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# Using exponential time-varying gains for sampled-data stabilization and estimation<sup>\*</sup>



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## [Tarek Ahmed-Ali](#page-6-0)ª, Emili[a](#page-0-1) Fridman <sup>[b](#page-0-2)</sup>, [Fouad Giri](#page-7-0)ª, [Laurent Burlion](#page-7-1) <sup>[c](#page-0-3)</sup>, [Françoise Lamnabhi-Lagarrigue](#page-7-2) [d](#page-0-4)

<span id="page-0-1"></span>a *Laboratoire GREYC, UMR CNRS 6072, Université de Caen Basse-Normandie, ENSICAEN, Caen, France*

<span id="page-0-2"></span>b *School of Electrical Engineering, Tel-Aviv University, Tel-Aviv 69978, Israel*

<span id="page-0-3"></span><sup>c</sup> *ONERA The French Aerospace Lab, 31055 Toulouse, France*

<span id="page-0-4"></span><sup>d</sup> *CNRS-INS2I, Laboratoire des Signaux et Systemes (L2S), European Embedded Control Institute, EECI Supelec,*

*3 rue Joliot Curie, 91190 Gif-sur-Yvette, France*

#### a r t i c l e i n f o

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#### **1. Introduction**

Designing sampled-data observers and controllers has been a hot topic in recent years, see e.g. [Fridman](#page-6-2) [\(2010\)](#page-6-2) and reference list therein. In this regard, a long standing issue is how to enlarge the sampling intervals [\(Heemels,](#page-6-3) [Johansson,](#page-6-3) [&](#page-6-3) [Tabuada,](#page-6-3) [2012\)](#page-6-3) while ensuring global exponential stability. In a recent paper [\(Cacace,](#page-6-4) [Germani,](#page-6-4) [&](#page-6-4) [Manes,](#page-6-4) [2014\)](#page-6-4), it has been shown that the introduction of time-varying gains in a specific class of observers improves their exponential convergence properties in presence of measurement delay. Presently, these properties are investigated in presence of measurement sampling. To this end, we consider two classes of sampled-data systems and analyze their exponential stability. The

*E-mail addresses:* [tarek.ahmed-ali@ensicaen.fr](mailto:tarek.ahmed-ali@ensicaen.fr) (T. Ahmed-Ali), [emilia@eng.tau.ac.il](mailto:emilia@eng.tau.ac.il) (E. Fridman), [fouad.giri@unicaen.fr](mailto:fouad.giri@unicaen.fr) (F. Giri), [Laurent.Burlion@onera.fr](mailto:Laurent.Burlion@onera.fr) (L. Burlion), [lamnabhi@lss.supelec.fr](mailto:lamnabhi@lss.supelec.fr) (F. Lamnabhi-Lagarrigue).

### A B S T R A C T

This paper provides exponential stability results for two system classes. The first class includes a family of nonlinear ODE systems while the second consists of semi-linear parabolic PDEs. A common feature of both classes is that the systems they include involve sampled-data states and a time-varying gain. Sufficient conditions ensuring global exponential stability are established in terms of Linear Matrix Inequalities (LMIs) derived on the basis of Lyapunov–Krasovskii functionals. The established stability results prove to be useful in designing exponentially convergent observers based on sampled-data measurements. It is shown throughout simulated examples from the literature that the introduction of time-varying gains is beneficial to the enlargement of sampling intervals while preserving the stability of the system.

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considered classes are respectively consisting of nonlinear globally Lipschitz ODEs and semi-linear parabolic PDEs. A common feature of both classes is that the systems they include are allowed to involve a time-varying gain of the form  $e^{-\eta(t-t_k)}$  with  $\eta > 0$  a tuning parameter, where  $t_k$  ( $k = 0, 1, \ldots$ ) are sampling instants. It turns out that, the first family, including ODE systems, is a generalization of that dealt with in [Cacace](#page-6-4) [et al.\(2014\)](#page-6-4). For both classes of systems, we establish sufficient conditions for global exponential stability in terms of Linear Matrix Inequalities (LMIs) derived from Lyapunov–Krasovskii functionals. Then, it is shown that these stability results are useful in designing sampled-data observers with timevarying gains. As the established LMIs conditions involve both the tuning parameter  $\eta$  and the maximum sampling interval  $h$ , these parameters can then be used to improve the observer convergence properties. Actually, it is checked through several simulated examples that the utilization of the above time-varying gain entails significant enlargement of the maximum sampling interval, compared with the constant gain case (corresponding to  $\eta = 0$ ). It is worth noting that, the present theoretical stability results can also be used in sampled-data control design improving exponential stability properties and enlarging sampling intervals. A part of the present results, namely those concerning ODEs, have been presented in our conference paper [\(Ahmed-Ali](#page-6-5) [et al.,](#page-6-5) [2015\)](#page-6-5).



<span id="page-0-0"></span> $\overrightarrow{x}$  This work was partially supported by Israel Science Foundation (grant no. 1128/14). The material in this paper was partially presented at the 12th IFAC Workshop on Time Delay Systems, June 28–30, 2015, Ann Arbor, MI, USA [\(Ahmed-Ali,](#page-6-5) [Fridman,](#page-6-5) [Giri,](#page-6-5) [Burlion,](#page-6-5) [&](#page-6-5) [Lamnabhi-Lagarrigue,](#page-6-5) [2015\)](#page-6-5). This paper was recommended for publication in revised form by Associate Editor Rafael Vazquez under the direction of Editor Miroslav Krstic.

The paper is organized as follows: in Section [2,](#page-1-0) the first stability result, concerning nonlinear ODE systems, is stated and applied to observer design; in Section [3,](#page-3-0) the second stability result, concerning semi-linear PDE systems, is stated and applied to observer design; a conclusion and reference list end the paper. Some technical proofs are appended.

#### *Notations and preliminaries*

Throughout the paper the superscript *T* stands for matrix transposition, **R** *<sup>n</sup>* denotes the *n*-dimensional Euclidean space with vector norm |.|,  $\mathbf{R}^{n \times m}$  is the set of all  $n \times m$  real matrices, and the notation  $P > 0$ , for  $P \in \mathbb{R}^{n \times n}$ , means that P is symmetric and positive definite. In Symmetric matrices, symmetric terms are denoted  $*$ ;  $\lambda_{min}(P)$  (resp.  $\lambda_{max}(P)$ ) denotes the smallest (resp. largest) eigenvalue. The notation  $(t_k)_{k>0}$  refers to a strictly increasing sequence such that  $t_0 = 0$  and  $\lim_{k \to \infty} t_k = \infty$ . The sampling periods are bounded i.e.  $0 < t_{k+1} - t_k < h$  for some scalar  $0 < h < \infty$  and all  $k = 0, 1, \ldots, \infty$ . We also define the variable  $\tau(t) = t - t_k, t \in [t_k, t_{k+1})$ .  $\mathcal{H}^1(0, l)$  is the Sobolev space of absolutely continuous functions  $z : (0, l) \rightarrow \mathbf{R}$  with the square integrable derivative  $\frac{d}{dx}$ . Given a two-argument function  $u(x, t)$ , its partial derivatives are denoted  $u_t = \frac{\partial u}{\partial t}$ ,  $u_{xx} = \frac{\partial^2 u}{\partial x^2}$  $\frac{\partial^{-u}}{\partial x^2}$ .

