Predictor-Based Control of Systems With State Multiplicative Noise

E. Gershon, E. Fridman, and U. Shaked

***Abstract*—The problem of H_∞ state-feedback control of linear continuous-time systems with state multiplicative noise in the presence of input delay is investigated. A predictor-based control is applied, for the first time, to these systems. A new condition for stability is derived in a form of a linear matrix inequality. The latter condition is extended to one that guarantees a prescribed $\mathcal{L}_2$-gain bound for the stochastic system. Solutions are obtained for both constant and time-varying delays. Because of the multiplicative noise, the predictor-based control cannot stabilize the system for arbitrarily large delay. It admits, however, delays that are significantly larger than the delays that can be treated by the corresponding non-predictive state-feedback control. The theoretical results are demonstrated by two examples. The first example shows the advantage of the predictor-based controller and the second one demonstrates the applicability of the theory to process control systems.**

***Index Terms*—Predictor-control, stochastic H_∞ control, time-delay.**

I. Introduction

H_∞ analysis and design of linear control systems with stochastic state multiplicative uncertainties have matured over the last three decades (see for example [1]–[3]). Such systems are encountered in many areas of control engineering including altitude and tracking control, to name a few (see [3] for a comprehensive study). H_∞ stability analysis and control of these systems have been extended, in the last decade, to include time-delay systems of various types (i.e constant time-delay, slow and fast varying delay) and they have become a central issue in the theory of stochastic state-multiplicative systems (see [4]–[10] for continuous-time systems and [11]–[13] for the discrete-time counterpart). Many of the results that have been derived for the stability and control of deterministic retarded systems, since the 90's ([14]–[23], see [24] for a comprehensive study), have also been applied to the stochastic case, mainly for continuous-time systems (see [10] and the references therein).

In the continuous-time stochastic setting the predominant tool for the solution of the traditional control and estimation stochastic problems is the Lyapunov-Krasovskii (L-K) approach. For example, in [6] and [7], the (L-K) approach is applied to systems with constant delays, and stability criteria are derived for cases with norm-bounded uncertainties. The H_∞ state-feedback control problem for systems with time-varying delay is treated also in [4] and [5] where the latter work treats also the H_∞ estimation of time delay systems. An alternative and relatively simpler approach for studying stochastic systems, is the *input-output* approach (see [10] for a comprehensive study). The *input-output* approach is based on the representation of the system's delay action by delay-free linear operators which allows one to replace the underlying system with an equivalent one that possesses norm-bounded uncertainty, and may therefore be treated by the theory of such uncertain non-retarded systems with state-multiplicative noise [10]. Albeit its simplicity and the convenience of its application to various control and estimation problems, the input-output approach is quite conservative since it inherently entails over-design [10]. We note that in [9], a robust delay-dependent state-feedback solution is obtained for systems with both: state and input delay.

The predictor-based control that transforms the problem of stabilizing a system with constant input delay to one that seeks a stabilizing controller for a corresponding delay-free system is an efficient classical control design method [27], [28]. The reduction of the problem of controlling systems with input time delay to one of finding a controller to a delay free counterpart system can be achieved by using the model reduction approach which is based on a change of the state variable [29], [30]. This reduction approach has been extended to linear systems with norm-bounded and delay uncertainties [31] as well as to sampled-data control [32]. However, this approach has not been studied yet in the stochastic framework, in general, and it has not been used in cases where the systems encounter parameter uncertainty of a stochastic type, in particular.

In the present work we consider continuous-time linear systems with stochastic uncertainty in their dynamics and a control input that is either constantly delayed or is subject to a delay that is fast varying in time. The stochastic uncertainty is modelled as white multiplicative noise in the state space description of the system. We extend the reduction approach to stochastic systems which is based on a transformed predictor type state space description [24]. We also derive a modified criterion for bounding the $\mathcal{L}_2$-gain of the considered system and present a design method for minimizing this bound by applying a predictor state-feedback control. The theory developed is demonstrated by two examples where the second example is taken from the field of process control where a predictor-based state-feedback controller is derived for a given input time-delay that is caused by a transport process.

Notation: Throughout the paper the superscript "T" stands for matrix transposition, $\mathcal{R}^n$ denotes the n dimensional Euclidean space and $\mathcal{R}^{n\times m}$ is the set of all $n \times m$ real matrices. For a symmetric $P \in \mathcal{R}^{n\times n}$, $P > 0$ means that it is positive definite. We denote expectation by $\mathcal{E}\{\cdot\}$ and the trace of a matrix by $\mathrm{Tr}\{\cdot\}$. In this paper we provide all spaces $\mathcal{R}^k, k \geq 1$ with the usual inner product $\langle \cdot, \cdot \rangle$ and with the standard Euclidean norm $\|\cdot\|$. The space of vector functions that are square integrable over $[0\ \infty)$ is denoted by $\mathcal{L}_2$ and $col\{a, b\}$ implies $[a^T\ b^T]^T$. We denote by $L^2(\Omega, \mathcal{R}^k)$ the space of square-integrable $\mathcal{R}^k$-valued functions on the probability space $(\Omega, \mathcal{F}, \mathcal{P})$, where Ω is the sample space, $\mathcal{F}$ is a σ algebra of a subset of Ω called events and $\mathcal{P}$ is the probability measure on $\mathcal{F}$. By $(\mathcal{F}_t)_{t>0}$ we denote an increasing

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family of σ-algebras $\mathcal{F}_t \subset \mathcal{F}$. We also denote by $\tilde{L}^2([0,T);\mathcal{R}^k)$ the space of nonanticipative stochastic process $f(\cdot) = (f(t))_{t\in[0,T]}$ in $\mathcal{R}^k$ with respect to $(\mathcal{F}_t)_{t\in[0,T)}$ satisfying

$$\|f(\cdot)\|_{\tilde{L}_2}^2 = \mathcal{E}\left\{\int_0^T \|f(t)\|^2\, dt\right\} = \int_0^T \mathcal{E}\left\{\|f(t)\|^2\right\} dt < \infty.$$

Stochastic differential equations will be interpreted to be of *Itô* type [25].

