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Delay-dependent stability and H_∞ control: constant and time-varying delays

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Three main model transformations were used in the past for delay-dependent stability. Recently a new (descriptor) model transformation has been introduced. In the present paper, we compare methods under different transformations and show the advantages of the descriptor one. We obtain new delay-dependent stability conditions for systems with time-varying delays in terms of linear matrix inequalities. We also refine recent results on delay-dependent H_∞control and extend them to the case of time-varying delays. Numerical examples illustrate the effectiveness of our method.

1. Introduction

Time-delay often appears in many control systems (such as aircraft, chemical or process control systems) and, in many cases, delay is a source of instability (see, for example, Hale and Lunel 1993). The stability issue of systems with delay is, therefore, of theoretical and practical importance. It is well-known (see, for example Kolmanovskii and Richard 1999) that the choice of an appropriate Lyapunov–Krasovskii functional is crucial for deriving stability conditions. The general form of this functional leads to a complicated system of partial differential equations (see, for example, Malek-Zavarei and Jamshidi 1987). Special forms of Lyapunov–Krasovskii functionals lead to simpler delay-independent (Boyd *et al*. 1994, Verriest and Niculescu 1998, Kolmanovskii and Richard 1999) and (less conservative) delay-dependent conditions (Niculescu *et al*. 1995, Li and de Souza 1997, Kolmanovskii *et al*. 1999, Kolmanovskii and Richard 1999, Park 1999, Lien *et al*. 2000, Niculescu 2001 a). Note that the latter simpler conditions are appropriate in the case of unknown delay, either unbounded (delay-independent conditions) or bounded by a known upper bound (delay-dependent conditions).

Delay-dependent stability conditions in terms of linear matrix inequalities (LMIs) have been obtained for retarded and neutral type systems. These conditions are based on three main model transformations of the original system (see Kolmanovskii and Richard 1999).

Recently a new *descriptor* model transformation was introduced for delay-dependent stability of neutral systems (Fridman 2001) and of a more general class of differential and algebraic (descriptor) system with delay (Fridman 2002 b). Unlike previous transformations, the descriptor model leads to a system which is equivalent to the original one, it does not depend on additional assumptions for stability of the transformed system and requires bounding of fewer cross-terms.

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Two main approaches for dealing with the stability of systems with time-varying delays have been suggested in the past. The first is based on Lyapunov–Krasovskii functionals and the second is based on Razumikhin theory. Two main cases of time-varying delays have been considered:

(1) differentiable uniformly bounded delays with delay-derivatives bounded by $d < 1$; and
(2) continuous uniformly bounded delays.

To the best of our knowledge, the Razumikhin approach was the only one that was able to cope with the second case, which allows fast time-varying delays.

In the present paper, we shed more light on the conservatism of the various model transformations and show the advantages of the descriptor one. We improve the delay-dependent stability conditions of Fridman (2001), that were based on descriptor model transformation, by applying tighter bounding of the cross terms introduced in Park (1999). We extend the results of Fridman (2001) to the case of systems with polytopic uncertainties and time-varying delays. We consider both the above-mentioned cases of time-varying delays. Our method based on the Lyapunov–Krasovskii functional seems to be the first of this type for the second case. In the first case our results significantly improve the existing ones (see Kim 2001 and references therein). Numerical example shows that our method, even for the more robust second case, leads to less restrictive results than those of Kim (2001) which were obtained for the first case on the basis of the first transformation. The new conditions are given in terms of LMIs.

A descriptor model transformation has been applied recently for H_∞ control problem Fridman and Shaked (2002 a). We refine the results of Fridman and Shaked (2002 a) and extend them to the time-varying case. Numerical examples are given which illustrate the advantages of our method.

Notation: Throughout the paper the superscript 'T' stands for matrix transposition, $\mathcal{R}^n$ denotes the n dimensional Euclidean space with vector norm $|\cdot|$, $\mathcal{R}^{n\times m}$ is the

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set of all $n \times m$ real matrices, and the notation $P > 0$, for $P \in \mathcal{R}^{n\times n}$, means that P is symmetric and positive definite. Let $C_n[a, b]$ denote the space of continuous functions ϕ: $[a, b] \to \mathcal{R}^n$ with the supremum norm $|\cdot|$. We also denote $x_t(\theta) = x(t+\theta)$ $(\theta \in [-h, 0])$.

2. On delay-dependent conditions under four model transformations

In this section we analyse the sources for the conservatism of the delay-dependent methods under different model transformations, which transform a system with discrete delays into one with distributed delays. It is well-known that this conservatism either stems from additional dynamics in the transformed system, in the case of transformation I below (Kharitonov and Melchor-Aguilar 2000, Gu and Niculescu 2000), or from additional assumptions that are made to guarantee the stability of the transformed system, in the case of transformation II below (Niculescu 2001 a). A frequency domain interpretation of stability conditions under transformations I and III was given recently in Zhang *et al.* (2001).

We show below that the additional terms in the derivative of Lyapunov–Krasovskii functional, which have to be treated as in the delay-independent case, lead to overdesign in the case of transformation III. Moreover, in all the model transformations the results are restrictive due to the bounding of cross terms. A minimum number of such terms and tighter bounding of them should certainly lead to better results. We also discuss the case of time-varying delays. For simplicity we consider a retarded linear system with a single delay

$$\dot{x}(t) = A_0x(t) + A_1x(t-h) \tag{1}$$

where $x(t) \in R^n$, $h > 0$, $A_0, A_1 \in \mathcal{R}^{n\times n}$.

2.1. *On the conservatism of delay-dependent conditions*

I. *The first transformation*

$$\dot{x}(t) = [A_0 + A_1]x(t) - A_1\int_{t-h}^{t}[A_0x(s) + A_1x(s-h)]\mathrm{d}s \tag{2}$$

It is well-known (see, for example Kharitonov and Melchor-Aguilar 2000, Gu and Niculescu 2000) that (2) is not equivalent to the original one and has additional dynamics. By choosing a Lyapunov–Krasovskii functional of the form (see, for example Kolmanovskii *et al.* 1999)

$$V(t) \triangleq x^{\mathrm{T}}(t)P_1x(t) + V_2 + V_3, \qquad P_1 = P_1^{\mathrm{T}} > 0 \tag{3}$$

with

$$\left.\begin{aligned} V_2 &= \int_{-h}^{0}\int_{t+\theta}^{t}x^{\mathrm{T}}(s)R_0x(s)\mathrm{d}s\,\mathrm{d}\theta \\ V_3 &= \int_{-2h}^{-h}\int_{t+\theta}^{t}x^{\mathrm{T}}(s)R_1x(s)\mathrm{d}s\,\mathrm{d}\theta,\ R_0 > 0,\ R_1 > 0 \end{aligned}\right\} \tag{4 a, b}$$

it is found that

$$\dot{V} = x^{\mathrm{T}}[(A_0 + A_1)^{\mathrm{T}}P_1 + P_1(A_0 + A_1)]x + \eta_1 + \eta_2 + \dot{V}_2 + \dot{V}_3$$

where

$$\eta_1(t) \triangleq -2\int_{t-h}^{t}x^{\mathrm{T}}(t)PA_1A_0x(s)\mathrm{d}s,$$

$$\eta_2(t) \triangleq -2\int_{t-h}^{t}x^{\mathrm{T}}(t)PA_1A_1x(s-h)\mathrm{d}s$$

Two cross terms η_1 and η_2 should then be bounded. In the case of m delays, $2m$ cross terms have to be bounded. Note that the terms V_2 and V_3 correspond to delay-dependent conditions and they compensate the terms that emerge when bounding the cross terms η_1 and η_2.

