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Improved stability conditions for discrete-time systems under dynamic network protocols

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Summary
This paper deals with the stability of discrete-time networked systems with multiple sensor nodes under dynamic scheduling protocols. Access to the communication medium is orchestrated by a weighted try-once-discard or by an independent and identically-distributed stochastic protocol that determines which sensor node can access the network at each sampling instant and transmit its corresponding data. Through a time-delay approach, a unified discrete-time hybrid system with time-varying delays in the dynamics and in the reset conditions is formulated under both scheduling protocols. Then, a new stability criterion for discrete-time systems with time-varying delays is proposed by the discrete counterpart of the second-order Bessel-Legendre integral inequality. The developed approach is applied to guarantee the stability of the resulting discrete-time hybrid system model with respect to the full state under try-once-discard or independent and identically-distributed scheduling protocol. The communication delays can be larger than the sampling intervals. Finally, the efficiency of the presented approach is illustrated by a cart-pendulum system.

KEYWORDS
discrete-time networked control systems, dynamic protocols, Lyapunov method, multiple sensors

1 | INTRODUCTION

With the development of communication techniques, network topologies, and control methods, networked control systems (NCSs) have received increasing attention in the past decades due to its widespread applications.[1] Meanwhile, because the network is usually shared by multiple sensor, controller and actuator nodes, these distributed nodes will compete for access to the network as a result of bandwidth limitations and interference channels. There is a need for network protocols to address communication constraints, which prohibit that sensor, controller, or actuator nodes transmits their corresponding values simultaneously. In the literature, there are two basic types of scheduling protocols, namely, static and dynamic protocols.

Static protocols correspond to the situation where the order of the activated nodes is chosen initially and remains fixed at each transmission instant. The well-known Round-Robin (RR) communication protocol[2] is one of the static protocols, where the nodes take turns transmitting its corresponding data in a predetermined and cyclic manner. Compared to static protocols, the dynamic protocols,[3] such as the often used try-once-discard (TOD) protocol and stochastic protocol, usually achieve better system performance than the static ones.[4] In the TOD protocol, the node, corresponding to the largest error between the current value and the last transmitted value, has the highest priority to use the communication medium. While the stochastic protocol usually determines the transmitted node through a Bernoulli or a Markov chain process.

Both static and dynamic protocols have been widely adopted to communication and signal processing problems in different frameworks. Stability of NCSs under RR and TOD protocols was studied in the context of hybrid systems in the works of Nesic and Liberzon[2] Heemels et al[5] and in the framework of discretization-based systems in the works of Cloosterman et al[6] and Donkers et al.[7] It is noted that, in the presence of scheduling protocols, both hybrid systems and discretization-based systems approaches do not allow large communication delays (that are larger than the sampling intervals).

To incorporate communication delays larger than the sampling interval without increasing the model complexity, a time-delay system approach was introduced for the stabilization of continuous-time networked systems with two sensor nodes under the RR protocol[8] and under the TOD protocol.[9] The closed-loop system was modeled as a switched system with multiple and ordered time-varying delays under RR scheduling or as a hybrid system with time-varying delays in the dynamics and in the reset equations under TOD scheduling. For continuous-time NCSs, the extension to multiple sensor nodes was presented in the works of Liu et al[10,11] in which a hybrid time-delay system model was given under the TOD/RR protocols and under the stochastic protocol, respectively, for the closed-loop system that contains time-varying delays in the continuous dynamics and in the reset conditions. Furthermore, from a time-delay system perspective, the RR protocol was considered in the work of Ugrinovskii and Fridman,[12] where distributed estimation with H_∞ consensus was analyzed. Recently, by taking into account TOD/RR protocols, a time-delay system approach was developed in the work of Freirich and Fridman[13] for the decentralized exponential stabilization of large-scale NCSs in the presence of asynchronous sampling of local networks. Moreover, the stochastic scheduling and RR scheduling protocols were used in the work of Wen et al[14] to achieve master-salve synchronization.

Although the problem of communication constraints has been widely studied, the aforementioned works on scheduling protocols are concerned with continuous-time systems, and the corresponding results for discrete-time systems are relatively few. In fact, discrete-time systems have already been applied in a wide range of areas, such as time-series analysis, image processing, quadratic optimization problem, and queuing analysis. Recently, RR and TOD protocols were utilized to the set-membership filtering problem in the work of Zou et al[15] for a class of discrete time-varying system. A stochastic communication protocol was applied in another work of Zou et al[16] to observer-based H_∞ control of networked discrete-time systems. However, the communication delays were not considered in these works. The time-delay approach was extended in the work of Liu and Fridman[17] to the stability analysis of discrete-time networked systems with actuator constraints and two sensor nodes under the TOD scheduling protocol in the presence of large communication delays. A time-dependent Lyapunov functional was introduced and only partial stability of the resulting hybrid delayed system was guaranteed. The extension from two to multiple sensor nodes is far from being straightforward. It has the following challenges.

1. The time-dependent Lyapunov functional of Liu and Fridman[17] is not applicable any more.
2. It is important to guarantee stability of the resulting closed-loop system with respect to the full state and not only to the partial state.

In this paper, we deal with the stability problem of discrete-time networked systems with multiple sensor nodes under dynamic scheduling protocols, aiming at presenting an improved stability criterion to find the maximum allowable sampling intervals and transmission delays such that the resulting closed-loop system is exponentially stable with respect to the full state. A unified discrete-time hybrid system model is formulated with time-varying delays in the dynamics and in the reset equations under both TOD and independent and identically-distributed (iid) scheduling protocols. The main contributions of this paper are as follows.

1. A new stability criterion for discrete-time systems with time-varying delays is proposed by virtue of the developed reciprocally convex combination inequality proposed in the work of Zhang et al[18] and the discrete counterpart of the augmented Lyapunov functional provided in the work of Liu et al[19] for the stability analysis of continuous-time systems with time-varying delays.
2. The proposed approach allows obtaining efficient stability criterion for the resulting discrete-time hybrid system model with respect to the full state under TOD or iid scheduling protocol.

The efficiency of the presented approach is illustrated by a cart-pendulum system. Preliminary results on the stabilization of networked discrete-time system with multiple sensor nodes under the TOD protocol have been presented in the work of Liu et al,[20] where the condition was obtained by the Jensen inequality and the reciprocally convex combination approach.[21]

Notation. Throughout the paper, the superscript "T" stands for matrix transposition. By $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$, we denote the n dimensional Euclidean space with vector norm $|\cdot|$ and the set of all $n \times m$ real matrices, respectively. The set $\mathbb{S}_+^n$ refers to the set of symmetric positive definite matrices. For $P \in \mathbb{R}^{n\times n}$, the notation $P > 0$ means that P is symmetric and positive definite. The symmetric elements of the symmetric matrix will be denoted by $*$, $\lambda_{\min}(P)$ denotes the smallest eigenvalue of matrix P. $\mathbb{Z}$, $\mathbb{Z}^+$, and $\mathbb{N}$ denote the set of integers, nonnegative integers, and positive integers, respectively. For integers a and b with $b > a$, the notation $\mathbb{Z}[a, b]$ stands for all integers in the interval $[a, b]$.

2 | NCS MODEL AND PRELIMINARIES

In this section, we first demonstrate the discrete-time description of the NCS model and then introduce the dynamic scheduling protocols to be adopted in this paper.

2.1 | Description of system data

Consider the networked control scheme shown in Figure 1, where a linear discrete-time plant, N distributed sensors, a controller node, and an actuator node are all connected via communication networks. The linear time-invariant discrete-time plant is given by

$$\begin{cases} x(k+1) &= Ax(k) + Bu(k), \quad k \in \mathbb{Z}^+, \\ y_i(k) &= C_i x(k), \quad i = 1, \dots, N, \end{cases} \tag{1}$$

where $x(k) \in \mathbb{R}^n$ denotes the state of the plant; $u(k) \in \mathbb{R}^{n_u}$ the control input; and $y_i(k) \in \mathbb{R}^{n_i}$ ($i = 1, \dots, N$) the measurement outputs of the plant; and A, B, and C_i, $i = 1, \dots, N$, are the system matrices with appropriate dimensions. These matrices can be uncertain with polytopic type uncertainty. The initial condition is given by $x(k) = x_0$. We denote $C = [\, C_1^T \ \cdots \ C_N^T \,]^T$, $y(k) = [\, y_1^T(k) \ \cdots \ y_N^T(k) \,]^T \in \mathbb{R}^{n_y}$, and $\sum_{i=1}^N n_i = n_y$.

The sequence of sampling instants $0 = s_0, s_1, s_2, \dots$ is strictly increasing in the sense that $s_{p+1} - s_p \le \text{MATI}$, where $\{s_p\}$ is a subsequence of $\mathbb{Z}^+$ and MATI denotes the maximum allowable transmission interval. Denote by t_p the updating time instant of the zero-order holder. Suppose that the updating data at the instant t_p on the actuator side has experienced an uncertain transmission delay $h_p = t_p - s_p$ as it is transmitted through the network (both from the sensor to the controller and from the controller to the actuator). The delays may be either smaller or larger than the sampling interval provided

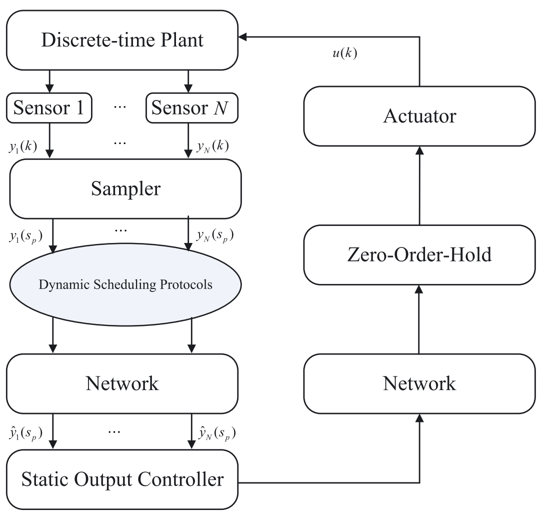

FIGURE 1 The architecture of networked control systems under dynamic scheduling protocols [Colour figure can be viewed at wileyonlinelibrary.com]

that the transmission order of data packets is maintained.[22] Assume that the network-induced delay h_p is bounded with the interval $[h_m, h_M]$, where h_m and h_M are known nonnegative integers.

Denote by $\hat{y}(s_p) = [\hat{y}_1^T(s_p) \cdots \hat{y}_N^T(s_p)]^T \in \mathbb{R}^{n_y}$ the output information submitted to the dynamic scheduling protocol. At each sampling instant s_p, one of the outputs $y_i(s_p) \in \mathbb{R}^{n_i}$ is transmitted over the network, that is, one of the $\hat{y}_i(s_p)$ values is updated with the recent state $y_i(s_p)$. Let $i_p^* \in \mathcal{I}_N = \{1, \ldots, N\}$ denote the active output node at the sampling instant s_p. Then,

$$\hat{y}_i(s_p) = \begin{cases} y_i(s_p), & i = i_p^*, \\ \hat{y}_i(s_{p-1}), & i \neq i_p^*. \end{cases} \tag{2}$$

We denote $e(k)$ by the error between the system output $y(s_p)$ and the latest available information $\hat{y}(s_{p-1})$, ie,

$$\begin{aligned} & e(k) = \text{col}\{e_1(k), \ldots, e_N(k)\} \equiv \hat{y}(s_{p-1}) - y(s_p), \\ & k \in [t_p, t_{p+1} - 1], \quad k \in \mathbb{Z}^+, \quad \hat{y}(s_{-1}) \triangleq 0, \quad e(k) \in \mathbb{R}^{n_y}. \end{aligned} \tag{3}$$

The choice of i_p^* will depend on the transmission error and will be chosen according to the scheduling protocols, which are defined in the following.