#### <span id="page-1-0"></span>**2. Sampled-data globally Lipschitz nonlinear ODEs**

#### *2.1. System description and stability result*

We are considering a class of sampled-data nonlinear systems described by the following equation:

$$
\dot{x}(t) = A_0 x(t) + A_1 e^{-\eta(t - t_k)} x(t_k) + \phi(x(t)), \quad t \in [t_k, t_{k+1}) \tag{1}
$$

where  $x(t) \in \mathbb{R}^n$ ; the scalar  $\eta \geq 0$ ;  $A_0, A_1$  are constant matrices with appropriate dimensions. As in [Bar](#page-6-6) [Am](#page-6-6) [and](#page-6-6) [Fridman](#page-6-6) [\(2014\)](#page-6-6), the function  $\phi$  is supposed to be class  $\mathcal{C}^1$  with uniformly bounded Jacobian  $\phi_x$ , satisfying  $\phi(0) = 0$  and

$$
\phi_x^T(x)\phi_x(x) \le M \quad \forall x \tag{2}
$$

for some positive constant  $n \times n$ -matrix *M*. Using Jensen's inequality it is readily checked that [\(2\)](#page-1-1) implies the following inequality:

$$
\int_0^1 \phi_x^T(sx)ds \int_0^1 \phi_x(sx)ds \leq M.
$$

**Remark 1.** Eq. [\(1\)](#page-1-2) may represent a networked control system described by

$$
\dot{x}(t) = A_0 x(t) + \phi(x(t)) + Bu(t),
$$

with the communication network placed between the sensor and the controller (but no network is placed between the controller and the actuator). Assuming that the discrete-time state measurements  $x(t_k)$  are transmitted through the communication network from the sensor to controller, consider the state-feedback,

$$
u(t) = e^{-\eta(t-t_k)} Kx(t_k), \quad t \in [t_k, t_{k+1}),
$$

where *K* is a gain and  $\eta > 0$  is a scalar. It turns out that the resulting closed-loop system fits Eq.  $(1)$  with  $A_1 = BK$ .

As in [Cacace](#page-6-4) [et al.](#page-6-4) [\(2014\)](#page-6-4), introduce the following change of coordinates  $z(t) = e^{\eta t} x(t)$  with  $\eta > 0$ . Then one gets

$$
\dot{z}(t) = \eta z(t) + A_0 z(t) + A_1 z(t_k) \n+ \left[ \int_0^1 \phi_x(sx(t)) ds \right] e^{\eta t} x(t), \quad t \in [t_k, t_{k+1})
$$
\n(3)

which is rewritten as follows:

<span id="page-1-3"></span>
$$
\dot{z}(t) = (\eta I_n + A_0) z(t) + A_1 z(t_k) + \left[ \int_0^1 \phi_x(sx(t)) ds \right] z(t), \quad t \in [t_k, t_{k+1}).
$$
\n(4)

Following [Fridman](#page-6-2) [\(2010\)](#page-6-2), consider the following Lyapunov– Krasovskii functional for [\(4\):](#page-1-3)

<span id="page-1-4"></span>
$$
V(t) = \bar{V}(t) + V_X(t)
$$
\n(5)

with

$$
\bar{V}(t) = z^{T}(t)Pz(t) + (t_{k+1} - t) \int_{t_{k}}^{t} \dot{z}^{T}(s)U\dot{z}(s)ds,
$$
  
\n
$$
P > 0, U > 0, t \in [t_{k}, t_{k+1})
$$

and

$$
V_X(t) = (t_{k+1} - t)\xi^T \begin{bmatrix} \frac{X + X^T}{2} & -X + X_1 \\ * & -X_1 - X_1^T + \frac{X + X^T}{2} \end{bmatrix} \xi,
$$

where  $\xi(t) = \text{col}\{z(t), z(t_k)\}\$ , *X* and *X*<sub>1</sub> are  $n \times n$  matrices. The positiveness of [\(5\)](#page-1-4) is ensured if the following LMI holds [\(Fridman,](#page-6-2) [2010\)](#page-6-2):

<span id="page-1-6"></span>
$$
\begin{bmatrix} P + h \frac{X + X^{T}}{2} & hX_{1} - hX \\ * & -hX_{1} - hX_{1}^{T} + h \frac{X + X^{T}}{2} \end{bmatrix} > 0.
$$
 (6)

<span id="page-1-2"></span>Using the definition of  $z(t)$ , we can see that the exponential stability of system  $(1)$  is guaranteed if:

$$
\dot{V}(t) + 2\alpha V(t) \le 0 \quad t \in [t_k, t_{k+1})
$$
\n
$$
(7)
$$

<span id="page-1-5"></span>for some scalar  $\alpha \in (-\eta, 0]$  (note that the scalar  $\alpha$  is allowed to be negative). Indeed, if  $(7)$  is satisfied one has,

<span id="page-1-1"></span>
$$
\dot{V}(t) \leq -2\alpha V(t) \Longrightarrow |z(t)| \leq \left(\frac{\sqrt{V_{|t=0}}}{\sqrt{\lambda_{\min}(P)}}\right) e^{-\alpha t}.
$$

Then, using the fact that  $z(t) = e^{\eta t}x(t)$ , one gets:

$$
|x(t)| \leq \left(\frac{\sqrt{V_{|t=0}}}{\sqrt{\lambda_{\min}(P)}}\right)e^{-(\eta+\alpha)t}.
$$

From the above inequality, one sees that the exponential convergence is guaranteed if  $\eta + \alpha > 0$ . Since the parameter  $\eta$  is positive and free, it is sufficient to let  $\alpha \in (-\eta, 0]$  for ensuring an exponential convergence with a decay rate  $\eta + \alpha$ . In the following proposition, it is shown that the property  $(7)$ , and resulting exponential stability with a decay rate  $\eta + \alpha > 0$ , are actually ensured under well established sufficient conditions, expressed in terms of LMIs.

<span id="page-1-7"></span>**Proposition 1.** *Consider the system* [\(1\)](#page-1-2) *with possibly varying sampling-intervals subject to*  $t_{k+1} - t_k \leq h$  with some scalar  $h > 0$ . *Given*  $\eta$  > 0 *and*  $\alpha$   $\in$   $(-\eta, 0]$ *, let there exist*  $n \times n$  *matrices*  $P > 0, U > 0, X, X_1, P_2, P_3, T, Y_1, Y_2$  *and a scalar*  $\lambda > 0$  *that satisfy the LMI* [\(6\)](#page-1-6) *and the following LMIs:*

<span id="page-1-8"></span>
$$
\Psi(0) \triangleq \begin{bmatrix}\n\Phi_{11} - X_{\alpha} & \Phi_{12} + X_{\tau(t)} & \Phi_{13} + X_{1\alpha} & P_2^T \\
* & \Phi_{22} + hU & \Phi_{23} - X_{1\tau(t)} & P_3^T \\
* & * & \Phi_{33} - X_{2\alpha} & 0 \\
* & * & * & -\lambda I_n\n\end{bmatrix}_{\tau(t)=0}
$$
\n
$$
< 0
$$
\n(8)

*and*

$$
\Psi(h) \triangleq \begin{bmatrix}\n\Phi_{11} - X_{\alpha_1} & \Phi_{12} & \Phi_{13} + X_{1\alpha} & hY_1^T & P_2^T \\
* & \Phi_{22} & \Phi_{23} & hY_2^T & p_3^T \\
* &* & \Phi_{33} - X_{2\alpha} & hT^T & 0 \\
* &* & * & -hU & 0 \\
* &* & * & * & -\lambda I_n\n\end{bmatrix}_{\tau(t) = h} \tag{9}
$$