In the case of a system with time-varying delay

$$\dot{x}(t) = A_0x(t) + A_1x(t-\tau(t))$$
$$0 \le \tau \le h, \quad \dot{\tau}(t) \le d < 1 \tag{5}$$

the Lyapunov–Krasovskii functional has the form of (3) with V_2 given by (4 a) and V_3 of the form (Kim 2001)

$$V_3 = \int_{-\tau-h}^{-\tau}\int_{t+\theta}^{t}x^{\mathrm{T}}(s)R_1x(s)\mathrm{d}s\,\mathrm{d}\theta$$

Differentiation of V_3 leads to additional terms due to the fact that the limits of the integral ($-\tau$ and $-\tau-h$) depend on t.

II. *The second* (*neutral type*) *transformation* brings (1) to the equivalent system

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[x(t) + A_1\int_{t-h}^{t}x(s)\mathrm{d}s\right] = (A_0 + A_1)x(t)$$

A Lyapunov–Krasovskii functional for this case has the form (see e.g. Niculescu 2001)

$$V(t) \triangleq \mathcal{D}^{\mathrm{T}}(x_t)P_1\mathcal{D}(x_t) + V_2, \qquad P_1 = P_1^{\mathrm{T}} > 0$$

where V_2 is given by (4 *a*)

$$\mathcal{D}(x_t) = x(t) + A_1\int_{t-h}^{t}x(s)\mathrm{d}s$$

Under an additional assumption on the stability of $\mathcal{D}$ (i.e. asymptotic stability of the equation $\mathcal{D}(x_t) = 0$) the system is asymptotically stable if $\dot{V}(t) < 0$. This assumption is difficult to verify. Sufficient condition for the stability of $\mathcal{D}$ is that $h|A_1| < 1$, where $|\cdot|$ is any matrix norm, but the latter may lead to conservative results (see Example 2 in Fridman 2001).

In the case of transformation II

$$\dot{V} = x^{\mathrm{T}}(t)[(A_0 + A_1)^{\mathrm{T}} P_1 + P_1(A_0 + A_1)]x(t) + \eta_1 + \dot{V}_2$$

where

$$\eta_1 = 2\int_{t-h}^{t} x^{\mathrm{T}}(t)(A_0 + A_1)^{\mathrm{T}} P A_1 x(s)\mathrm{d}s$$

The number of cross terms that have to be bounded in this case is thus half of those obtained by applying the first transformation.

Transformation II is not appropriate for the case of time-varying delay.

III. *The third transformation* leads to the system

$$\dot{x}(t) = (A_0 + A_1)x(t) - A_1\int_{t-h}^{t} \dot{x}(s)\mathrm{d}s$$

which is equivalent to the original system (for absolutely continuous initial functions). Here the Lyapunov–Krasovskii functional has the form (Kolmanovskii and Richard 1999, Park 1999)

$$V(t) \triangleq x^{\mathrm{T}}(t)P_1x(t) + \tilde{V}_2 + V_3, \qquad P_1 = P_1^{\mathrm{T}} > 0$$

where

$$\tilde{V}_2 = \int_{-h}^{0}\int_{t+\theta}^{t} \dot{x}^{\mathrm{T}}(s)A_1^{\mathrm{T}}RA_1\dot{x}(s)\mathrm{d}s\mathrm{d}\theta$$

$$V_3 = \int_{t-h}^{t} x^{\mathrm{T}}(s)Sx(s)\mathrm{d}s$$

Note that term V_3 corresponds to delay-independent stability. It is found that

$$\begin{aligned}\dot{V} &= x^{\mathrm{T}}[(A_0 + A_1)^{\mathrm{T}} P_1 + P_1(A_0 + A_1)]x + \eta_1 \\ &\quad + h\dot{x}^{\mathrm{T}}(t)A_1^{\mathrm{T}}RA_1\dot{x}(t) \\ &\quad - \int_{-h}^{0} \dot{x}^{\mathrm{T}}(t+\theta)A_1^{\mathrm{T}}RA_1\dot{x}(t+\theta)\mathrm{d}\theta + \dot{V}_3\end{aligned}$$

where

$$\eta_1(t) \triangleq -2\int_{t-h}^{t} x^{\mathrm{T}}(t)P_1A_1\dot{x}(s)\mathrm{d}s$$

The additional positive term $h\dot{x}^{\mathrm{T}}(t)A_1^{\mathrm{T}}RA_1\dot{x}(t)$ in the expression for $\dot{V}$ is treated then as

$$\begin{aligned}h\dot{x}^{\mathrm{T}}(t)A_1^{\mathrm{T}}RA_1\dot{x}(t) = h(A_0x(t) \\ + A_1x(t-h))^{\mathrm{T}}A_1^{\mathrm{T}}RA_1(A_0x(t) \\ + A_1x(t-h))\end{aligned}$$

The vectors $x(t)$ and $x(t-h)$ are considered further to be independent and they are treated as in the delay-independent case. This leads to an additional conservatism.

Transformation III may be adapted to treat the first case of time-varying delays.

IV. *The fourth* (*descriptor*) *transformation* was introduced in Fridman (2001)

$$\dot{x}(t) = y(t), \qquad y(t) = A_0x(t) + A_1x(t-h) \tag{6}$$

The latter can be represented in the form of a descriptor system with distributed delay in the 'fast variable' y for $t \geq h$

$$\dot{x}(t) = y(t), \qquad y(t) = (A_0 + A_1)x(t) - A_1\int_{t-h}^{t} y(s)\mathrm{d}s \tag{7}$$

Conversely, every solution of (7) satisfies (1) for $t \geq h$. Hence, the linear systems (7) and (1) are equivalent from the point of view of stability, i.e. stable or unstable simultaneously.

The Lyapunov–Krasovskii functional for the latter system has the form:

$$V(t) = [x^{\mathrm{T}}(t)\ y^{\mathrm{T}}(t)]EP\begin{bmatrix} x(t) \\ y(t)\end{bmatrix} + V_2 \tag{8}$$

where

$$E = \begin{bmatrix} I & 0 \\ 0 & 0\end{bmatrix}, \quad P = \begin{bmatrix} P_1 & 0 \\ P_2 & P_3\end{bmatrix}, \quad P_1 = P_1^{\mathrm{T}} > 0 \tag{9}$$

$$V_2 = \int_{-h}^{0}\int_{t+\theta}^{t} y^{\mathrm{T}}(s)Ry(s)\mathrm{d}s\mathrm{d}\theta, \qquad R > 0 \tag{10}$$

In this case

$$\dot{V} = \xi^{\mathrm{T}}\Psi\xi + \eta_1 + \dot{V}_2$$

where $\xi = \mathrm{col}\{x, y\}$

$$\Psi \triangleq P^{\mathrm{T}}\begin{bmatrix} 0 & I \\ A_0 + A_1 & -I\end{bmatrix} + \begin{bmatrix} 0 & (A_0 + A_1)^{\mathrm{T}} \\ I & -I\end{bmatrix}P$$

$$\eta_1(t) \triangleq -2\int_{t-h}^{t}[x^{\mathrm{T}}(t)\ y^{\mathrm{T}}(t)]P^{\mathrm{T}}\begin{bmatrix} 0 \\ A_1\end{bmatrix}y(s)\mathrm{d}s$$

It will be shown in the next section that the same results remain true in the case of continuous and bounded delay $\tau \leq h$.

The delay-dependent results are conservative under all transformations due to the bounding of the cross terms. In this sense transformations II–IV lead to less restrictive methods having fewer cross terms. In the case of multiple delays or when additional terms appear in the right-hand side of (1) the difference between the number of the cross terms in transformations I and those in II–IV increases. While transformation I leads to a system that is not equivalent to the original one, transformation II requires an additional restrictive assumption, transformation III introduces additional terms in $\dot{V}$, that increases the overdesign, and transformation IV does not require additional assumptions or terms and is, therefore, the least conservative.