II. Problem Formulation and Preliminaries

We consider the following system:

$$\begin{aligned} dx(t) &= [Ax(t) + B_1 w(t)]\, dt + Gx(t)d\beta(t) + B_2 u(t-\tau_0)dt \\ x(t) &= 0, \quad t \in [-\tau_0,\ 0) \\ \bar{z}(t) &= C_1 x(t) + D_{12} u(t) \end{aligned} \tag{1a-c}$$

where $x(t) \in R^n$ is the state vector, $w(t) \in \tilde{L}^2_{\mathcal{F}_t}([0,\infty);\mathcal{R}^q)$ is an exogenous disturbance, $\bar{z}(t) \in R^m$ is the objective vector and $u(t) \in R^\ell$ is the control input signal. τ_0 is a known time-delay. For the sake of clarity we take $C_1^T D_{12} = 0$. The zero-mean real scalar Wiener process $\beta(t)$ satisfies

$$\mathcal{E}\left\{\beta(t)\beta(s)\right\} = \min(t,s).$$

We treat the following problem:

Given the system of (1a-c) we seek a state-feedback control law $u(t) = Kz(t)$ that stabilizes the system and guarantees a prescribed bound on the $\mathcal{L}_2$-gain of the resulting closed-loop.

In order to solve the above problem, we introduce the following modified state-vector [30]:

$$z(t) = e^{A\tau_0}x(t) + \int_{t-\tau_0}^{t} e^{A(t-s)}B_2 u(s)ds \tag{2}$$

and obtain that

$$\begin{aligned} dz(t) &= e^{A\tau_0}dx(t) + A\left[z(t) - e^{A\tau_0}x(t)\right]dt + B_2 u dt \\ &\quad - e^{A\tau_0}B_2 u(t-\tau_0)dt \\ &= Az(t)dt + e^{A\tau_0}Gx(t)d\beta(t) + e^{A\tau_0}B_1 w(t)dt \\ &\quad + B_2 u(t)dt \\ &= Az(t)dt + \tilde{G}\left[z - \int_{t-\tau_0}^{t} e^{A(t-s)}B_2 u(s)ds\right]d\beta(t) \\ &\quad + e^{A\tau_0}B_1 w(t)dt + B_2 u(t)dt \\ &= \left[Adt + \tilde{G}d\beta(t)\right]z - \tilde{G}\left[\int_{t-\tau_0}^{t} e^{A(t-s)}B_2 u(s)ds\right]d\beta(t) \\ &\quad + e^{A\tau_0}B_1 w(t)dt + B_2 u(t)dt \end{aligned} \tag{3}$$

where

$$\tilde{G} \triangleq e^{A\tau_0}Ge^{-A\tau_0}.$$

We seek first a state-feedback gain matrix K such that

$$u(t) = Kz(t) \tag{4}$$

stabilizes the system and then

$$\begin{aligned} dz(t) &= \left[(A + B_2 K)dt + \tilde{G}d\beta(t)\right]z(t) + e^{A\tau_0}B_1 w(t)dt \\ &\quad - \tilde{G}\left[\int_{t-\tau_0}^{t} e^{A(t-s)}B_2 K z(s)ds\right]d\beta(t). \end{aligned} \tag{5}$$

We use the *Itô* lemma [25]. By this lemma, taking $w(t) \equiv 0$, if one has

$$dz(t) = f(z(t),t)dt + g(z(t),t)d\beta(t)$$

then for a scalar function $V(z,t)$

$$dV = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial z}dz + \frac{1}{2}g^T V_{z,z} g dt. \tag{6}$$

In our case $f = (A + B_2 K)z$ and $g = \tilde{G}[z - \int_{t-\tau_0}^{t} e^{A(t-s)} B_2 K z(s)ds]$.

We consider the following Lyapunov function:

$$V(t) = V_1(t,z) + V_2(t) \tag{7}$$

where

$$\begin{aligned} V_1 &= z^T Q z(t), \quad Q > 0 \in R^{n\times n} \\ V_2 &= \int_{-\tau_0}^{0}\int_{t+\theta}^{t} z^T(s)K^T B_2^T e^{-A^T\theta} R e^{-A\theta} B_2 K z(s)ds\, d\theta \\ R &> 0 \in R^{n\times n}. \end{aligned} \tag{8}$$

For $w(t) \equiv 0$, we obtain, using (6)

$$\begin{aligned} &\{(LV)(t)\} \\ &= 2\mathcal{E}\left\{\langle Qz(t), (A + B_2 K)z(t)\rangle\right\} \\ &\quad + \Bigg\{ Tr\Bigg\{ Q\tilde{G}\left[z - \int_{t-\tau_0}^{t} e^{A(t-s)}B_2 K z(s)ds\right] \\ &\qquad \times \left[\tilde{G}\left(z - \int_{t-\tau_0}^{t} e^{A(t-s)}B_2 K z(s)ds\right)\right]^T\Bigg\} - \Gamma(\tau_0) \\ &\qquad + z^T(t)\left[\int_{-\tau_0}^{0} K^T B_2^T e^{-A^T\theta} R e^{-A\theta} B_2 K d\theta\right] z(t)\Bigg\} \end{aligned} \tag{9}$$

where $(LV)(t)$ is the infinitesimal generator [25] of the system with respect to V and

$$\Gamma(\tau) \triangleq \int_{-\tau}^{0} z^T(t+\theta)K^T B_2^T e^{-A^T\theta} R e^{-A\theta} B_2 K z(t+\theta)d\theta.$$