*where*

$$
\Phi_{11} = (A_0 + \eta I_n)^T P_2 + P_2^T (A_0 + \eta I_n)
$$
  
+  $2\alpha P - Y_1 - Y_1^T + \lambda M$   

$$
\Phi_{12} = P - P_2^T + (A_0 + \eta I_n)^T P_3 - Y_2
$$
  

$$
\Phi_{13} = Y_1^T + P_2^T A_1 - T
$$
  

$$
\Phi_{22} = -P_3 - P_3^T
$$
  

$$
\Phi_{23} = Y_2^T + P_3^T A_1
$$
  

$$
\Phi_{33} = T + T^T
$$
  

$$
X_{\tau(t)} = (h - \tau(t)) \frac{X + X^T}{2}
$$
  

$$
X_{\alpha} = (1 - 2\alpha(h - \tau(t))) \frac{X + X^T}{2}
$$
  

$$
X_{1\alpha} = (1 - 2\alpha(h - \tau(t)))(X - X_1)
$$
  

$$
X_{1\alpha} = (1 - 2\alpha(h - \tau(t)))(X - X_1)
$$
  

$$
X_{2\alpha} = (1 - 2\alpha(h - \tau(t))) \frac{X + X^T - 2X_1 - 2X_1^T}{2}.
$$

*Then, the system* [\(1\)](#page-1-2) *is exponentially stable with a decay rate*  $n + \alpha$ *. Moreover, if the above LMIs hold with*  $\alpha = -\eta$ *, then the system is exponentially stable with a small enough decay rate in* (0, ϵ)*for some*  $\epsilon \in (0, -\alpha)$ .

See [Appendix A](#page-5-0) for the proof.

<span id="page-2-3"></span>**Remark 2.** Practically, the following procedure is used to get suitable values of the design parameters involved in [Proposition 1.](#page-1-7) The search procedure is started by taking a small value of  $\eta$  (e.g.  $\eta = 0$ ). Then, one proceeds as follows:

(1) Let  $\alpha = -\eta$ .

- (2) Take a small value of *h* and check the LMIs.
- (3) Increase *h* until the LMIs are no longer feasible. Then, retain the value of *h*.
- (4) Increase the value of  $\eta$  and repeat Steps (1)–(3).
- (5) The above procedure is stopped when the increase of  $\eta$  does not entail an increase of *h*.

**Remark 3.** [Proposition 1](#page-1-7) clearly shows that the maximum sampling interval *h* that preserves the exponential stability is made dependent on the design parameters  $\eta$  and  $\alpha$ . Additional highlights will be provided in Example 1 (given hereafter) where it is shown that the maximum *h* preserving stability may be significantly enlarged by tuning  $\eta$ . On the other hand, [Proposition 1](#page-1-7) also confirms the somewhat obvious fact that the exponential decay rate  $\eta + \alpha < \eta$  is also depending on  $\eta$  and  $\alpha$ .

**Remark 4.** In the linear case,

$$
\dot{x}(t) = A_0 x(t) + A_1 e^{-\eta(t - t_k)} x(t_k), \quad t \in [t_k, t_{k+1})
$$

<span id="page-2-1"></span>





<span id="page-2-0"></span>the function  $\phi$  (see [\(1\)\)](#page-1-2) is zero and  $M = 0$  in [\(2\).](#page-1-1) Then, it is sufficient to verify the feasibility of the smaller-order  $\lambda$ -independent LMIs [\(8\)](#page-1-8) and  $(9)$  with the deleted last column and row. Indeed, the feasibility of the smaller-order LMIs implies the feasibility of the full-order LMIs with some  $\lambda > 0$ .

**Remark 5.** If in [\(1\)](#page-1-2) the matrix  $A_1 = BK$  depends on the unknown gain  $K$ , the inequalities  $(8)$  and  $(9)$  become nonlinear. However, by using the standard arguments for the controller design via the descriptor method [\(Fridman,](#page-6-7) [2014\)](#page-6-7), where it is assumed that  $P_2 = \varepsilon P_3$  with a tuning parameter  $\varepsilon$ , one can easily derive LMIs for finding *K*.

#### *Example 1*

To illustrate the result of [Proposition 1,](#page-1-7) consider the following system

$$
\dot{x}(t) = -x(t) + \sin(x(t)) - Ke^{-\eta(t - t_k)}x(t_k), \quad t \in [t_k, t_{k+1}) \tag{10}
$$

<span id="page-2-2"></span>with  $K = 1$ . Here we have  $A_0 = -1$ ,  $A_1 = -K = -1$ , and *M* = 1. Then, application of [Proposition 1](#page-1-7) with  $\alpha = -\eta$  with various values of η leads to different values of *h* that preserve the exponential stability of the system (see [Table 1\)](#page-2-1). Thus,  $(10)$ with  $\eta = 1$  achieves exponential stability for more than twice larger sampling interval compared with the constant gain (i.e. case  $\eta = 0$ ). As a matter of fact, the parameter  $\eta$  cannot be infinitely increased. Presently, the maximum value of  $\eta$  that still yields an increase of the sampling interval is  $\eta = 1$ .

*Example 2*

Consider the following much studied system (see e.g. [Fridman,](#page-6-2) [2010,](#page-6-2) [Zhang,](#page-6-8) [Branicky,](#page-6-8) [&](#page-6-8) [Phillips,](#page-6-8) [2001\)](#page-6-8):

$$
\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & -0.1 \end{bmatrix} x(t) + e^{-\eta(t-t_k)} \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} K x(t_k) \quad t \in [t_k, t_{k+1})
$$

with  $K = -[3.75, 11.5]$ . It is well-known that the above system with  $\eta = 0$  remains stable under constant sampling with  $h < 1.72$ and becomes unstable for constant sampling periods with  $h >$ 1.73. Applying the result of [Proposition 1,](#page-1-7) one obtains for  $\eta = 0.8$ and  $\alpha = -0.8$  a maximum sampling interval value of  $h = 2.43$ that preserves the stability.

#### *Example 3*

<span id="page-2-4"></span>Consider the following closed loop system:

$$
\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-\eta(t-t_k)} K x(t_k) \quad t \in [t_k, t_{k+1}) \quad (11)
$$

with  $Ke^{-\eta(t-t_k)} = \begin{bmatrix} -1 & 1 \end{bmatrix} e^{-\eta(t-t_k)}$  is the regulator gain. Applying the procedure of [Remark 2](#page-2-3) to system  $(11)$ , one obtains the results of [Table 2.](#page-3-1) This shows that the time-varying nature of the regulator gain is actually beneficial to the enlargement of the sampling interval. The analysis of the closed loop system [\(11\)](#page-2-4) in the case of a constant regulator gain (i.e.  $\eta = 0$ ) has been made in several previous studies using various approaches yielding different maximum sampling intervals. A comparison of the obtained maximum sampling intervals is made in [Kao](#page-6-9) [and](#page-6-9) [Wu](#page-6-9) [\(2014\)](#page-6-9), see Example 2 in Section 5 therein. The highest maximum sampling interval, obtained in [Kao](#page-6-9) [and](#page-6-9) [Wu](#page-6-9) [\(2014\)](#page-6-9), is  $h = 1.728$ . Clearly, this value is much smaller than the maximum sampling interval achieved by using a time-varying regulator gain.