2.2. *Parametrized model transformation*

All the tranformations presented above are *fixed model transformations* (Niculescu 2001 b), that rewrite the delayed term $A_1x(t-h)$ via integration.

A different idea is the application of a parmetrized model transformation with a new matrix parameter C

$$\dot{x}(t) = (A_0 + C)x(t) + (A_1 - C)x(t-h) + Cx(t-h) \tag{11}$$

The term $(A_1 - C)x(t-h)$ is then treated as in the delay-independent case, while the term $Cx(t-h)$ is rewritten via a fixed model transformation. The combination of the parametrized transformation with transformation I was used by Zhang *et al.* (2002) and the combination with transformation II appears in Han (2002).

In the present paper we concentrate on applying the descriptor model transformation together with the bounding method of Park (1999). We found that the parametrized model transformation does not lead to less conservative results.

2.3. *Park's inequality for bounding of cross terms*

In the delay-dependent case the following inequality was used for bounding the cross terms: given $a, b \in R^n$

$$2a^{\mathrm{T}}b \le a^{\mathrm{T}}Ra + b^{\mathrm{T}}R^{-1}b, \qquad R \in \mathcal{R}^{n\times n},\ R > 0 \tag{12}$$

This bound was significantly improved in Park (1999) and applied to delay-dependent stability results that were based on transformation III.

Park introduced a free matrix $M \in \mathcal{R}^{n\times n}$ and obtained the new inequality

$$-2a^{\mathrm{T}}b \le \begin{bmatrix} a \\ b \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} R & RM \\ M^{\mathrm{T}}R & \Upsilon \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \tag{13}$$

for $a \in R^n, b \in \mathcal{R}^n,\ R \in \mathcal{R}^{n\times n},\ R > 0$. Here $\Upsilon = (M^{\mathrm{T}}R + I)R^{-1}(RM + I)$. Note that (12) is obtained from (13) for $M = 0$.

The delay-independent result that was obtained in the past stems from the delay-dependent method under transformation III and (13) by taking $R = -M^{-1} = -\varepsilon I$ with $\varepsilon \to 0^+$ and $a = A_1x(s),\ b = Px(t)$. The following LMI is obtained

$$\Gamma + O(\varepsilon) < 0, \qquad \Gamma = \begin{bmatrix} P_1A_0 + A_0^{\mathrm{T}}P_1 + S & P_1A_1 \\ A_1^{\mathrm{T}}P_1 & -S \end{bmatrix}$$

The well-known delay-independent condition $\Gamma < 0$ thus follows. Unlike methods based on the restrictive bounding (12), Park's condition generalizes the delay-independent condition $\Gamma < 0$ and the delay-dependent condition of Kolmanovskii and Richard (1999), based on transformation III and (12). It has been shown recently, using the frequency domain approach (Zhang *et al.* 2001), that the condition of Park (1999) generalizes also the delay-dependent condition of Li and de Souza (1997), which is based on transformation I. An example where Park's condition holds for all h, whereas the conditions that are based on (12) are feasible only for small $h \ge 0$, is given in Zhang *et al.* (2001).

3. Stability of neutral systems under descriptor model transformation with Park's bounding: time-varying delays

Application of the less conservative model transformation with tighter bounding of the cross terms leads to efficient sufficient conditions. Consider the following system with time-varying delays

$$\left.\begin{aligned} \dot{x}(t) - \sum_{i=1}^{2} F_i\dot{x}(t-g_i) &= \sum_{i=0}^{2} A_ix(t-\tau_i(t)) \\ x(t) &= \phi(t), \quad t \in [-h, 0] \end{aligned}\right\} \tag{14}$$

where $g_i \ge 0,\ i = 1, 2$ and $x(t) \in \mathcal{R}^n,\ \tau_0 \equiv 0,\ A_i$ and F_i are constant $n \times n$-matrices, ϕ is a continuously differentiable initial function. We consider two different cases:

A1: $\tau_i(t)$ are differentiable functions, satisfying for all $t \ge 0$

$$0 \le \tau_i(t) \le h_i, \qquad \dot{\tau}_i(t) \le d_i, \quad i = 1, 2 \tag{15}$$

or

A2: $\tau_i(t)$ are continuous functions, satisfying for all $t \ge 0,\ 0 \le \tau_i(t) \le h_i,\ i = 1, 2$.

Taking in (14) $h = \max\{h_1, h_2, g_1, g_2\}$, we are looking for stability criteria, delay-independent with respect to g_i and dependent on h_i and d_i. We consider, for simplicity, two delays $g_1,\ g_2$ and $\tau_1,\ \tau_2$, but all the results are easily generalized for the case of any finite number of delays. Representing (14) in the descriptor form

$$\begin{aligned} \dot{x}(t) = y(t),\ y(t) - \sum_{i=0}^{2} F_iy(t-g_i) \\ = \left[\sum_{i=0}^{2} A_i\right]x(t) - \sum_{i=1}^{2} A_i\int_{t-\tau_i(t)}^{t} y(s)\,\mathrm{d}s \end{aligned} \tag{16}$$

and denoting

$$\bar{x}(t) = \mathrm{col}\{x(t),\ y(t)\}$$

consider the following Lyapunov–Krasovskii functional

$$V(t) = \bar{x}^{\mathrm{T}}(t)EP\bar{x}(t) + V_2 + V_3 + V_4 \tag{17}$$

where

$$V_2 = \sum_{i=1}^{2}\int_{-h_i}^{0}\int_{t+\theta}^{t} y^{\mathrm{T}}(s)A_i^{\mathrm{T}}R_iA_iy(s)\mathrm{d}s\mathrm{d}\theta$$

$$V_3 = \sum_{i=1}^{2}\int_{t-g_i}^{t} y^{\mathrm{T}}(s)U_iy(s)\mathrm{d}s$$

$$V_4 = \sum_{i=1}^{2}\int_{t-\tau_i(t)}^{t} x^{\mathrm{T}}(s)S_ix(s)\mathrm{d}s$$

The new term V_3 that appears in (17) (comparatively to (8)) is due to the neutral type system. The term V_4 is used in order to apply Park's inequality. The following result is based on (13).

Theorem 1: *Under A1 the neutral system* (14) *is stable if there exist* $n \times n$ *matrices* $0 < P_1,\ P_2,\ P_3,\ S_i = S_i^{\mathrm{T}},$ $U_i = U_i^{\mathrm{T}},\ W_{i1},\ W_{i2}$ *and* $R_i = R_i^{\mathrm{T}},\ i = 1,2$ *that satisfy the LMI:*

$$\begin{bmatrix} \Psi_1 & \Psi_2 & h_1\Phi_{11} & h_2\Phi_{21} & -W_{11}^{\mathrm{T}}A_1 & -W_{21}^{\mathrm{T}}A_2 & P_2^{\mathrm{T}}F_1 & P_2^{\mathrm{T}}F_2 \\ * & \Psi_3 & h_1\Phi_{12} & h_2\Phi_{22} & -W_{12}^{\mathrm{T}}A_1 & -W_{22}^{\mathrm{T}}A_2 & P_3^{\mathrm{T}}F_1 & P_3^{\mathrm{T}}F_2 \\ * & * & -h_1R_1 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -h_2R_2 & 0 & 0 & 0 & 0 \\ * & * & * & * & -S_1(1-d_1) & 0 & 0 & 0 \\ * & * & * & * & * & -S_2(1-d_2) & 0 & 0 \\ * & * & * & * & * & * & -U_1 & 0 \\ * & * & * & * & * & * & * & -U_2 \end{bmatrix} < 0 \tag{18}$$

where

$$\left.\begin{aligned} \Psi_1 &= \left(\sum_{i=0}^{2}A_i^{\mathrm{T}}\right)P_2 + P_2^{\mathrm{T}}\left(\sum_{i=0}^{2}A_i\right) \\ &\quad + \sum_{i=1}^{2}(W_{i1}^{\mathrm{T}}A_i + A_i^{\mathrm{T}}W_{i1}) + \sum_{i=1}^{2}S_i \\ \Psi_2 &= P_1 - P_2^{\mathrm{T}} + \left(\sum_{i=0}^{2}A_i^{\mathrm{T}}\right)P_3 + \sum_{i=1}^{2}A_i^{\mathrm{T}}W_{i1} \\ \Psi_3 &= -P_3 - P_3^{\mathrm{T}} + \sum_{i=1}^{2}(U_i + h_iA_i^{\mathrm{T}}R_iA_i) \\ \Phi_{i1} &= [W_{i1}^{\mathrm{T}} + P_2^{\mathrm{T}}],\ \Phi_{i2} = [W_{i2}^{\mathrm{T}} + P_3^{\mathrm{T}}], \quad i = 1,2 \end{aligned}\right\} \tag{19}$$