Using Jensen inequality [26] we obtain

$$-\Gamma(\tau_0) \le -\frac{1}{\tau_0}\left[\int_{-\tau_0}^{0} z^T(t+\theta)K^T B_2^T e^{-A^T\theta}d\theta R \times \int_{-\tau_0}^{0} e^{-A\theta}B_2 K z(t+\theta)d\theta\right]. \quad (10)$$

Thus, choosing

$$R = rI \quad (11)$$

where r is a positive scalar, we obtain

$$\begin{aligned}\{(LV)(t)\} \le\, & 2\langle Qz(t), (A+B_2K)z(t)\rangle \\ &+\left[\tilde{G}\left(z-\int_{t-\tau_0}^{t} e^{A(t-s)}B_2Kz(s)ds\right)\right]^T Q \\ &\times\left[\tilde{G}\left(z-\int_{t-\tau_0}^{t} e^{A(t-s)}B_2Kz(s)ds\right)\right] \\ &+z^T(t)\left[\int_{-\tau_0}^{0} K^TB_2^Te^{-A^T\theta}Re^{-A\theta}B_2Kd\theta\right]z(t) \\ &-\int_{-\tau_0}^{0} z^T(t+\theta)K^TB_2^Te^{-A^T\theta}d\theta\frac{R}{\tau_0} \\ &\times\int_{-\tau_0}^{0} e^{-A\theta}B_2Kz(t+\theta)d\theta \\ =&\left[z^T(t)\int_{-\tau_0}^{0} z^T(t+\theta)K^TB_2^Te^{-A^T\theta}d\theta\right] \\ &\times\Psi(Q,r,K)\left[z^T(t)\int_{-\tau_0}^{0} z^T(t+\theta)K^TB_2^Te^{-A^T\theta}d\theta\right]^T\end{aligned} \quad (12)$$

where

$$\Psi(Q,r,K) = \begin{bmatrix}\Psi_{1,1} & -\tilde{G}^TQ \\ * & \tilde{G}^TQ\tilde{G}-\frac{r}{\tau_0}I\end{bmatrix}$$

$$\begin{aligned}\Psi_{1,1} =\, & Q(A+B_2K)+(A+B_2K)^TQ+\tilde{G}^TQ\tilde{G} \\ &+r\left[\int_{-\tau_0}^{0} K^TB_2^Te^{-A^T\theta}e^{-A\theta}B_2Kd\theta\right].\end{aligned} \quad (13)$$

We thus obtain the following result:

Theorem 1: The system (1a,b) is stabilized by the feedback control (4) if there exist matrices $P>0$, Y, and a tuning parameter $r>0$ that satisfy $\hat{\Psi}<0$ where

$$\hat{\Psi}=\begin{bmatrix}AP+B_2Y+(AP+B_2Y)^T & 0 & P\tilde{G}^T & rY^TB_2^T\Phi_A \\ * & -\frac{r}{\tau_0}I & -\tilde{G}^T & 0 \\ * & * & -P & 0 \\ * & * & * & -r\Phi_A\end{bmatrix}. \quad (14)$$

The controller gain is then given by $K=YP^{-1}$.

Proof: It follows from (12), (13) that stability is guaranteed if:

$$\Psi(Q,r,K)<0. \quad (15)$$

Noting that

$$\begin{aligned}&\Psi(Q,r,K) \\ &= \text{diag}\left\{Q(A+B_2K)+(A+B_2K)^TQ +r\left[\int_{-\tau_0}^{0} K^TB_2^Te^{-A^T\theta}e^{-A\theta}B_2Kd\theta\right], -\frac{r}{\tau_0}I\right\} \\ &+\begin{bmatrix}\tilde{G}^T \\ -\tilde{G}^T\end{bmatrix}Q[\tilde{G} \quad -\tilde{G}]\end{aligned}$$

we obtain, applying Schur's complements, that the requirement of (15) becomes

$$\bar{\Psi}=\begin{bmatrix}\bar{\Psi}_{1,1} & 0 & \tilde{G}^T\tilde{Q} \\ * & -\frac{r}{\tau_0}I_n & -\tilde{G}^TQ \\ * & * & -Q\end{bmatrix}<0,$$

$$\begin{aligned}\bar{\Psi}_{1,1} =\, & Q(A+B_2K)+(A+B_2K)^TQ \\ &+r\left[\int_{-\tau_0}^{0} K^TB_2^Te^{-A^T\theta}Qe^{-A\theta}B_2Kd\theta\right].\end{aligned}$$

Defining $P=Q^{-1}$ and $Y=KP$, and multiplying the latter inequality, from both sides, by diag$\{P,I,P\}$ we obtain the following condition for stability:

$$\tilde{\Psi}=\begin{bmatrix}\tilde{\Psi}_{1,1} & 0 & P\tilde{G}^T \\ * & -\frac{r}{\tau_0}I & -\tilde{G}^T \\ * & * & -P\end{bmatrix}<0$$

$$\begin{aligned}\tilde{\Psi}_{1,1} =\, & (AP+B_2Y)+(AP+B_2Y)^T \\ &+r\left[\int_{-\tau_0}^{0} Y^TB_2^Te^{-A^T\theta}e^{-A\theta}B_2Yd\theta\right].\end{aligned} \quad (16)$$

Denoting $\Phi_A=\int_{-\tau_0}^{0} e^{-(A^T+A)\theta}d\theta$ we obtain the result of (14).

