Sampled-data H_∞ state-feedback control of systems with state delays

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The finite horizon piecewise-constant state-feedback H_∞ control of time-invariant linear systems with a finite number of point and distributed time-delays is considered. For the controller, coupled Riccati type partial differential equations between sampling and updating the terminal conditions are derived. For small time delays the solutions and the resulting controllers are approximated by series expansions in powers of the largest delay. Unlike the infinite horizon case, these approximations possess both regular and boundary layer terms. It is shown that the controller obtained by high-order approximations improves the performance of the system. The performance of the closed-loop system under the memoryless zero-approximation controller is analysed. Bounded Real Lemmas for state-delay systems with jumps are obtained.

1. Introduction

Continuous-time H_∞ control problem of state-delay systems has been studied by Bensoussan *et al.* (1992), van Keulen *et al.* (1993), van Keulen (1993), McMillan and Triggiani (1993), Lee *et al.* (1994), Ge *et al.* (1996), Fridman and Shaked (1998, 2000). Bensoussan *et al.* (1992), van Keulen *et al.* (1993), van Keulen (1993) and McMillan and Triggiani (1993) have obtained the controller by solving Riccati operator equations. Lee *et al.* (1994) and Ge *et al.* (1996) have designed a delay-independent controller. Fridman and Shaked (1998, 2000) have derived the controller from Riccati type partial differential equations (RPDEs) or inequalities, and the solution of the RPDEs has been approximated by expansions in the powers of the delay. In Shaked *et al.* (1998) a bounded real lemma has been obtained in terms of differential linear matrix inequalities.

Sampled-data H_∞ control of systems without delay has been studied by Bamieh and Pearson (1992), Toivonen (1992), Khargonekar *et al.* (1993), Sivashankar and Khargonekar (1994), Basar and Bernard (1995), Sagfors and Toivonen (1997). Two main approaches have been used. The first one is based on the lifting technique in which the problem is transformed into equivalent finite-dimensional discrete H_∞ problem (Bamieh and Pearson 1992, Toivonen 1992). The second, more direct, approach is based on the representation of the system in the form of hybrid discrete/continuous one and the solution is obtained in terms of Riccati equations with jumps (Khargonekar *et al.* 1993, Sivashankar and Khargonekar 1994, Basar and Bernard 1995, Sagfors and Toivonen 1997).

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In the present paper, we adopt the second approach to the finite horizon sampled-data state-delay H_∞-control problem. Note that in the case of state-delays the lifting technique leads to complicated discrete system with infinite-dimensional state variable. We obtain the piecewise-constant state-feedback controllers by solving coupled RPDEs between sampling and updating the terminal conditions at the sampling instants. We derive an asymptotic approximation to the solution of these RPDEs by expanding it in the powers of the largest delay. The resulting approximation is obtained by solving uncoupled low-order partial differential equations. The performance of the system with the controller that has been obtained using the zero approximation (the one that corresponds to zero delay) is analysed when the open-loop system possesses a non-zero delay. Bounded real lemmas for systems with state delays and jumps are obtained.

2. Problem formulation

Throughout this paper we denote by $|\cdot|$ the Euclidean norm of a vector or the appropriate norm of a matrix. Given $t_f > 0$, let $L_2[0, t_f]$ be the space of the square integrable functions with the norm $\|\cdot\|_{L_2}$ and let $C[a,b]$ be the space of the continuous functions on $[a,b]$ with the norm $|\cdot|_c$. We denote $x_t = x(t+\theta)$, $y^t = y(t-\theta)$, $\theta \in [-h, 0]$ and $x(t^-) = \lim_{0<s\to 0} x(t-s)$, and $x_{t^-} = x((t+\theta)^-)$. Prime denotes the transpose of a matrix and col $\{x,y\}$ denotes a column vector with components x and y.

Consider the system

$$\left.\begin{aligned} \dot{x}(t) &= L(x_t(\cdot)) + Bu(t) + Dw(t) \\ z(t) &= \text{col}\,\{Cx(t), u(t)\} \end{aligned}\right\} \qquad (1)$$

where $x(t) \in \mathbf{R}^n$ is the state vector, $u(t) \in \mathbf{R}^l$ is the control signal, $w(t) \in \mathbf{R}^q$ is the exogenous disturbance, and $z(t) \in \mathbf{R}^p$ is the observation vector, B, C and D are constant matrices of appropriate dimensions. The R^n-valued

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function $L(\cdot)$ which carries R^n-valued functions on $[-h,0]$ into R^n is defined as

$$L(x_t(\cdot)) = \sum_{i=0}^{r} A_i x_t(-h_i) + \int_{-h}^{0} A_{01}(s) x_t(s)\,\mathrm{d}s \tag{2}$$

where $-h = -h_r < -h_{r-1} < \cdots < -h_1 < -h_0 = 0$, $A_0, A_1, \ldots, A_r$ are constant matrices and $A_{01}(s)$ is a smooth enough matrix function.

Consider the 'structural operator' $F: L_2[-h,0] \to L_2[-h,0]$ given by

$$F(x_t)(\xi) = \sum_{i=1}^{r} A_i x_t(-h_i - \xi)\chi_i(\xi) + \int_{-h}^{\xi} A_{01}(p) x_t(p-\xi)\,\mathrm{dp} \tag{3}$$

where χ_i is the indicator function for the set $[-h_i, 0]$, i.e. $\chi_i(\xi) = 1$ if $\xi \in [-h_i, 0]$ and $\chi_i(\xi) = 0$ otherwise. This operator was introduced by Delfour (1986) and the use of this operator in the Lyapunov–Krasovskii functional leads to simplified RPDEs for H_2 and H_∞ design (Delfour 1986; Fridman and Shaked 2000).

Given $\gamma > 0$, and assuming that $w \in L_2[0, t_f]$, we consider the following performance index

$$J = ||z||^2_{L_2} - \gamma^2 ||w||^2_{L_2} + E(x_{t_f}) \tag{4}$$

where

$$E(x_{t_f}) = x'(t_f) P_f x(t_f) + 2x'(t_f) \int_{-h}^{0} Q_f(\xi) F(x_{t_f})(\xi)\,\mathrm{d}\xi + \int_{-h}^{0}\int_{-h}^{0} F'(x_{t_f})(s) R_f(s,\xi) F(x_{t_f})(\xi)\,\mathrm{d}s\,\mathrm{d}\xi \tag{5}$$

and $P_f = P_f' \geq 0$. The form of $E(x_{t_f})$ stems from the form of Lyapunov–Krasovskii functional. We assume that the matrix-functions Q_f and R_f are continuous and piecewise continuously differentiable functions of their arguments, that satisfy the relations:

$$P_f = Q_f(0), \quad Q_f(\xi) = R_f(0, \xi) \tag{6}$$

We are looking for the piecewise-constant state-feedback controller of the form

$$u(t) = \mu(x_{t_k}, \cdots, x_{t_1}, x_0), \quad t_k \leq t < t_{k+1}, \quad k = 0,\ 1\ ,\cdots\ , K-1 \tag{7}$$

where $0 < t_1 < \cdots < t_K = t_f$ are the sampling instants. This type of control is encountered in many practical problems where a zero-order-hold is used to provide an analog input signal at the output of a digital controller.

The problem is to find a state-feedback controller of (7) which ensures that $J \leq 0$ for all $w \in L_2[0, t_f]$ and for the zero initial conditions $x(\tau) = 0$, $\tau \leq 0$. This means that the H_∞-norm of (1), which is defined by the supremum over $w \in L_2[0, t_f]$ of the ratio between $\{||z||^2_{L_2} + E(x_{t_f})\}^{1/2}$ and $||w||_{L_2}$, is not greater than γ.

3. Piecewise-constant controller design

Following Basar and Bernard (1995) and Toivonen and Sagfors (1997) we denote

$$\left.\begin{aligned} &\bar{A}_0 = \begin{bmatrix} A_0 & B \\ 0 & 0 \end{bmatrix}, \bar{A}_i = \begin{bmatrix} A_i & 0 \\ 0 & 0 \end{bmatrix}, \bar{A}_{01} = \begin{bmatrix} A_{01} & 0 \\ 0 & 0 \end{bmatrix} \\ &\bar{D} = \begin{bmatrix} D \\ 0 \end{bmatrix}, \quad \bar{C} = \begin{bmatrix} C & 0 \\ 0 & I_l \end{bmatrix} \\ &A_d = \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix}, \quad B_d = \begin{bmatrix} 0 \\ I_l \end{bmatrix} \end{aligned}\right\} \tag{8}$$

The system of (1) attains then the form

$$\dot{\bar{x}}(t) = \sum_{i=0}^{r} \bar{A}_i \bar{x}(t-h_i) + \int_{-h}^{0} \bar{A}_{01}(s)\bar{x}(t+s)\,\mathrm{d}s + \bar{D}w(t),$$

$$z(t) = \bar{C}\bar{x}(t), \quad t_k \leq t < t_{k+1} \tag{9}$$

$$\bar{x}(t_k) = A_d \bar{x}(t_k^-) + B_d u_k \tag{10}$$

where $\bar{x}(t) = \mathrm{col}\,\{x(t), u(t)\}$. Similarly to (3) we denote

$$\bar{F}(\bar{x}_t)(\xi) = \sum_{i=1}^{r} \bar{A}_i \bar{x}_t(-h_i - \xi)\chi_i(\xi) + \int_{-h}^{\xi} \bar{A}_{01}(p)\bar{x}_t(p-\xi)\,\mathrm{d}p \tag{11}$$

clearly $\bar{F}(\bar{x}_t)(\xi) = \mathrm{col}\,\{F(x_t)(\xi),\ 0\}$. The Lyapunov–Krasovskii functional for the system of (9) is given by Delfour (1986)

$$\bar{V}(t, \bar{x}_t) = \bar{x}(t)'\bar{P}(t)\bar{x}(t) + 2\bar{x}'(t)\int_{-h}^{0}\bar{Q}(t,\xi)\bar{F}(\bar{x}_t)(\xi)\,\mathrm{d}\xi + \int_{-h}^{0}\int_{-h}^{0}\bar{F}'(\bar{x}_t)(s)\bar{R}(t,s,\xi)\bar{F}(\bar{x}_t)(\xi)\,\mathrm{d}s\,\mathrm{d}\xi \tag{12}$$

where

$$\bar{P} = \begin{bmatrix} P & M \\ M' & U \end{bmatrix}, \quad \bar{Q} = \begin{bmatrix} Q & 0 \\ N' & 0 \end{bmatrix}, \quad \bar{R} = \begin{bmatrix} R & 0 \\ 0 & 0 \end{bmatrix} \tag{13}$$

In the latter formulas the zero-blocks stem from the zero delay in u.