2.2 | Dynamic scheduling protocols

2.2.1 | TOD protocol

In the TOD protocol, the output node $i_p^* \in \mathcal{I}_N$ with the greatest weighted error will be granted the access to the network.

Let $Q_i > 0, i = 1, \ldots, N$, be some weighting matrices that will be computed in Theorem 2 as follows. At the sampling instant s_p, the weighted TOD protocol is a protocol in which the active output node with the index i_p^* is defined as any index that satisfies

$$\left|\sqrt{Q}_{i_p^*} e_{i_p^*}(k)\right|^2 \geq \left|\sqrt{Q}_i e_i(k)\right|^2, k \in [t_p, t_{p+1} - 1], i \in \mathcal{I}_N \setminus \left\{i_p^*\right\}. \tag{4}$$

A possible selection of i_p^* is given by

$$i_p^* = \min\left\{\arg \max_{i \in \{1, \ldots, N\}} \left|\sqrt{Q}_i \left(\hat{y}_i(s_{p-1}) - y_i(s_p)\right)\right|^2\right\}.$$

2.2.2 | Iid scheduling

The selection of i_p^* is assumed to be iid with the probabilities given by

$$\text{Prob}\left\{i_p^* = i\right\} = \beta_i, \quad i \in \mathcal{I}_N, \tag{5}$$

where $\beta_i \in \mathcal{I}_N$ are nonnegative scalars and $\sum_{i=1}^N \beta_i = 1$. Here, $\beta_j, j = 1, \ldots, N$, are the probabilities of the state $x_j(s_p)$ to be transmitted at s_p.

Remark 1. The iid protocol, as one of dynamic protocols, is adopted to schedule which sensor node can access to the communication medium at each sampling instant. It has been applied to describe probabilistic measurements missing,[23] stochastic sampling intervals,[24] stochastic interval time-delays,[25] and the probability of a switched system staying in each subsystem.[26,27]

3 | A DISCRETE-TIME HYBRID SYSTEM MODEL

Consider (1) under the static output feedback control. In the following, we propose a hybrid system model for the closed-loop system of NCS provided earlier.

The controller and the actuator are supposed to be event driven. The most recent output information on the controller side is denoted by $\hat{y}(s_p)$. Assume that there exists a matrix $K = [K_1 \cdots K_N]$, $K_i \in \mathbb{R}^{n_u \times n_i}$ such that $A + BKC$ is Schur stable. Consider the static output feedback controller

$$u(k) = K\hat{y}(s_p), \quad k \in [t_p, t_{p+1} - 1], \quad k \in \mathbb{Z}^+. \tag{6}$$

From (2), it follows that the controller (6) can be rewritten as

$$u(k) = K_{i_p^*} y_{i_p^*}(s_p) + \sum_{i=1, i\neq i_p^*}^{N} K_i \hat{y}_i(s_{p-1}), \tag{7}$$

for $k \in [t_p, t_{p+1} - 1]$, where i_p^* is the index of the active node at s_p.

Therefore, from (1), (3), and (7), we obtain the following augmented closed-loop system for $k \in [t_p, t_{p+1} - 2],\ k \in \mathbb{Z}^+$:

$$\begin{cases} x(k+1) = Ax(k) + A_1 x(s_p) + \sum_{i=1, i\neq i_p^*}^{N} B_i e_i(t_p), \\ e(k+1) = e(k), \end{cases} \tag{8}$$

with the delayed reset system for $k = t_{p+1} - 1$

$$\begin{cases} x(t_{p+1}) = Ax(t_{p+1} - 1) + A_1 x(s_p) + \sum_{i=1, i\neq i_p^*}^{N} B_i e_i(t_p), \\ e_i(t_{p+1}) = C_i \left[x(s_p) - x(s_{p+1})\right], \quad i = i_p^*, \\ e_i(t_{p+1}) = e_i(t_p) + C_i \left[x(s_p) - x(s_{p+1})\right], \quad i \neq i_p^*, \end{cases} \tag{9}$$

where the augmented state is $\text{col}\{x(k), e(k)\}$, $e(k)$ the error between the system output $y(s_p)$ and the latest available information $\hat{y}(s_{p-1})$ is defined in (3), and

$$A_1 = BKC, \quad K = [K_1 \cdots K_N], \quad B_i = BK_i, \qquad i = 1, \dots, N.$$

For $k \in [t_p, t_{p+1} - 1]$, denote $h(k) = k - s_p$. Then $h_m \leq h(k) \leq \text{MATI} + h_M - 1 \triangleq \tau_M$. Therefore, (8)-(9) can be considered as a discrete-time hybrid system with a time-varying interval delay.

Remark 2. For continuous-time NCSs, the closed-loop system under RR protocol was presented in the work of Liu et al[10] as a hybrid time-delay system model, which leads to complicated Lyapunov-based analysis. A more accurate model of the closed-loop system under the RR protocol was given in a different work of Liu et al[8] in the form of switched N subsystems with multiple and ordered time-varying delays. In the works of Freirich and Fridman,[13,28] the closed-loop system was simplified to a model with one system instead of N and independent multiple delays in the continuous and discrete cases, respectively.

The objective of the present paper is to provide improved stability criteria to find MATI and the maximum allowable transmission delay h_M such that the closed-loop system (8)-(9) under dynamic protocol (4) (or (5)) is exponentially stable (exponentially mean-square stable) with respect to the full state. To do so, in the following, we first establish a new stability criterion for discrete-time systems with time-varying delays by the discrete counterpart of the augmented Lyapunov functional provided in the work of Liu et al[19] for the stability analysis of continuous-time systems with time-varying delays. Then, the proposed approach is applied to guarantee the stability of the resulting discrete-time hybrid system model with respect to the full state under TOD and iid scheduling protocols, respectively.

4 | PRELIMINARY RESULTS ON DISCRETE-TIME SYSTEMS WITH TIME-VARYING DELAYS

In this section, we propose a new stability condition for the linear discrete-time system with time-varying delays. This class of system is governed by

$$\begin{cases} x(k+1) &= Ax(k) + A_1 x(k - h(k)), \quad k \geq 0, \quad k \in \mathbb{Z}^+, \\ x(k) &= \phi(k),\ k \in [-h_2, 0] \cap \mathbb{Z}, \end{cases} \tag{10}$$

where $x(k) \in \mathbb{R}^n$ is the state vector, $A \in \mathbb{R}^{n\times n}$, $A_1 \in \mathbb{R}^{n\times n}$ are constant matrices, and $\phi(k)$ is an initial condition. The time-varying delay $h(k)$ is a positive integer satisfying $h_1 \le h(k) \le h_2,\ h_{12} \overset{\Delta}{=} h_2 - h_1$, where h_1 and h_2 are known positive integers.

This study indeed represents the first stage toward the stability analysis of system (8)-(9) because it could represent the situation, where $e(k) = 0, \forall k \ge 0, k \in \mathbb{Z}$. This preliminary result will then be extended to address the objective of the present paper, ie, exponential stability (exponential mean-square stability) of (8)-(9) under dynamic protocol (4) (or (5)).

Let us first recall the discrete counterpart of the second-order Bessel-Legendre integral inequality[29] (ie, the Bessel-Legendre inequality with the degree of Legendre polynomials equal to 2). The same inequality was also proposed in the work of Hien and Trinh.[30] In this inequality, an improvement of Abel lemma–based inequality[31] or Wirtinger-type inequality[32] has been achieved.

Lemma 1. *For a given matrix $R \in \mathbb{S}_+^n$, integers a and b with $b > a$, any vector function $x: \mathbb{Z}[a,b] \to \mathbb{R}^n$, the inequality*

$$\sum_{s=a}^{b} \omega^T(s) R \omega(s) \ge \frac{1}{b-a+1} \Omega^T \mathrm{diag}(R, 3R, 5R)\Omega \tag{11}$$

holds, where $\omega(s) = x(s) - x(s-1)$ and

$$\Omega = \begin{bmatrix} x(b) - x(a-1) \\ x(b) + x(a-1) - \frac{2}{b-a+2}\sum_{s=a-1}^{b} x(s) \\ x(b) - x(a-1) - \frac{6}{b-a+2}\sum_{s=a-1}^{b} \delta_{a,b}(s)x(s) \end{bmatrix},$$

$$\delta_{a,b}(s) = 2\left(\frac{s-a}{b-a}\right) - 1.$$

In the following, based on Lemma 1 together with a newly developed delay-dependent reciprocally convex combination lemma,[18] a novel stability criterion is provided for discrete-time system (10) with time-varying delays. For simplicity of presentation, in this section, we denote by $\varrho_i (i = 1, \dots, 14)$ the block row vectors of the identity matrix I_{14n} and use the following notations:

$$\begin{aligned}
G_1 &= \begin{bmatrix} \varrho_1^T & -\varrho_1^T + (h_1+1)\varrho_5^T & -\varrho_2^T - \varrho_3^T + \varrho_7^T + \varrho_9^T - \varrho_1^T + (h_1+1)\varrho_6^T & \hat{G}_1^T \end{bmatrix}^T, \\
\hat{G}_1 &= -(h_{12}-1)\varrho_2 + (h_{12}+1)\varrho_3 - \varrho_{12} - \varrho_{14}, \\
G(h) &= \begin{bmatrix} 0 & 0 & (h-h_1)\varrho_7^T + (h_2-h)\varrho_9^T & 0 & \hat{G}(h)^T \end{bmatrix}^T, \\
\hat{G}(h) &= -2(h_2-h)\varrho_3 + (h_2-h)(\varrho_{11}+\varrho_{14}) + (h-h_1)(\varrho_{12}-\varrho_{13}), \\
G_2 &= \begin{bmatrix} \varrho_1^T - \varrho_2^T & \varrho_1^T + \varrho_2^T - 2\varrho_5^T & \varrho_1^T - \varrho_2^T - 6\varrho_6^T \end{bmatrix}^T, \\
G_3 &= \begin{bmatrix} \varrho_2^T - \varrho_3^T & \varrho_2^T + \varrho_3^T - 2\varrho_7^T & \varrho_2^T - \varrho_3^T - 6\varrho_8^T \end{bmatrix}^T, \\
G_4 &= \begin{bmatrix} \varrho_3^T - \varrho_4^T & \varrho_3^T + \varrho_4^T - 2\varrho_9^T & \varrho_3^T - \varrho_4^T - 6\varrho_{10}^T \end{bmatrix}^T, \\
\Gamma &= \begin{bmatrix} G_3^T & G_4^T \end{bmatrix}^T, \quad \Sigma = A\varrho_1 + A_1\varrho_3, \\
G_0 &= \begin{bmatrix} \Sigma^T & -\varrho_2^T + (h_1+1)\varrho_5^T & -\varrho_3^T - \varrho_4^T + \varrho_7^T + \varrho_9^T \hat{G}_{01}^T & \hat{G}_{02}^T \end{bmatrix}^T, \\
\hat{G}_{01} &= \left(1 + \frac{4}{h_1 - 1}\right)\varrho_2 - \frac{2(h_1+1)}{h_1}\varrho_5 + (h_1+1)\varrho_6, \\
\hat{G}_{02} &= (h_{12}+3)(\varrho_3+\varrho_4) - 2(\varrho_{11}+\varrho_{13}) - (\varrho_{12}+\varrho_{14}),
\end{aligned} \tag{12}$$