Remark 1: The condition of Theorem 1 is sufficient only. The conservatism of this condition stems from the use of Jensen inequality in (10) and requiring R to be a scalar matrix. The latter restriction can be eliminated by choosing a full matrix R and solving the resulting inequality as a Bilinear Matrix Inequality (BMI).

The above stabilization method has a little meaning in the case where the system is open-loop stable. In the case where a disturbance attenuation is the objective, the above should be extended to establish conditions for achieving $\mathcal{L}_2$-gain bound on the closed-loop system.

III. The $\mathcal{L}_2$-Gain Bound

In the case where $w(t)$ is not zero we consider the following performance index:

$$J \triangleq \int_{0}^{\infty} J_1(s)ds$$

$$\text{where } J_1(t) \triangleq \mathcal{E}\{\|C_1x(t)\|^2+\|D_{12}u(t)\|-\gamma^2\|w(t)\|^2\} \quad (17)$$

and where γ is a prescribed positive scalar. We obtain the following result:

Theorem 2: The $\mathcal{L}_2$-gain of the system (1a-c) with the feedback law (4) is less than a prescribed scalar $\gamma > 0$ if there exist matrices $P > 0$, Y, and a tuning parameter $r > 0$ that satisfy $\tilde{\Phi} < 0$ where

$$\tilde{\Phi} \triangleq \begin{bmatrix} \tilde{\Phi}_{1,1} & 0 & Y^T\hat{\Phi}_A(r) & e^{A\tau_0}B_1 & P\tilde{G}^T & P\tilde{C}_1^T \\ * & -\frac{r}{\tau_0}I & 0 & 0 & -\tilde{G}^T & -\tilde{C}_1^T \\ * & * & -\hat{\Phi}_A(r) & 0 & 0 & 0 \\ * & * & * & -\gamma^2 I & 0 & 0 \\ * & * & * & * & -P & 0 \\ * & * & * & * & * & -I \end{bmatrix}$$

where $\tilde{\Phi}_{1,1} = AP + B_2Y + PA^T + Y^TB_2^T$, $\hat{\Phi}_A(r) = rB_2^T\Phi_AB_2 + \bar{R}$, $\bar{R} = D_{12}^TD_{12}$ and where Φ_A is defined following (16). The controller gain that achieves the $\mathcal{L}_2$-gain bound is given by $u(t) = Kz(t)$ where $K = YP^{-1}$.

Proof: It follows from (9) that:

$$\begin{aligned} &\{(LV)(t)\} \\ &= 2\mathcal{E}\left\{\left\langle Qz(t), (A + B_2K)z(t) + e^{A\tau_0}B_1w(t)\right\rangle\right\} \\ &+\left\{\mathrm{Tr}\left\{Q\tilde{G}\left[z - \int_{t-\tau_0}^{t} e^{A(t-s)}B_2Kz(s)ds\right]\right.\right. \\ &\left.\left.\times\left[\tilde{G}\left(z - \int_{t-\tau_0}^{t} e^{A(t-s)}B_2Kz(s)ds\right)\right]^T\right\} - \Gamma(\tau_0)\right. \\ &\left. + z^T(t)\left[\int_{-\tau_0}^{0} K^TB_2^Te^{-A^T\theta}Re^{-A\theta}B_2Kd\theta\right]z(t)\right\}. \end{aligned} \tag{18}$$

Requiring then that for all $t \geq 0$

$$\{(LV)(t)\} + J_1(t) \leq 0 \quad \forall w(t) \in \tilde{L}^2_{\mathcal{F}_t}([0,\infty);\mathcal{R}^q)$$

stability and the desired disturbance attenuation are achieved.

Applying the steps that lead to (12) and choosing, again, $R = rI$ the following sufficient condition is obtained:

$$\begin{aligned} &\left[z^T(t) \int_{-\tau_0}^{0} z^T(t+\theta)K^TB_2^Te^{-A^T\theta}d\theta\right]\Psi(Q,r,K) \\ &\left[z^T(t) \int_{-\tau_0}^{0} z^T(t+\theta)K^TB_2^Te^{-A^T\theta}d\theta\right]^T \\ &+ 2\left\langle Qz(t), e^{A\tau_0}B_1w(t)\right\rangle + J_1(t) - \gamma^2\|w\|^2 \leq 0. \end{aligned}$$

Denoting $\xi(t) = col\{z(t), \int_{-\tau_0}^{0} e^{-A\theta}B_2Kz(t+\theta)d\theta, w(t)\}$, the latter inequality becomes

$$\xi^T(t)\Phi(Q, z(t), r, K)\xi(t) < 0 \tag{19}$$

where

$$\Phi \triangleq \begin{bmatrix} \Phi_{1,1} & -\tilde{G}^TQ\tilde{G} - \tilde{C}_1^T\tilde{C}_1 & Qe^{A\tau_0}B_1 \\ * & \tilde{G}^TQ\tilde{G} - \frac{r}{\tau_0}I + \tilde{C}_1^T\tilde{C}_1 & 0 \\ * & * & -\gamma^2 I \end{bmatrix}$$

$$\begin{aligned} \Phi_{1,1} &= Q(A + B_2K) + (A + B_2K)^TQ + \tilde{C}_1^T\tilde{C}_1 + \tilde{G}^TQ\tilde{G} \\ &+ K^T(rB_2^T\Phi_AB_2 + \bar{R})K \end{aligned} \tag{20}$$

and where $\tilde{C} \triangleq C_1e^{-A\tau_0}$. Inequality (19) can then be written, applying Schur's complements as

$$\bar{\Phi} = \begin{bmatrix} \bar{\Phi}_{1,1} & 0 & Qe^{A\tau_0}B_1 & \tilde{G}^TQ & \tilde{C}_1^T \\ * & -\frac{r}{\tau_0}I & 0 & -\tilde{G}^TQ & -\tilde{C}_1^T \\ * & * & -\gamma^2 I & 0 & 0 \\ * & * & * & -Q & 0 \\ * & * & * & * & -I \end{bmatrix} < 0$$

$$\bar{\Phi}_{1,1} = Q(A+B_2K)+(A+B_2K)^TQ+K^T(rB_2^T\Phi_AB_2+\bar{R})K. \tag{21}$$

Denoting $P = Q^{-1}$, $Y = KP$ and multiplying (21), from both sides, by $\mathrm{diag}\{P, I, I, P, I\}$ we obtain the result of Theorem 2.