Proof: Since

$$\bar{x}^{\mathrm{T}}(t)EP\bar{x}(t) = x^{\mathrm{T}}(t)P_1x(t)$$

differentiating the first term of (17) with respect to t we have:

$$\frac{\mathrm{d}}{\mathrm{d}t}\bar{x}^{\mathrm{T}}(t)\bar{E}P\bar{x}(t) = 2x^{\mathrm{T}}(t)P_1\dot{x}(t) = 2\bar{x}^{\mathrm{T}}(t)P^{\mathrm{T}}\begin{bmatrix}\dot{x}(t)\\ 0\end{bmatrix} \tag{20}$$

Substituting (16) into (20) we obtain

$$\begin{aligned} \frac{\mathrm{d}V(x_t)}{\mathrm{d}t} &\le \xi^{\mathrm{T}}\begin{bmatrix} \Psi & P^{\mathrm{T}}\begin{bmatrix}0\\ F_1\end{bmatrix} & P^{\mathrm{T}}\begin{bmatrix}0\\ F_2\end{bmatrix} \\ * & -U_1 & 0 \\ * & * & -U_2 \end{bmatrix}\xi + \sum_{i=1}^{2}\eta_i \\ &\quad - \sum_{i=1}^{2}[(1-d_i)x^{\mathrm{T}}(t-\tau_i)S_ix(t-\tau_i) \\ &\quad + y^{\mathrm{T}}(t-g_i)U_iy(t-g_i)] \\ &\quad - \sum_{i=1}^{2}\int_{t-h_i}^{t} y^{\mathrm{T}}(s)A_i^{\mathrm{T}}R_iA_iy(s)\,\mathrm{d}s \end{aligned} \tag{21}$$

where $\xi \triangleq \mathrm{col}\{\bar{x}(t)\ , y(t-g_1),\ y(t-g_2)\}$

$$\left.\begin{aligned} \Psi &\triangleq P^{\mathrm{T}}\begin{bmatrix} 0 & I \\ \left(\sum_{i=0}^{2}A_i\right) & -I \end{bmatrix} + \begin{bmatrix} 0 & \left(\sum_{i=0}^{2}A_i^{\mathrm{T}}\right) \\ I & -I \end{bmatrix}P \\ &\quad + \begin{bmatrix} \sum_{i=1}^{2}S_i & 0 \\ 0 & \sum_{i=1}^{2}(U_i + h_iA_i^{\mathrm{T}}R_iA_i) \end{bmatrix} \\ \eta_i(t) &\triangleq -2\int_{t-\tau_i}^{t}\bar{x}^{\mathrm{T}}(t)P^{\mathrm{T}}\begin{bmatrix}0\\ I_n\end{bmatrix}A_iy(s)\mathrm{d}s, \quad i = 1,2 \end{aligned}\right\} \tag{22}$$

Applying (13), where $R = R_i$, $M = M_i$, $a(s) = A_iy(s)$ and $b = [P_2\ \ P_3]\bar{x}(t)$, and denoting $\bar{W}_i = [W_{i1}\ \ W_{i2}] = R_iM_i[P_2\ \ P_3]$, we obtain for $i = 1,2$

$$\begin{aligned} \eta_i(t) &\le h_i\bar{x}^{\mathrm{T}}(t)(\bar{W}_i^{\mathrm{T}} + [P_2\ \ P_3]^{\mathrm{T}})R_i^{-1}(\bar{W}_i + [P_2\ \ P_3])\bar{x}(t) \\ &\quad + 2(x^{\mathrm{T}}(t) - x^{\mathrm{T}}(t-\tau_i))A_i^{\mathrm{T}}\bar{W}_i\bar{x}(t) \\ &\quad + \int_{t-h_i}^{t} y^{\mathrm{T}}(s)A_i^{\mathrm{T}}R_iA_iy(s)\mathrm{d}s \end{aligned} \tag{23}$$

Substituting (23) into (21) and using Schur complements we find $\dot{V} < 0$. Note that (18) implies the feasibility of the LMI

$$\begin{bmatrix} -P_3^T - P_3 + \sum_{i=1}^{2} U_i & P_3^T A_1 & P_3^T A_2 \\ * & -U_1 & 0 \\ * & * & -U_2 \end{bmatrix} < 0$$

and thus also the delay-independent stability of the difference operator $\mathcal{D}x_t = x(t) - F_1 x(t-g_1) - F_2 x(t-g_2)$ (Fridman 2002 b). Therefore, equation (14) is asymptotically stable. □

Corollary 1: *Assume* A2. *The neutral system* (14) *is stable if there exist* $n \times n$ *matrices* $0 < P_1,\ P_2,\ P_3,\ U_i = U_i^T$ *and* $R_i = R_i^T,\ i = 1, 2$ *that satisfy the following LMI*

$$\begin{bmatrix} \Psi_1 & \Psi_2 & h_1 P_2^T & h_2 P_2^T & P_2^T F_1 & P_2^T F_2 \\ * & \Psi_3 & h_1 P_3^T & h_2 P_3^T & P_3^T F_1 & P_3^T F_2 \\ * & * & -h_1 R_1 & 0 & 0 & 0 \\ * & * & * & -h_2 R_2 & 0 & 0 \\ * & * & * & * & -U_1 & 0 \\ * & * & * & * & * & -U_2 \end{bmatrix} < 0 \tag{24}$$

where $W_{i1} = W_{i2} = 0,\ S_i = 0,\ i = 1, 2.$

Proof: The proof follows from the proof of Theorem 1 by choosing $M = 0$ in (13). LMI (24) is similar to Fridman (2001). □

Remark 1: Note that in Fridman and Shaked (2002 a), Park's inequality was used with $a(s) = \mathrm{col}\{0,\ A_i y(s)\}$ and $b = Px(t)$, that lead to $2n \times 2n$ matrices R_i and $W_i = R_i M_i P$. As a result, a more complicated form of LMI was derived. The latter LMI led to conservative conditions in the case of state-feedback controller design, where it was assumed that $W_i = \varepsilon_i P, \varepsilon_i \in \mathcal{R}$.