Then $E(x_{t_f}) = \bar{E}(\bar{x}_{t_f})$, where

$$\bar{E}(\bar{x}_{t_f}) = \bar{x}'(t_f)\bar{P}_f\bar{x}(t_f) + 2\bar{x}'(t_f)\int_{-h}^{0}\bar{Q}_f(\xi)\bar{F}(\bar{x}_{t_f})(\xi)\,\mathrm{d}\xi + \int_{-h}^{0}\int_{-h}^{0}\bar{F}'(\bar{x}_{t_f})(s)\bar{R}_f(s,\xi)\bar{F}(\bar{x}_{t_f})(\xi)\,\mathrm{d}s\,\mathrm{d}\xi \tag{14}$$

and where

$$\bar{P}_f = \begin{bmatrix} P_f & 0 \\ 0 & 0 \end{bmatrix}, \quad \bar{Q}_f = \begin{bmatrix} Q_f & 0 \\ 0 & 0 \end{bmatrix}, \quad \bar{R}_f = \begin{bmatrix} R_f & 0 \\ 0 & 0 \end{bmatrix} \tag{15}$$

Denote

$$S = \gamma^{-2} D D', \quad \bar{S} = \gamma^{-2} \bar{D} \bar{D}'$$

Consider the following RPDEs with respect to the $(n+l) \times (n+l)$-matrices $\bar{P}(t), \bar{Q}(t,\xi)$ and $\bar{R}(t,\xi,s)$

$$\dot{\bar{P}}(t) + \bar{A}_0' \bar{P}(t) + \bar{P}(t)\bar{A}_0 + \sum_{i=1}^{r} \bar{A}_i' \bar{Q}'(t,-h_i) + \sum_{i=1}^{r} \bar{Q}(t,-h_i)\bar{A}_i + \bar{P}(t)\bar{S}\bar{P}(t) + \bar{C}'\bar{C} + \int_{-h}^{0} \bar{Q}(t,\theta)\bar{A}_{01}(\theta)\,\mathrm{d}\theta + \int_{-h}^{0} \bar{A}_{01}'(\theta)\bar{Q}'(t,\theta)\,\mathrm{d}\theta = 0 \tag{16}$$

$$\frac{\partial}{\partial t}\bar{Q}(t,\xi) + \frac{\partial}{\partial \xi}\bar{Q}(t,\xi) = -[\bar{A}_0' + \bar{P}\bar{S}]\bar{Q}(t,\xi) - \sum_{i=1}^{r} \bar{A}_i' \bar{R}(t,-h_i,\xi) - \int_{-h}^{0} \bar{A}_{01}'(s)\bar{R}(t,s,\xi)\,\mathrm{d}s \tag{17}$$

$$\frac{\partial}{\partial t}\bar{R}(t,\xi,s) + \frac{\partial}{\partial \xi}\bar{R}(t,\xi,s) + \frac{\partial}{\partial s}\bar{R}(t,\xi,s) = -\bar{Q}'(t,\xi)\bar{S}\bar{Q}(t,s) \tag{18}$$

$$\left.\begin{aligned} P(t) = Q(t,0), \quad & M(t) = N(t,0) \\ Q(t,\xi) = R(t,0,\xi), \quad & R(t,\xi,s) = R'(t,s,\xi) \end{aligned}\right\} \tag{19}$$

$$\left.\begin{aligned} & P(t_f) = P_f, \quad U(t_f) = 0, \quad M(t_f) = 0 \\ & N(t_f,s) = 0, \quad Q(t_f,\xi) = Q_f(\xi), \quad R(t_f,\xi,s) = R_f(\xi,s) \end{aligned}\right\} \tag{20}$$

A solution of (16)–(20) on $[t_{K-1}, t_f]$ is a triple of $(n+l) \times (n+l)$-matrices $\{\bar{P}(t), \bar{Q}(t,\xi), \bar{R}(t,\xi,s)\}$ $t \in [t_{K-1}, t_f], \xi \in [-h,0], s \in [-h,0]$, where $\bar{P}(t)$, $\bar{Q}(t,\xi)$ and $\bar{R}(t,\xi,s)$ are continuous and piecewise continuously differentiable functions of their arguments that have a form of (13), satisfy initial and terminal conditions (19), (20) for every t, ξ and s and satisfy partial differential equations (16)–(18) for almost every t, ξ and s.

Lemma 1: *The supremum value with respect to w of*

$$J_K = \bar{E}(\bar{x}_{t_f}) + \int_{t_{K-1}}^{t_f} [|\bar{z}|^2 - \gamma^2 |w|^2]\,\mathrm{d}t$$

for the system of (9) is given by

$$\sup_w J_K = J_K^* = V(t_{K-1}, x_{t_{K-1}}) \tag{21}$$

where the matrix-functions $\bar{P}$, $\bar{Q}$ and $\bar{R}$ of the form (13) satisfy (16)–(20).

The proof of Lemma 1 is given in the Appendix. Writing (16)–(18) for the various blocks of (13), we obtain

$$\dot{P}(t) + A_0' P(t) + P(t)A_0 + \sum_{i=1}^{r} A_i' Q'(t,-h_i) + \sum_{i=1}^{r} Q(t,-h_i)A_i + P(t)SP(t) + C'C + \int_{-h}^{0} Q(t,\theta)A_{01}(\theta)\,\mathrm{d}\theta + \int_{-h}^{0} A_{01}'(\theta)Q'(t,\theta)\,\mathrm{d}\theta = 0 \tag{22}$$

$$\frac{\partial}{\partial t}Q(t,\xi) + \frac{\partial}{\partial \xi}Q(t,\xi) = -[A_0' + PS]Q(t,\xi) - \sum_{i=1}^{r} A_i' R(t,-h_i,\xi) - \int_{-h}^{0} A_{01}'(s)R(t,s,\xi)\,\mathrm{d}s \tag{23}$$

$$\frac{\partial}{\partial t}R(t,\xi,s) + \frac{\partial}{\partial \xi}R(t,\xi,s) + \frac{\partial}{\partial s}R(t,\xi,s) = -Q'(t,\xi)SQ(t,s) \tag{24}$$

$$\dot{M}(t) + [A_0' + P(t)S]M(t) + P(t)B + \sum_{i=1}^{r} A_i' N(t,-h_i) + \int_{-h}^{0} A_{01}'(\theta)N(t,\theta)\,\mathrm{d}\theta = 0 \tag{25}$$

$$\frac{\partial}{\partial t}N(t,\xi) + \frac{\partial}{\partial \xi}N(t,\xi) = -Q'(t,\xi)[B + SM(t)] \tag{26}$$

and

$$\dot{U}(t) + B'M(t) + M'(t)B + M'(t)SM(t) + I_l = 0 \tag{27}$$

We obtain the following result.

Theorem 1: *Assume that for every $k = K, \ldots, 1$ there exists a solution to (16)–(19) on $[t_{k-1}, t_k]$ with the following terminal conditions: (20) for $k = K$ and for $k < K$*

$$\left.\begin{aligned} P(t_k^-) &= P(t_k) - M(t_k)U^{-1}(t_k)M'(t_k) \\ U(t_k^-) &= 0, \quad N(t_k^-,\xi) = 0, \quad M(t_k^-) = 0 \\ Q(t_k^-,\xi) &= Q(t_k,\xi) - M(t_k)U^{-1}(t_k)N'(t_k,\xi) \\ R(t_k^-,s,\xi) &= R(t_k,s,\xi) - N(t_k,s)U^{-1}(t_k)N'(t_k,\xi) \end{aligned}\right\} \quad (28)$$

Then, the controller

$$u(t) = -U^{-1}(t_k)\left[M'(t_k)x(t_k) + \int_{-h}^{0} N'(t_k,s)F(x_{t_k})(s)\mathrm{d}s\right],$$
$$t_k \le t < t_{k+1},\ k = K-1,\ldots,0 \quad (29)$$

solves the sampled-data state-feedback H_∞-control problem.

Proof: We shall apply dynamic programming arguments, by adapting the dynamic programming equation to the sampling intervals:

$$\inf_u \sup_w J = \inf_u \sup_w \left\{\int_0^{t_1}[|z|^2 - \gamma^2|w|^2]\,\mathrm{d}t + \inf_u \sup_w \left\{\cdots + \inf_u \sup_w \left\{E(x_{t_f}) + \int_{t_{K-1}}^{t_f}[|z|^2 - \gamma^2|w|^2]\,\mathrm{d}t\right\}\cdots\right\}\right\}$$

We first consider stage K. By Lemma 1 the supremum value of

$$J_K = E(x_{t_f}) + \int_{t_{K-1}}^{t_f}[|z|^2 - \gamma^2|w|^2]\,\mathrm{d}t$$

with respect to w is given by

$$\begin{aligned} \sup_w J_K = J_K^* &= \bar{V}(t_{K-1},\bar{x}_{t_{K-1}}) \\ &= V(t_{K-1},x_{t_{K-1}}) + u'(t_{K-1})U(t_{K-1})u(t_{K-1}) \\ &\quad + 2x'(t_{K-1})M(t_{K-1})u(t_{K-1}) \\ &\quad + 2u'(t_{K-1})\int_{-h}^{0} N'(t_{K-1},\xi)F(x_{t_{K-1}})(\xi)\,\mathrm{d}\xi \end{aligned}$$

where

$$V(t,x_t) = x(t)'P(t)x(t) + 2x'(t)\int_{-h}^{0} Q(t,\xi)F(x_t)(\xi)\,\mathrm{d}\xi + \int_{-h}^{0}\int_{-h}^{0} F'(x_t)(s)R(t,s,\xi)F(x_t)(\xi)\,\mathrm{d}s\,\mathrm{d}\xi$$

and where the matrix-functions $\bar{P}$, $\bar{Q}$ and $\bar{R}$ of the form (13) satisfy (16)–(18) and (19).