The condition of Theorem 2, if satisfied, guarantees an $\mathcal{L}_2$-gain bound of γ. The definition of the $\mathcal{L}_2$-gain is based, however, on the performance index J_E of (17). Since the control u is delayed by τ_0 seconds, the controller cannot affect the system in the time interval $[0, \tau_0]$ and a more useful criterion would be to achieve

$$\begin{aligned} J_z \triangleq \mathcal{E}&\left\{\int_{\tau_0}^{\infty} \left(\|C_1x(t)\|^2 + u(t)^T\bar{R}u(t)\right)dt\right. \\ &\left. -\gamma^2 \int_0^{\infty} \|w(t)\|^2 dt\right\} < 0 \; \forall\, w(t) \in \tilde{L}^2_{\mathcal{F}_t}([0,\infty);\mathcal{R}^q). \end{aligned} \tag{22}$$

It follows from the definition (2) that $z(t)$ is the predictor value of $x(t)$ which is based on the control input $u(t)$ that is delayed by τ_0 seconds, where the effect of $w(t)$ on $x(\tau)$ in the interval $\tau \in [t - \tau_0, t]$ is discarded. Thus, adopting the predictor type performance index

$$\begin{aligned} J_P \triangleq \mathcal{E}&\left\{\int_{\tau_0}^{\infty} \left(\|C_1z(t)\|^2 + u(t)^T\bar{R}u(t)\right)dt\right. \\ &\left. -\gamma^2\left[\lim_{T\to\infty} \int_0^{T-\tau_0} \|w(t)\|^2\right]dt\right\} \end{aligned} \tag{23}$$

the contribution of $w(t)$ at the interval $[T - \tau_0, T]$ for T that tends to infinity is ignored. The latter is negligible when T tends to infinity and thus J_z and J_P have the same physical meaning. We thus obtain the following from the arguments that led to (21).

Corollary 1: J_P will be negative for all $w(t) \in \tilde{L}^2_{\mathcal{F}_t}([0,\infty);\mathcal{R}^q)$, using the control input (4), if there exist matrices $P > 0$ and Y, and a tuning parameter $r > 0$ that satisfy the following inequality:

$$\Theta = \begin{bmatrix} \Theta_{1,1} & 0 & Y^T\hat{\Phi}_A(r) & e^{A\tau_0}B_1 & P\tilde{G}^T & PC_1^T \\ * & -\frac{r}{\tau_0}I & 0 & 0 & -\tilde{G}^T & 0 \\ * & * & -\hat{\Phi}_A(r) & 0 & 0 & 0 \\ * & * & * & -\gamma^2 I & 0 & 0 \\ * & * & * & * & -P & 0 \\ * & * & * & * & * & -I \end{bmatrix} < 0$$

$$\Theta_{1,1} = AP + B_2Y + PA^T + Y^TB_2^T. \tag{24}$$

IV. Time-Varying Delay

In the above, the delay τ_0 was assumed to be a known constant. The results can be readily extended, however, also to the case of time-varying delay

$$\tau(t) = \tau_0 + \eta(t), \text{ where } |\eta(t)| \leq \mu \leq \tau_0. \tag{25}$$

We first obtain the following stability condition:

Theorem 3: The system (1a,b) with the delay (25) is stabilized by the feedback control (4) if there exist matrices $P > 0$ and Y, and tuning parameters $r, \varepsilon > 0$ that satisfy the following:

$$\begin{bmatrix} \tilde{\Upsilon}_{1,1} & 0 & P\tilde{G}^T & -e^{A\tau_0}B_2Y & \tilde{\Upsilon}_{1,5} & rY^TB_2^T\Phi_A \\ * & -\frac{r}{\tau_0}I & -\tilde{G}^T & 0 & 0 & 0 \\ * & * & -P & 0 & 0 & 0 \\ * & * & * & -\varepsilon P & \tilde{\Upsilon}_{4,5} & 0 \\ * & * & * & * & -\varepsilon P & 0 \\ * & * & * & * & * & -r\Phi_A \end{bmatrix} < 0$$

$$\tilde{\Upsilon}_{1,1} = (AP + B_2Y) + (AP + B_2Y)^T$$

$$\tilde{\Upsilon}_{1,5} = \varepsilon\bar{\tau}_0\left(PA^T + Y^TB_2^T\right)$$

$$\tilde{\Upsilon}_{4,5} = -\varepsilon\bar{\tau}_0 Y^TB_2^Te^{A^T\tau_0} \tag{26}$$

where Φ_A is defined following (16). The controller gain is then given by $K = YP^{-1}$.

