'R_i\dot{e}_i(t) - \bar{\mathbf{e}}_i'\bar{\Psi}_i\bar{\mathbf{e}}_i. \quad (30)$$

Then, the statement of the lemma follows from the definition of the matrix $\tilde{\Psi}$ and inequality (30).

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