and

$$\begin{aligned}
\eta_0(k) &= \begin{bmatrix} x^T(k) & x^T(k-h_1) & x^T(k-h(k)) & x^T(k-h_2) \end{bmatrix}^T,\\
\eta_1(k) &= \frac{1}{h_1+1}\left[\sum_{s=k-h_1}^{k} x^T(s) \quad \sum_{s=k-h_1}^{k} \delta_1(k,s)x^T(s) \right]^T,\\
\eta_2(k) &= \frac{1}{h-h_1+1}\left[\sum_{s=k-h}^{k-h_1} x^T(s) \quad \sum_{s=k-h}^{k-h_1} \delta_2(k,s)x^T(s) \right]^T,\\
\eta_3(k) &= \frac{1}{h_2-h+1}\left[\sum_{s=k-h_2}^{k-h} x^T(s) \quad \sum_{s=k-h_2}^{k-h} \delta_3(k,s)x^T(s) \right]^T,\\
\eta_4(k) &= (h-h_1+1)\eta_2(k), \quad \eta_5(k) = (h_2-h+1)\eta_3(k),\\
\eta_6(k) &= \left[\sum_{s=k-h_2}^{k-h_1} x^T(s) \;\; h_{12} \sum_{s=k-h_2}^{k-h_1} \delta_4(k,s)x^T(s) \right]^T,
\end{aligned} \tag{13}$$

and where the functions δ_i, for $i = 1, \dots, 4$, which refer to the functions $\delta_{a,b}$ given in Lemma 1, are given by

$$\begin{aligned}
\delta_1(k,s) &= 2\left(\frac{s-k+h_1-1}{h_1-1}\right)-1, \qquad \delta_2(k,s) = 2\left(\frac{s-k+h-1}{h-h_1-1}\right)-1,\\
\delta_3(k,s) &= 2\left(\frac{s-k+h_2-1}{h_2-h-1}\right)-1, \qquad \delta_4(k,s) = 2\left(\frac{s-k+h_2-1}{h_{12}-1}\right)-1.
\end{aligned} \tag{14}$$

The following theorem gives sufficient conditions for exponential stability of system (10).

Theorem 1. *The system (10) is exponentially stable with the decay rate $\lambda \in (0,1)$ for all time-varying delays $h(k) \in [h_1, h_2]$ if there exist matrices $P \in \mathbb{S}_+^{5n}$, $S_1, S_2, R_1, R_2 \in \mathbb{S}_+^n$, $N_1, N_2 \in \mathbb{R}^{14n\times 2n}$, and a matrix $X \in \mathbb{R}^{3n\times 3n}$ such that the matrix inequalities*

$$\begin{bmatrix} \tilde{\Phi}_i & \begin{bmatrix} (\Sigma-\varrho_1)^T H \\ 0 \end{bmatrix} \\ * & -H \end{bmatrix} < 0, \quad i = 1,2, \tag{15}$$

hold, where

$$\begin{aligned}
\tilde{\Phi}_1 &= \begin{bmatrix} \Phi_0(h_1) - \lambda^{h_2}\Gamma^T\Psi(h_1)\Gamma & G_3^T X & \sqrt{1-\lambda}G^T(h)P \\ * & -\lambda^{-h_2}\tilde{R}_2 & 0 \\ * & * & -P \end{bmatrix},\\
\tilde{\Phi}_2 &= \begin{bmatrix} \Phi_0(h_2) - \lambda^{h_2}\Gamma^T\Psi(h_2)\Gamma & G_4^T X^T & \sqrt{1-\lambda}G^T(h)P \\ * & -\lambda^{-h_2}\tilde{R}_2 & 0 \\ * & * & -P \end{bmatrix},
\end{aligned} \tag{16}$$

and, for any θ in $\mathbb{R}$,

$$\begin{aligned}
\Phi_0(\theta) =& G_0^T P G_0 - \lambda G_1^T P G_1 + \mathrm{He}\left(G^T(h)P(G_0-\lambda G_1)\right.\\
&\left. + N_1 g_1(\theta) + N_2 g_2(\theta)\right) + \hat{S} - G_2^T \tilde{R}_1 G_2,\\
\hat{S} =& \mathrm{diag}\left(S_1, -\lambda^{h_1}(S_1-S_2), 0_{n\times n}, -\lambda^{h_2}S_2, 0_{10n\times 10n}\right),\\
\tilde{R}_i =& \mathrm{diag}(R_i, 3R_i, 5R_i), \quad i=1,2,\\
H =& h_1^2 R_1 + h_{12}^2 R_2,
\end{aligned} \tag{17}$$

and

$$\begin{aligned}
\Psi(\theta) &= \begin{bmatrix} (2-(h-h_1)/h_{12})\tilde{R}_2 & X \\ X^T & (1+(h-h_1)/h_{12})\tilde{R}_2 \end{bmatrix},\\
g_1(\theta) &= (\theta-h_1+1)\begin{bmatrix} \varrho_7 \\ \varrho_8 \end{bmatrix} - \begin{bmatrix} \varrho_{11} \\ \varrho_{12} \end{bmatrix}, \qquad g_2(\theta) = (h_2-\theta+1)\begin{bmatrix} \varrho_9 \\ \varrho_{10} \end{bmatrix} - \begin{bmatrix} \varrho_{13} \\ \varrho_{14} \end{bmatrix}.
\end{aligned} \tag{18}$$

Proof. Consider a Lyapunov functional given by

$$\begin{aligned}
V(k) &= V_1(k) + V_2(k) + V_3(k),\\
V_1(k) &= \tilde{x}^T(k)P\tilde{x}(k),\\
V_2(k) &= \sum_{s=k-h_1}^{k-1} \lambda^{k-s-1}\left|\sqrt{S_1}x(s)\right|^2 + \sum_{s=k-h_2}^{k-h_1-1} \lambda^{k-s-1}\left|\sqrt{S_2}x(s)\right|^2,\\
V_3(k) &= h_1 \sum_{j=-h_1+1}^{0}\sum_{s=k+j}^{k} \lambda^{k-s}\left|\sqrt{R_1}\eta(s)\right|^2 + h_{12}\sum_{j=-h_2+1}^{-h_1}\sum_{s=k+j}^{k} \lambda^{k-s}\left|\sqrt{R_2}\eta(s)\right|^2,
\end{aligned} \tag{19}$$

where $\eta(k) = x(k) - x(k-1)$ and

$$\tilde{x}(k) = \begin{bmatrix} x(k) \\ \sum_{s=k-h_1}^{k-1} x(s) \\ \sum_{s=k-h_2}^{k-h_1-1} x(s) \\ \sum_{s=k-h_1}^{k-1} \delta_1(k,s)x(s) \\ (h_{12}-1)\sum_{s=k-h_2}^{k-h_1-1} \delta_4(k,s)x(s) \end{bmatrix},$$

with $\delta_1(k,s)$ and $\delta_4(k,s)$ given in (14).

The objective of the next development consists in finding an upper bound of the difference of $V(k+1) - \lambda V(k)$ along the trajectories of system (10) using the augmented vector

$$\zeta(k) = \mathrm{col}\left\{\eta_0(k), \eta_1(k), \eta_2(k), \eta_3(k), \eta_4(k), \eta_5(k)\right\}, \tag{20}$$

where $\eta_i(k)$, for $i = 0, \dots, 5$, are given in (13).

To express $\Delta V_1(k) = \tilde{x}^T(k+1)P\tilde{x}(k+1) - \lambda\tilde{x}^T(k)P\tilde{x}(k)$ in terms of the augmented vector $\zeta(k)$, we need to express $\tilde{x}(k+1)$ and $\tilde{x}(k)$ using $\zeta(k)$. On one hand, we note that the first four components of $\tilde{x}(k)$ and $\tilde{x}(k+1)$ can be straightforwardly expressed as the components of $\zeta(k)$. Simple calculations show that

$$\begin{aligned}
x(k) &= \varrho_1\zeta(k),\\
\sum_{s=k-h_1}^{k-1} x(s) &= -x(k) + \sum_{s=k-h_1}^{k} x(s) = (-\varrho_1 + (h_1+1)\varrho_5)\,\zeta(k),\\
\sum_{s=k-h_2}^{k-h_1-1} x(s) &= -x(k-h_1) - x(k-h) + \sum_{s=k-h}^{k-h_1} x(s) + \sum_{s=k-h_2}^{k-h} x(s)\\
&= (-\varrho_2 - \varrho_3 + (h-h_1+1)\varrho_7 + (h_2-h+1)\varrho_9)\,\zeta(k),\\
\sum_{s=k-h_1}^{k-1} \delta_1(k,s)x(s) &= -x(k) + \sum_{s=k-h_1}^{k} \delta_1(k,s)x(s) = (-\varrho_1 + (h_1+1)\varrho_6)\,\zeta(k)
\end{aligned}$$

and

$$
\begin{aligned}
x(k+1) &= \Sigma\zeta(k),\\
\sum_{s=k+1-h_1}^{k} x(s) &= -x(k-h_1) + \sum_{s=k-h_1}^{k} x(s) = (-\varrho_2 + (h_1+1)\varrho_5)\zeta(k),\\
\sum_{s=k+1-h_2}^{k-h_1} x(s) &= -x(k-h) - x(k-h_2) + \sum_{s=k-h}^{k-h_1} x(s) + \sum_{s=k-h_2}^{k-h} x(s)\\
&= \left(-\varrho_3 - \varrho_4 + (h-h_1+1)\varrho_7^T + (h_2-h+1)\varrho_9^T\right)\zeta(k),\\
\sum_{s=k+1-h_1}^{k} \delta_1(k+1,s)x(s) &= \left(1+\frac{4}{h_1-1}\right)x(k-h_1) - \frac{2}{h_1-1}\sum_{s=k-h_1}^{k} x(s) + \sum_{s=k-h_1}^{k} \delta_1(k,s)x(s)\\
&= \left[\left(1+\frac{4}{h_1-1}\right)\varrho_2 - \frac{2(h_1+1)}{h_1-1}\varrho_5 + (h_1+1)\varrho_6\right]\zeta(k),
\end{aligned}
$$

where the matrix Σ is given in (12). The last component of $\tilde{x}(k)$ and $\tilde{x}(k+1)$ requires a more dedicated development. To achieve this goal, we first note that

$$
\begin{aligned}
(h_{12}-1)\sum_{s=k-h_2}^{k-h_1-1} \delta_4(k,s)x(s) =&(h_{12}-1)\sum_{s=k-h}^{k-h_1} \delta_4(k,s)x(s) + (h_{12}-1)\sum_{s=k-h_2}^{k-h} \delta_4(k,s)x(s) - (h_{12}-1)x(k-h_1)\\
&- (2(h_2-h-1) - h_{12}+1)x(k-h).
\end{aligned}
\tag{21}
$$