The unique minimizing u for J_K^* is given by (29), where $k = K-1$, and the corresponding minimum value of J_K^* is given by

$$\begin{aligned} \inf_u J_k^* &= V(t_{k-1},x_{t_{k-1}}) - x'(t_{k-1})M(t_{k-1}) \\ &\quad \times U^{-1}(t_{k-1})M'(t_{k-1})x(t_{k-1}) \\ &\quad - \int_{-h}^{0} F'(x_{t_{k-1}})(\xi)N(t_{k-1},\xi)\,d\xi\, U^{-1}(t_{k-1}) \\ &\quad \times \left[\int_{-h}^{0} N'(t_{k-1},\xi)F(x_{t_{k-1}})(\xi)\,\mathrm{d}\xi + 2M'(t_{k-1})x(t_{k-1})\right] \end{aligned} \quad (30)$$

where $k = K$. Therefore at the stage $K-1$ we obtain the performance cost

$$J_{K-1} = \inf_u J_K^* + \int_{t_{K-2}}^{t_{K-1}}[|z|^2 - \gamma^2|w|^2]\,\mathrm{d}t$$

By the same arguments and due to (28) for $k = K-1$ we see that $\inf_u J_{K-1}^*$ is given by (30), where $k = K-1$. Similarly it can be shown that (30) holds for all $k = K-1,\ldots,1$. Then $\inf_u J \le \inf_u J_1^* = 0$ since $x_0 = 0$. □

4. Asymptotic approximation of the H_∞ controller

4.1. *Asymptotic solutions to the RPDEs*

As we have seen the H_∞ controller has been found by solving a set of coupled RPDEs. Finding a solution to the latter is not an easy task and we are, therefore, looking for a solution to the RPDEs on $[t_k,t_{k+1}]$, $k = K-1,\ldots,0$ in a form of asymptotic expansion in the powers of the delay h

$$\left.\begin{aligned} \bar{P}(t) &= \bar{P}_0(t) + h[\bar{P}_1(t) + \Pi_{1\bar{P}}(\tau)] \\ &\quad + h^2[\bar{P}_2(t) + \Pi_{2\bar{P}}(\tau)] + \cdots \\ \bar{Q}(t,h\zeta) &= \bar{Q}_0(t,\zeta) + h[\bar{Q}_1(t,\zeta) + \Pi_{1\bar{Q}}(\tau,\zeta)] \\ &\quad + h^2[\bar{Q}_2(t,\zeta) + \Pi_{2\bar{Q}}(\tau,\zeta)] + \cdots \\ R(t,h\zeta,h\theta) &= R_0(t,\zeta,\theta) + h[R_1(t,\zeta,\theta) \\ &\quad + \Pi_{1R}(\tau,\zeta,\theta)] + h^2[R_2(t,\zeta,\theta) \\ &\quad + \Pi_{2R}(\tau,\zeta,\theta)] + \cdots \\ t &\in [t_k,t_{k+1}],\ k = K-1,\ldots,0 \\ \tau &= \frac{t_{k+1}-t}{h},\ \zeta \in [-1,0],\ \theta \in [-1,0] \end{aligned}\right\} \quad (31)$$

Note that we consider the asymptotic expansion of R instead of $\bar{R}$ since R is the only non-zero block of $\bar{R}$. The 'outer expansion' terms $\{\bar{P}_i,\bar{Q}_i,R_i\}$, $i = 0,1,\ldots$ constitute the major part of the solution that satisfies (16)–(19) for $t \in [0,t_f]$, $t \ne t_{k+1}$, $\theta \in [-1,0]$, $\zeta \in [-1,0]$. The boundary-layer correction terms

$\Pi_{i\bar{P}}$, $\Pi_{i\bar{Q}}$ and Π_{iR} will be chosen such that (31) satisfies the terminal conditions of (20) and that

$$|\Pi_{i\bar{P}}(\tau)| + \sup_{\zeta\in[-1,0]} |\Pi_{i\bar{Q}}(\tau,\zeta)| + \sup_{\zeta,\theta\in[-1,0]} |\Pi_{iR}(\tau,\zeta,\theta)| \to 0 \quad \text{as} \quad \tau \to \infty \qquad (32)$$

Since τ is a stretched-time variable around $t = t_{k+1}$, (32) asserts that $\Pi_{i\bar{P}}, \Pi_{i\bar{Q}}$ and Π_{iR} are essential only around $t = t_{k+1}$ and they thus provide a correction to the outer expansion at the terminal point $t = t_{k+1}$.

In the sampled-data case we approximate the solution to the RPDEs on each sampling segment: $[t_{K-1}, t_f], \ldots, [0, t_1]$. We assume that the terminal values P_f, Q_f and R_f are approximated as:

$$\left.\begin{aligned} \bar{P}_f &= \bar{P}_{f0} + \sum_{i=1}^{m+1} h^i \bar{P}_{fi} \\ \bar{Q}_f(h\zeta) &= \bar{Q}_{f0} + \sum_{i=1}^{m+1} h^i \bar{Q}_{fi}(\zeta) \\ \bar{Q}_{fi}(0) &= \bar{P}_{fi} \\ R_f(h\zeta, h\theta) &= P_{f0} + \sum_{i=1}^{m+1} h^i R_i(\zeta,\theta) \\ R_{fi}(0,\theta) &= Q_{fi}(\theta) \end{aligned}\right\} \qquad (33)$$

where $\bar{Q}_{fi}$ and R_{fi} are continuous and piecewise-continuously differentiable functions of ζ and θ. The remainders $\bar{P}_{f,m+1}$, $\bar{Q}_{f,m+1}$ and $R_{f,m+1}$ may depend on h and are uniformly bounded functions. We realize that, in practical cases there is a little sense in taking delay-dependent $\bar{P}_f$. We nevertheless consider the general case of nonzero $\bar{P}_{fi}$ since this will be required in our analysis on the sampling segments $[t_{k-1}, t_k]$, $k < K$.

For simplicity we assume that $A_{01} = 0$. Note that A_{01} appears in the integral terms of RPDEs and the latter terms are of the order of $O(h)$. That is why in the case of $A_{01} \neq 0$ the asymptotic approximation is similar. We substitute (31) in (16)–(19) and equate, separately, outer expansion and boundary-layer correction terms with the same powers of h. We notice that for $t = t_{k+1} - h\tau$, $\xi = h\zeta$ and $s = h\theta$ we have

$$\frac{\partial}{\partial t} = -h^{-1}\frac{\partial}{\partial \tau}, \quad \frac{\partial}{\partial \xi} = h^{-1}\frac{\partial}{\partial \zeta}, \quad \frac{\partial}{\partial s} = h^{-1}\frac{\partial}{\partial \theta}$$

Thus, for the zero-order terms we obtain from (23), (24) and (19)

$$\left.\begin{aligned} \bar{Q}_0(t,\zeta) &= \begin{bmatrix} P_0 & 0 \\ M_0' & 0 \end{bmatrix}, \quad R_0(t,\zeta,\theta) = P_0(t) \\ \bar{P}_0 &= \begin{bmatrix} P_0 & M_0 \\ M_0' & U_0 \end{bmatrix} \end{aligned}\right\} \qquad (34)$$

Then, from (22), (25) and (27), we obtain the differential equations

$$\dot{P}_0(t) + \sum_{i=0}^{r} A_i' P_0(t) + \sum_{i=0}^{r} P_0(t) A_i + P_0(t) S P_0(t) + C'C = 0 \qquad (35)$$

$$\left.\begin{aligned} \dot{M}_0(t) + \left[\sum_{i=0}^{r} A_i' + P_0(t) S\right] M_0(t) + P_0(t) B = 0 \\ \dot{U}_0(t) + B' M_0(t) + M_0'(t) B + M_0'(t) S M_0(t) + I_l = 0 \end{aligned}\right\} \qquad (36)$$

with the terminal conditions

$$\left.\begin{aligned} & P_0(t_K^-) = P_{f0} \\ & P_0(t_k^-) = P_0(t_k) - M_0(t_k) U_0^{-1}(t_k) M_0'(t_k) \\ & \qquad\qquad k = K-1, \cdots, 1 \\ & M_0(t_k^-) = 0, \quad U_0(t_k^-) = 0, \quad k = K, \ldots, 1 \end{aligned}\right\} \qquad (37)$$

These equations correspond to the sampled-data H_∞-control of systems without delay (Basar and Bernard 1995). We make here the following assumption:

Assumption 1: *For a specified value of γ, the DREs (35), (36) and (37) possess a bounded solution on $[0, t_f]$.*

Assumption 1 means that the H_∞ state-feedback sampled-data control problem for (1) without delay has a solution. If this were not the case, even $\bar{P}_0$, the zero-order term in (31), would not exist.

To determine the first-order terms we start with $[t_{K-1}, t_f]$ and with the equations for $\bar{Q}_1$:

$$\left.\begin{aligned} \frac{\partial}{\partial \zeta} \bar{Q}_1(t,\zeta) &= -\mathcal{M}'(t) \bar{Q}_0(t) - \dot{\bar{Q}}_0(t) \\ \bar{Q}_1(t,0) &= \bar{P}_1(t) \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix} \\ \mathcal{M} &= \sum_{i=0}^{r} \bar{A}_i + \bar{S}\bar{P}_0 \end{aligned}\right\} \qquad (38)$$

Then

$$\bar{Q}_1(t,\zeta) = \bar{Q}_1(t,0) - [\mathcal{M}'(t)\bar{Q}_0(t) + \dot{\bar{Q}}_0(t)]\zeta$$

Substituting this expression into the equation for $\bar{P}_1$, we obtain

$$\left.\begin{aligned} & \dot{\bar{P}}_1 + \mathcal{M}'\bar{P}_1 + \bar{P}_1\mathcal{M} + \sum_{i=1}^{r} g_i \bar{A}_i' (\bar{Q}_0'\mathcal{M} + \dot{\bar{Q}}_0') \\ & \qquad + \sum_{i=1}^{r} g_i (\mathcal{M}'\bar{Q}_0 + \dot{\bar{Q}}_0) \bar{A}_i = 0 \\ & \bar{P}_1(t_f) + \Pi_{1\bar{P}}(0) = \bar{P}_{f1}, \quad g_i = h_i/h \end{aligned}\right\} \qquad (39)$$

It follows from (22) that $\dot{\Pi}_{1\bar{P}}(\tau) = 0$. Since $\Pi_{1\bar{P}}$ vanishes for $\tau \to \infty$, we have $\Pi_{1\bar{P}}(\tau) \equiv 0$, $\tau \geq 0$. Hence, $\bar{P}_1(t_f) = \bar{P}_{f1}$, and $\bar{P}_1$ is a solution to the linear differential equation (39) with the latter terminal condition.