<span id="page-3-1"></span>

*h* 2.03 3.38 4.69 5.60 The above examples illustrate the fact that, when using a time

varying gain, the maximum sampling interval *h* that preserves the stability can be much larger compared to the constant gain case  $(\eta = 0).$ 

#### *2.2. Application to sampled-data observer design*

Consider the class of nonlinear systems

$$
\begin{cases}\n\dot{x}(t) = Ax(t) + f(x(t)) \\
y(t) = Cx(t)\n\end{cases}
$$
\n(12)

where  $x \in \mathbb{R}^n$  and A, C are matrices with appropriate dimensions. It is supposed that the function *f* satisfies assumption [\(2\)](#page-1-1) and *y* is accessible to measurements only at instants  $t_k$ . We also suppose that the pair (*A*, *C*) is detectable. The following observer is then proposed:

$$
\dot{\hat{x}}(t) = A\hat{x}(t) + f(\hat{x}(t)) - Ke^{-\eta(t - t_k)}(C\hat{x}(t_k) - y(t_k))
$$
  
\n
$$
t \in [t_k, t_{k+1}),
$$
\n(13)

with  $\eta > 0$  and *K* is a matrix gain of appropriate dimension. Our goal is to determine a matrix *K* and a maximum sampling period *h* so that, the observation error  $\tilde{x}(t) = \hat{x}(t) - x(t)$  is globally exponentially stable. It is readily checked that this error satisfies the following equation:

$$
\dot{\tilde{x}}(t) = A\tilde{x}(t) + f(\hat{x}(t)) - f(x(t)) \n- Ke^{-\eta(t-t_k)}(\hat{C}\hat{x}(t_k) - y(t_k)) \quad t \in [t_k, t_{k+1}).
$$

Considering the change of coordinates  $z_1(t) = e^{\eta t} \tilde{x}(t)$ , one gets the following equation

$$
\dot{z}_1(t) = (\eta I_n + A)z_1(t) - KCz_1(t_k) + e^{\eta t}(f(\hat{x}(t)) - f(x(t))). \quad (14)
$$

Using the fact that,

$$
f(\hat{\mathbf{x}}(t)) - f(\mathbf{x}(t)) = \left[ \int_0^1 f_{\mathbf{x}}(\mathbf{x}(t) + s(\hat{\mathbf{x}}(t) - \mathbf{x}(t))) \, ds \right] \tilde{\mathbf{x}}(t),
$$

Eq. [\(14\)](#page-3-2) can be rewritten as follows:

$$
\dot{z}_1(t) = (\eta I_n + A)z_1(t) + A_1 z_1(t_k) \n+ \left[ \int_0^1 f_x(x(t) + s(\hat{x}(t) - x(t))) ds \right] z_1(t), \quad t \in [t_k, t_{k+1}) \quad (15)
$$

with  $A_1 = -KC$ . Clearly, Eq. [\(15\)](#page-3-3) falls in the class of systems described by Eq.  $(4)$ . Therefore, the result of [Proposition 1](#page-1-7) can directly be applied to get sufficient conditions for the observer [\(13\)](#page-3-4) to be exponentially convergent. This is illustrated in the next example.

#### *Example 4*

Consider the following nonlinear system

$$
\begin{cases}\n\dot{x}_1(t) = x_2(t) \\
\dot{x}_2(t) = -2x_1(t) - x_2(t) + 0.2\sin(x_2(t)) \\
y(t) = Cx(t) = x_1(t).\n\end{cases}
$$

Clearly, this system is of the form [\(12\)](#page-3-5) with  $A = \begin{pmatrix} 0 & 1 \ -2 & -1 \end{pmatrix}$ ,  $C =$  $(1, 0)$ , and  $f(x(t)) = (0 \quad 0.2 \sin(x_2(t)))^T$ .

<span id="page-3-6"></span>**Table 3**

Max. values of $h$ vs. $\eta$ preserving the stability of Example 4.	
--	--



It is readily checked that the function *f* satisfies inequality [\(2\)](#page-1-1) with  $M = diag\{0, 0.04\}$ . Then the observer  $(13)$  is written as follows:

$$
\begin{cases} \dot{\hat{x}}_1(t) = \hat{x}_2(t) - K^1 e^{-\eta(t - t_k)} (\hat{x}_1(t_k) - y(t_k)) \\ \dot{\hat{x}}_2(t) = -2\hat{x}_1(t) - \hat{x}_2(t) + 0.2 \sin(\hat{x}_2(t)) \\ -K^2 e^{-\eta(t - t_k)} (\hat{x}_1(t_k) - y(t_k)) \end{cases}
$$

[w](#page-1-7)here  $K = [K_1K_2]$  and  $A_1 = -KC = \begin{pmatrix} -K^1 & 0 \\ -K^2 & 0 \end{pmatrix}$ . Applying [Propo](#page-1-7)[sition 1](#page-1-7) with  $K = [1.65, -0.85]^T$  and  $\alpha = -\eta$ , one obtains the results of [Table 3.](#page-3-6)

<span id="page-3-5"></span>This clearly shows that the maximum sampling interval *h* can be significantly increased thanks to the time-varying gain feature of the observer.

<span id="page-3-4"></span>**Remark 6.** Note that high gain observers are particular cases of the class of observers described by  $(13)$ . They correspond to  $K =$  $\theta \Delta^{-1}L$  where  $\theta \geq 1$ ,  $\Delta = \text{diag}\{1, \ldots, 1/\theta^{n-1}\}\$  and  $\overline{L}$  is such that *A* − *LC* is Hurwitz. Then the function in [\(12\)](#page-3-5) must be triangular [\(Gauthier,](#page-6-10) [Hammouri,](#page-6-10) [&](#page-6-10) [Othman,](#page-6-10) [1992\)](#page-6-10). Compared to the timevarying gain observer proposed in [Farza](#page-6-11) [et al.](#page-6-11) [\(2014\)](#page-6-11), our observer [\(13\)](#page-3-4) enjoys a simpler time-varying gain (because it is scalar) and offers more flexibility because the parameter  $\eta$  is independent on the gain *K*. In this respect, one might also note that the timeinvariant-gain version of the observer proposed in [Farza](#page-6-11) [et al.](#page-6-11) [\(2014\)](#page-6-11) is nothing other than an obvious rewriting in impulsive form of the original observer of [Karafyllis](#page-6-12) [and](#page-6-12) [Kravaris](#page-6-12) [\(2009\)](#page-6-12).

#### <span id="page-3-0"></span>**3. Sampled-data semi-linear parabolic PDEs**

#### *3.1. System description and stability result*

<span id="page-3-2"></span>In this section, it is shown that results similar to those of Section [2](#page-1-0) can be established for infinite-dimensional systems. Let us consider the class of systems governed by the following partial differential equation:

<span id="page-3-7"></span>
$$
u_t(x, t) = u_{xx}(x, t) + \phi(x, u(x, t), t)u(x, t)
$$
  
- $Ke^{-\eta(t-t_k)}u(\bar{x}_j, t_k)$   $t \in [t_k, t_{k+1}), x \in [x_j, x_{j+1}),$  (16)

with  $\bar{x}_j = \frac{x_{j+1}+x_j}{2}$   $(j = 0, \ldots, N-1)$ , where the points  $x_j$  divide the interval  $[0, l]$  such that  $0 = x_0 < \cdots < x_N = l$ , under Neumann boundary conditions

<span id="page-3-3"></span>
$$
u_x(l, t) = u_x(0, t) = 0, \quad t \ge 0.
$$
\n(17)