We need to find two expressions of $\delta_4(k,s)$, which depend on $\delta_2(k,s)$ and $\delta_3(k,s)$, respectively. Some calculations show

$$
\begin{aligned}
\delta_4(k,s) =&2\left(\frac{s-k+h_2-1}{h_{12}-1}\right) - 1 = 2\frac{(s-k+h-1)+(h_2-h)}{h_{12}-1} - \frac{(h_2-h)+(h-h_1-1)}{h_{12}-1}\\
=&2\frac{s-k+h-1}{h-h_1-1}\frac{h-h_1-1}{h_{12}-1} - \frac{h-h_1-1}{h_{12}-1} + \frac{h_2-h}{h_{12}-1}\\
=&\frac{h-h_1-1}{h_{12}-1}\delta_2(k,s) + \frac{h_2-h}{h_{12}-1},
\end{aligned}
\tag{22}
$$

and, similarly,

$$
\begin{aligned}
\delta_4(k,s) =&2\frac{h_2-h-1}{h_{12}-1}\frac{s-k+h_2-1}{h_2-h-1} - \frac{(h_2-h-1)+(h-h_1)}{h_{12}-1}\\
=&\frac{h_2-h-1}{h_{12}-1}\delta_3(k,s) - \frac{h-h_1}{h_{12}-1}.
\end{aligned}
\tag{23}
$$

Reinjecting (22) and (23) into (21) leads to

$$
\begin{aligned}
(h_{12}-1)\sum_{s=k-h_2}^{k-h_1-1} \delta_4(k,s)x(s) =&(h-h_1-1)\sum_{s=k-h}^{k-h_1} \delta_2(k,s)x(s) + (h_2-h)\sum_{s=k-h}^{k-h_1} x(s)\\
&+ (h_2-h-1)\sum_{s=k-h_2}^{k-h} \delta_3(k,s)x(s) - (h-h_1)\sum_{s=k-h_2}^{k-h} x(s) - (h_{12}-1)x(k-h_1)\\
&- (2(h_2-h-1) - h_{12}+1)x(k-h)\\
=&[0_{n\times 4n}\, I_n](G_1 + G(h))\zeta(k).
\end{aligned}
$$

In the same way, the last component of $\tilde{x}(k+1)$ can be presented as

$$
\begin{aligned}
(h_{12}-1)\sum_{s=k+1-h_2}^{k-h_1}\delta_4(k+1,s)x(s) =&(h_{12}-1)\sum_{s=k+1-h_2}^{k-h_1}\delta_4(k,s)x(s)-2\sum_{s=k+1-h_2}^{k-h_1}x(s)\\
=&(h_{12}-1)\sum_{s=k-h}^{k-h_1}\delta_4(k,s)x(s)+(h_{12}-1)\sum_{s=k-h_2}^{k-h}\delta_4(k,s)x(s)\\
&-(2(h_2-h-1)-h_{12}-1)\,x(k-h)+(h_{12}+3)x(k-h_2)-2\sum_{s=k-h}^{k-h_1}x(s)-2\sum_{s=k-h_2}^{k-h}x(s)\\
=&(h-h_1-1)\sum_{s=k-h}^{k-h_1}\delta_2(k,s)x(s)+(h_2-h-2)\sum_{s=k-h}^{k-h_1}x(s)\\
&+(h_2-h-1)\sum_{s=k-h_2}^{k-h}\delta_3(k,s)x(s)-(h-h_1+2)\sum_{s=k-h_2}^{k-h}x(s)\\
&-(2(h_2-h-1)-h_{12}-1)\,x(k-h)+(h_{12}+3)x(k-h_2)\\
=&[0_{n\times 4n}\, I_n](G_0+G(h))\zeta(k).
\end{aligned}
\tag{24}
$$

Hence, we finally obtain that $\tilde{x}(k)=(G_1+G(h))\zeta(k)$ and $\tilde{x}(k+1)=(G_0+G(h))\zeta(k)$. Thus, $\Delta V_1(k)$ writes

$$
\begin{aligned}
\Delta V_1(k) =&\tilde{x}^T(k+1)P\tilde{x}(k+1)-\lambda\tilde{x}^T(k)P\tilde{x}(k)\\
=&\zeta^T(k)\left[G_0^TPG_0-\lambda G_1^TPG_1+(1-\lambda)G^T(h)PG(h)+\mathrm{He}\left(G^T(h)P(G_0-\lambda G_1)\right)\right]\zeta(k).
\end{aligned}
\tag{25}
$$

Moreover, from the definition of the augmented vector $\zeta(k)$, one can see that the last four components can be seen as linear combination of the other components of $\zeta(k)$ because the relations $\eta_4(k) = (h-h_1+1)\eta_2(k)$ and $\eta_5(k)=(h_2-h+1)\eta_3(k)$ hold. Then, for any matrices N_1, N_2 in $\mathbb{R}^{14n\times 2n}$, it holds that

$$
2\zeta^T(k)\left(N_1g_1(h)+N_2g_2(h)\right)\zeta(k)=0. \tag{26}
$$

The computation of $\Delta V_2(k)$ and $\Delta V_3(k)$ yields

$$
\begin{aligned}
\Delta V_2(k) \le& x^T(k)S_1x(k)-\lambda^{h_1}x^T(k-h_1)(S_1-S_2)x(k-h_1)-\lambda^{h_2}x^T(k-h_2)S_2x(k-h_2)\\
=&\zeta^T(k)\hat{S}\zeta(k)
\end{aligned}
\tag{27}
$$

and

$$
\begin{aligned}
\Delta V_3(k) \le& \eta^T(k+1)\left(h_1^2R_1+h_{12}^2R_2\right)\eta(k+1)\\
&-h_1\lambda^{h_1}\sum_{s=k-h_1+1}^{k}\left|\sqrt{R_1}\eta(s)\right|^2-\lambda^{h_2}h_{12}\sum_{s=k-h_2+1}^{k-h_1}\left|\sqrt{R_2}\eta(s)\right|^2.
\end{aligned}
\tag{28}
$$

Then, by Lemma 1 and the definition of the matrices G_2 and $\tilde{R}_1$ in (12) and (17), respectively, we arrive to the following upper bound of the first summation in (28):

$$
-h_1\lambda^{h_1}\sum_{s=k-h_1+1}^{k}\left|\sqrt{R_1}\eta(s)\right|^2\le-\lambda^{h_1}\zeta^T(k)G_2^T\tilde{R}_1G_2\zeta(k). \tag{29}
$$

Applying again Lemma 1 to the last summation term of (28) yields

$$
\begin{aligned}
-\lambda^{h_2}h_{12}\sum_{s=k-h_2+1}^{k-h_1}\left|\sqrt{R_2}\eta(s)\right|^2 =&-\lambda^{h_2}h_{12}\sum_{s=k-h+1}^{k-h_1}\left|\sqrt{R_2}\eta(s)\right|^2-\lambda^{h_2}h_{12}\sum_{s=k-h_2+1}^{k-h}\left|\sqrt{R_2}\eta(s)\right|^2\\
\le&-\lambda^{h_2}\zeta^T(k)\Gamma^T\begin{bmatrix}\frac{h_{12}}{h-h_1}\tilde{R}_2 & 0\\ 0 & \frac{h_{12}}{h_2-h}\tilde{R}_2\end{bmatrix}\Gamma\zeta(k)\\
\le&-\zeta^T(k)\Phi_1(h(k))\zeta(k),
\end{aligned}
\tag{30}
$$

TABLE 1 Example 1: admissible upper bound of h_2 for different h_1

h_1	2	4	6	10	15	20	25	NoVs
Feng et al[33]	21	21	21	22	24	27	31	$9.5n^2 + 5.5n$
Kwon et al[34]	22	22	22	23	25	28	32	$27n^2 + 9n$
Hien and Trinh[30]	26	27	28	31	34	35	36	$20n^2 + 5n$
Theorem 1	27	29	30	33	35	37	38	$79.5n^2 + 4.5n$

where

$$\Phi_1(h(k)) = \lambda^{h_2}\Gamma^T\left(\begin{bmatrix} \tilde{R}_2 & X \\ X^T & \tilde{R}_2 \end{bmatrix} + \begin{bmatrix} \frac{h_2-h}{h_{12}}T_1 & 0 \\ 0 & \frac{h-h_1}{h_{12}}T_2 \end{bmatrix}\right)\Gamma, \tag{31}$$
$$T_1 = \tilde{R}_2 - X\tilde{R}_2^{-1}X^T, \qquad T_2 = \tilde{R}_2 - X^T\tilde{R}_2^{-1}X.$$

The latter inequality is guaranteed due to the refined reciprocally convex combination lemma.[18]
From (25)-(30), it follows that

$$\Delta V(k) \leq \zeta^T(k)\Phi(h)\zeta(k),$$

where $\Phi(h) = \Phi_0(h) - \Phi_1(h) + (1-\lambda)G^T(h)PG(h) + (\Sigma - \varrho_1)^T H(\Sigma - \varrho_1)$ with $\Phi_0(h)$ and $\Phi_1(h)$ given in (17) and (31), respectively. By Schur complement, the two matrix inequalities of (15) are equivalent to $\Phi(h_i) \prec 0, i = 1, 2$, and thus guarantee $\Delta V(k) < 0$, implying exponential stability with the decay rate λ of system (10) for all time-varying delays in the interval $[h_1, h_2]$. □

Remark 3. In order to fully benefit from the summation inequality of (11), the augmented term V_1 in (19) not only includes the signals $x(k)$, $\sum_{s=k-h_1}^{k-1} x(s)$ and $\sum_{s=k-h_2}^{k-h_1-1} x(s)$ that were adopted in the work of Seuret et al[32] but also includes two additional signals $\sum_{s=k-h_1}^{k-1} \delta_1(k,s)x(s)$ and $(h_{12}-1)\sum_{s=k-h_2}^{k-h_1-1} \delta_4(k,s)x(s)$. This state augmentation allows achieving less conservative stability criteria. More recently, new summation inequalities in double form has been proposed in the work of Hien and Trinh.[30] Therefore, the condition of Theorem 1 could be further improved by employing generalized summation inequalities and Lyapunov functional with triple summation terms.[30]

A widely used numerical example is taken from the literature to make a comparison with some results recently reported in the existing works. Consider the following much-studied system (10) with:

$$A = \begin{bmatrix} 0.8 & 0.0 \\ 0.05 & 0.9 \end{bmatrix}, \quad A_1 = \begin{bmatrix} -0.1 & 0.0 \\ -0.2 & -0.1 \end{bmatrix}.$$

Table 1 shows that the maximum allowable delays h_2 for several values of h_1 obtained by Theorem 1 with $\lambda = 1$ are less conservative than those obtained by various recent methods from the literature. As usual, the reduction of the conservatism of Theorem 1 over existing results is at the price of additional decision variables, showing again a trade-off between the improvement and the numerical complexity.[19]

5 | NCSs UNDER TOD SCHEDULING PROTOCOL

Based on the exponential delay-dependent analysis of Theorem 1, in this section, we derive the exponential stability criteria of (8)-(9) under (4), the definition of which is given as follows.