For $\Pi_{1\bar{Q}}$, R_1 and Π_{1R} we obtain from (17), (18) and (34)

$$\left.\begin{aligned} &\frac{\partial}{\partial\tau}\Pi_{1\bar{Q}}(\tau,\zeta) - \frac{\partial}{\partial\zeta}\Pi_{1\bar{Q}}(\tau,\zeta) = 0 \\ &\bar{Q}_1(t_f,\zeta) + \Pi_{1\bar{Q}}(0,\zeta) = \bar{Q}_{f1}(\zeta) \\ &\Pi_{1\bar{Q}}(\tau,0) = \Pi_{1\bar{P}}(\tau) = 0 \\ &\frac{\partial}{\partial\zeta}R_1(t,\zeta,\theta) + \frac{\partial}{\partial\theta}R_1(t,\zeta,\theta) = -P_0(t)SP_0(t) - \dot{P}_0(t) \\ &R_1(\tau,0,\theta) = Q_1(\tau,\theta) \end{aligned}\right\} \tag{40}$$

and

$$\left.\begin{aligned} &\frac{\partial}{\partial\tau}\Pi_{1R}(\tau,\zeta,\theta) - \frac{\partial}{\partial\zeta}\Pi_{1R}(\tau,\zeta,\theta) - \frac{\partial}{\partial\theta}\Pi_{1R}(\tau,\zeta,\theta) = 0 \\ &R_1(t_f,\zeta,\theta) + \Pi_{1R}(0,\zeta,\theta) = R_{f1}(\zeta,\theta) \\ &\Pi_{1R}(\tau,0,\theta) = \Pi_{1Q}(\tau,\theta) \end{aligned}\right\} \tag{41}$$

where

$$\bar{Q}_1 = \begin{bmatrix} Q_1 & 0 \\ N_1' & 0 \end{bmatrix}, \quad \Pi_{\bar{Q}_1} = \begin{bmatrix} \Pi_{Q_1} & 0 \\ \Pi_{N_1}' & 0 \end{bmatrix}$$

Then, for $\tau \geq 0$ and $t \in [t_{K-1}, t_f]$, we find successively

$$\Pi_{1\bar{Q}}(\tau,\zeta) = \begin{cases} \bar{Q}_{f1}(\zeta+\tau) - \bar{Q}_1(t_f,\zeta+\tau), & \text{if } \tau \leq -\zeta \\ 0, & \text{if } \tau > -\zeta \end{cases}$$

$$\begin{aligned} R_1(t,\zeta,\theta) &= R_1'(t,\theta,\zeta) \\ &= -\zeta[P_0(t)SP_0(t) + \dot{P}_0(t)] + Q_1(t,\theta-\zeta), \\ &\qquad \zeta \geq \theta \end{aligned}$$

$$\begin{aligned} \Pi_{1R}(0,\zeta,\theta) = R_{f1}(\zeta,\theta) + \zeta(P_{f0}SP_{f0} + \dot{P}_0(t_f)) \\ - Q_1(t_f,\theta-\zeta) \end{aligned}$$

$$\Pi_{1R}(\tau,\zeta,\theta) = \Pi_{1R}'(\tau,\theta,\zeta) = \begin{cases} \Pi_{1R}(0,\zeta+\tau,\theta+\tau) \\ \qquad \text{if } \tau \leq -\zeta,\ \theta \leq \zeta \\ \Pi_{1Q}(\tau+\zeta,\theta-\zeta) \\ \qquad \text{if } \tau > -\zeta, \theta \leq \zeta \end{cases}$$

Therefore

$$\begin{gathered} \Pi_{1\bar{Q}}(\tau,\zeta) = 0, \quad \tau+\zeta > 0 \\ \Pi_{1R}(\tau,\zeta,\theta) = \Pi_{1Q}(\tau+\zeta,\theta-\zeta) = 0 \\ \tau+\theta > 0,\ \theta \leq \zeta \end{gathered}$$

The first-order and the higher order terms of the expansions can be similarly found on all sampling segments.

4.2. *Near-optimal piecewise-constant H_∞-control*

For small enough values of h the approximations of the controller contain the outer expansion terms only, i.e. all the boundary-layer terms Π_i vanish. This is due to the fact that we need an approximation of the solution to RPDEs only on the left end of the sampling segment. In the Appendix we prove the following.

Theorem 2: *Under Assumption 1 the following holds for all small enough delays h:*

(1) *The system of* (16)–(20), (22)–(28) *has a solution. This solution is approximated for any integer m by*

$$\left.\begin{aligned} \bar{P}(t) &= \bar{P}_0(t) + \sum_{i=1}^{m} h^i[\bar{P}_i(t) \\ &\quad + \Pi_{i\bar{P}}(\tau)] + O(h^{m+1}) \\ \bar{Q}(t,h\zeta) &= \bar{Q}_0(t) + \sum_{i=1}^{m} h^i[\bar{Q}_i(t,\zeta) \\ &\quad + \Pi_{i\bar{Q}}(\tau,\zeta)] + O(h^{m+1}) \\ R(t,h\zeta,h\theta) &= P_0(t) + \sum_{i=1}^{m} h^i[R_i(t,\zeta,\theta) \\ &\quad + \Pi_{iR}(\tau,\zeta,\theta)] + O(h^{m+1}) \\ \tau &= \frac{t_f - t}{h},\ \zeta \in [-1,0],\ \theta \in [-1,0] \end{aligned}\right\} \tag{42}$$

where the boundary-layer terms satisfy

$$\begin{gathered} \Pi_{i\bar{P}}(\tau) = 0, \quad \tau > i-1; \quad \Pi_{i\bar{Q}}(\tau,\zeta) = 0, \quad \tau+\zeta > i-1 \\ \Pi_{iR}(\tau,\zeta,\theta) = 0, \quad \tau+\theta > i-1,\ \theta \leq \zeta \end{gathered}$$

and $|O(h^{m+1})| \leq ch^{m+1}$, *where c is a positive scalar which is independent of* h, t, ζ *and* θ. *The matrices* $\bar{P}_i$ *and* $\bar{Q}_i$ *are taken with bars and have the structure of* (13), *where all the components are taken with index i.*

(2) *Denote by* Y_i, $i = 0, \ldots, m$ *the terms of the expansions of* U^{-1} *in the powers of h, i.e.*

$$\left(\sum_{i=0}^{m} h^i U_i(t)\right)^{-1} = \sum_{i=0}^{m} h^i Y_i(t) + O(h^{m+1}),$$

$$Y_0(t) = U_0^{-1}(t)$$

Then the controller of (29) *is approximated by*

$$\left.\begin{aligned} u(t) &= u_m(t) + O(h^{m+1}) \\ u_m(t) &= -\sum_{i=0}^{m}\sum_{j=0}^{m-i} h^{i+j} Y_i(t_k) \\ &\quad \times \left[M_j'(t_k)x(t_k) + \int_{-1}^{0} N_{j-1}'(t_k,s)F(x_{t_k})(hs)\,\mathrm{d}s \right] \\ t_k &\le t < t_{k+1},\ k = K-1, \cdots, 0 \end{aligned}\right\} \tag{43}$$

where $N_{-1} = 0$ and $N_0 = M_0$. The approximate controller u_m guarantees an attenuation level of $\gamma + O(h^{m+1})$.

5. The zero-order controller performance

We study the performance of the system under the zero-order controller

$$u_0(t) = -U_0^{-1}(t_k)M_0'(t_k)x(t_k)$$

which solves the H_∞-control problem for (1) without delay.

5.1. L_2-gain of state-delay systems with jumps

In order to study the performance under the zero-order controller in the sampled-data case we note that the closed-loop system of (1), where $u = u_0(t)$, has the form of (9) with the jumps condition

$$\bar{x}(t_k) = G_k \bar{x}(t_k^-), \quad k = 1, \ldots, K-1 \tag{44}$$

where

$$G_k = \begin{bmatrix} I & 0 \\ -U_0^{-1}(t_k)M_0'(t_k) & 0 \end{bmatrix} \tag{45}$$

We derive first the relevant bounded real lemma for (9) with (44), for any given $\bar{A}_i$, $\bar{A}_{01}$, $\bar{D}$, $\bar{C}$ and G_k of the appropriate dimensions. Consider

$$J = ||\bar{z}||_{L_2}^2 - \gamma^2 ||w||_{L_2}^2 + \bar{E}(\bar{x}_{t_f}) \tag{46}$$

where $\bar{E}$ is given by (14), $\bar{F}$ is defined by (11) and $\bar{P}_f, \bar{Q}_f$ and $\bar{R}_f$ are any continuous and piecewise continuously differentiable functions of their arguments. We then apply the lemma to the special matrices of (8) and (45).

Lemma 2: *Assume that for every $k = K, \ldots, 1$ there exists a solution on $[t_{k-1}, t_k]$ to* (16)–(19) *such that*

$$\bar{P}(t) = \bar{Q}(t,0), \quad \bar{Q}(t,\xi) = \bar{R}(t,0,\xi) \tag{47}$$

with the following terminal conditions

$$\left.\begin{aligned} &\bar{P}(t_K) = \bar{P}_f, \quad \bar{Q}(t_K,\xi) = \bar{Q}_f(\xi), \quad \text{and} \\ &\bar{R}(t_K,s,\xi) = \bar{R}_f(s,\xi), \quad \bar{P}(t_k^-) = G_k'\bar{P}(t_k)G_k \\ &\bar{Q}(t_k^-,\xi) = G_k'\bar{Q}(t_k,\xi), \quad \text{and} \\ &\bar{R}(t_k^-,s,\xi) = \bar{R}(t_k,s,\xi), \quad k < K \end{aligned}\right\} \tag{48}$$

Then the performance index J of (46) *for the system of* (9) *with the jumps condition* (44) *is non-negative for all $w \in L_2[0,t_f]$ and $\bar{x}_0 = 0$.*

Proof: Similarly to Theorem 1 we use the dynamic programming argument

$$\sup_w J = \sup_w \left\{ \int_0^{t_1} [|\bar{z}|^2 - \gamma^2|w|^2]\,\mathrm{d}t + \sup_w \left\{ \cdots + \sup_w \left\{ \bar{E}(\bar{x}_{t_f}) + \int_{t_{K-1}}^{t_f} [|\bar{z}|^2 - \gamma^2|w|^2]\,\mathrm{d}t \right\} \cdots \right\} \right\}$$

We first consider stage K. By Lemma 1 the supremum value of

$$J_K = \bar{E}(\bar{x}_{t_f}) + \int_{t_{K-1}}^{t_f} [|\bar{z}|^2 - \gamma^2|w|^2]\,\mathrm{d}t$$

with respect to w is given by

$$\begin{aligned} \sup_w J_k &= \bar{V}(t_{k-1}, \bar{x}_{t_{k-1}}) \\ &= \bar{x}(t_{k-1})'\bar{P}(t_{k-1})\bar{x}(t_{k-1}) + 2\bar{x}'(t_{k-1}) \\ &\quad \times \int_{-h}^{0} \bar{Q}(t_{k-1},\xi)\bar{F}(\bar{x}_{t_{k-1}})(\xi)\,\mathrm{d}\xi \\ &\quad + \int_{-h}^{0}\int_{-h}^{0} \bar{F}'(\bar{x}_{t_{k-1}})(s)\bar{R}(t,s,\xi)\bar{F}(\bar{x}_{t_{k-1}})(\xi)\,\mathrm{d}s\,\mathrm{d}\xi \end{aligned} \tag{49}$$

The matrix-functions $\bar{P}$, $\bar{Q}$ and $\bar{R}$ satisfy (16)–(18) and (47). Substituting now (44) into (49) we obtain the following performance cost for the stage $K-1$

$$\begin{aligned} J_{K-1} &= \bar{x}(t_{K-1})'G_{K-1}'\bar{P}(t_{K-1})G_{K-1}\bar{x}(t_{K-1}) \\ &\quad + 2\bar{x}'(t_{K-1})G_{K-1}'\int_{-h}^{0} \bar{Q}(t_{K-1},\xi)\bar{F}(\bar{x}_{t_{K-1}})(\xi)\,\mathrm{d}\xi \\ &\quad + \int_{-h}^{0}\int_{-h}^{0} \bar{F}'(\bar{x}_{t_{K-1}})(s)\bar{R}(t,s,\xi)\bar{F}(\bar{x}_{t_{K-1}})(\xi)\,\mathrm{d}s d\xi \\ &\quad + \int_{t_{K-2}}^{t_{K-1}} [|\bar{z}|^2 - \gamma^2|w|^2]\,\mathrm{d}t \end{aligned}$$

By Lemma 1 this cost has a supremum value given by (49), where $k = K-1$, since the corresponding RPDEs with terminal conditions (48) have no escape point.