For $W_{i1} = -P_2,\ W_{i2} = -P_3,\ R_i = \varepsilon I_n / h_i,\ i = 1, 2,$ LMI (18) implies for $\varepsilon \to 0^+$ the delay-independent/delay-derivative-dependent LMI

$$\begin{bmatrix} \Phi & P^T \begin{bmatrix} 0 \\ A_1 \end{bmatrix} & P^T \begin{bmatrix} 0 \\ A_2 \end{bmatrix} & P^T \begin{bmatrix} 0 \\ F_1 \end{bmatrix} & P^T \begin{bmatrix} 0 \\ F_2 \end{bmatrix} \\ * & -S_1(1-d_1) & 0 & 0 & 0 \\ * & * & -S_2(1-d_2) & 0 & 0 \\ * & * & * & -U_1 & 0 \\ * & * & * & * & -U_2 \end{bmatrix} < 0 \tag{25}$$

where

$$\Phi = P^T \begin{bmatrix} 0 & I \\ A_1 & -I_{n_1} \end{bmatrix} + \begin{bmatrix} 0 & I \\ A_1 & -I_{n_1} \end{bmatrix}^T P + \sum_{i=1}^{2} \begin{bmatrix} S_i & 0 \\ 0 & U_i \end{bmatrix}$$

If LMI (25) is feasible then (18) is feasible for a small enough $\varepsilon > 0$ and for R_i and $\bar{W}_i$ given above. Thus, Theorem 1 implies the following delay-independent/delay-derivative-dependent conditions.

Corollary 2: *Under* **A1** *the system of* (14) *is asymptotically stable for all* $h_i \geq 0,\ g_i \geq 0,\ i = 1, 2$ *if there exist* $0 < P_1 = P_1^T,\ P_2,\ P_3,\ \ U_i = U_i^T$ *and* $S_i = S_i^T,\ i = 1, 2$ *that satisfy* (25).

Similar to the case of Park (1999), the conditions of Theorem 1 are feasible for all $h_i \geq 0$ if (25) holds. Corollary 2 and the existing delay-independent conditions (Verriest and Niculescu 1998) lead to complementary results: in Fridman (2002 b) two examples are given, for one of them the conditions of Verriest and Niculescu (1998) are feasible whereas those of Corollary 2 are not. In the second example, the opposite situation occurs.

3.1. *Stability in the case of polytopic uncertainty*

The LMI of (18) is affine in the system matrices, therefore Theorem 1 can be used to derive a criterion that will guarantee the stability in the case where the system matrices are not exactly known and they reside within a given polytope.

Denoting

$$\Omega = [\, A_i \quad F_i, \qquad i = 1, 2 \quad A_0 \,]$$

we assume that $\Omega \in \mathcal{C}o\{\Omega_j,\ j = 1, \ldots, N\}$, namely

$$\Omega = \sum_{j=1}^{N} f_j \Omega_j \qquad \text{for some} \quad 0 \leq f_j \leq 1, \quad \sum_{j=1}^{N} f_j = 1$$

where the N vertices of the polytope are described by

$$\Omega_j = [\,A_i^{(j)} \quad F_i^{(j)}, \quad i=1,2 \quad A_0^{(j)}\,]$$

We readily obtain the following.

Corollary 3: *Assume* A1. *Consider the system of* (14), *where the system matrices reside within the polytope* Ω. *This system is asymptotically stable for all positive delays* g_1, g_2 *if there exist* $n \times n$*-matrices* $0 < P_1^{(j)}$, $j=1,\dots,N$, W_{i1}, W_{i2}, P_2, P_3, *and* R_i, $U_i^{(j)}$, $S_i^{(j)}$ $i=1,2$, $j=1,\dots,N$ *that satisfy* (18) *for* $j=1,\dots,N$, *where the matrices*

$$A_0, A_i, F_i, P_1, U_i, S_i, \qquad i=1,2$$

are taken with the superscript j.

Corollary 1 can be similarly generalized to the case of polytopic uncertainty.

Corollary 4: *Assume* A2. *Consider the system of* (14), *where the system matrices reside within the polytope* Ω. *This system is asymptotically stable for all positive delays* g_1, g_2 *if there exist* $n \times n$*-matrices* $0 < P_1^{(j)}$, $j=1,\dots,N$, P_2, P_3, *and* R_i, $U_i^{(j)}, i=1,2,\ j=1,\dots,N$ *that satisfy* (24) *for* $j=1,\dots,N$, *where the matrices*

$$A_0,\ A_i,\ F_i,\ P_1, U_i, \qquad i=1,2$$

are taken with the superscript j.

3.2. *Examples*

Example 1a (Li and de Souza 1997)**:** Consider the system with constant delay

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-h) \tag{26}$$

where

$$D_1 = 0,\ A_0 = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, \qquad A_1 = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}$$

In Park (1999) (transformation III) it was found that the system is asymptotically stable for $h \le 4.36$ and this bound is less conservative than the bound $h \le 0.99$ that follows from the conditions of Li and de Souza (1997) (transformation I) and Niculescu (2001 a) (transformation II), and slightly better than the bound $h \le 4.35$ that follows from the conditions of Zhang *et al.* (2002) (transformation I with parametrizing one and Park's inequality) and of Han (2002) (transformation II with parametrizing one). By Theorem 1 we obtain the less restrictive bound $h \le 4.47$.

The exact upper bound h^*, such that for $h > h^*$ the system is unstable, is found by Nyquist criterion to be $h^* = 6.1726$. Note that the descritized Lyapunov functional technique of Gu (1999) leads to less conservative results than the model tranformation methods. However, it is computationally more complicated and it is not appropriate for the case of time-varying delay A2.

Example 1b: By changing the coefficient -0.9 in A_0 to 0.9 we obtain the following results: $h \le 0.05$ by Li and de Souza (1997), $h \le 0.22$ by Park (1999), $h \le 0.23$ by Zhang (2002) and $h \le 0.99$ by Niculescu (2001 a), and Han (2002). By Theorem 1 we obtain an improvement: $h \le 1.025$. In this case $h^* = 1.21$.

The results of Examples 1a and 1b are summarized in tables 1 and 2, respectively. We see that in both examples the most conservative condition was obtained via transformation I and the least conservative was achieved via transformation IV. The results via transformations II and III (with Park's inequality) do not show a consistent advantage of one of these transformations over the other. Note that transformation II with parametrizing one (Han, 2002) leads to less conservative results than transformation III (with Park's inequality) results, but one has to choose appropriately C in (11) and then to use the conditions of Han (2002), otherwise these conditions are non-linear matrix inequalities. The results via transformation I with parametrizing transformation and via transformation III (with Park's inequality) are approximately the same.

Example 2 (Kim 2001)**:** We consider

Transformation	I	II	III	I + param.	II + param.	IV	Nyquist
Max h	0.99	0.99	4.36	4.35	4.35	4.47	6.17

Table 1. Example 1a.

Transformation	I	II	III	I + param.	II + param.	IV	Nyquist
Max h	0.05	0.99	0.22	0.23	0.99	1.025	1.21

Table 2. Example 1b.

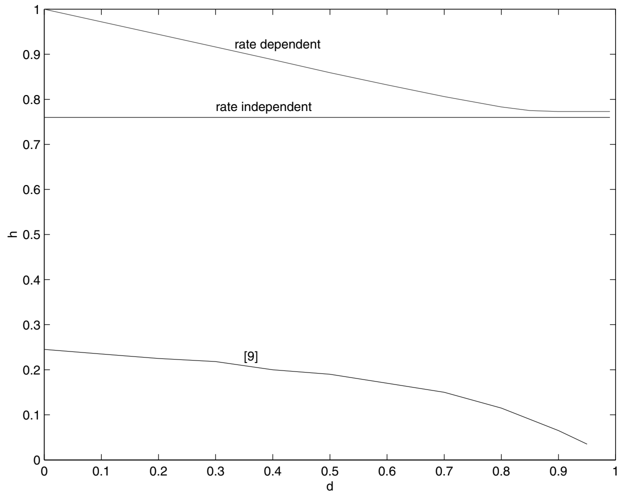

Figure 1. Stability bounds of the time-delay h as a function of d.

$$\dot{x}(t) = \begin{bmatrix} -2+\delta_1 & 0 \\ 0 & -1+\delta_2 \end{bmatrix} x(t) + \begin{bmatrix} -1+\gamma_1 & 0 \\ -1 & -1+\gamma_2 \end{bmatrix} x(t-\tau(t))$$

$$|\delta_1| \le 1.6, \quad |\delta_2| \le 0.05, \quad |\gamma_1| \le 0.1, \quad |\gamma_2| \le 0.3$$

In Kim (2001), where the case A1 (with $0 \le \tau \le h$, $\dot{\tau} \le d < 1$) was treated via transformation I, the maximum values of h for which stability is secured was found as a function of the bound d on the delay rate of change. For $d = 0$ the maximum value of $h = 0.2412$ was reported and compared with previous results in the literature. Applying the method of Corollary 3 we obtained for $d = 0$ that the system is asymptotically stable for the maximum value of $h = 1$. Our results are favourably compared to those in Kim (2001) also for $d \ne 0$. In figure 1 we depict the maximum achievable h as a function of d. There, we bring the results that stem from Corollary 3 (the rate dependent case A1) and the extension of Corollary 1 to the rate independent polytopic uncertainty case A2 with $0 \le \tau \le h$. These results are compared to those reported in Kim (2001). We see that our results even in the case A2 are significantly better than those of Kim (2001).