Definition 1. For any initial condition $x_{t_0} \in \underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{\tau_M+1 \text{ times}}$, if there exist constants $b > 0$ and $0 < \kappa < 1$ such that the solutions of the hybrid system (8)-(9) under (4) satisfy

$$|x(k)|^2 \leq b\kappa^{2(k-t_0)}\left\{\|x_{t_0}\|_c^2 + |e(t_0)|^2\right\},$$

and

$$\left\{|e(k)|^2\right\} \leq b\kappa^{2(k-t_0)}\left\{\|x_{t_0}\|_c^2 + |e(t_0)|^2\right\},$$

where $\|x_{t_0}\|_c = \sup_{t_0-\tau_M \leq s \leq t_0} |x(s)|$, then the hybrid system (8)-(9) under (4) is said to be exponentially stable.

We apply the following discrete-time Lyapunov functional to system (8)-(9) under (4) for the exponential stability of systems with time-varying delays from the maximum delay interval $[h_m, \tau_M]$:

$$\begin{aligned} V_e(k) =& \tilde{V}(k) + \sum_{i=1}^{N} e_i^T(t_p) Q_i e_i(t_p), \\ \tilde{V}(k) =& V(k) + V_G(k), \quad k \in [t_p, t_{p+1} - 1], \quad k, p \in \mathbb{Z}^+, \end{aligned} \tag{32}$$

where $V(k)$ is given by (19) with h_1, h_2, and h_{12} replaced by h_m, τ_M, and $\tau_M - h_m$, respectively, and

$$V_G(k) = (\tau_M - h_m) \sum_{i=1}^{N} \sum_{s=s_p}^{k} \lambda^{k-s} \left| \sqrt{G_i} C_i \eta(s) \right|^2,$$

with $\eta(k) = x(k) - x(k-1)$, $0 < \lambda < 1$, $G_i > 0$, $Q_i > 0$, $i = 1, \ldots, N$.

Remark 4. In order to reduce the numerical complexity of resulting stability conditions, one may follow the work of Freirich and Fridman[13] and include C in the integral terms of Lyapunov functional (32) with reduced-order matrices $S_i, R_i, i = 1, 2$, which will be the decision variables of the resulting matrix inequalities.

The term V_G was introduced in the work of Liu et al[10] to deal with the delays in the reset conditions

$$V_G(t_{p+1}) - \lambda V_G(t_{p+1} - 1) \leq \sum_{i=1}^{N} \left[(\tau_M - h_m) \left| \sqrt{G_i} C_i \eta(t_{p+1}) \right|^2 \right] - \sum_{i=1}^{N} \lambda^{\tau_M} \left| \sqrt{G_i} C_i \left[x(s_{p+1}) - x(s_p) \right] \right|^2.$$

To ease the presentation, for $i \in \mathcal{I}_N$, we will use in this section the following notations:

$$\begin{aligned} \tilde{\Sigma}^i =& \begin{bmatrix} A - I & 0_{n\times n} & A_1 & 0_{n\times 19n} & \tilde{F}_0^i \end{bmatrix}, \\ \tilde{F}_0^i =& \begin{bmatrix} B_1 & \cdots & B_{j\,|\,j\neq i} \cdots B_N \end{bmatrix}, \\ \tilde{H} =& h_m^2 R_0 + (\tau_M - h_m)^2 R_1 + (\tau_M - h_m) \sum_{l=1}^{N} C_l^T G_l C_l. \end{aligned} \tag{33}$$

Thanks to Theorem 1 for the exponential delay-dependent analysis, we prove the following theorem on the exponential stability of (8)-(9) under (4).

Theorem 2. *For any given scalar $0 < \lambda < 1$, integers $0 \leq h_m < \tau_M$, and K_i, $i = 1, \ldots, N$, assume that there exist matrices $P \in \mathbb{S}_+^{5n}$, $S_1, S_2, R_1, R_2 \in \mathbb{S}_+^n$, $Q_i, U_i, G_i \in \mathbb{S}_+^{n_i}$, $i = 1, \ldots, N$, $N_1, N_2 \in \mathbb{R}^{14n\times 2n}$, and a matrix $X \in \mathbb{R}^{3n\times 3n}$ such that the following matrix inequalities are feasible:*

$$\Omega_i \triangleq \begin{bmatrix} \Gamma_i & Q_i \\ * & Q_i - \lambda^{\tau_M} G_i \end{bmatrix} < 0, \qquad i = 1, \ldots, N, \tag{34}$$

$$\begin{bmatrix} \begin{bmatrix} \tilde{\Phi}_j & 0 \\ * & \phi_i \end{bmatrix} & (\tilde{\Sigma}^i)^T \tilde{H} \\ * & -\tilde{H} \end{bmatrix} < 0, \quad j = 1, 2, \quad i = 1, \ldots, N, \tag{35}$$

with $\tilde{\Phi}_j$, $j = 1, 2$, defined in (16) and

$$\begin{aligned} \Gamma_i =& \frac{-\lambda - (1-\lambda)(\tau_M - h_m)}{N-1} Q_i + \left(1 + \frac{1}{\tau_M - h_m}\right) U_i, \\ \phi_i =& \operatorname{diag}\left\{ W_1, \ldots, W_{j\,|\,j\neq i}, \ldots, W_N \right\}, \\ W_i =& -\frac{1}{\tau_M - h_m} U_i + (1-\lambda) Q_i, \end{aligned} \tag{36}$$

and other notations given by (33). Then, we have

(i) $V_e(k)$ satisfies the following inequalities along (8)-(9) for $k \in [t_p, t_{p+1} - 2]$:

$$\Theta_1(k) \triangleq V_e(k+1) - \lambda V_e(k) - \frac{1}{\tau_M - h_m} \sum_{i=1, i\neq i_p^*}^{N} \left| \sqrt{U_i} e_i(t_p) \right|^2 - (1-\lambda) \left| \sqrt{Q_{i_p^*}} e_{i_p^*}(t_p) \right|^2 \leq 0; \tag{37}$$

(ii) at $k = t_{p+1} - 1$,

$$\Theta_2 \triangleq V_e(t_{p+1}) - \lambda V_e(t_{p+1}-1) + \sum_{i=1, i\neq i_p^*}^{N} \left|\sqrt{U_i} e_i(t_p)\right|^2 + (1-\lambda)(\tau_M - h_m)\left|\sqrt{Q_{i_p^*}} e_{i_p^*}(t_p)\right|^2 \leq 0; \tag{38}$$

(iii) the following bounds

$$\lambda_{\min}(P)|x(k)|^2 \leq \tilde{V}(k) \leq V_e(k) \leq \lambda^{k-t_0} V_e(t_0), \tag{39}$$

and

$$\sum_{i=1}^{N} \left|\sqrt{Q_i} e_i(k)\right|^2 \leq \tilde{c}\lambda^{k-t_0} V_e(t_0), \quad k \geq t_0, \quad k \in \mathbb{N}, \tag{40}$$

with $V_e(t_0) = \tilde{V}(t_0) + \sum_{i=1}^{N} |\sqrt{Q_i} e_i(t_0)|^2$ *and* $\tilde{c} = \lambda^{-(\tau_M - h_m)}$*, are valid for the solutions of (4), (8), and (9) initialized by* $x_{t_0} \in \underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{\tau_M+1 \quad \text{times}}$, $e(t_0) \in \mathbb{R}^{n_y}$.

Moreover, the hybrid system (8)-(9) under (4) is exponentially stable.

Proof. The proof consists in proving each item separately.

Proof of (i). First, from functional (32), it holds that, for $k \in [t_p, t_{p+1} - 2]$,

$$\Theta_1(k) \leq \tilde{V}(k+1) - \lambda\tilde{V}(k) + (\tau_M - h_m)\sum_{i=1}^{N} |\sqrt{G_i} C_i \eta(k+1)|^2 + \sum_{i=1, i\neq i_p^*}^{N} |\sqrt{W_i} e_i(t_p)|^2 \triangleq \Psi(k). \tag{41}$$

Therefore, $\Theta_1(k) \leq 0$ of (37) is satisfied if $\Psi(k) \leq 0$ for $k \in [t_p, t_{p+1} - 2]$.

Let $i_p^* = i \in \mathcal{I}_N$ and define $\xi_i(k) = [\zeta^T(k), \ \bar{\xi}_i^T(k)]^T$, where $\zeta(k)$ is defined in (20) and $\bar{\xi}_i(k) = \text{col}\{e_1(k), \ldots, e_j(k)_{|j\neq i}, \ldots, e_N(k)\}$, $i = 1, \ldots, N$. Following the arguments of Theorem 1 for the exponential delay-dependent analysis, we arrive at $\Psi(k) \leq 0$ for $k \in [t_p, t_{p+1} - 2]$ if (35) is feasible.

This completes the proof of (i).

Proof of (ii). From (32) and (41), we have

$$\begin{aligned}
\Theta_2 \leq & \tilde{V}(t_{p+1}) - \lambda\tilde{V}(t_{p+1}-1) + (\tau_M - h_m)\sum_{i=1}^{N}\left|\sqrt{G_i} C_i \eta(t_{p+1})\right|^2 \\
& - \sum_{i=1}^{N} \lambda^{\tau_M}\left|\sqrt{G_i} C_i \left[x(s_{p+1}) - x(s_p)\right]\right|^2 + \sum_{i=1}^{N}\left[\left|\sqrt{Q_i} e_i(t_{p+1})\right|^2 - \lambda\left|\sqrt{Q_i} e_i(t_p)\right|^2\right] \\
& + \sum_{i=1, i\neq i_p^*}^{N}\left|\sqrt{U_i} e_i(t_p)\right|^2 + (1-\lambda)(\tau_M - h_m)\left|\sqrt{Q_{i_p^*}} e_{i_p^*}(t_p)\right|^2 \\
\leq & \Psi(t_{p+1}-1) + \frac{1}{\tau_M - h_m}\sum_{i=1, i\neq i_p^*}^{N}\left|\sqrt{U_i} e_i(t_p)\right|^2 - (1-\lambda)\sum_{i=1, i\neq i_p^*}^{N}\left|\sqrt{Q_i} e_i(t_p)\right|^2 \\
& - \sum_{i=1}^{N} \lambda^{\tau_M}\left|\sqrt{G_i} C_i \left[x(s_{p+1}) - x(s_p)\right]\right|^2 + \sum_{i=1}^{N}\left[\left|\sqrt{Q_i} e_i(t_{p+1})\right|^2 - \lambda\left|\sqrt{Q_i} e_i(t_p)\right|^2\right] \\
& + \sum_{i=1, i\neq i_p^*}^{N}\left|\sqrt{U_i} e_i(t_p)\right|^2 + (1-\lambda)(\tau_M - h_m)\left|\sqrt{Q_{i_p^*}} e_{i_p^*}(t_p)\right|^2.
\end{aligned}$$

Note that, under the TOD protocol,

$$-\left|\sqrt{Q_{i_p^*}} e_{i_p^*}(t_p)\right|^2 \leq -\frac{1}{N-1}\sum_{i=1, i\neq i_p^*}^{N}\left|\sqrt{Q_i} e_i(t_p)\right|^2.$$

From (41) and (34), we have $\Psi(t_{p+1}-1) \le 0$ and $\lambda^{\tau_M} G_{i_p^*} - Q_{i_p^*} C_{i_p^*} > 0$, respectively. Denote $\varsigma_i = \text{col}\{e_i(t_p), C_i[x(s_{p+1}) - x(s_p)]\}$. Then, employing (9), we arrive at

$$\Theta_2 \le \Psi(t_{p+1}-1) - \left|\sqrt{\lambda^{\tau_M} G_{i_p^*} - Q_{i_p^*}} C_{i_p^*} \left[x(s_{p+1}) - x(s_p)\right]\right|^2 + \sum_{i=1, i\neq i_p^*}^{N} \varsigma_i^T \Omega_i \varsigma_i \le 0,$$

which yields (38). This completes the proof of (ii).