Similarly it can be shown that (49) holds for all $k = K-1,\dots,1$. Then $\sup_w J = \sup_w J_1 \le 0$ due to $\bar{x}_0 = 0$. □

Note that for $h \to 0$, $\bar{P} \to \bar{P}_0$ and $\bar{Q} \to \bar{Q}_0$ (cf. (42)). Therefore the latter lemma coincides with the corresponding results in Sivashankar and Khargonekar (1994).

5.2. *Robustness of the performance under u_0*

Theorem 3: *Under Assumption 1 the controller u_0 for all small enough h leads to a performance level of γ.*

Proof: Applying u_0 to (1), we obtain (9) and (44). By Lemma 2 this closed-loop system has an induced L_2-gain less or equal to γ if the corresponding RPDEs of (16)–(19) with the terminal conditions of (48) have a bounded solution. The existence of a solution to these RPDEs, approximated by (42) with $m = 0$, can be proved similarly to (i) of Theorem 2. □

Given $\gamma > 0$ and h, one should verify that the corresponding RPDEs have a solution in order to make certain that u_0 leads to a performance level of γ. This is not an easy task. That is why one may resort to more conservative, but computationally simpler, conditions in terms of differential linear matrix inequalities (DLMI) or Riccati differential inequalities (RDI) that were formulated for the case of one delay in Shaked *et al.* (1998). Generalization of these lemmas to the case of r delays and jumps in the system is given below in Lemmas 3 and 4.

5.3. *Bounded real lemmas for state-delay systems with jumps using DLMI*

Consider (9) with (44), where $\bar{A}_i$, $\bar{D}$, $\bar{C}$ and G_k are any given matrices of appropriate dimensions. For simplicity we assume that $\bar{A}_{01} = 0$. Consider

$$J = ||\bar{z}||^2_{L_2} - \gamma^2 ||w||^2_{L_2} + \bar{x}'(t_f)\bar{P}_f\bar{x}(t_f) \tag{50}$$

where $\bar{P}_f$ is any matrix.

Lemma 3: *The performance index J of (50) for the system of (9) with the jumps condition of (44) is non-positive for all $w \in L_2[0,t_f]$, and $\bar{x}_0 = 0$, if there exist square integrable matrices $Q_i(t) = Q_i'(t)$, $i = 1,\dots,r$ that are positive definite on $[t_f - h_i, t_f)$ and allow an absolutely continuous solution $P(t) = P'(t) \ge 0$ on $t \in [t_{k-1}, t_k)$, $k = 1,\dots,K$ to the following DLMI*

$$\mathcal{N}(t) = \begin{bmatrix} \Psi(t) & P(t)\bar{A}_1 & P(t)\bar{A}_2 & \cdots & P(t)\bar{A}_r & P(t)\bar{D} \\ \bar{A}_1'P(t) & -Q_1(t-h_1) & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \bar{A}_r'P(t) & 0 & 0 & \cdots & -Q_r(t-h_r) & 0 \\ \bar{D}'P(t) & 0 & 0 & \cdots & 0 & -\gamma^2 I \end{bmatrix} \le 0 \tag{51}$$

$$\Psi(t) \triangleq \dot{P}(t) + \bar{A}_0'P(t) + P(t)\bar{A}_0 + \bar{C}'\bar{C} + \sum_{i=1}^r Q_i(t)$$

with the terminal conditions

$$\left.\begin{aligned} &P(t_K^-) = \bar{P}_f \quad \text{and} \\ &P(t_k^-) = G_k'P(t_k)G_k, \ k = K-1,\dots,1 \end{aligned}\right\} \tag{52}$$

Proof: We consider the function

$$V(t,\bar{x}_t) \triangleq \bar{x}'(t)P(t)\bar{x}(t) + \sum_{i=1}^r \int_{t-h_i}^t \bar{x}'(\tau)Q_i(\tau)\bar{x}(\tau)\,d\tau \tag{53}$$

Using (9), it can be easily established that for $t \ne t_k$

$$\begin{aligned} \frac{d}{dt}V(t,\bar{x}_t) = {} & \bar{x}'(t)[\dot{P}(t) + \bar{A}_0'P(t) + P(t)\bar{A}_0 + \bar{C}'\bar{C} \\ & + \sum_{i=1}^r Q_i(t)]\bar{x}(t) + 2\bar{x}'(t)P\sum_{i=1}^r A_i y_i(t) \\ & - \bar{x}'(t)\bar{C}'\bar{C}\bar{x}(t) + \sum_{i=1}^r y_i'(t)Q_i(t-h_i)y_i(t) \\ & + 2\bar{x}'(t)P(t)\bar{D}w(t) \end{aligned} \tag{54}$$

where $y_i(t) \triangleq x(t-h_i)$. Denoting $\xi \triangleq [\bar{x}' \ y_1' \ \cdots \ y_r' \ w']'$ it follows from (54) that

$$\frac{dV}{dt} = \xi'\mathcal{N}\xi + \gamma^2|w|^2 - |\bar{z}|^2, \quad t \in [t_k, t_{k+1}) \tag{55}$$

From (52), (53) and the fact that $\bar{x}(t) = 0$, $\forall\, t \le 0$, it is readily obtained that

$$\begin{aligned} \int_0^{t_f} \frac{dV}{d\tau}d\tau &= V(t_f,\bar{x}_{t_f}) - V(t_{K-1},\bar{x}_{t_{K-1}}) + V(t_{K-1}^-,\bar{x}_{t_{K-1}}^-) \\ &\quad - \cdots + V(t_1^-,\bar{x}_{t_1}^-) \\ &= V(t_f,\bar{x}_{t_f}) \\ &= \bar{E}(\bar{x}_{t_f}) + \sum_{i=1}^r \int_{t_f-h_i}^{t_f} \bar{x}'(\tau)Q_i(\tau)\bar{x}(\tau)\,d\tau \end{aligned}$$

By considering (55), this implies that

$$J = \int_0^{t_f} \xi'\mathcal{N}\xi d\tau - \sum_{i=1}^r \int_{t_f-h_i}^{t_f} \bar{x}'Q_i\bar{x}d\tau \le 0 \quad \forall w \in L_2[0,t_f]$$

□

Lemma 3 provides a bounded real criterion in terms of the DLMI of (51). Note that if $Q(t) > 0$, $t \in [0,t_f]$, we obtain using Schur complements, that (51) is equivalent to the RDI

$$\dot{P}(t) + \bar{A}_0' P(t) + P(t)\bar{A}_0 + P(t)\left[\gamma^{-2}\bar{D}\bar{D}' + \sum_{i=1}^{r} \bar{A}_i Q_i^{-1}(t-h_i)\bar{A}_i'\right]P(t) + \bar{C}'\bar{C} + \sum_{i=1}^{r} Q_i(t) \le 0 \qquad (56)$$

The result of Lemma 3 is most powerful, in the sense that it establishes a condition for the performance index J to be non-positive almost independently of the delay length. It is thus too conservative. A less conservative condition which explicitly depends on the length of the delay is derived in the next lemma.

Lemma 4: *Assume that $h \le \min_{k=1,\dots,K}(t_k - t_{k-1})$. The performance index J of (50) for the system of (9) with the jumps condition of (44) is nonpositive for all $w \in L_2[0, t_f]$ if for every $k = K-1, \dots, 0$ there exist constant symmetric positive definite matrices P_{1i}, P_{2i}, P_{3i} and P_{4i}, $i = 1, \dots, r$ with $I - \sum_{i=1}^{r} h_i P_{3i} > 0$, that allow a solution $P(t) = P'(t) \ge 0$, that is absolutely continuous for $t \ne t_k$, to the following RDI for $k = K-1, \dots, 1$*

$$\dot{P}(t) + \sum_{i=0}^{r} \bar{A}_i' P(t) + P(t)\sum_{i=0}^{r} \bar{A}_i + \sum_{i=1}^{r} h_i\left[\bar{A}_0' P_{1i}\bar{A}_0 + \left(\sum_{j=1}^{r}\bar{A}_j'\right)P_{2i}\left(\sum_{j=1}^{r}\bar{A}_j\right)\right] + \bar{C}'\bar{C} + P(t)\left[\gamma^{-2}\bar{D}(I - \sum_{i=1}^{r} h_i P_{3i})^{-1}\bar{D}' + \sum_{i=1}^{r}[\Gamma_i + \gamma^{-2}\bar{A}_i H_i \bar{A}_i' + P_{4i}^{-1}\chi_{ik}(t)]\right]P(t) \le 0$$

$$t \in [t_k, t_{k+1}) \qquad (57)$$

and to the following RDI for $k = 0$

$$\dot{P}(t) + \sum_{i=0}^{r} \bar{A}_i' P(t) + P(t)\sum_{i=0}^{r} \bar{A}_i + \sum_{i=1}^{r} h_i\left[\bar{A}_0' P_{1i}\bar{A}_0 + \left(\sum_{j=1}^{r}\bar{A}_j'\right)P_{2i}\left(\sum_{j=1}^{r}\bar{A}_j\right)\right] + \bar{C}'\bar{C} + P(t)\left[\gamma^{-2}\bar{D}\left(I - \sum_{i=1}^{r} h_i P_{3i}\right)^{-1}\bar{D}' + \sum_{i=1}^{r}[\Gamma_i + \gamma^{-2}\bar{A}_i H_i \bar{A}_i']\right]P(t) \le 0$$

$$t \in [t_k, t_{k+1}) \qquad (58)$$

with the terminal conditions

$P(t_K^-) = \bar{P}_f$ *and*

$$P(t_k^-) = G_k' P(t_k) G_k + (I - G_k')\left[\sum_{i=1}^{r} h_i \bar{A}_i' P_{4i}\bar{A}_i\right](I - G_k)$$

$$k = K-1, \dots, 1 \qquad (59)$$

where

$$H_i(t) \triangleq h_i \bar{D} P_{3i}^{-1}\bar{D}', \quad \Gamma_i \triangleq h_i \bar{A}_i (P_{1i}^{-1} + P_{2i}^{-1})\bar{A}_i' \qquad (60)$$

and where $\chi_{ik}(t) \triangleq \chi_i(t - t_k - h_i)$, i.e. $\chi_{ik}(t) = 1$ if $t \in [t_k, t_k + h_i]$ and $\chi_{ik}(t) = 0$ otherwise.