Proof: Applying the definition of $z(t)$ as in (2) the differential change $dz(t)$ of (2) will now have an additional term $e^{A\tau_0}B_2[u(t-\tau(t)) - u(t-\tau_0)]dt$ and for $u(t)$ defined in (4) we obtain

$$\begin{aligned} dz(t) = &\left[(A + B_2K)dt + \tilde{G}d\beta(t)\right]z(t) \\ &- \tilde{G}\left[\int_{t-\tau_0}^{t} e^{A(t-s)}B_2Kz(s)ds\right]d\beta(t) + e^{A\tau_0}B_1w(t)dt \\ &+ e^{A\tau_0}B_2K\left[z\left(t-\tau(t)\right) - z(t-\tau_0)\right]dt. \end{aligned} \tag{27}$$

In order to avoid the use of the derivative of z (since only $dz(t)$ is allowed), we apply the input-output approach for delayed systems (see [10], Chapter 2). Thus, Eq. (27) can be written as

$$\begin{aligned} dz(t) = &\left[(A + B_2K)dt + \tilde{G}d\beta(t)\right]z(t) \\ &- \tilde{G}\left[\int_{t-\tau_0}^{t} e^{A(t-s)}B_2Kz(s)ds\right]d\beta(t) + e^{A\tau_0}B_1w(t)dt \\ &- e^{A\tau_0}B_2K\left(\int_{t-\tau}^{t-\tau_0}\bar{y}(s)ds\right)dt - \Gamma_\beta dt. \end{aligned} \tag{28}$$

where

$$\bar{y}(t) = (A + B_2K)z(t) - e^{A\tau_0}B_2Kw_2(t) - \Gamma_\beta + e^{A\tau_0}B_1w(t)$$

and where we define

$$w_2(t) = (\Delta_2\bar{y})(t), \quad (\Delta_2 g)(t) \triangleq \int_{t-\tau(t)}^{t-\tau_0} g(s)ds$$

and denote

$$\Gamma_\beta = \int_{t-\tau}^{t-\tau_0}\tilde{G}\left[z(t) - \int_{t-\tau_0}^{t} e^{A(t-s)}B_2Kz(s)ds\right]d\beta(t).$$

It is well known [33] that

$$\bar{y}^T(t)\Delta_2^T\Delta_2\bar{y}(t) \le \bar{\tau}_0^2\left\|\bar{y}(t)\right\|^2, \quad \text{where} \quad \bar{\tau}_0 = \frac{7}{4}\tau_0.$$

Replacing then the right side of (12) by

$$\begin{aligned} &2\left\langle Qz(t), (A + B_2K)z(t) - e^{A\tau_0}B_2K(\Delta_2\bar{y})(t)\right\rangle \\ &+ \left[\tilde{G}\left(z - \int_{t-\tau_0}^{t} e^{A(t-s)}B_2Kz(s)ds\right)\right]^T \\ &\times Q\left[\tilde{G}\left(z - \int_{t-\tau_0}^{t} e^{A(t-s)}B_2Kz(s)ds\right)\right] \\ &+ z^T(t)\left[\int_{-\tau_0}^{0} K^TB_2^Te^{-A^T\theta}Re^{-A\theta}B_2Kd\theta\right]z(t) \\ &- \frac{1}{\tau_0}\int_{-\tau_0}^{0} z^T(t+\theta)K^TB_2^Te^{-A^T\theta}d\theta R\int_{-\tau_0}^{0} e^{-A\theta}B_2Kz(t+\theta)d\theta \\ &= \left[z^T(t)\int_{-\tau_0}^{0} z^T(t+\theta)K^TB_2^Te^{-A^T\theta}d\theta\right]\Psi(Q, r, K) \\ &\quad\times\left[z^T(t+\theta)\int_{-\tau_0}^{0} z^T(t+\theta)K^TB_2^Te^{-A^T\theta}d\theta\right]^T - 2z^T(t)Q\Delta_2\bar{y}(t) \end{aligned}$$

and using the fact that

$$-w_2^T(t)R_2w_2(t) + \bar{\tau}_0^2\bar{y}^T(t)R_2\bar{y}(t) \ge 0$$

where R_2 is a positive definite matrix, the condition for stability becomes

$$\bar{\xi}^T(t)\bar{\Phi}\left(Q, z(t), r, K\right)\bar{\xi}(t) + \bar{\tau}_0^2\bar{y}^T(t)R_2\bar{y}(t) < 0 \tag{29}$$

where $\bar{\xi}(t) = col\{z(t), \int_{-\tau_0}^{0} e^{-A\theta}B_2Kz(t+\theta)d\theta, w_2(t)\}$, and where

$$\bar{\Phi} \triangleq \begin{bmatrix} \bar{\Phi}_{1,1} & -\tilde{G}^TQ\tilde{G} & -Qe^{A\tau_0}B_2K \\ * & \tilde{G}^TQ\tilde{G} - \frac{r}{\tau_0}I & 0 \\ * & * & -R_2 \end{bmatrix}$$

$$\begin{aligned} \bar{\Phi}_{1,1} = &Q(A + B_2K) + (A + B_2K)^TQ + \tilde{G}^TQ\tilde{G} \\ &+ rK^TB_2^T\Phi_AB_2K. \end{aligned} \tag{30}$$

We thus require that $\tilde{\Phi} =$

$$\begin{bmatrix} \tilde{\Phi}_{1,1} & -\tilde{G}^TQ\tilde{G} & -Qe^{A\tau_0}B_2K & \bar{\tau}_0(A + B_2K)^TR_2 \\ * & \tilde{G}^TQ\tilde{G} - \frac{r}{\tau_0}I & 0 & 0 \\ * & * & -R_2 & -\bar{\tau}_0K^TB_2^Te^{A^T\tau_0}R_2 \\ * & * & * & -R_2 \end{bmatrix} < 0$$

$$\begin{aligned} \tilde{\Phi}_{1,1} = &Q(A + B_2K) + (A + B_2K)^TQ + \tilde{G}^TQ\tilde{G} \\ &+ rK^TB_2^T\Phi_AB_2K. \end{aligned} \tag{31}$$