Remark 2: The results under transformation IV, being the least conservative, have relatively complicated form. In the case when sufficient conditions are sought for robust stability with respect to small delays (without maximizing the size of the delay), one can use the simplest transformation which is appropriate for the problem. Thus, in Fridman (2002 a) sufficient conditions for robustness of stability of singularly perturbed system with respect to small values of delay and of the singular perturbation parameter were derived via transformation I.

4. H_∞ control of systems with time-varying state delays

In this section we improve results of Fridman and Shaked (2001, 2002 a), based on transformation IV, and extend them to the case of time-varying delay. To the best of our knowledge, H_∞ control was treated in the past only via transformation I in the case of constant delays (see, for example, de Souza and Li 1999). In the case where the time-delay appears only in the input and it is constant, the solution of the H_∞control problem was obtained in Tadmor (2000). In the case of constant known delay in the state the H_∞ control problem was solved in Fridman and Shaked (1998) via general Lyapunov–Krasovskii functional.

4.1. *Delay-dependent bounded real lemma* (*BRL*)

Given the system

$$\left.\begin{aligned} \dot{x}(t) - \sum_{i=1}^{2} F_i \dot{x}(t-g_i) &= \sum_{i=0}^{2} A_i x(t-\tau_i(t)) + B_1 w(t) \\ x(t) &= 0, \quad t \le 0 \\ z(t) &= Cx(t) \end{aligned}\right\} \tag{27}$$

where $x(t) \in \mathcal{R}^n$ is the system state vector, $w(t) \in \mathcal{L}_2^q[0, \infty]$ is the exogenous disturbance signal and $z(t) \in \mathcal{R}^p$ is the state combination (objective function signal) to be attenuated. The time delays are defined in § 3. The matrices $A_i,\ i = 0, \ldots, 2,\ F_i,\ i = 1, 2,\ B_1$ and C are constant matrices of appropriate dimensions. For a prescribed scalar $\gamma > 0$, we define the performance index

$$J(w) = \int_0^\infty (z^{\mathrm{T}} z - \gamma^2 w^{\mathrm{T}} w)\, \mathrm{d}s \tag{28}$$

Using the argument of the previous section we obtain the following BRL.

Lemma 1: *Consider the system of* (27). *Assume* A1. *For a prescribed* $\gamma > 0$, *the cost function* (28) *achieves* $J(w) < 0$ *for all non-zero* $w \in \mathcal{L}_2^q[0, \infty)$ *and for all positive delays* g_1, g_2, *if there exist* P *of* (9)

$$W_i = \begin{bmatrix} -P_1 & 0 \\ W_{i1} & W_{i2} \end{bmatrix} \tag{29}$$

and $n \times n$*-matrices* $S_i = S_i^{\mathrm{T}},\ U_i = U_i^{\mathrm{T}},\ R_i = R_i^{\mathrm{T}}$ *that satisfy the LMI:* (see (30) at bottom of next page) *where for* $i = 1, 2$

$$\bar{\Psi} \triangleq P^{\mathrm{T}} \begin{bmatrix} 0 & I \\ \left(\sum_{i=0}^{2} A_i\right) & -I \end{bmatrix} + \begin{bmatrix} 0 & \left(\sum_{i=0}^{2} A_i^{\mathrm{T}}\right) \\ I & -I \end{bmatrix} P + \begin{bmatrix} \sum_{i=1}^{2} S_i & 0 \\ 0 & \sum_{i=1}^{2}(U_i + h_i A_i^{\mathrm{T}} R_i A_i) \end{bmatrix} + \sum_{i=1}^{2} W_i^{\mathrm{T}} \begin{bmatrix} 0 & 0 \\ A_i & 0 \end{bmatrix} + \sum_{i=1}^{2} \begin{bmatrix} 0 & A_i^{\mathrm{T}} \\ 0 & 0 \end{bmatrix} W_i$$

$$\Phi_i^{\mathrm{T}} = [0 \;\; I_n][W_i + P]$$

Similarly to Corollary 1, the rate-independent result is obtained by choosing $M_i = 0$ and thus $W_{i1} = W_{i2} = 0$.

Corollary 5: *Assume* A2. *For a prescribed* $\gamma > 0$, *the cost function* (28) *achieves* $J(w) < 0$ *for all non-zero* $w \in \mathcal{L}_2^q[0, \infty)$ *and for all positive delays* g_1, g_2, *if there exist* P *of* (9), *and* $n \times n$*-matrices* $U_i = U_i^{\mathrm{T}}$, $R_i = R_i^{\mathrm{T}}$ *that satisfy the LMI:*

$$\begin{bmatrix} \Psi & P^{\mathrm{T}}\begin{bmatrix} 0 \\ B_1 \end{bmatrix} & h_1\Phi_1 & h_2\Phi_2 & P^{\mathrm{T}}\begin{bmatrix} 0 \\ F_1 \end{bmatrix} & P^{\mathrm{T}}\begin{bmatrix} 0 \\ F_2 \end{bmatrix} & \begin{bmatrix} C^{\mathrm{T}} \\ 0 \end{bmatrix} \\ * & -\gamma^2 I & 0 & 0 & 0 & 0 & 0 \\ * & * & -h_1R_1 & 0 & 0 & 0 & 0 \\ * & * & * & -h_2R_2 & 0 & 0 & 0 \\ * & * & * & * & -U_1 & 0 & 0 \\ * & * & * & * & * & -U_2 & 0 \\ * & * & * & * & * & * & -I_p \end{bmatrix} < 0 \tag{31}$$

where $\Phi_i = [P_2 \;\; P_3]^{\mathrm{T}}$, $i = 1, 2$ *and*

$$\Psi = P^{\mathrm{T}} \begin{bmatrix} 0 & I \\ \left(\sum_{i=0}^{2} A_i\right) & -I \end{bmatrix} + \begin{bmatrix} 0 & \left(\sum_{i=0}^{2} A_i^{\mathrm{T}}\right) \\ I & -I \end{bmatrix} P + \begin{bmatrix} 0 & 0 \\ 0 & \sum_{i=1}^{2}(U_i + h_i A_i^{\mathrm{T}} R_i A_i) \end{bmatrix}$$

4.2. *State-feedback* H_∞ *control.*

Given the system

$$\begin{aligned} \dot{x}(t) - \sum_{i=1}^{2} F_i \dot{x}(t - g_i) &= \sum_{i=0}^{2} \bar{A}_i x(t - \tau_i(t)) + B_1 w(t) \\ &\quad + B_2 u(t) \\ z(t) &= \mathrm{col}\{\bar{C}x(t), \; D_{12}u(t)\} \\ x(t) &= 0 \;\forall t \leq 0 \end{aligned} \tag{32}$$

where $u \in \mathcal{R}^{\ell}$ is the control input, F_1, F_2, $\bar{A}_0$, $\bar{A}_1$, $\bar{A}_2$, B_1, B_2 are constant matrices of appropriate dimension, z is the objective vector, $\bar{C} \in \mathcal{R}^{p \times n}$ and $D_{12} \in \mathcal{R}^{r \times \ell}$. We look for a state-feedback gain matrix K which, via the control law

$$u(t) = Kx(t) \tag{33}$$

achieves $J(w) < 0$ for all non-zero $w \in \mathcal{L}_2^q[0, \infty)$.