We also suppose that the function  $\phi$  is  $\mathcal{C}^1$  and is possibly unknown but  $\phi_m \leq \phi \leq \phi_M$ , where  $\phi_m$  and  $\phi_M$  are known bounds. The sampling interval in space may be variable but bounded

$$
x_{j+1}-x_j\leq \Delta\leq l.
$$

By arguments developed in [Fridman](#page-6-13) [and](#page-6-13) [Blighovsky](#page-6-13) [\(2012\)](#page-6-13), there exists a unique strong solution of  $(16)$  initialized by

<span id="page-3-8"></span>
$$
u(\cdot,0) \in \mathcal{H}^1(0,l) : u_x(0,0) = u_x(l,0) = 0. \tag{18}
$$

<span id="page-3-9"></span>**Proposition 2.** *Consider the class of systems described by* [\(16\)](#page-3-7) *and the positive scalars*  $\eta$  > 0*, h* > 0*, R* > 0*,*  $2\eta$  >  $\delta_1$  > 0*,*  $2\delta$   $\in$  $(\delta_1 - 2\eta, 0]$ *. Choose K* >  $\phi_M + \eta - \frac{\pi^2}{l^2}$ *l* <sup>2</sup> *and suppose that there exist scalars*  $p_i > 0$ *,*  $r > 0$ *, and*  $y_i$  *(i = 1, 2, 3), satisfying the inequality* 

<span id="page-3-10"></span>
$$
\Delta KR^{-1}(p_2 + p_3) \leq \pi \delta_1 p_3 \tag{19}
$$

*and the following LMIs:*

$$
\Phi^i_{|\phi=\phi_m} < 0, \qquad \Phi^i_{|\phi=\phi_M} < 0, \quad i = 0, 1
$$
\nwhere

\n(20)

$$
\varPhi^{0} \triangleq \begin{bmatrix} \chi_{11} - \lambda & \chi_{12} & \chi_{13} \\ * & hr + \chi_{22} & \chi_{23} \\ * & * & \chi_{33} \end{bmatrix}
$$

$$
\varPhi^{1} \triangleq \begin{bmatrix} \chi_{11} - \lambda & \chi_{12} & \chi_{13} & hy_{1} \\ * & \chi_{22} & \chi_{23} & hy_{2} \\ * & * & \chi_{33} & hy_{3} \\ * & * & * & -hr \end{bmatrix}
$$

*with*

 $\cdot$ <sup>*l*</sup>

$$
\chi_{11} = 2\delta p_1 + 2p_2 \left(\phi + \eta + \frac{\Delta KR}{2\pi}\right) - 2y_1
$$
  
\n
$$
\chi_{12} = p_1 - p_2 + p_3(\phi + \eta) - y_2
$$
  
\n
$$
\chi_{13} = y_1 - y_3 - Kp_2
$$
  
\n
$$
\chi_{22} = -2p_3 + \frac{\Delta KRp_3}{\pi}
$$
  
\n
$$
\chi_{23} = y_2 - Kp_3
$$
  
\n
$$
\lambda = 2\frac{\pi^2}{l^2}(p_2 - \delta p_3)
$$
  
\n
$$
\chi_{33} = 2y_3 - \delta_1 p_1.
$$

*Then, the unique strong solution of the system* [\(16\)](#page-3-7) *initiated by* [\(18\)](#page-3-8) *satisfies the following bound, for all*  $t > 0$ *:* 

$$
\int_0^t [p_1 u^2(x, t) + p_3 u_x^2(x, t)] dx
$$
  
\n
$$
\le e^{(-2(\delta + \eta) + \delta_1)t} \int_0^l [p_1 u^2(x, 0) + p_3 u_x^2(x, 0)] dx,
$$
\n(21)

*accordingly the system* [\(16\)](#page-3-7) *is exponentially stable with a decay rate*  $(\delta + \eta) - \delta_1/2$ *. Moreover, if the LMIs* [\(20\)](#page-4-0) *hold with*  $2\delta = \delta_1 - 2\eta$ *, then the system* [\(16\)](#page-3-7) *is exponentially stable with a decay rate in*  $(0, \epsilon)$ *for some*  $\epsilon \in (0, \delta_1)$ *.* 

See the proof placed in [Appendix B.](#page-5-1)

**Remark 7.** A practical selection procedure similar to the ODE case [\(Remark 2\)](#page-2-3) is used to find out suitable values of the design parameters of [Proposition 2.](#page-3-9) Again, the search is started by considering a small value of  $\eta$ , (e.g.  $\eta = 0$ ).

- (1) Choose  $\delta_1$  arbitrarily in the interval (0, 2η) and let  $\delta = \delta_1/2 \eta$ .
- (2) Take a small value of *h* and check the LMIs.
- (3) Increase *h* until the LMIs are no longer feasible. Then, retain the value of *h*.
- (4) Increase the value of  $\eta$  and repeat Steps (1)–(3).
- (5) The above procedure is stopped when the increase of  $\eta$  entails no increase of *h*.

*Example 5*

Consider the following system:

$$
u_t(x, t) = u_{xx}(x, t) - Ke^{-\eta(t - t_k)}u(\bar{x}_j, t_k)
$$
  
 
$$
x \in [x_j, x_{j+1}), t \in [t_k, t_{k+1})
$$

under the Neumann boundary conditions. This system fits system structure [\(16\),](#page-3-7) with  $\phi(x, u) = 0$ . Applying [Proposition 2](#page-3-9) with  $l = \pi$ ,  $K = 1$ ,  $\Delta = \pi/7$ ,  $R = 1$ , one gets for different values of  $\eta$  the maximum sampling interval that preserves the exponential stability. The results thus obtained are summarized in [Table 4](#page-4-1) which shows that the maximum sampling interval *h* can be substantially enlarged by tuning  $\eta$ .

<span id="page-4-1"></span>**Table 4**

Max.values of *h* vs. η that preserve the stability of Example 5.

<span id="page-4-0"></span>

#### *3.2. Application to sampled-data observer design*

Consider the following semi-linear diffusion equation:

$$
u_t(x, t) = u_{xx}(x, t) + f(u(x, t), x, t)
$$
\n(22)

with Neumann condition  $u_x(0, 0) = u_x(l, 0) = 0$ . The system output,  $y(t_k) = u(\bar{x}_i, t_k)$ , for some *j*, is only available at sampling instants  $t_k$ . It is supposed that the function f is known, of class  $\mathcal{C}^1$ , and satisfying  $f_m \leq f_u \leq f_M$ , for some scalar constants  $f_m$  and  $f_M$ . The following observer structure is considered:

<span id="page-4-3"></span>
$$
\hat{u}_t(x, t) = \hat{u}_{xx}(x, t) + f(\hat{u}(x, t), x, t) \n- Le^{-\eta(t - t_k)}(\hat{u}(\bar{x}_j, t_k) - y(t_k)) \n x \in [\bar{x}_j, \bar{x}_{j+1}), t \in [t_k, t_{k+1})
$$
\n(23)

with  $\hat{u}_x(0, t) = \hat{u}_x(l, t) = 0$ . It is readily checked that the observation error  $e(x, t) = \hat{u}(x, t) - u(x, t)$  undergoes the following equation:

<span id="page-4-2"></span>
$$
e_t(x, t) = e_{xx}(x, t) + \phi(e(x, t), x, t)e(x, t)
$$
  
- 
$$
Le^{-\eta(t-t_k)}e(\bar{x}_j, t_k), \quad x \in [\bar{x}_j, \bar{x}_{j+1}), \ t \in [t_k, t_{k+1})
$$
 (24)