Following the steps that led to (14) the following requirement is then obtained.

$$\begin{bmatrix} \Upsilon_{1,1} & 0 & P\tilde{G}^T & -e^{A\tau_0}B_2K & \Upsilon_{1,5} & rY^TB_2^T\Phi_A \\ * & -\frac{r}{\tau_0}I & -\tilde{G}^T & 0 & 0 & 0 \\ * & * & -P & 0 & 0 & 0 \\ * & * & * & -R_2 & \Upsilon_{4,5} & 0 \\ * & * & * & * & -R_2^{-1} & 0 \\ * & * & * & * & * & -r\Phi_A \end{bmatrix} < 0$$

$$\Upsilon_{1,1} = (AP + B_2Y) + (AP + B_2Y)^T$$

$$\Upsilon_{1,5} = \bar{\tau}_0(PA^T + Y^TB_2^T), \quad \Upsilon_{4,5} = -\bar{\tau}_0K^TB_2^Te^{A^T\tau_0}. \tag{32}$$

Choosing $R_2 = \varepsilon Q$, where ε is a tuning positive scalar, and multiplying the 4th column and row block in (32) by P, the result of (26) is obtained.

Adopting the performance index (23), the problem of securing a prescribed bound on the $\mathcal{L}_2$-gain of the closed-loop system can be solved by applying similar arguments to those that led to the LMI condition of (24). The following result is obtained:

Theorem 4: The $\mathcal{L}_2$-gain of the system (1a-c) with the feedback law (4) is less than a prescribed scalar $\gamma > 0$, for the time-varying delay (25), if there exist matrices $P > 0$ and Y, and tuning parameters $r > 0$ and $\varepsilon > 0$ that satisfy $\hat{\Upsilon} < 0$ where $\hat{\Upsilon} =$

$$\begin{bmatrix} \Upsilon_{1,1} & 0 & P\tilde{G}^T & \hat{\Upsilon}_{1,4} & \tilde{\Upsilon}_{1,5} & Y^T\hat{\Phi}_A(r) & e^{A\tau_0}B_1 & PC_1^T \\ * & -\frac{r}{\tau_0}I & -\tilde{G}^T & 0 & 0 & 0 & 0 & 0 \\ * & * & -P & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -\varepsilon P & \tilde{\Upsilon}_{4,5} & 0 & 0 & 0 \\ * & * & * & * & -\varepsilon P & 0 & e^{A\tau_0}B_1 & 0 \\ * & * & * & * & * & -\hat{\Phi}_A(r) & 0 & 0 \\ * & * & * & * & * & * & -\gamma^2I & 0 \\ * & * & * & * & * & * & * & -I \end{bmatrix}$$

$$\hat{\Upsilon}_{1,4} = -e^{A\tau_0}B_2Y, \; \hat{\Upsilon}_{1,5} = \epsilon\bar{\tau}_0(PA^T + Y^TB_2^T)$$

$$\hat{\Upsilon}_{4,5} = -\epsilon\bar{\tau}_0K^TB_2^Te^{A^T\tau_0}. \tag{33}$$

The controller gain that achieves the $\mathcal{L}_2$-gain bound is given by $u(t) = Kz(t)$ where $K = YP^{-1}$.

Remark 2: We note that a simple strategy is applied for the numerical solution of (33) which also applies to (26). The solution of (33) involves a search for two scalar variables: r and ε. One may start by taking arbitrary values for the latter two parameters and seek, using line searching, values for these tuning parameters that leads to a stabilizing controller of minimum γ. Once such a controller is obtained, standard optimization techniques can be used, say Matlab function "fminsearch", which seek the combination of the two scalar parameters that bring γ to a local minimum.

V. Examples

A. Stability and $\mathcal{L}_2$-Gain

We consider the system of (1a-c) with

$$A = \begin{bmatrix} -1 & 1 \\ 0.3 & -0.08 \end{bmatrix}, \quad G = \begin{bmatrix} 0 & 0.08 \\ 0 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ and } C_1 = \begin{bmatrix} -0.5 & 0.4 \\ 0 & 0 \end{bmatrix}$$

where $D_{12} = [0 \;\; 0.1]^T$. Applying Theorem 1 we find that the system can be stabilized for $\tau_0 \leq 2.93$ secs. For $\tau_0 = 2.93$ sec, a stabilizing feedback gain $K = [0.0104 \;\; -0.4684]$ is obtained for $r = 1$. An

TABLE I
Range of Delay and $\mathcal{L}_2$-Gain Bounds

method	max delay	γ
Th. 1, Cor. 1	$\tau_0 = 2.93$ (constant)	2.01 (for $\tau_0 = 2.8$)
[10]	$\tau_0 = 1.5$ (constant)	void (for $\tau_0 = 2.8$)
Th. 3, (33)	$\tau(t) \in [0, 1.86]$	6.56 (for $\tau(t) = [0, 1.80]$)

upper-bound of $\gamma = 2.01$ is guaranteed for the predictor-based $\mathcal{L}_2$-gain of the closed-loop system using the condition of Corollary 1 for $\tau_0 = 2.8$ and $r = 0.1$. We note that one can apply the method of [9] for deriving the $\mathcal{L}_2$-gain of this example, however there is no clear indication on how to optimize the obtained results there [see Theorem 2 and Remark 4 in [9]].