Proof of (iii). The next step is to prove (39) and (40). By the comparison principle, for $k \in [t_p, t_{p+1}-1]$, the inequality (37) implies

$$V_e(k) \le \lambda^{k-t_p} V_e(t_p) + \sum_{i=1, i\neq i_p^*}^{N} \left\{\left|\sqrt{U_i} e_i(t_p)\right|^2\right\} + (1-\lambda)(\tau_M - h_m)\left|\sqrt{Q_{i_p^*}} e_{i_p^*}(t_p)\right|^2. \tag{42}$$

Note that (34) guarantees $0 < (1-\lambda)(\tau_M - h_m) < \lambda < 1$ and $U_i < (1 + \frac{1}{\tau_M - h_m})U_i < \frac{\lambda-(1-\lambda)(\tau_M-h_m)}{N-1} Q_i < Q_i$, $i = 1, \dots, N$. Hence,

$$\tilde{V}(k) \le \lambda^{k-t_p} V_e(t_p), \quad k \in [t_p, t_{p+1}-1]. \tag{43}$$

On the other hand, the inequalities (38) and (42) with $k = t_{p+1} - 1$ imply

$$\begin{aligned} V_e(t_{p+1}) \le & \lambda V_e(t_{p+1}-1) - \sum_{i=1, i\neq i_p^*}^{N} \left|\sqrt{U_i} e_i(t_p)\right|^2 - (1-\lambda)(\tau_M - h_m)\left|\sqrt{Q_{i_p^*}} e_{i_p^*}(t_p)\right|^2 \\ \le & \lambda^{t_{p+1}-t_p} V_e(t_p) - (1-\lambda) \sum_{i=1, i\neq i_p^*}^{N} \left|\sqrt{U_i} e_i(t_p)\right|^2 - (1-\lambda)^2(\tau_M - h_m)\left|\sqrt{Q_{i_p^*}} e_{i_p^*}(t_p)\right|^2 \\ \le & \lambda^{t_{p+1}-t_p} V_e(t_p). \end{aligned}$$

Then, a recursive argument allows us to conclude that, for all $p \in \mathbb{Z}$, we have

$$V_e(t_{p+1}) \le \lambda^{t_{p+1}-t_{p-1}} V_e(t_{p-1}) \le \lambda^{t_{p+1}-t_0} V_e(t_0). \tag{44}$$

Substituting in (44) $p + 1$ for p and taking into account (43), we arrive at (39), which yields exponential stability of (8)-(9) under (4) because $\lambda_{\min}(P)|x(k)|^2 \le \tilde{V}(k)$, $V(t_0) \le \delta \|x_{t_0}\|_c^2$ for some scalar $\delta > 0$. Moreover, (44) with $p + 1$ replaced by p implies (40) because $\lambda^{t_p - t_0} = \lambda^{k-t_0} \lambda^{t_p - k} \le \tilde{c} \lambda^{k-t_0}$ for $k \in [t_p, t_{p+1}-1]$. This completes the proof of (iii). □

Remark 5. The exponential stability of system (8)-(9) under (4) can be alternatively analyzed via the Lyapunov functional $\tilde{V}_e(k) = V_e(k) + V_W(k)$, where $V_e(k)$ is given by (32) and

$$V_W(k) = (1-\lambda)(t_p - k)\left|\sqrt{Q_{i_p^*}} e_{i_p^*}(t_p)\right|^2 + \frac{t_p - k}{t_{p+1} - t_p} \sum_{i=1, i\neq i_p^*}^{N} \left|\sqrt{U_i} e_i(t_p)\right|^2, \quad k \in [t_p, t_{p+1}-1].$$

The negative term $V_W(k)$ is a discrete-time counterpart of piecewise continuous in time term that was recently employed in the work of Freirich and Fridman[13] to simplify the exponential stability analysis of the hybrid system. Under the conditions (34) and (35), it holds that $\tilde{V}_e(k)$ is positive for $k \ge t_0, k \in \mathbb{Z}^+$, ie, $\tilde{V}_e(k) \ge \rho(|x(k)|^2 + |e(k)|^2)$ with some $\rho > 0$, and that $\tilde{V}_e(k+1) - \lambda \tilde{V}_e(k) \le 0$, $k \in [t_p, t_{p+1}-1]$. These two inequalities imply the exponential stability of system (8)-(9) under (4).

Remark 6. For discrete-time NCSs under TOD scheduling protocol, lemma 2 of the work of Liu and Fridman[17] guarantees only partial stability of the closed-loop system with $N = 2$ sensor nodes, whereas Theorem 2 in this paper guarantees that (39) gives a bound not only on $x(k)$ but also on $e_i(k)$, $i = 1, \dots, N$. That is why Theorem 2 assesses stability of system (8)-(9) under (4) with respect to the full state $\text{col}\{x(k), e(k)\}$.

6 | NCSs UNDER IID SCHEDULING PROTOCOL

In this section, we first reformulate system (8) and (9) under iid scheduling protocol (5) as a stochastic impulsive system with the system matrices having stochastic parameters with Bernoulli distributions and then derive exponential mean-square stability criteria by virtue of Theorem 1.

6.1 | Stochastic hybrid time-delay model with Bernoulli distributed parameters

Following the work of Yue et al,[25] we introduce the indicator functions

$$\pi_{\{\sigma_p^*=i\}} = \begin{cases} 1, & \sigma_p^* = i \\ 0, & \sigma_p^* \neq i, \end{cases} \quad i \in \mathcal{I}_N, \quad p \in \mathbb{Z}^+.$$

Thus, from (5), it follows that

$$\mathbb{E}\{\pi_{\{\sigma_p^*=i\}}\} = \mathbb{E}\left\{[\pi_{\{\sigma_p^*=i\}}]^2\right\} = \text{Prob}\left\{\sigma_p^* = i\right\} = \beta_i,$$

$$\mathbb{E}\left\{[\pi_{\{\sigma_p^*=i\}} - \beta_i]\left[\pi_{\{\sigma_p^*=j\}} - \beta_j\right]\right\} = \begin{cases} -\beta_i\beta_j, & i \neq j, \\ \beta_i(1-\beta_i), & i = j. \end{cases}$$

Therefore, the stochastic impulsive system model (8)-(9) under (5) can be rewritten as

$$\begin{cases} x(k+1) = Ax(k) + A_1 x(s_p) + \sum_{i=1}^{N}(1 - \pi_{\{\sigma_p^*=i\}})B_i e_i(t_p), \\ e(k+1) = e(k), \quad k \in [t_p, t_{p+1}-2], \quad k \in \mathbb{Z}^+, \end{cases} \tag{45}$$

with the delayed reset system for $k = t_{p+1} - 1$

$$\begin{cases} x(t_{p+1}) = Ax(t_{p+1}-1) + A_1 x(s_p) + \sum_{i=1}^{N}(1 - \pi_{\{\sigma_p^*=i\}})B_i e_i(t_p), \\ e_i(t_{p+1}) = (1 - \pi_{\{\sigma_p^*=i\}})e_i(t_p) + C_i\left[x(s_p) - x(s_{p+1})\right], \quad i = 1, \dots, N. \end{cases} \tag{46}$$

Definition 2. The hybrid system (45)-(46) is said to be exponentially mean-square stable if there exist constants $b > 0$ and $0 < \kappa < 1$ such that, for initial condition $x_{t_0} \in \underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{\tau_M+1 \text{ times}}$, the solutions of the hybrid system (45)-(46) satisfy

$$\mathbb{E}\left\{|x(k)|^2\right\} \leq b\kappa^{2(k-t_0)}\mathbb{E}\left\{\|x_{t_0}\|_c^2 + |e(t_0)|^2\right\}, \quad k \geq t_0,$$

and

$$\mathbb{E}\left\{|e(k)|^2\right\} \leq b\kappa^{2(k-t_0)}\mathbb{E}\left\{\|x_{t_0}\|_c^2 + |e(t_0)|^2\right\}, \quad k \geq t_0.$$

The objective of this section is to derive condition for exponential mean-square stability of the hybrid system (45)-(46).

6.2 | Exponential mean-square stability of NCSs under iid scheduling protocol

The stability analysis of (45)-(46) will be based on discrete-time Lyapunov functional (32).

The term V_G satisfies for $k = t_{p+1} - 1$

$$\mathbb{E}\left\{V_G(t_{p+1}) - \lambda V_G(t_{p+1}-1)\right\} \leq (\tau_M - h_m)\sum_{i=1}^{N}\mathbb{E}\left\{\left|\sqrt{G_i}C_i\eta(t_{p+1})\right|^2\right\} - \sum_{i=1}^{N}\lambda^{\tau_M}\mathbb{E}\left\{\left|\sqrt{G_i}C_i\left[x(s_{p+1}) - x(s_p)\right]\right|^2\right\}.$$

Following the arguments for network-based stabilization under TOD scheduling protocol, we arrive at the following.