The proof is given in the Appendix. Using Schur complements we can rewrite (57) in the form of equivalent DLMI (see (61) below) where

$$\Phi = \dot{P}(t) + \sum_{i=0}^{r} \bar{A}_i' P(t) + P(t)\sum_{i=0}^{r} \bar{A}_i + \sum_{i=1}^{r} h_i\left[\bar{A}_0' P_{1i}\bar{A}_0 + \left(\sum_{j=1}^{r}\bar{A}_j'\right)P_{2i}\left(\sum_{j=1}^{r}\bar{A}_j\right)\right] + \bar{C}'\bar{C}$$

The RDI (58) can also be brought into the form of (61), where the rows and the columns containing $P_{4i}, i = 1, \dots, r$ should be deleted.

$$\begin{bmatrix}
\Phi & P\bar{D} & h_1 P\bar{A}_1 & h_1 P\bar{A}_1 & h_1 P\bar{A}_1\bar{D} & P\chi_{1k} & \cdots & h_r P\bar{A}_r & h_r P\bar{A}_r & h_r P\bar{A}_r\bar{D} & P\chi_{rk} \\
\bar{D}'P & -\gamma^2\sum_{i=1}^{r}(I - h_i P_{3i}) & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
h_1\bar{A}_1'P & 0 & -h_1 P_{11} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
h_1\bar{A}_1'P & 0 & 0 & -h_1 P_{21} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
h_1\bar{D}'\bar{A}_1'P & 0 & 0 & 0 & -\gamma^2 h_1 P_{31} & 0 & \cdots & 0 & 0 & 0 & 0 \\
P\chi_{1k} & 0 & 0 & 0 & 0 & -P_{41} & \cdots & 0 & 0 & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
h_r\bar{A}_r'P & 0 & 0 & 0 & 0 & 0 & \cdots & -h_r P_{1r} & 0 & 0 & 0 \\
h_r\bar{A}_r'P & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -h_r P_{2r} & 0 & 0 \\
h_r\bar{D}'\bar{A}_r'P & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & -\gamma^2 h_r P_{3r} & 0 \\
P\chi_{rk} & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & -P_{4r}
\end{bmatrix} \le 0 \qquad (61)$$

Remark 1: In the case when $\bar{A}_i(I - G_k) = 0$ for some $i > 0$ we set $P_{4i} = P_{4i}^{-1} = 0$ in (57) and (59) and we delete the row and the column containing $P_{4i}, i = 1, \ldots, r$ in (61).

The above result is obtained for any matrices in (9), (44). In the special case of matrices of the form (8), (45) we obtain that $\bar{A}_i(I - G_k) = 0$ for all $i = 1, \ldots, r$ and hence for all $k = K - 1, \ldots, 0$ we solve the RDI of (58) with the terminal conditions given by (52). We obtain the following corollary.

Corollary 1: *Assume that for every* $k = K - 1, \ldots, 0$ *there exist constant symmetric positive definite matrices* P_{1i}, P_{2i} *and* P_{3i} $i = 1, \ldots, r$ *with* $I - \sum_{i=1}^{r} h_i P_{3i} > 0$, *that allow a solution* $P(t) = P'(t) \geq 0$, *that is absolutely continuous for* $t \in [t_k, t_{k+1})$, *to the RDI of* (58) (*to the DLMI of* (61), *where the row and the column containing* $P_{4i}, i = 1, \ldots, r$ *is deleted*), *with the terminal conditions of* (52), *where* $\bar{A}_i, \bar{D}, \bar{C}, G_k$ *are given by* (8), (45) *and* $\bar{P}_f = 0$. *Then the performance index J of* (4) *with* $E = 0$ *for the closed-loop system of* (1), *where* $u = u_0(x_t)$, *is nonpositive for all* $w \in L_2[0, t_f]$ *and* $x_0 = 0$.

Remark 2: Note that the case of $A_{01} \neq 0$ requires modification of technique used in lemmas 3 and 4 and leads to more complicated DLMIs and RDIs.

Example: Consider the system

$$\dot{x}(t) = x(t) - x(t - h) + u + w, \quad z = \text{col}\,\{x, u\} \quad (62)$$

and $P_f = Q_f = R_f = 0$. For $h = 0$ this example coincides with the one in Basar and Bernard (1995, p. 135). We chose $t_f = 1$ s, $K = 2$, $t_1 = 0.5$ s and $\gamma = 1$. We verified that (35) (and, hence, linear equations (36)) with terminal conditions (37) had bounded solutions on $[0, 1]$:

$$P_0(t) = U_0(t) = \tan(1 - t), \; M_0(t) = \cos^{-1}(1 - t) - 1 \quad 0.5 \leq t \leq 1$$

$$P_0(t) = \tan(0.9721 - t), \quad 0 \leq t < 0.5$$

Therefore, Assumption 1 holds and

$$u_0(t) = 0, \quad 0 \leq t < 0.5; \quad u_0(t) = -0.2553x(0.5) \quad 0.5 \leq t < 1$$

From (39) and (43) we obtained

$$P_1(0.5) = -1.1955, \quad M_1(0.5) = -0.3113$$

$$U_1(0.5) = -0.1029$$

$$u_1(t) = 0, \quad 0 \leq t < 0.5$$

$$u_1(t) = u_0(t) - h\left[3.0253x(0.5) - 0.2553\int_{-1}^{0} x(0.5 + hs)\,ds\right], \quad 0.5 \leq t \leq 1$$

Considering the system of (62), where $u = u_0$, and applying the RDI of Lemma 3, we were not able to find a time-invariant $Q > 0$ for which the RDI of (56) holds. If there existed such a Q, then u_0 would lead to a closed-loop performance level of $\gamma = 1$ for all h. Applying therefore Lemma 4 and choosing $P_{11} = P_{21} = P_{31} = P_{41} = I$ we find that the RDI of (58) with terminal condition of (59) has a solution for all $0 \leq h \leq 0.17$ s. For $h = 0.18$ s the solution of (58), with the equality sign and (59), has an escape point (see Figure 1). The control law of $u = u_0$ thus guarantees $\gamma = 1$ for all $0 \leq h \leq 0.17$s, whereas for $h = 0.18$ our theory, which only provides a sufficient condition cannot guarantee the performance level of $\gamma = 1$.

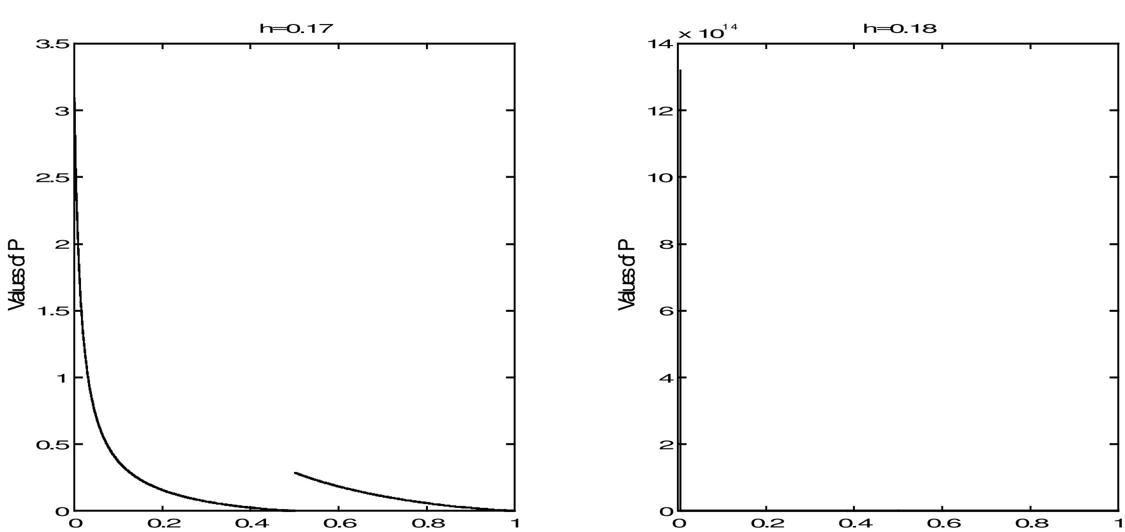

Figure 1. The (1, 2) component of the solution P to (57) for $h = 0.17$ s and $h = 0.18$ s.

w	1	$\sin 2\pi t$	$\cos 2\pi t$	$\sin 4\pi t$	$\cos 4\pi t$	$\sin 6\pi t$	$\cos 6\pi t$
J	−0.5701	−0.4446	−0.4827	−0.4874	−0.4924	−0.4946	−0.4986

Table 1. The values of J for $h = 0.17$ s.

The behaviour of (62) for $h = 0.17$ under the controller $u = u_0$ and the disturbances $w = 1$, $w = \sin 2\pi nt$, $w = \cos 2\pi nt$, $n = 1, 2, \ldots$ (that constitute the orthogonal basis in $L_2[0, 1]$) has been simulated. We see from Table 1 that indeed the resulting $J \le 0$ for all values of w under consideration.

6. Conclusions

The paper presents a solution to the sampled-data state-feedback H_∞ control of linear time-invariant systems with state time-delays in the finite horizon case. Similarly to sampled data H_∞-control of systems without delay (Khargonekar *et al.* 1993, Basar and Bernard 1995, Sagfors and Toivonen 1997), our solution includes two steps: one of solving the continuous-time problem between sampling and the second is an updating the terminal conditions at the sampling instants. One additional outcome that stems from the theory of this paper is the derivation of new bounded real lemmas for invariant-time systems with state-delays and jumps.