<span id="page-4-4"></span>where  $\phi = \int_0^l f_u(\hat{u} + \theta e, x, t) d\theta$ , with boundary conditions  $e_x(0, t) = e_x(l, t) = 0$ . Clearly, Eq. [\(24\)](#page-4-2) is of the form [\(16\).](#page-3-7) Therefore, one can make use of [Proposition 2](#page-3-9) to determine the observer gain *L* as well as a suitable value of the maximum *h*, while guaran-teeing the exponential convergence of the observer [\(23\).](#page-4-3)

#### *3.3. Example 6*

Heat/mass transfer systems with heat generation or volumetric chemical reactions, e.g. chemical tubular reactor, are examples of systems that can be described by [\(16\),](#page-3-7) see [Boskovic](#page-6-14) [and](#page-6-14) [Krstic](#page-6-14) [\(2002\)](#page-6-14) and references therein. Indeed, such systems undergo a heat equation with non-constant coefficient i.e.

$$
u_t = \epsilon_u u_{xx} + \lambda_{\alpha\beta}(x)u(x,t), \quad x \in (0,1)
$$
\n(25)

with  $u_x(0, t) = u_x(1, t) = 0$ , where

$$
\lambda_{\alpha\beta}(x) = \frac{2}{\cosh^2(\alpha x - \beta)}.\tag{26}
$$

Let  $\epsilon_u = 1$ ,  $\alpha = 4$  and  $\beta = 2$  and consider the following observer:  $\hat{u}_t = \hat{u} + 1$  (*w*) $\hat{u}_t \neq 0$ 

$$
t = \ddot{u}_{xx} + \lambda_{\alpha\beta}(x)\dot{u}(x, t) - Le^{-\eta(t-t_k)}(\hat{u}(\bar{x}_j, t_k) - y(t_k)), \quad x \in (0, 1)
$$
 (27)

with  $\hat{u}_x(0, t) = \hat{u}_x(1, t) = 0$ . It is readily checked that the observation error  $e(x, t) = \hat{u}(x, t) - u(x, t)$  undergoes Eq. [\(24\)](#page-4-2) with  $\phi$ (*e*, *x*, *t*) =  $\phi$ (*x*) =  $\lambda_{\alpha\beta}$ (*x*) and the boundary conditions  $e_x(0, t) = e_x(1, t) = 0$ . It can be shown that one presently has  $\phi_M = 30$  and  $\phi_m = 0$ . Applying the above procedure with  $R = 1, \Delta = 0.001, L = 0.11 + \phi_M + \eta - \pi^2$ , several possible sampling interval values, corresponding to different values of the parameter  $\eta$ , have been obtained, see [Table 5.](#page-5-2) It is readily seen that the observer time-varying gain associated with  $\eta = 2$  provides a maximum sampling interval that is nearly 10 times larger than the case of fixed observer gain (i.e. case  $\eta = 0$ ).

<span id="page-5-2"></span>**Table 5** Max. values of *h* vs. *n* that preserve the stability of Example 6.

n		0.5	0.75		
$\delta_1$	0.5	0.5	0.5	0.5	0.5
	0.25	$-0.25$	$-0.5$	$-0.75$	$-1.75$
$h \times 10^3$		16.5	19.9	22.6	29

#### **4. Conclusion**

Novel stability results, stated in [Propositions 1](#page-1-7) and [2,](#page-3-9) are established for two classes of systems respectively described by ODEs and PDEs. Both studied classes involve the exponentially decaying term *e* −η(*t*−*t<sup>k</sup>* ) and all stability conditions are expressed in terms of LMIs. The new results conditions are shown to be useful in designing sampled-data observers with exponentially decaying gains. Furthermore, it is checked through several numerical examples that the introduction of exponentially decaying gains may lead to a substantial enlargement of the maximum sampling intervals while preserving stability. A theoretical proof that, timevarying gain entails sampling interval enlargement is yet to be found. Finally, the extension of the results presented in the present paper, to the case of disturbances is a topic for our future research.

#### <span id="page-5-0"></span>**Appendix A. Proof of [Proposition 1](#page-1-7)**

Let  $\alpha \in (-\eta, 0]$  and differentiate  $V(t)$ . After some simple computations one obtains:

$$
\dot{V}(t) + 2\alpha V(t) + \lambda z^{T}(t)
$$
\n
$$
\times \left(M - \int_{0}^{1} \phi_{x}^{T}(sx)ds \int_{0}^{1} \phi_{x}(sx)ds\right) z(t)
$$
\n
$$
\leq 2z^{T}(t)P\dot{z}(t) + (t_{k+1} - t)\dot{z}^{T}(t)U\dot{z}(t) - \int_{t_{k}}^{t} \dot{z}^{T}(s)U\dot{z}(s)ds
$$
\n
$$
-\xi^{T}\begin{bmatrix} \frac{X + X^{T}}{2} & -X + X_{1} \\ * & -X_{1} - X_{1}^{T} + \frac{X + X^{T}}{2} \end{bmatrix} \xi
$$
\n
$$
+ (t_{k+1} - t)[\dot{z}^{T}(t)(X + X^{T})z(t)
$$
\n
$$
+ 2\dot{z}^{T}(t)(-X + X_{1})z(t_{k})] + 2\alpha z^{T}(t)Pz(t)
$$
\n
$$
+ 2\alpha(t_{k+1} - t)\xi^{T}\begin{bmatrix} \frac{X + X^{T}}{2} & -X + X_{1} \\ * & -X_{1} - X_{1}^{T} + \frac{X + X^{T}}{2} \end{bmatrix} \xi
$$
\n
$$
+ \lambda z^{T}(t)Mz(t)
$$
\n
$$
- \lambda z^{T}(t)\left(\int_{0}^{1} \phi_{x}^{T}(sx)ds \int_{0}^{1} \phi_{x}(sx)ds\right) z(t)
$$
\n(A.1)

using the well known Jensen's inequality, one gets,

$$
\int_{t_k}^t \dot{z}^T(s)U\dot{z}(s)ds \ge (t - t_k)v_1^T U v_1
$$

with  $v_1 = \frac{1}{t-t_k} \int_{t_k}^t \dot{z}^T(s)$ . According to the descriptor approach [\(Fridman,](#page-6-15) [2001\)](#page-6-15), the left-hand sides of the equations,

$$
0 = 2[zT(t)Y1T + zT(t)Y2T + zT(tk)TT]\times [-z(t) + z(tk) + (t - tk)v1]
$$

$$
0 = 2 \left[ z^{T}(t) P_{2}^{T} + \dot{z}^{T}(t) P_{3}^{T} \right] \times \left[ (A_{0} + \eta I_{n}) z(t) + A_{1} z(t_{k}) + \left( \int_{0}^{1} \phi_{x}(sx) ds \right) z(t) - \dot{z} \right]
$$
\n(A.2)

<span id="page-5-3"></span>can be added to  $\dot{V}(t) + 2\alpha V(t)$  where  $Y_1, Y_2, T, P_2, P_3$  are free matrices. Let us define the augmented vector,

$$
\mu = \text{col}\left\{z(t), \dot{z}(t), z(t_k), v_1, \left(\int_0^1 \phi_x(sx)ds\right)z(t)\right\}.
$$

Then, combining  $(A.2)$  and  $(A.1)$ , one gets:

<span id="page-5-5"></span>
$$
\dot{V}(t) + 2\alpha V(t) + \lambda z^{T}(t) \left( M - \int_{0}^{1} \phi_{x}^{T}(sx) ds \int_{0}^{1} \phi_{x}(sx) ds \right) z(t)
$$
\n
$$
\leq \mu^{T} \Psi \mu < 0
$$
\n(A.3)

provided that

$$
\Psi \triangleq \begin{bmatrix}\n\Phi_{11} - X_{\alpha} & \Phi_{12} + X_{\tau(t)} & \Phi_{13} + X_{1\alpha} & \tau(t)Y_1^T & P_2^T \\
* & \Phi_{22} + s.U & \Phi_{23} - X_{1\tau(t)} & \tau(t)Y_2^T & p_3^T \\
* &* & \Phi_{33} - X_{2\alpha} & \tau(t)T^T & 0 \\
* &* &* & -\tau(t)U & 0 \\
* &* &* & * & -\lambda I_n\n\end{bmatrix}
$$
\n
$$
< 0.
$$
\n(A.4)

with  $s = t_{k+1} - t$ . Writing the last matrix inequality for  $\tau(t) \to 0$ and  $\tau(t) \rightarrow h$  leads to LMIs [\(8\)](#page-1-8) and [\(9\),](#page-2-0) respectively. Now, using arguments of Lemma 2 in [Fridman](#page-6-2) [\(2010\)](#page-6-2), it is not difficult to see that, if the LMIs  $(6)$ ,  $(8)$  and  $(9)$  are feasible for some  $h > 0$ , then they are also feasible for all  $\bar{h} \in (0, h]$ .

Now, consider the case where  $\alpha = -\eta$ . If the LMIs [\(6\),](#page-1-6) [\(8\)](#page-1-8) and [\(9\)](#page-2-0) are feasible, then by continuity, they are also feasible with the same  $\eta$  and  $\bar{\alpha} = -\eta + \epsilon$  for small enough  $\epsilon > 0$ . Then, an inequality like [\(A.3\)](#page-5-5) holds with  $\alpha$  being replaced by  $\bar{\alpha}$ . Then, one gets

$$
\dot{V}(t) + \bar{\alpha}V \leq 0
$$

which yields

$$
|x(t)| \leq \left(\frac{\sqrt{V_{|t=0}}}{\sqrt{\lambda_{\min}(P)}}\right)e^{-(\eta+\bar{\alpha})t} = \left(\frac{\sqrt{V_{|t=0}}}{\sqrt{\lambda_{\min}(P)}}\right)e^{-\epsilon t}.
$$

This completes the proof of [Proposition 1.](#page-1-7)

#### <span id="page-5-1"></span>**Appendix B. Sketch of proof of [Proposition 2](#page-3-9)**

Let us introduce the change of coordinates  $\xi(x, t) = e^{\eta t} u(x, t)$ with  $\eta > 0$ . Then it is checked that  $\xi(x, t)$  undergoes the equation:

<span id="page-5-6"></span><span id="page-5-4"></span>
$$
\xi_t(x, t) = \xi_{xx}(x, t) + (\phi(x, u(x, t), t) + \eta)\xi(x, t) \n- K\xi(\bar{x}_j, t_k) \quad t \in [t_k, t_{k+1}), x \in [\bar{x}_j, \bar{x}_{j+1}).
$$
\n(B.1)

Consider the following Lyapunov-functional inspired by [Fridman](#page-6-13) [and](#page-6-13) [Blighovsky](#page-6-13) [\(2012\)](#page-6-13):

$$
V_1(t) = p_1 \int_0^l \xi(x, t)^2 dx + \int_0^l \left[ p_3 \xi_x^2(x, t) + r(t_{k+1} - t) \right. \\
 \times \int_{t_k}^t e^{2\delta(s-t)} z_s^2(x, s) ds \right] dx, \quad t \in [t_k, t_{k+1}).
$$

Notice that the above functional is continuous i.e. it satisfies  $V_1(t_k) = V_1(t_k^-)$ . The rest of the proof matches mutatis mutandis

a similar proof in [Fridman](#page-6-13) [and](#page-6-13) [Blighovsky](#page-6-13) [\(2012\)](#page-6-13). Relying on the descriptor technique, the two following equalities are considered:

$$
0 = 2 \int_0^l [p_2\xi(x, t) + p_3\xi_t(x, t)] \times [-\xi_t(x, t) + \xi_{xx}(x, t) + (\phi(x, u(x, t)) + \eta)\xi(x, t) - K\xi(x, t_k)]dx
$$
  
+ 2K  $\sum_{j=1}^{N-1} \int_{x_j}^{x_{j+1}} [p_2\xi(x, t) + p_3\xi_t(x, t)]$   
 $\times \int_{\bar{x}_j}^{x} \xi_{\zeta}(\zeta, t_k) d\zeta dx$  (B.2)  

$$
0 = 2 \int_0^l [y_1\xi(x, t) + y_2\xi_t(x, t) + y_3\xi(x, t_k)]
$$

$$
\begin{array}{l}\n\int_0^1 (2x(t) + 2x(t) + 2x(t) + 2x(t)) \, dx \\
\times \left[ -\xi(x, t) + \xi(x, t_k) + (t - t_k) v_2(x, t) \right] dx\n\end{array} \tag{B.3}
$$

with

$$
v_2 = \frac{1}{t - t_k} \int_{t_k}^t \xi_s(x, s) ds
$$

where the first equality is directly got from  $(B,1)$  while the second is obtained using Leibnitz–Newton formula. Adding the right sides of equalities [\(B.2\),](#page-6-16) [\(B.3\)](#page-6-17) to  $\dot{V}_1(t) + 2\delta V_1(t) - \delta_1 V_1(t_k)$ , one obtains after some computations the following inequality:

$$
\dot{V}_1(t) + 2\delta V_1(t) - \delta_1 V_1(t_k) \le \int_0^l \eta_2^T \bar{\Phi}_s \eta_2 dx \n+ \left[ \frac{\Delta}{\pi} K R^{-1} (p_2 + p_3) - \delta_1 p_3 \right] \int_0^l \xi_x^2(x, t_k) dx
$$

with

$$
\eta_2 = \text{col}\{\xi(x, t), \xi_t(x, t), \xi(x, t_k), v_2\}
$$
  
and

$$
\bar{\Phi}_s = \begin{bmatrix} \chi_{11} & \chi_{12} & \chi_{13} & (t - t_k)y_1 \\ * & (t_{k+1} - t)r + \chi_{22} & \chi_{23} & (t - t_k)y_2 \\ * & * & \chi_{33} & (t - t_k)y_3 \\ * & * & * & -(t - t_k) \end{bmatrix}.
$$

Just as in [Fridman](#page-6-13) [and](#page-6-13) [Blighovsky](#page-6-13) [\(2012\)](#page-6-13), it can be seen that if the LMIs [\(20\)](#page-4-0) are feasible, then  $\bar{\varPhi}_{\rm s} \leq 0$ . If further [\(19\)](#page-3-10) is fulfilled, then the following inequality holds:

$$
\dot{V}_1(t) + 2\delta V_1(t) - \delta_1 V_1(t_k) \leq 0 \quad t \in [t_k, t_{k+1}).
$$

Since  $2\delta < 0$ , one easily gets

$$
|\dot{V}_1(t)| \leq (-2\delta + \delta_1) \sup_{t_k \leq s < t} (V_1(s)) \quad t \in [t_k, t_{k+1}).
$$

Now, let us define the following continuous functional:

$$
\psi(t) = \sup_{t_k \le s < t} (V_1(s)), \quad t \in [t_k, t_{k+1}).
$$

It is readily checked that

$$
\lim_{h\to 0^+}\left(\sup\frac{\psi(t+h)-\psi(t)}{h}\right)\leq (-2\delta+\delta_1)\psi(t).
$$

Then, by using the comparison Lemma 2.12 in [Karafyllis](#page-6-18) [and](#page-6-18) [Jiang](#page-6-18) [\(2011\)](#page-6-18), one obtains

$$
V_1(t) \le \sup_{t_k \le s < t} (V_1(s)) \le V_1(0)e^{(-2\delta + \delta_1)t}, \quad t \ge 0.
$$
 (B.4)

This, together with the following inequality

$$
\int_0^l e^{2\eta t} [p_1 u^2(x, t) + p_3 u_x^2(x, t)] dx \le V_1(t)
$$
  
\n
$$
\le \left[ \int_0^l [p_1 u^2(x, 0) + p_3 u_x^2(x, 0)] dx \right] e^{(-2\delta + \delta_1)t}, \quad t \ge 0
$$
  
\nyields (21).