Theorem 3. *For any given scalar* $0 < \lambda < 1$, *integers* $0 \leq h_m < \tau_M$, *and* K_i, $i = 1, \dots, N$, *assume that there exist matrices* $P \in \mathbb{S}_+^{5n}$, $S_1, S_2, R_1, R_2 \in \mathbb{S}_+^n$, $Q_i, U_i, G_i \in \mathbb{S}_+^{n_i}$, $i = 1, \dots, N$, $N_1, N_2 \in \mathbb{R}^{14n\times 2n}$, *and a matrix* $X \in \mathbb{R}^{3n\times 3n}$ *such that the following matrix inequalities are feasible for* $k \in [t_p, t_{p+1} - 1]$:

$$\hat{\Omega}_i \triangleq \begin{bmatrix} \hat{\Gamma}_i & (1-\beta_i)Q_i \\ * & Q_i - \lambda^{\tau_M}G_i \end{bmatrix} < 0, \qquad i = 1, \dots, N, \tag{47}$$

$$\begin{bmatrix} \begin{bmatrix} \tilde{\Phi}_j & 0 \\ * & -\psi \end{bmatrix} & \Xi^T \tilde{H} & \hat{\Xi}^T \tilde{H} \\ * & -\tilde{H} & 0 \\ * & * & -\beta \tilde{H} \end{bmatrix} < 0, \quad j = 1, 2, \tag{48}$$

are feasible, where

$$\begin{aligned} \hat{\Gamma}_i &= -\lambda Q_i + (1-\beta_i) Q_i + \left(1 + \frac{1}{\tau_M - h_m}\right) U_i. \\ \Xi &= [A - I \quad 0_{n\times n} \quad A_1 \quad 0_{n\times 19n} \quad \Xi_0], \ \Xi_0 = [(1-\beta_1)B_1 \quad \cdots \quad (1-\beta_N)B_N], \\ \Xi_1 &= \left[0_{n\times 22n} - B_1 \quad 0_{n\times(n_y - n_1)}\right], \ \Xi_2 = \left[0_{n\times(22n+n_1)} - B_2 \quad 0_{n\times(n_y-n_1-n_2)}\right], \ \dots, \\ \Xi_N &= \left[0_{n\times(22n+n_y-n_N)} - B_N\right], \ \Xi_j \in \mathbb{R}^{n\times(22n+n_y)}, \quad \hat{\Xi} = \left[\Xi_1^T \quad \cdots \quad \Xi_N^T\right]^T, \\ \beta &= \mathrm{diag}\left\{\beta_1^{-1}, \dots, \beta_N^{-1}\right\}, \ \psi = \frac{1}{\tau_M - h_m} \mathrm{diag}\{U_1, \dots, U_N\}, \end{aligned} \tag{49}$$

with the notations $\tilde{\Phi}_j, j = 1, 2$, and $\tilde{H}$ given by (16) and (33), respectively. Then, we have the following.

(i) For $k \in [t_p, t_{p+1} - 2]$, $V_e(k)$ satisfies the following inequalities along (45)-(46):

$$\hat{\Theta}_1(k) \triangleq \mathbb{E}\left\{ V_e(k+1) - \lambda V_e(k) - \frac{1}{\tau_M - h_m} \sum_{i=1}^{N} \left| \sqrt{U_i} e_i(t_p) \right|^2 \right\} \le 0. \tag{50}$$

(ii) At $k = t_{p+1} - 1$,

$$\hat{\Theta}_2 \triangleq \mathbb{E}\left\{ V_e(t_{p+1}) - \lambda V_e(t_{p+1} - 1) + \sum_{i=1}^{N} \left| \sqrt{U_i} e_i(t_p) \right|^2 \right\} \le 0. \tag{51}$$

(iii) For the solutions of (45)-(46) initialized by $x_{t_0} \in \underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{\tau_M + 1 \ \text{times}}$, $e(t_0) \in \mathbb{R}^{n_y}$, the following bounds

$$\mathbb{E}\{\tilde{V}(k)\} \le \mathbb{E}\{V_e(k)\} \le \lambda^{k-t_0} \mathbb{E}\{V_e(t_0)\}, \quad k \ge t_0, \quad k \in \mathbb{N} \tag{52}$$

and

$$\sum_{i=1}^{N} \mathbb{E}\left\{ \left| \sqrt{Q_i} e_i(k) \right|^2 \right\} \le \tilde{c} \lambda^{k-t_0} \mathbb{E}\{V_e(t_0)\} \tag{53}$$

hold with $V_e(t_0) = \tilde{V}(t_0) + \sum_{i=1}^{N} |\sqrt{Q_i} e_i(t_0)|^2$ and $\tilde{c} = \lambda^{-(\tau_M - h_m)}$.

Consequently, the exponential mean-square stability of (45)-(46) is guaranteed.

Proof. Proof of (i). First, for $k \in [t_p, t_{p+1} - 2]$, it holds that, from $V_e(k)$ of (32),

$$\hat{\Theta}_1(k) \le \mathbb{E}\left\{ \tilde{V}(k+1) - \lambda \tilde{V}(k) + (\tau_M - h_m) \sum_{i=1}^{N} \left| \sqrt{G_i} C_i \eta(k+1) \right|^2 - \frac{1}{\tau_M - h_m} \sum_{i=1}^{N} \left| \sqrt{U_i} e_i(t_p) \right|^2 \right\} \triangleq \hat{\Psi}(k). \tag{54}$$

Therefore, $\hat{\Theta}_1(k) \le 0$ (ie, (50)) holds if $\hat{\Psi}(k) \le 0$ is satisfied.

Consider $k \in [t_p, t_{p+1} - 1]$, $\quad p \in \mathbb{Z}^+$, and define $\hat{\xi}(k) = [\zeta^T(k), 0_{1\times 8n}, e_1^T(k), \dots, e_N^T(k)]^T$. It can be shown from (45) that

$$\begin{aligned} \eta(k+1) =& \Xi \hat{\xi}(k) + \sum_{i=1}^{N} \left[\pi_{\{\sigma_p^* = i\}} - \beta_i \right] \Xi_i \hat{\xi}(k), \\ \left| \sqrt{\tilde{H}} \eta(k+1) \right|^2 =& \left| \sqrt{\tilde{H}} \Xi \hat{\xi}(k) \right|^2 + 2 \sum_{i=1}^{N} \left[\pi_{\{\sigma_p^* = i\}} - \beta_i \right] \hat{\xi}^T(k) \Xi^T \tilde{H} \Xi_i \hat{\xi}(k) \\ &+ \sum_{i,j=1, i\ne j}^{N} \left[\pi_{\{\sigma_p^* = i\}} - \beta_i \right] \left[\pi_{\{\sigma_p^* = j\}} - \beta_j \right] \hat{\xi}^T(k) \Xi_i^T \tilde{H} \Xi_j \hat{\xi}(k) + \sum_{i=1}^{N} \left[\pi_{\{\sigma_p^* = i\}} - \beta_i \right]^2 \left| \sqrt{\tilde{H}} \Xi_i \hat{\xi}(k) \right|^2, \end{aligned} \tag{55}$$

with Ξ and $\Xi_i, i = 1, \dots, N$ given by (49).

Following the proof of Theorem 2, calculating the difference $\tilde{V}(k+1) - \lambda\tilde{V}(k)$ of along (45) and taking the mathematical expectation, we have $\mathbb{E}\{\hat{\Psi}(k)\} \leq 0$ for $k \in [t_p, t_{p+1} - 2]$ if the matrix inequalities (48) with $j = 1, 2$ hold. This completes the proof of (i).

Proof of (ii). From $V_e(k)$ of (32) and (54), it follows that

$$\begin{aligned}\hat{\Theta}_2 \leq & \mathbb{E}\left\{ V(t_{p+1}) - \lambda V(t_{p+1}-1) + (\tau_M - h_m)\sum_{i=1}^{N}\left|\sqrt{G_i}C_i\eta(t_{p+1})\right|^2 - \sum_{i=1}^{N}\lambda^{\tau_M}\left|\sqrt{G_i}C_i\left[x(s_{p+1}) - x(s_p)\right]\right|^2 \right.\\ & \left. + \sum_{i=1}^{N}\left[\left|\sqrt{Q_i}e_i(t_{p+1})\right|^2 - \lambda\left|\sqrt{Q_i}e_i(t_p)\right|^2\right] + \sum_{i=1}^{N}\left|\sqrt{U_i}e_i(t_p)\right|^2\right\}\\ = & \hat{\Psi}(t_{p+1}-1) + \mathbb{E}\left\{\frac{1}{\tau_M - h_m}\sum_{i=1}^{N}\left|\sqrt{U_i}e_i(t_p)\right|^2 - \sum_{i=1}^{N}\lambda^{\tau_M}\left|\sqrt{G_i}C_i\left[x(s_{p+1}) - x(s_p)\right]\right|^2\right.\\ & \left. + \sum_{i=1}^{N}\left[\left|\sqrt{Q_i}e_i(t_{p+1})\right|^2 - \lambda\left|\sqrt{Q_i}e_i(t_p)\right|^2\right] + \sum_{i=1}^{N}\left|\sqrt{U_i}e_i(t_p)\right|^2\right\}.\end{aligned}$$

Note that $\hat{\Psi}(t_{p+1}-1) \leq 0$ and

$$\begin{aligned}&\mathbb{E}\left\{e_i^T(t_{p+1})Q_ie_i(t_{p+1})\right\}\\ =&\mathbb{E}\left\{\left|\sqrt{Q_i}\left[\left(1 - \pi_{\{\sigma_p^*=i\}}\right)e_i(t_p) + C_ix(s_p) - C_ix(s_{p+1})\right]\right|^2\right\}\\ =&\mathbb{E}\left\{(1-\beta_i)e_i^T(t_p)Q_ie_i(t_p) + 2(1-\beta_i)e_i^T(t_p)Q_iC_i\left[x(s_p) - x(s_{p+1})\right]\right.\\ &\left. + \left|\sqrt{Q_i}C_i\left[x(s_p) - x(s_{p+1})\right]\right|^2\right\}, \quad i = 1, \ldots, N.\end{aligned}$$

Denote $\hat{\varsigma}_i = \text{col}\{e_i(t_p), C_i[x(s_p) - x(s_{p+1})]\}$. Then, employing (46), we arrive at

$$\hat{\Theta}_2 \leq \hat{\Psi}(t_{p+1}-1) + \sum_{i=1}^{N}\mathbb{E}\left\{\hat{\varsigma}_i^T\hat{\Omega}_i\hat{\varsigma}_i\right\} \leq 0,$$

which yields (51). This completes the proof of (ii).