Substituting (33) into (32), we obtain the structure of (14) with

$$\begin{aligned} A_0 &= \bar{A}_0 + B_2 K, \quad A_i = \bar{A}_i, \; i = 1, 2 \\ C^{\mathrm{T}} C &= \bar{C}^{\mathrm{T}} \bar{C} + K^{\mathrm{T}} D_{12}^{\mathrm{T}} D_{12} K \end{aligned} \tag{34}$$

$$\begin{bmatrix} \bar{\Psi} & P^{\mathrm{T}}\begin{bmatrix} 0 \\ B_1 \end{bmatrix} & h_1\Phi_1 & h_2\Phi_2 & -W_1^{\mathrm{T}}\begin{bmatrix} 0 \\ A_1 \end{bmatrix} & -W_2^{\mathrm{T}}\begin{bmatrix} 0 \\ A_2 \end{bmatrix} & P^{\mathrm{T}}\begin{bmatrix} 0 \\ F_1 \end{bmatrix} & P^{\mathrm{T}}\begin{bmatrix} 0 \\ F_2 \end{bmatrix} & \begin{bmatrix} C^{\mathrm{T}} \\ 0 \end{bmatrix} \\ * & -\gamma^2 I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & -h_1R_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -h_2R_2 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & -(1-d_1)S_1 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & -(1-d_2)S_2 & 0 & 0 & 0 \\ * & * & * & * & * & * & -U_1 & 0 & 0 \\ * & * & * & * & * & * & * & -U_2 & 0 \\ * & * & * & * & * & * & * & * & -I_p \end{bmatrix} < 0 \tag{30}$$

Applying the BRL of §4.1 to the above matrices, a non-linear matrix inequality is obtained due to the terms $P_2^{\mathrm{T}} B_2 K$ and $P_3^{\mathrm{T}} B_2 K$.

In order to obtain a LMI we restrict ourselves to the case of

$$W_i = \mathrm{diag}\,\{-I_n,\ \varepsilon_i\}P, \qquad i = 1,2$$

where $\varepsilon_i \in \mathcal{R}^{n\times n}$ is a diagonal matrix. Such a choice for W_i is less conservative than the one in Fridman and Shaked (2002 a), where $W_i = \varepsilon_i P$ for a scalar ε_i. For $\varepsilon_i = -I_n$ (30) yields the delay-independent condition.

It is obvious from the requirement of $0 < P_1$, and the fact that in (30) $-(P_3 + P_3^{\mathrm{T}})$ must be negative definite, that P is non-singular. Defining

$$P^{-1} = Q = \begin{bmatrix} Q_1 & 0 \\ Q_2 & Q_3 \end{bmatrix} \quad \text{and} \quad \Delta = \mathrm{diag}\{Q,\ I\} \tag{35a–b}$$

we multiply (30) by Δ^{T} and Δ, on the left and on the right, respectively. Applying Schur formula to the quadratic term in Q, and denoting $\bar{S}_i = S_i^{-1}$, $\bar{U}_i = U_i^{-1}$ and $\bar{R}_i = R_i^{-1}$, $i = 1,2$ we obtain, similarly to Fridman and Shaked (2002 a), the following.

Theorem 2: *Assume* A1. *Consider the system of* (32) *and the cost function of* (28). *For a prescribed* $0 < \gamma$, *the state-feedback law of* (33) *achieves* $J(w) < 0$ *for all non-zero* $w \in \mathcal{L}_2^q[0,\ \infty)$ *if for some diagonal matrices* $\varepsilon_1,\ \varepsilon_2 \in \mathcal{R}^{n\times n}$, *there exist* $Q_1 > 0$, $\bar{S}_1$, $\bar{S}_2$, $\bar{U}_1$, $\bar{U}_2$, Q_2, Q_3, $\bar{R}_1$, $\bar{R}_2 \in \mathcal{R}^{n\times n}$ *and* $Y \in \mathcal{R}^{\ell\times n}$ *that satisfy the LMI given in* (36) (*see bottom of page*), *where*

$$\Xi = Q_3 - Q_2^{\mathrm{T}} + Q_1\left(\sum_{i=0}^{2}\bar{A}_i^{\mathrm{T}} + \sum_{i=1}^{2}\bar{A}_i^{\mathrm{T}}\varepsilon_i\right) + Y^{\mathrm{T}}B_2^{\mathrm{T}}$$

The state-feedback gain is then given by

$$K = YQ_1^{-1} \tag{37}$$

Choosing $\varepsilon_i = 0$, we obtain the counterpart of the Theorem 2 for the case A2

Corollary 6: *Assume* A2. *Consider the system of* (32) *and the cost function of* (28). *For a prescribed* $0 < \gamma$, *the state-feedback law of* (33) *achieves* $J(w) < 0$ *for all non-zero* $w \in \mathcal{L}_2^q[0,\ \infty)$ *if there exist* $Q_1 > 0$, $\bar{U}_1$, $\bar{U}_2$, Q_2, Q_3, $\bar{R}_1$, $\bar{R}_2 \in \mathcal{R}^{n\times n}$ *and* $Y \in \mathcal{R}^{\ell\times n}$ *that satisfy the LMI given in* (38) (*see bottom of next page*)

$$\left[\begin{array}{ccccccccccccccccc}
Q_2+Q_2^{\mathrm{T}} & \Xi & 0 & 0 & 0 & 0 & 0 & Q_1 & Q_1 & Q_1\bar{C}^{\mathrm{T}} & Y^{\mathrm{T}}D_{12}^{\mathrm{T}} & 0 & Q_2^{\mathrm{T}} & 0 & Q_2^{\mathrm{T}} & h_1Q_2^{\mathrm{T}}A_1^{\mathrm{T}} & h_2Q_2^{\mathrm{T}}A_2^{\mathrm{T}} \\
* & -Q_3-Q_3^{\mathrm{T}} & B_1 & h_1(\varepsilon_1+I_n)\bar{R}_1 & h_2(\varepsilon_2+I_n)\bar{R}_2 & \varepsilon_1A_1\bar{S}_1 & \varepsilon_2A_2\bar{S}_2 & 0 & 0 & 0 & 0 & F_1\bar{U}_1 & Q_3^{\mathrm{T}} & F_2\bar{U}_2 & Q_3^{\mathrm{T}} & h_1Q_3^{\mathrm{T}}A_1^{\mathrm{T}} & h_2Q_3^{\mathrm{T}}A_2^{\mathrm{T}} \\
* & * & -\gamma I_q & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & -h_1\bar{R}_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & -h_2\bar{R}_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & -(1-d_1)\bar{S}_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & -(1-d_2)\bar{S}_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & -\bar{S}_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & -\bar{S}_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & -I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & -I & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & * & -\bar{U}_1 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & * & * & -\bar{U}_1 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & * & * & * & -\bar{U}_2 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & * & * & * & * & -\bar{U}_2 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & * & * & * & * & * & -h_1\bar{R}_1 & 0 \\
* & * & * & * & * & * & * & * & * & * & * & * & * & * & * & * & -h_2\bar{R}_2
\end{array}\right] < 0 \tag{36}$$

where

$$\Xi = Q_3 - Q_2^{\mathrm{T}} + Q_1\left(\sum_{i=0}^{2} A_i^{\mathrm{T}}\right) + Y^{\mathrm{T}} B_2^{\mathrm{T}}$$

The state-feedback gain is then given by (37).