The theory that has been developed here shows that for small time delays, similarly to the case of singularly perturbed systems (Pan and Basar 1993, Fridman, 1996), our controllers are affected by the boundary-layer phenomenon. This fact requires evaluation of both, outer expansion and boundary-layer corrections.

One can adopt the Riccati operator equations approach of Bensoussan *et al.* (1992), van Keulen *et al.* (1993) and McMillan and Triggiani (1993) to solve sampled-data state-delay case. This leads to differential Riccati operator equations with jumps. The performance that is obtained in this case can be analysed, similarly to Ndiaye and Sorine (2000), only for the zero-order approximation.

Appendix

Proof of Lemma 1: Let $\bar{x}(t)$ be a solution of (9). Then, differentiating $\bar{V}(t, \bar{x}_t)$ with respect to t we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\bar{V}(t,\bar{x}_t) &= 2[\bar{L}(\bar{x}_t(\cdot)) + \bar{D}w]' \\ &\quad\times\left[\bar{P}(t)\bar{x}(t) + \int_{-h}^{0}\frac{\partial}{\partial t}\bar{Q}(t,\xi)\bar{F}(\bar{x}_t)(\xi)\,\mathrm{d}\xi\right] \\ &\quad+\bar{x}'(t)\left[\dot{\bar{P}}(t)\bar{x}(t) + 2\int_{-h}^{0}\frac{\partial}{\partial t}\bar{Q}(t,\xi)\bar{F}(\bar{x}_t)(\xi)\,\mathrm{d}\xi\right] \\ &\quad+2\bar{x}'(t)\int_{-h}^{0}\bar{Q}(t,\xi)\frac{\mathrm{d}}{\mathrm{d}t}[\bar{F}(\bar{x}_t)](\xi)\,\mathrm{d}\xi \\ &\quad+\int_{-h}^{0}\int_{-h}^{0}\frac{d}{dt}[\bar{F}'(\bar{x}_t)(s)\bar{R}(t,s,\xi)\bar{F}(\bar{x}_t)(\xi)]\,\mathrm{d}s\,\mathrm{d}\xi\end{aligned} \tag{63}$$

where $\bar{L}(\bar{x}_t(\cdot))$ is defined by (2) with all matrices taken with bars. Denote by

$$w^* = \gamma^{-2}\bar{D}'\left[\bar{P}\bar{x}(t) + \int_{-h}^{0}\bar{Q}(t,\xi)\bar{F}(\bar{x}_t)(\xi)\,\mathrm{d}\xi\right]$$

Then, integrating by parts in (63) e.g.

$$\begin{aligned}\int_{-h}^{0}\bar{Q}(t,\xi)\frac{\mathrm{d}}{\mathrm{d}t}\bar{F}(\bar{x}_t)(\xi)\,\mathrm{d}\xi &= -\int_{-h}^{0}\bar{Q}(t,\xi)\left\{\frac{\mathrm{d}}{\mathrm{d}\xi}[\bar{F}(\bar{x}_t)(\xi)] - \bar{A}_{01}(\xi)\bar{x}(t)\right\}\mathrm{d}\xi \\ &= -\bar{Q}(t,0)[\bar{L}(\bar{x}_t(\cdot)) - \bar{A}_0\bar{x}(t)] \\ &\quad+\sum_{i=1}^{r}\bar{Q}(t,-h_i)\bar{A}_i\bar{x}(t) \\ &\quad+\int_{-h}^{0}\frac{\partial}{\partial\xi}\bar{Q}(t,\xi)\bar{F}(\bar{x}_t)(\xi)\,\mathrm{d}\xi \\ &\quad+\int_{-h}^{0}\bar{Q}(t,\xi)\bar{A}_{01}(\xi)\,\mathrm{d}\xi\;\bar{x}(t)\end{aligned}$$

where due to (19)

$$\bar{Q}(t,0)[\bar{L}(\bar{x}_t(\cdot)) - \bar{A}_0\bar{x}(t)] = \bar{P}(t)[\bar{L}(\bar{x}_t(\cdot)) - \bar{A}_0\bar{x}(t)]$$

and applying (16)–(19) we get (64)

$$\begin{aligned}\frac{\mathrm{d}\bar{V}}{\mathrm{d}t} &= \bar{x}'(t)[\dot{\bar{P}}(t) + \bar{A}_0'\bar{P} + \bar{P}\bar{A}_0 \\ &\quad+\sum_{i=1}^{r}\bar{A}_i'\bar{Q}'(t,-h_i) + \sum_{i=1}^{k}\bar{Q}(t,-h_i)\bar{A}_i \\ &\quad+\int_{-h}^{0}\bar{A}_{01}'\bar{Q}'(t,\theta)\,\mathrm{d}\theta + \int_{-h}^{0}\bar{Q}(t,\theta)\bar{A}_{01}(\theta)\,\mathrm{d}\theta]\bar{x}(t) \\ &\quad+2\bar{x}'(t)\int_{-h}^{0}\left[\frac{\partial}{\partial t}\bar{Q}(t,\xi) + \frac{\partial}{\partial\xi}\bar{Q}(t,\xi)\right. \\ &\quad+\bar{A}_0'\bar{Q}(t,\xi) + \sum_{i=1}^{r}\bar{A}_i'\bar{R}(t,-h_i,\xi) \\ &\quad\left.+\int_{-h}^{0}\bar{A}_{01}'(\theta)\bar{R}(t,\theta,\xi)\,\mathrm{d}\theta\right]\bar{F}(\bar{x}_t)(\xi)\,\mathrm{d}\xi\end{aligned}$$

$$+\int_{-h}^{0}\int_{-h}^{0}\bar{F}'(\bar{x}_t)(s)\left[\frac{\partial}{\partial\xi}\bar{R}(t,\xi,s)+\frac{\partial}{\partial s}\bar{R}(t,\xi,s)+\frac{\partial}{\partial t}\bar{R}(t,\xi,s)\right]\bar{F}(\bar{x}_t)(\xi)\,d\xi\,ds$$
$$+2\gamma^2 w'w^* = \bar{x}'(t)(-\bar{C}'\bar{C}-\bar{P}\bar{S}\bar{P})\bar{x}(t)$$
$$+2\gamma^2 w'w^* - 2\bar{x}'(t)\int_{-h}^{0}\bar{P}\bar{S}\bar{Q}(t,\xi)\bar{F}(\bar{x}_t)(\xi)\,d\xi$$
$$-\int_{-h}^{0}\int_{-h}^{0}\bar{F}'(\bar{x}_t)(s)\bar{Q}'(t,\xi)\bar{S}\bar{Q}(t,s)\bar{F}(\bar{x}_t)(\xi)\,d\xi\,ds$$
$$= -\bar{x}'(t)\bar{C}'\bar{C}\bar{x}(t)-\gamma^2(w-w^*)'(w-w^*)+\gamma^2 w'w$$

It follows from (64) that

$$\bar{E}(\bar{x}_{t_f})-\bar{V}(t_{K-1},\bar{x}_{t_{K-1}})+\int_{t_{K-1}}^{t_f}[|z|^2-\gamma^2|w|^2]\,dt = -\gamma^2\|w-w^*\|_{L_2}$$

The latter relation implies (21).

Proof of Theorem 2: The proof of (i) is similar to Fridman and Shaked (1998). Equation (43) follows from (42). We prove only that u_m leads to an attenuation level of $\gamma+O(h^{m+1})$. We apply u and u_m on (9) and (10). We obtain correspondingly $\bar{x}$ and x_m that satisfy (9) and the jump conditions

$$\bar{x}(t_k)=W_k\bar{x}_{t_k^-},\quad x_{mt_k}=[W_k+O(h^{m+1})]x_{mt_k^-}$$

where

$$W_k\bar{x}_t=\mathrm{col}\left\{x(t),-U^{-1}(t_k)\left[M'(t_k)x(t)+\int_{-h}^{0}N'(t_k,s)F(x_t)(s)ds\right]\right\}$$

Thus, $y_t=\bar{x}_t-x_{mt}$ satisfies

$$\dot{y}(t)=\sum_{i=0}^{r}\bar{A}_i y(t-h_i)+\int_{-h}^{0}\bar{A}_{01}(s)y(t+s)\,ds$$
$$\bar{z}(t)=\bar{C}\bar{x}(t),\quad t_k\le t<t_{k+1} \tag{65}$$
$$y(t_k)=W_k y_{t_k^-}+O(h^{m+1})x_{mt_k^-} \tag{66}$$

We find that

$$\|z_m-z\|_{L_2}^2\le c_0(|y_{t_f}|_c^2+h^{m+1}|x_{t_f}|_c^2)+\int_0^{t_f}c_1[|y_t|_c^2+h^{m+1}|\bar{x}_t|_c^2]\,dt$$
$$\le c_0(|y_{t_f}|_c^2+h^{m+1}|x_{t_f}|_c^2)+c_1[\|y\|_{L_2}^2+h^{m+1}\|\bar{x}_t\|_{L_2}^2],\quad c_1>0 \tag{67}$$

where $\|x_t\|_{L_2}=\|\max_{\theta\in[-h,0]}x_t(\theta)\|_{L_2}$. Evidently, $\|x(t)\|_{L_2}\le\|x_t\|_{L_2}$.