Proof of (iii). The objective of the next step is to prove (52) and (53). By the comparison principle, for $k \in [t_p, t_{p+1}-1]$, the inequality (50) implies

$$\mathbb{E}\{V_e(k) \leq \lambda^{k-t_p}\mathbb{E}\{V_e(t_p) + \sum_{i=1}^{N}\mathbb{E}\left\{\left|\sqrt{U_i}e_i(t_p)\right|^2\right\}. \tag{56}$$

Note that (47) yields

$$\begin{aligned}U_i &< (1 + \frac{1}{\tau_M - h_m})U_i\\ &< [\lambda - (1-\beta_i)]\,Q_i < Q_i, \quad i = 1, \ldots, N.\end{aligned}$$

Hence, for $k \in [t_p, t_{p+1} - 1]$, it holds that

$$\mathbb{E}\left\{\tilde{V}(k)\right\} \leq \lambda^{k-t_p}\mathbb{E}\left\{V_e(t_p)\right\}. \tag{57}$$

On the other hand, inequalities (51) and (56) with $k = t_{p+1} - 1$ imply

$$\begin{aligned}\mathbb{E}\left\{V_e(t_{p+1})\right\} \leq & \lambda\mathbb{E}\left\{V_e(t_{p+1}-1)\right\} - \sum_{i=1}^{N}\mathbb{E}\left\{\left|\sqrt{U_i}e_i(t_p)\right|^2\right\}\\ \leq & \lambda^{t_{p+1}-t_p}\mathbb{E}\left\{V_e(t_p)\right\} - (1-\lambda)\sum_{i=1}^{N}\mathbb{E}\left\{\left|\sqrt{U_i}e_i(t_p)\right|^2\right\}\\ \leq & \lambda^{t_{p+1}-t_p}\mathbb{E}\left\{V_e(t_p)\right\}.\end{aligned}$$

Then, we have

$$\mathbb{E}\left\{V_e(t_{p+1})\right\} \leq \lambda^{t_{p+1}-t_{p-1}}\mathbb{E}\left\{V_e(t_{p-1})\right\} \leq \lambda^{t_{p+1}-t_0}\mathbb{E}\left\{V_e(t_0)\right\}. \tag{58}$$

Replacing in (58) $p+1$ by p and using (57), we arrive at (52), which yields the exponential mean-square stability of (45)-(46) because the inequalities

$$\lambda_{\min}(P)\mathbb{E}\left\{|x(k)|^2\right\} \leq \mathbb{E}\left\{\tilde{V}(k)\right\}, \quad \mathbb{E}\left\{V(t_0)\right\} \leq \delta\mathbb{E}\left\{\|x_{t_0}\|_c^2\right\}$$

hold for some scalar $\delta > 0$. Moreover, (58) with $p+1$ replaced by p implies (53) because $\lambda^{t_p-t_0} = \lambda^{k-t_0}\lambda^{t_p-k} \leq \tilde{c}\lambda^{k-t_0}$ for $k \in [t_p, t_{p+1}-1]$. This completes the proof of (iii). □

Remark 7. To simplify the exponential mean-square stability analysis of the hybrid system (45)-(46), we can follow Remark 5 and adopt Lyapunov functionals of the form $\hat{V}_e(k) = V_e(k) + V_U(k)$, where $V_e(k)$ is shown in (32) and the term $V_U(k)$ is negative and is given by

$$V_U(k) = \frac{t_p - k}{t_{p+1} - t_p}\sum_{i=1}^{N}\left|\sqrt{U_i}e_i(t_p)\right|^2, \quad k \in \left[t_p, t_{p+1}-1\right].$$

Then, the same conditions (47) and (48) guarantee that $\hat{V}_e(k)$ is positive for $k \geq t_0, k \in \mathbb{Z}$, ie, $\mathbb{E}\{\hat{V}_e(k)\} \geq \hat{\rho} \quad \mathbb{E}\{|x(k)|^2 + |e(k)|^2\}$ with some $\hat{\rho} > 0$ and that $\mathbb{E}\{\hat{V}_e(k+1) - \lambda\hat{V}_e(k)\} \leq 0, k \in [t_p, t_{p+1}-1]$, which can substitute for the conditions (i) and (ii) of Theorem 3.

Remark 8. To enlarge MATI and the maximum allowable delay h_M, one can resort to Markovian scheduling (see, eg, the works of Donkers et al[3] and Liu et al[11]) instead of iid scheduling. In this protocol, the value of σ_p^* is determined through a Markov Chain and the closed-loop system can be modeled as a stochastic Markovian jump discrete-time impulsive system.

Remark 9. The modeling and stability analysis of discrete-time networked systems with multiple sensor nodes under TOD and iid protocols follows exactly the discrete-time counterpart of the results obtained in the works of Liu et al[10,11] for TOD and iid protocols, respectively. Note that the conditions in the aforementioned works[10,11] were derived by Jensen inequality and the reciprocally convex combination approach,[21] whereas Theorems 2 and 3 were obtained by the virtue of the developed reciprocally convex combination inequality proposed in the work of Zhang et al[18] and the discrete counterpart of the augmented Lyapunov functional provided in the work of Liu et al[19] for the stability analysis of continuous-time systems with time-varying delays.

Moreover, the stability analysis of discrete-time system (8)-(9) with time-varying interval delays under scheduling protocols can be alternatively analyzed by substituting the switched system transformation approach for the Lyapunov method. More details can be found in the work of Hetel et al.[35]

Remark 10. The conditions of Theorems 2 and 3 are easily adapted to the decentralized networked control of large-scale interconnected systems with local independent networks,[13] where every plant is controlled under TOD or under iid stochastic scheduling protocol.

7 | ILLUSTRATIVE EXAMPLE

In this section, we will verify the efficiency of the derived conditions in Theorems 2 and 3 through the widely used inverted pendulum system. Denote the cart position coordinate and the pendulum angle from vertical by x and θ, respectively. Then, the dynamics of the inverted pendulum on a cart shown in Figure 2 can be described in the following as in, eg, the work of Zhang et al[36]:

$$\begin{bmatrix}\dot{x}\\ \ddot{x}\\ \dot{\theta}\\ \ddot{\theta}\end{bmatrix} = \begin{bmatrix}0 & 1 & 0 & 0\\ 0 & -\frac{(a+ml^2)b}{a(M+m)+Mml^2} & \frac{m^2gl^2}{a(M+m)+Mml^2} & 0\\ 0 & 0 & 0 & 1\\ 0 & -\frac{mlb}{a(M+m)+Mml^2} & \frac{mgl(M+m)}{a(M+m)+Mml^2} & 0\end{bmatrix}\begin{bmatrix}x\\ \dot{x}\\ \theta\\ \dot{\theta}\end{bmatrix} + \begin{bmatrix}0\\ \frac{a+ml^2}{a(M+m)+Mml^2}\\ 0\\ \frac{ml}{a(M+m)+Mml^2}\end{bmatrix}u, \tag{59}$$

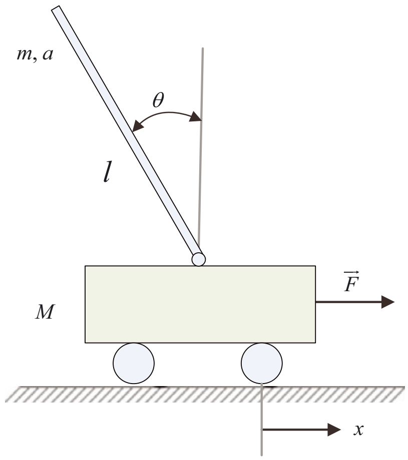

FIGURE 2 Geometry of the inverted pendulum system [Colour figure can be viewed at wileyonlinelibrary.com]

where the parameters a and b represent the friction of the cart and inertia of the pendulum, respectively, and are chosen as $a = 0.0034$ kg·m^2 and $b = 0.1$ N/m/sec; the mass of the cart M and the mass of the pendulum m are given by $M = 1.096$ kg and $m = 0.109$ kg, respectively; and the length of the pendulum l is chosen as 0.25 m, and $g = 9.8$ m/s^2 is the gravity acceleration.

We discretize system (59) with a time $T_s = 0.01$ s and obtain the following discrete-time system model:

$$x(k+1) = \begin{bmatrix} 1 & 0.01 & 0 & 0 \\ 0 & 0.9991 & 0.0063 & 0 \\ 0 & 0 & 1.0014 & 0.01 \\ 0 & -0.0024 & 0.2784 & 1.0014 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 0.0088 \\ 0.0001 \\ 0.0236 \end{bmatrix} u(k), \quad k \in \mathbb{Z}^+. \tag{60}$$

The pendulum can be stabilized by a state feedback $u(k) = Kx(k)$ with the gain $K = [K_1\ K_2]$

$$K = [K_1 \quad K_2], \qquad K_1 = \begin{bmatrix} 7.7606 & 14.6847 \end{bmatrix}, K_2 = \begin{bmatrix} -86.7306 & -26.3029 \end{bmatrix}, \tag{61}$$

which leads to the closed-loop system having eigenvalues $\{0.5374, 0.9860 + 0.0177i, 0.9860 - 0.0177i, 0.9924\}$. Suppose that the spatially distributed components of the state variables of system (60) are not accessible simultaneously. We start with the case of $N = 2$ and consider two measurements $y_i(k) = C_i x(k)$, $k \in \mathbb{Z}^+$, where

$$C_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

For the values of h_m given in Table 2, we apply Theorems 2 and 3 with $\lambda = 1$ and find the maximum allowable values of $\tau_M = \text{MATI} + h_M$ that preserve the stability of the hybrid time-delay system (8)-(9). From Table 2, it is observed that, under the TOD protocol, the conditions of Theorem 2 stabilize the system for larger τ_M than the results in the works of Liu et al.[17,20] Note that the suggested time-dependent Lyapunov approach in the work of Liu and Fridman[17] only guaranteed partial stability of the resulting hybrid delayed system. Moreover, it was restricted to $N = 2$ sensors and cannot be extended to the general case of $N \geq 2$. The condition of Liu et al[20] was obtained by Jensen inequality and the reciprocally convex combination approach.[21] The improvement of Theorem 2 compared to the works of Liu et al[17,20] is achieved due to the application of both Lemma 1 and the developed reciprocally convex combination inequality proposed in the work of Zhang et al.[18] Compared to the TOD protocol in the work of Liu and Fridman[17] and Theorem 2, the iid

TABLE 2 Example ($N = 2$): maximum value of $\tau_M = MATI + h_M$ for different h_m

$\tau_M \setminus h_m$	0	2	5	10	15
TOD by Liu and Fridman[17]	19	20	23	27	30
TOD by Liu et al[20]	17	19	22	25	28
Theorem 3 (iid, $\beta_1 = 0.3$)	18	20	23	26	29
TOD by Theorem 2	21	23	25	29	32

Abbreviations: iid, independent and identically-distributed; TOD, try-once-discard.

TABLE 3 Example ($N = 4$): max. value of $\tau_M = MATI + h_M$ for different h_m

$\tau_M \setminus h_m$	0	1	2	4	8	10
TOD by Liu et al[20]	1	2	3	5	9	–
Theorem 3 (iid)	2	3	4	6	10	11
TOD by Theorem 2	3	4	6	8	12	13

Abbreviations: iid, independent and identically-distributed; TOD, try-once-discard.

protocol of Theorem 3 with $\beta_1 = 0.3$ leads to conservative results but can easily include data packet dropouts or collisions in the presence of large communication delays.[11]

We proceed next with the case of $N = 4$, where $C_1, \dots, C_4$ are the rows of I_4 and $K_1, \dots, K_4$ are the entries of K given by (61). In this case, the conditions of the work of Liu and Fridman[17] are not applicable any more. By applying Theorems 2 and 3 with $\beta_i = 0.25, i = 1, \dots, 4$, Table 3 shows the maximum value of $\tau_M = MATI + h_M$ that preserves the exponential stability of the hybrid system (8)-(9). Furthermore, here, Theorem 2 achieves the least conservative results. Moreover, when $h_m > \frac{\tau_M}{2}(h_m = 8, 10)$, the proposed method is still feasible, representing the case where communication delays can be larger than the sampling intervals.

8 | CONCLUSIONS

This paper has addressed the stability problem of discrete-time NCSs under dynamic scheduling protocols in which the components communicate through a shared communication medium that introduces large but bounded time-varying transmission delays. The closed-loop system was modeled as a discrete-time hybrid system with time-varying delays in the dynamics and in the reset conditions. By a newly constructed augmented Lyapunov functional and the discrete counterpart of the second-order Bessel-Legendre integral inequality, an improved stability criterion to find the maximum allowable sampling intervals and transmission delays was derived such that the resulting closed-loop system is exponentially stable with respect to the full state. Numerical example illustrates the efficiency of our method.

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