Now, consider the case where  $\delta_1 = 2\delta + 2\eta$ . If the LMIs [\(20\)](#page-4-0) are feasible, then by continuity, they are also feasible with the same  $\eta$ and  $\bar{\delta}_1 = 2\delta + 2\eta - \epsilon$  for small enough  $\epsilon > 0$ . Then, an inequality like [\(B.4\)](#page-6-19) holds with  $\delta_1$  being replaced by  $\bar{\delta}_1$ . Then, one gets

<span id="page-6-16"></span>
$$
V_1(t) \le \sup_{t_k \le s < t} (V_1(s)) \le V_1(0)e^{(-2\delta + \bar{\delta}_1)t}, \quad t \ge 0
$$
 (B.5)

which yields

$$
\int_0^l [p_1 u^2(x,t) + p_3 u_x^2(x,t)] dx \le V_1(0) e^{-\epsilon t}.
$$

<span id="page-6-17"></span>This completes the proof of [Proposition 2.](#page-3-9)

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<span id="page-6-0"></span>**Tarek Ahmed-Ali** was born in Algeria in 1972. In 1994 he received an electrical engineering degree from Ecole Nationale Polytechnique d'Alger. In 1998 he received his Ph.D. degree from the University of Orsay-Paris Sud within the L2S-CNRS. In 2009, he was appointed as full professor at Ecole Nationale Supérieure des Ingénieurs de Caen (ENSICAEN). His main recent research interests include observer design, performance and robustness issues in nonlinear, hybrid, networked and infinite dimensional systems.

<span id="page-6-19"></span>

<span id="page-6-1"></span>**Emilia Fridman** received the M.Sc. degree from Kuibyshev State University, USSR, in 1981 and the Ph.D. degree from Voronezh State University, USSR, in 1986, all in mathematics. From 1986 to 1992 she was an Assistant and Associate Professor in the Department of Mathematics at Kuibyshev Institute of Railway Engineers, USSR. Since 1993 she has been at Tel Aviv University, where she is currently Professor of Electrical Engineering-Systems. She has held visiting positions at the Weierstrass Institute for Applied Analysis and Stochastics in Berlin (Germany), INRIA in Rocquencourt (France), Ecole Centrale de Lille (France),

Valenciennes University (France), Leicester University (UK), Kent University (UK), CINVESTAV (Mexico), Zhejiang University (China), St. Petersburg IPM (Russia), Melbourne University (Australia), Supelec (France), KTH (Sweden). Her research interests include time-delay systems, networked control systems, distributed parameter systems, robust control, singular perturbations and nonlinear control. She has published more than 100 articles in international scientific journals. She is the author of the monograph Introduction to Time-Delay Systems: Analysis and Control (Birkhauser, 2014). In 2014 she was nominated as a Highly Cited Researcher by Thomson ISI. Currently she serves as Associate Editor in Automatica and SIAM Journal on Control and Optimization.



<span id="page-7-0"></span>**Fouad Giri** obtained the Ph.D. degree in automatic control and the Accreditation to Supervise Researches, both from Grenoble Institut National Polytechnique (INP), Grenoble, France. Since 1982, he has been Professor successively with the University of Rabat, Morocco and the University of Caen Basse-Normandie, Caen, France. He has been recipient of the 'Research and Doctoral Supervising' award, granted by the French Ministry of High Education and Research, all along the period from 1998 to 2016. He is currently serving as the Chair of the IFAC Technical Committee TC1.2 (Adaptive and Learning Systems), over

the triennial 2014–2017. He has served as Vice-Chair of that TC during the triennial 2011–2014. He has served as the General Chair of the 11th IFAC International Workshop on Adaptation and learning in Control and Signal processing (ALCOSP'2013) and the 5th IFAC International Workshop on Periodic Systems Control (PSYCO'2013) both held in Caen, France, on 03–05 July 2013. He has been Associate Editor for several journals and conferences including Control Engineering Practice (2008–2014), IEEE CSS Conference Editorial Board (CEB) (2010 to present), IEEE Transactions on Control System Technology (2012–2014), Automatica (2014 to present). He has also served as Technical Associate Editor of the IFAC World Congress 2014 and as Associate Editor for the IFAC World Congress 2017. He has published over 85 journal articles, 200 conference papers, 15 book chapters. He has coauthored/*coedited the books: AC Electric Motors Control: Advanced Design Techniques and Applications, Wiley, 2013; Block-Oriented Nonlinear System Identification*, Springer, 2010 (co-edited by E.W. Bai); *Feedback Systems in Control and Regulation: Representations Analysis and Performances*, Eyrolles, 1993; *Feedback* *Systems in Control and Regulation: Synthesis, Applications and Instrumentation* (Eyrolles, 1994). He has supervised to completion 20 Ph.D. students.

> <span id="page-7-1"></span>**Laurent Burlion** received in 2003 the M.Sc. degree in Control Theory at the IRRCyN in Nantes and an engineering degree from ENSTA Bretagne (Ensieta). He obtained his Ph.D. degree from the University of Orsay-Paris Sud within the L2S-CNRS in 2007. Since 2010, he has been working as a research scientist at the French Aerospace Lab (ONERA). His research interests are focused on sampled-



data systems and nonlinear control theory applied to visual servoing, flexible modes attenuation or maneuver load control. He is a member of the IFAC technical committee on Aerospace. **Françoise Lamnabhi-Lagarrigue** is CNRS *Senior Researcher* since 1993 at the *Laboratoire des Signaux et Systémes*, France. She obtained the *Master of Mathematics* degree at the *University Paul Sabatier (Toulouse)* in 1976 and she held a CNRS researcher position in 1980. She re-

<span id="page-7-2"></span>ceived her *Ph.D.* and *Habilitation Doctorate degrees (Docteur d'état és Sciences Physiques)* from the *University Paris Sud* in 1980 and 1985, respectively. From 2004 to 2014, she has been the Scientific Manager of the HYCON and HY-CON2 European Networks of Excellence. She is Senior Editor of *International Journal of Control* (Taylor&Francis) and

Editor-in-Chief of *Annual Reviews in Control* (Elsevier). She is the chair of the EECI International Graduate School on Control. Since 2011, she is serving as Member of the IFAC Technical Board as Chair of the CC9 Social Systems. Her main recent research interests include observer design, performance and robustness issues in nonlinear, hybrid, networked and distributed control systems. She is the prizewinner of the 2008 *Michel Monpetit prize* of the French Academy of Science, of the 2010–2013 *Prime d'Excellence Scientifique* at CNRS and of the 2014 IFAC NMO France.