Let $X(t)$ be a transition matrix for (9), $X(0)=I$ and $X(t)=0$ for $t<0$. Denote $T(t)X_0=X_t$. By the variation of constants formula (Hale 1977)

$$\bar{x}_t=T(t-t_k)W_k\bar{x}_{t_k^-}+\int_{t_k}^{t}T(t-s)X_0Dw(s)\,ds\qquad t_k<t\le t_{k+1} \tag{68}$$

From the latter equation and the condition $\bar{x}_0=0$ we have $|x_{t_1}^-|_c\le c_1\|w\|_{L_2}$. By induction it is easy to obtain from (68) that

$$|\bar{x}_{t_k^-}|_c\le c'\|w\|_{L_2},\quad k=1,\ldots,K,\ c'>0 \tag{69}$$

Then from (68) and (69) we derive

$$\|\bar{x}_t\|_{L_2}^2=\sum_{k=0}^{K}\int_{t_k}^{t_{k+1}}|\bar{x}_s|_c^2ds\le c_3\left[\sum_{k=0}^{K}|\bar{x}_{t_k^-}|_c^2+\|w\|_{L_2}^2\right]\le c\|w\|_{L_2}^2$$

Similarly to the latter inequality, one can derive $\|x_{mt}\|_{L_2}^2\le c\|w\|_{L_2}^2$, and $\|y_t\|_{L_2}^2\le ch^{m+1}\|\bar{x}_t\|_{L_2}^2\le c^2h^{m+1}\|w\|_{L_2}^2$. The latter inequalities, together with (67), imply $\|z_m\|_{L_2}^2=\|z\|_{L_2}^2+O(h^{m+1})\|w\|_{L_2}^2$. Since $\|z\|_{L_2}^2\le\gamma^2\|w\|_{L_2}^2$, we derive $\|z_m\|_{L_2}^2\le[\gamma^2+O(h^{m+1})]\|w\|_{L_2}^2=[\gamma+O(h^{m+1})]^2\|w\|_{L_2}^2$. □

Proof of Lemma 4: Since $\bar{x}(t)=0$ and $w(t)=0$ for $t<0$ and $h_0=0$ we find that for $t\in[t_k,t_k+h_i)$

$$\bar{x}(t)=\bar{x}(t_k)+\int_{t_k}^{t}\left[\sum_{j=0}^{r}\bar{A}_j\bar{x}(\tau-h_j)\right]d\tau+\int_{t_k}^{t}\bar{D}w\,d\tau$$

and

$$\bar{x}(t_k^-)=\bar{x}(t-h_i)+\int_{t-h_i}^{t_k}\left[\sum_{j=0}^{r}\bar{A}_j\bar{x}(\tau-h_j)\right]d\tau+\int_{t-h_i}^{t_k}\bar{D}w\,d\tau$$

Summing the latter two equations and using (44) we obtain that for all $t\in[0,t_f]$

$$\bar{x}(t-h_i)=\bar{x}(t)-\int_{t-h_i}^{t}\left[\sum_{j=0}^{r}\bar{A}_j\bar{x}(\tau-h_j)\right]d\tau-\int_{t-h_i}^{t}\bar{D}w\,d\tau+(I-G_k)\bar{x}(t_k^-)\chi_{ik}(t)$$

Thus, for $t\in[0,t_f]$

$$\dot{\bar{x}}(t) = \sum_{i=0}^{r} \bar{A}_i \bar{x}(t) - \sum_{i=1}^{r} \bar{A}_i \int_{t-h_i}^{t} \left[\sum_{j=0}^{r} \bar{A}_j \bar{x}(\tau - h_j) \right] \mathrm{d}\tau - \sum_{i=1}^{r} \bar{A}_i \bar{D} \int_{t-h_i}^{t} w \, \mathrm{d}\tau + \bar{D} w(t) + \sum_{i=1}^{r} \bar{A}_i (I - G_k) \bar{x}(t_k^-) \chi_{ik}(t)$$

Defining the function $V(t, \bar{x}(t)) \triangleq \bar{x}'(t) P(t) x(t)$, we obtain that for almost all t

$$\frac{d}{dt} V(t, \bar{x}(t)) = \bar{x}'(t) \left[\dot{P}(t) + \sum_{j=0}^{r} \bar{A}_j' P(t) + P(t) \sum_{j=0}^{r} \bar{A}_j \right] \times \bar{x}(t) - 2(\eta_1 + \eta_2 + \eta_3 - \eta_4) \tag{70}$$

where

$$\eta_1(t) \triangleq \sum_{i=1}^{r} \int_{t-h_i}^{t} \bar{x}'(t) P(t) \bar{A}_i \bar{A}_0 \bar{x}(\tau) \, \mathrm{d}\tau$$

$$\eta_2(t) \triangleq \sum_{i=1}^{r} \int_{t-h_i}^{t} \bar{x}'(t) P(t) \bar{A}_i \left[\sum_{j=1}^{r} \bar{A}_j \bar{x}(\tau - h_j) \right] \mathrm{d}\tau$$

$$\eta_3(t) \triangleq \sum_{i=1}^{r} \int_{t-h_i}^{t} \bar{x}'(t) P(t) \bar{A}_i \bar{D} w(\tau) \, \mathrm{d}\tau - \bar{x}'(t) P(t) \bar{D} w(t)$$

$$\eta_4 \triangleq \bar{x}'(t) P(t) \sum_{i=1}^{r} [\bar{A}_i (I - G_k) \bar{x}(t_k^-) \chi_{ik}(t)]$$

Since for any z, $y_i \in R^n$, $i = 1, \ldots, r$ and for any symmetric positive definite matrices $X_i \in R^{n \times n}$

$$-2 \sum_{i=1}^{r} y_i' z \le \sum_{i=1}^{r} y_i' X_i^{-1} y_i + \sum_{i=1}^{r} z' X_i z$$

we find that for any $(n+l) \times (n+l)$ constant symmetric matrices $P_{1i} > 0, P_{2i} > 0, P_{3i}, P_{4i}$, $i = 0, \ldots, r$, and for any $q \times q$ symmetric constant matrix $P_5 > 0$

$$\left.\begin{aligned}
-2\eta_1 &\le \sum_{i=1}^{r} h_i \bar{x}' P \bar{A}_i P_{1i}^{-1} \bar{A}_i' P \bar{x} + \sum_{i=1}^{r} \int_{t-h_i}^{t} \bar{x}' \bar{A}_0' P_{1i} \bar{A}_0 \bar{x} \, \mathrm{d}\tau \\
-2\eta_2 &\le \sum_{i=1}^{r} h_i \bar{x}' P \bar{A}_i P_{2i}^{-1} \bar{A}_i' P \bar{x} + \sum_{i=1}^{r} \int_{t-h_i}^{t} \left[\sum_{j=1}^{r} \bar{x}'(\tau - h_j) \bar{A}_j' \right] \times P_{2i} \left[\sum_{j=1}^{r} \bar{A}_j \bar{x}(\tau - h_j) \right] \mathrm{d}\tau \\
-2\eta_3 &\le \gamma^2 \sum_{i=1}^{r} \int_{t-h_i}^{t} w' P_{3i} w \, \mathrm{d}\tau + \gamma^{-2} \sum_{i=1}^{r} \bar{x}' P \bar{A}_i H_i \bar{A}_i' P \bar{x} + \gamma^2 w' P_5 w + \gamma^{-2} \bar{x}' P \bar{D} P_5^{-1} \bar{D}' P \bar{x} \\
2\eta_4 &\le \bar{x}'(t) P(t) \left[\sum_{i=1}^{r} P_{4i}^{-1} \chi_{ik}(t) \right] P(t) \bar{x}(t) + \bar{x}'(t_k^-)(I - G_k') \left[\sum_{i=1}^{r} \bar{A}_i' P_{4i} \bar{A}_i \chi_{ik}(t) \right] \times (I - G_k) \bar{x}(t_k^-)
\end{aligned}\right\} \tag{71}$$

where H_i is defined in (60). Since

$$\int_0^{t_f} \frac{\mathrm{d}}{\mathrm{d}t} V(t, \bar{x}_t) \, \mathrm{d}t = \bar{E}(\bar{x}_{t_f}) - \sum_{k=1}^{K-1} [V(t_k, \bar{x}_{t_k}) - V(t_k^-, \bar{x}_{t_k}^-)]$$

we find, using (70)–(71), that

$$\begin{aligned}
J &= \int_0^{t_f} \frac{\mathrm{d}}{\mathrm{d}t} V(t, \bar{x}_t) \, \mathrm{d}t + \sum_{k=1}^{K-1} \left[V(t_k, \bar{x}_{t_k}) - V(t_k^-, \bar{x}_{t_k}^-) \right] + ||z||_{L_2}^2 - \gamma^2 ||w||_{L_2}^2 \\
&\le \sum_{k=1}^{K-1} \left\{ \bar{x}'(t_k) P(t_k) \bar{x}(t_k) - \bar{x}'(t_k^-) P(t_k^-) \bar{x}(t_k^-) + \bar{x}'(t_k^-)(I - G_k') \times \left[\sum_{i=1}^{r} (h_i \bar{A}_i' P_{4i} \bar{A}_i)(I - G_k) \bar{x}(t_k^-) \right] \right\} \\
&\quad + \int_0^{t_f} \bar{x}' S_1(t) \bar{x} \, \mathrm{d}t + \gamma^2 \int_0^{t_f} w'(P_5 - I) w \, \mathrm{d}t \\
&\quad + \int_0^{t_f} \sum_{i=1}^{r} \int_{t-h_i}^{t} \left\{ \bar{x}' \bar{A}_0' P_{1i} \bar{A}_0 \bar{x} + \left[\sum_{j=1}^{r} \bar{x}'(\tau - h_j) A_j' \right] \times P_{2i} \left[\sum_{j=1}^{r} \bar{A}_j \bar{x}(\tau - h_j) \right] + \gamma^2 w' P_{3i} w \right\} \mathrm{d}\tau \, \mathrm{d}t
\end{aligned} \tag{72}$$

where

$$S_1(t) \triangleq \dot{P} + \sum_{i=0}^{r} \bar{A}_i' P + P \sum_{i=0}^{r} \bar{A}_i + P \left[\sum_{i=1}^{r} (\Gamma_i + \gamma^{-2} \bar{A}_i H_i \bar{A}_i) + \sum_{i=1}^{r} P_{4i}^{-1} \chi_{ik}(t) + \gamma^{-2} \bar{D} P_5^{-1} \bar{D}' \right] P + \bar{C}' \bar{C} \tag{73}$$

and where Γ_i is defined in (60). Due to (44) and (59) the term in the second line of (72) vanishes. Then from the inequality

$$\int_0^{t_f}\left[\int_{t-h_i}^{t}\xi'(\tau)\xi(\tau)d\tau\right]\mathrm{d}t\le h_i\int_0^{t_f}\xi'(t)\xi(t)\,\mathrm{d}t,$$
$$\forall\,\xi(t)\,:\,\xi(t)=0,\forall\,t\le 0$$

and $\bar{x}_0=0$ and $w_0=0$, and from (72) and (59) it follows that

$$J\le\int_0^{t_f}\bar{x}'S(t)\bar{x}\,\mathrm{d}t+\int_0^{t_f}\gamma^2w'\left(\sum_{i=1}^{r}h_iP_{3i}+P_5-I\right)w\,\mathrm{d}t \tag{74}$$

where

$$S(t)=S_1(t)+\sum_{i=1}^{r}h_i\left[\bar{A}_0P_{1i}\bar{A}_0+\left(\sum_{j=1}^{r}\bar{A}_j'\right)P_{2i}\left(\sum_{j=1}^{r}\bar{A}_j\right)\right]$$

If $S(t)\le 0$ and $P_5=I-\sum_{i=1}^{r}h_iP_{3i}$, due to (74) $J\le 0$ for all $w\in L_2[0,t_f]$. Finally we note that, in view of (74) the conditions of (74) are equivalent to the RDI (57). □

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