In [10], an alternative method for solving the problem has been introduced. It is based on augmenting the system dynamics to include the delayed input as a delayed state vector and it applies the input-output approach for delayed state systems. Using the latter method, a near maximum constant delay $\tau_0 = 1.5$ sec was reached for which an upper-bound of $\gamma = 277.73$ was found on the $\mathcal{L}_2$-gain of the closed-loop system. For a time-varying delay, stability is guaranteed using Theorem 3 for maximum value of $\tau_0 = \mu = 0.93$ sec, namely for a fast varying delay interval [0, 1.86] sec, using $r = 0.005$, $\varepsilon = 0.6$ and the feedback gain matrix $K = [-0.1260 \;\; -0.2640]$. A bound of $\gamma_{\max} = 6.56$ is obtained for $\tau_0 = \mu = 0.9$ sec, using (33) with $r = 0.005$, $\varepsilon = 0.6$ and $K = [-0.1378 \;\; -0.2698]$. The results are summarized in Table I.

B. Process Control

Time delay is an inherent physical phenomenon in process control dynamics, which is usually caused by transport of heat or mass from a certain point in the physical plant [say, from the source of a certain substance] to another point [the main reactor]. Since, compared to electrical or electronic dynamical systems, chemical reactions are relatively slow, typical delays are of tens of seconds. The continuous stirred tank reactor (CSTR) is one of the fundamental physical plants in process control engineering systems. We consider the following irreversible reaction scheme which is known as the Van der Vusse reaction [34], [35]

$$\bar{A} \xrightarrow{k_1} \bar{B} \xrightarrow{k_2} \bar{C}, \quad 2\bar{A} \xrightarrow{k_3} \bar{D}$$

where $\bar{A}$ is the source material [Cyclopentadine] applied to the reactor, $\bar{B}$ is an intermediate substance [Cyclopentenol], $\bar{C}$ [Cyclopentaediol] and $\bar{D}$ [Dicyclopentadiene] are the competing products. The parameters k_1, k_2 and k_3 are the reaction rate constants. The linearized dynamics, around a set-point, of this reaction is described by the model of (1a-c) (excluding, in the meantime, the delay and the multiplicative noise) with the following matrices:

$$A = \begin{bmatrix} A_{11} & 0 \\ k_1 & -\frac{F_s}{V} - k_2 \end{bmatrix}, \; B_1 = \begin{bmatrix} -\frac{F_s}{V} \\ 0 \end{bmatrix}, \; B_2 = \begin{bmatrix} \bar{C}_{Afs} - \bar{C}_{As} \\ -\bar{C}_{Bs} \end{bmatrix}, \tag{34}$$

$C_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $D_{12} = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}$ where $A_{11} = (F_s/V) - k_1 - 2k_3\bar{C}_{As}$, $F_s/V = 0.5714$ min^{-1}, $\bar{C}_{As} = 3$ gmoll^{-1}, $\bar{C}_{Afs} = 10$ gmoll^{-1}, $\bar{C}_{Bs} = 1.117$ gmoll^{-1} and where $k_1 = 0.83$ min^{-1} and $k_3 = 0.166$ moll^{-1}min^{-1}. The rate k_2 depends on the fluctuating concentration of a catalyst. We model then k_2 as a white noise process with average of 2.43 min $^{-1}$ and standard deviation of 0.77 min $^{-1}$. In our model, the state vector is $x(t) \triangleq \begin{bmatrix} \bar{C}_A - \bar{C}_{As} \\ \bar{C}_B - \bar{C}_{Bs} \end{bmatrix}$, where $\bar{C}_A$ and $\bar{C}_B$ are the concentrations of $\bar{A}$ and $\bar{B}$, respectively and $\bar{C}_{As}$ and $\bar{C}_{Bs}$ are the steady-state concentrations of the the latter substances. The input to the system F/V is the flow of substance $\bar{A}$ into the reactor.

We consider the above Van der Vusse reaction scheme in an isothermal CSTR [i.e. with constant temperature] subject to transport time-delay of 0.5 minutes in the inflow of substance $\bar{A}$ [Cyclopentadine]. The statistical nature of k_2 leads to a state-multiplicative noisy system [k_2 appears in the dynamical matrix of (34a)]. Substituting for the above given system parameters, we obtain

$$A = \begin{bmatrix} -2.4 & 0 \\ 0.83 & -3.0 \end{bmatrix}, \quad G = \begin{bmatrix} 0 & 0 \\ 0 & -0.77 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 0.57 \\ 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 7 \\ -1.11 \end{bmatrix}.$$

Applying Corollary 1 and solving (24), for $r = 1.1$, a controller is found for a delay of $\tau_0 = 0.5$ minutes with an upper-bound of $\gamma = 0.040$. The corresponding stabilizing control law is $F/V = -[0.0171(\bar{C}_A - \bar{C}_{As}) + 0.0094(\bar{C}_B - \bar{C}_{Bs})]$.

VI. Conclusions

A predictor-type state-feedback control is applied to systems with stochastic multiplicative-type uncertainty and time delayed actuators. A new state vector is defined which, given the feedback gain matrix, predicts the value of the true state of the resulting closed-loop. A condition for closed-loop stability is derived for these systems which is used to find a stabilizing state-feedback control. Similar conditions are obtained that guarantee prescribed bounds on the $\mathcal{L}_2$-gain of the resulting closed-loop systems.

Two types of delays in the actuators are treated. The first is a constant known delay and the second is time-varying delay that may be fast varying. Two examples were given. In Example 1, an open-loop unstable system is considered. It is shown there that a much larger time-delay can be handled compared to a corresponding result that is obtained by another existing method. It is also shown in the example that the fact that the delay can be fast varying, reduces the maximum range of the delay for which stabilization cab be achieved. A practical control problem taken from the field of process control is treated in Example 2. The result that is obtained there may be used in the future to develop a new methodology in process control.

The results of this work are based on the input-output approach where by applying the norm of the delay operator a delay free stabilization problem is obtained and solved. A future work may be to properly apply the direct Lyapunov-Krasovskii approach in an attempt to reduce the encountered overdesign.

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