Remark 3: The conditions of Corollaries 5 and 6 in the case of $F_1 = F_2 = 0$ and constant delays are similar to those of Fridman and Shaked (2001).

The results of this section may be adapted to the case of systems with polytopic uncertainties similarly to § 3.2.

The case of output-feedback H_∞control for systems with time-varying delays can be treated similarly to Fridman and Shaked (2002 a) with corresponding modification of the first phase (state-feedback) as above.

Example 3: Li and de Souza (1997). We consider the system

$$\left.\begin{aligned} \dot{x}(t) &= \bar{A}_0 x(t) + \bar{A}_1 x(t-\tau) + B_1 w(t) + B_2 u(t) \\ z(t) &= \mathrm{col}\{\bar{C}x(t),\ D_{12}u(t)\} \end{aligned}\right\} \tag{39}$$

where

$$\left.\begin{aligned} &\bar{A}_0 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \bar{A}_1 = \begin{bmatrix} -1 & -1 \\ 0 & -0.9 \end{bmatrix} \\ &B_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &\bar{C} = [0 \quad 1], \qquad D_{12} = 0.1 \end{aligned}\right\} \tag{40}$$

Applying the method of Li and de Souza (1997) based on transformation I (Corollary 3.2 there) it was found that, for $\dot{\tau} \equiv 0$, the system is stabilizable for all $\tau < 1$. For, say, $\tau = 0.999$ a minimum value of $\gamma = 1.8822$ results for $K = -[0.104\,52 \quad 749\,058]$. Using the method of Fridman and Shaked (2001) (transformation IV with (12)) for $\dot{\tau} \equiv 0$, a minimum value of $\gamma = 0.228\,44$ was obtained for the same value of τ with a state-feedback gain of $K = [0 - 182\,194]$. By Corollary 6, the same γ and K are achieved in the case A2 of time-varying delay $\tau(t) \le 0.999$.

Consider now the case A1 with $0 \le \tau \le h$, $\dot{\tau} \le d < 1$. Applying, for $\dot{\tau} \equiv 0$ and $\epsilon = -0.3$, the method of Fridman and Shaked (2002 a) (Theorem 3.1 there), a maximum value of $h = 1.28$ was obtained for which a state-feedback controller stabilizes the system. The corresponding feedback gain was $K = [0 - 1.2091 \times 10^6]$. Using Theorem 2 of the present paper we obtain for $d = 0$ a maximum value of $h = 1.408$ for which there exists a state-feedback gain that stabilizes the system. The maximum values of h that still allow stabilizabilty via state-feedback are depicted in figure 2 as a function of d. In figure 3 we describe the minimum achievable value of γ as a function of d for $h = 1.38$ and for $\epsilon_1 = -0.29$ and $\epsilon_2 = -1$. The latter value of h is quite close to the maximum achievable value of $h = 1.408$.

5. Conclusions

Sources for the conservatism of delay-dependent stability methods have been revealed, and the advantages of the one under descriptor transformation have

$$\begin{bmatrix}
Q_2+Q_2^{\mathrm{T}} & \Xi & 0 & 0 & 0 & Q_1\bar{C}^{\mathrm{T}} & Y^{\mathrm{T}}D_{12}^{\mathrm{T}} & 0 & Q_2^{\mathrm{T}} & 0 & Q_2^{\mathrm{T}} & h_1Q_2^{\mathrm{T}}A_1^{\mathrm{T}} & h_2Q_2^{\mathrm{T}}A_2^{\mathrm{T}} \\
* & -Q_3-Q_3^{\mathrm{T}} & B_1 & h_1\bar{R}_1 & h_2\bar{R}_2 & 0 & 0 & F_1\bar{U}_1 & Q_3^{\mathrm{T}} & F_2\bar{U}_2 & Q_3^{\mathrm{T}} & h_1Q_3^{\mathrm{T}}A_1^{\mathrm{T}} & h_2Q_3^{\mathrm{T}}A_2^{\mathrm{T}} \\
* & * & -\gamma^2 I_q & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & -h_1\bar{R}_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & -h_2\bar{R}_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & -I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & -I & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & -\bar{U}_1 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & -\bar{U}_1 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & -\bar{U}_2 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & -\bar{U}_2 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & * & -h_1\bar{R}_1 & 0 \\
* & * & * & * & * & * & * & * & * & * & * & * & -h_2\bar{R}_2
\end{bmatrix} < 0 \tag{38}$$

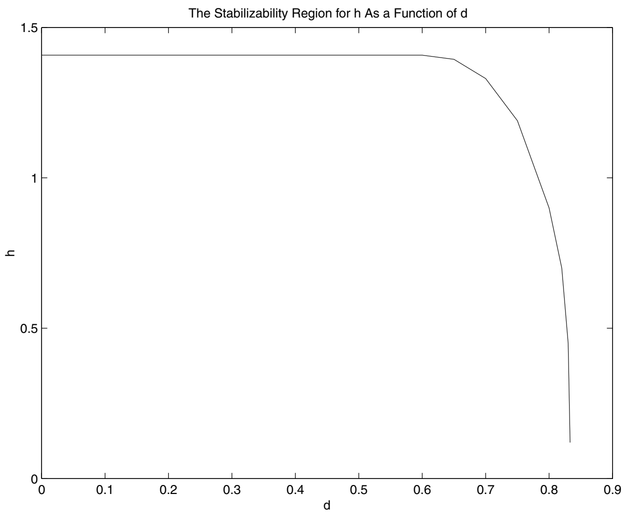

Figure 2. The stabilizability bounds for the time-delay h as a function of d.

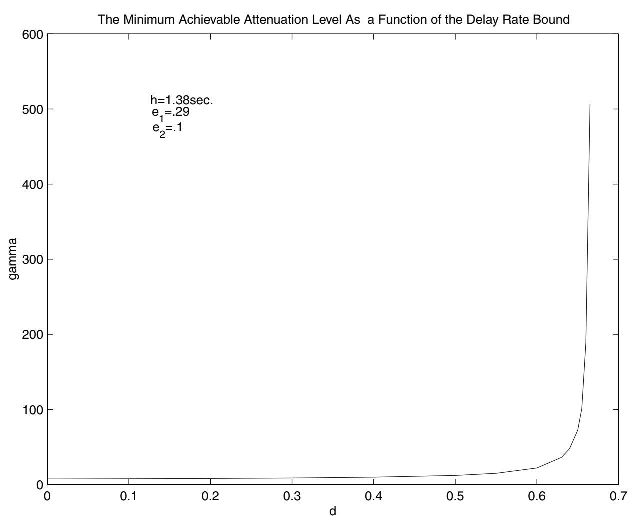

Figure 3. The minimum achievable attenuation level as a function of d for $h = 1.38$.

been demonstrated. A delay-dependent LMI solution is proposed for the problems of stability and H_∞control of linear systems with time-varying delays. This solution is based on the descriptor model transformation and Park's inequality for bounding of cross terms. Two types of results for systems with time-varying delays have been derived: delay-dependent/rate-dependent and delay-dependent/rate-independent. In both cases, the new stability results are less restrictive than the existing results (Kim 2001), obtained for the first (less robust) case via transformation I. Our results for the second case, which includes fast-varying delays, seem to be the first results that are based on Lyapunov–Krasovskii functionals. The conditions that we have obtained for H_∞control of systems with constant delays are less conservative than those of Fridman and Shaked (2002 a).

Recently an improved inequality for bounding of cross terms has been suggested in Moon *et al*. (2001). This new inequality introduces additional degrees of freedom in the solution which may be used to improve the synthesis part of § 4 (Fridman and Shaked 2002 b).

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