Variable Structure Control With Generalized Relays: A Simple Convex Optimization Approach

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***Abstract*—The article proposes a convex optimization approach for the design of relay feedback controllers. The case of linear systems is studied in the presence of matched perturbations. The system input is a generalized relay that may take values in a finite set of constant vectors. A simple design method is proposed using linear matrix inequalities (LMIs). Furthermore, the approach is used in the sampled-data case in order to guarantee (locally) the practical stabilization to a bounded ellipsoid of the order of the sampling interval. Time-varying uncertainties (in the state matrix and the sampling interval) can be easily included in the analysis.**

***Index Terms*—Linear matrix inequalities (LMIs), relay feedback control, sampled-data control, switched systems.**

I. Introduction

Relay feedback control is well known in a wide range of technical domains [13], [37]. It is simple to implement and has interesting robustness properties [14], [38]. However, analysis and design of relay control systems is a non trivial task even for the case of linear systems. The closed-loop system represents a *hybrid dynamical system* [9], [12], [28] which may describe complex behaviours: sliding modes [37], [38], Zeno solutions [36] or limit cycles [8], [20], [27]. Although relay feedback has been studied for many decades, it is still an open problem to choose the switching surfaces so as to optimize the system performances, the robustness properties or the size of the domain of attraction. Furthermore, in practical sampled-data implementations, relay actuators may induce oscillations and even instability [38]. For recent techniques on sampled-data control, we point to [17], [24], [29], [30], [32]. While some analytical approaches exist for the sampled-data implementation of sliding mode controllers [1], [4], [31], [33], very few optimization methods have been provided for the design of sampled-data relay feedback laws. This study may be related to the works in [18], [19], [21], [34], where the effect of input delay has been studied in relay feedback control.

The aim of this article is to propose a convex optimization approach to the design of relay feedback control in the case of linear systems. For the sake of generality, we assume that the system input may take values in a finite set of constant vectors, which includes as a particular case the classical control generated by sign functions. This control law may be related to the simplex method in [3], [5] and to the stabilization of switched affine systems [9], [12], [28]. We show how our approach can be used in the sampled-data case in order to guarantee (local) practical stabilization. Differently from the context of piecewise affine systems where switching surfaces are given (see [35] and the references within), our problem is to design the switching surfaces that ensure (locally) the stability of the closed-loop system. The main idea of the design procedure is to use the existence of an exponentially stabilizing linear state feedback as a reference control to be emulated by a relay feedback. The method is inspired by convex combination techniques used for switched systems [22], [28] and LMIs techniques for systems with bounded controls and saturation [7], [10], [25], [26]. It is based on simple convex optimization arguments and does not need any computation of normal forms. LMI conditions are proposed for dealing with robustness aspects as well as for estimating the maximum sampling interval that ensures (local) practical stabilization. Preliminary results have been presented in [23] where the uncertainty-free case has been treated without any study of finite time reachability.

This technical note is structured as follows: the problem formulation is given in Section II. Section III provides a design method for the generalized relay feedback. LMI design conditions are proposed in Section IV. Section V is dedicated to the sampled-data implementation of the control.

Notations: In this technical note we use standard notations. Given a set $\mathcal{S}$, $\operatorname{conv}\{\mathcal{S}\}$ denotes its closed convex hull and $\operatorname{Int}\{\mathcal{S}\}$ its interior. For $c > 0$ and $x \in \mathbb{R}^n$, $\mathcal{B}(x, c) := \{y \in \mathbb{R}^n : |x - y| < c\}$. For a convex polytope $\mathcal{S}$, $\alpha > 0$, we denote $\alpha\mathcal{S} := \{\alpha x,\ x \in \mathcal{S}\}$ and $\operatorname{vert}\{\mathcal{S}\}$ the set of vertices of $\mathcal{S}$. Given a compact set $\mathcal{S}$ and a continuous function $f : \mathcal{S} \to \mathbb{R}$, $\arg\min_{s\in\mathcal{S}} f(s) = \{y \in \mathcal{S} : f(y) \leq f(r), \forall r \in \mathcal{S}\}$. For $y \in \mathbb{R}$, $\operatorname{sign}(y)$ denotes the set-valued map taking the values $\{1\}$ if $y > 0$, $\{-1\}$ if $y < 0$ and $\{-1, 1\}$, for $y = 0$. For $y \in \mathbb{R}^n$, $\operatorname{sign}(y) = (\operatorname{sign}(y_1), \operatorname{sign}(y_2), \ldots, \operatorname{sign}(y_n))^T$. In a symmetric matrix, the symbol $*$ denotes a block that may be inferred by symmetry. For $N \in \mathbb{N}^+$, $\mathcal{I}_N$ denotes the set $\{1, 2, \ldots, N\}$.

II. Preliminaries and Problem Formulation

Consider $n, m \in \mathbb{N}^+$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and the system

$$\dot{x} = Ax + B(u + d) \tag{1}$$

where $x \in \mathbb{R}^n$ represents the system state, $u \in \mathbb{R}^m$ the input and $d \in \mathbb{R}^m$ a matched perturbation. We adopt the following assumptions:

- (A.1) The pair (A, B) is stabilizable.
- (A.2) The input u is a static state feedback constraint to take values in a finite set of constant vectors $\mathcal{V} := \{v_1, v_2, \ldots, v_N\} \subset \mathbb{R}^m$, where N is a positive integer, i.e., $u : \mathbb{R}^n \to \mathcal{V}$.
- (A.3) The perturbation d is a measurable function taking values in the cube $\mathcal{P}(d_{\max})$ where $d_{\max} \geq 0$ is a known scalar and $\mathcal{P}(c) := \{y \in \mathbb{R}^m : |y|_\infty \leq c\},\ \forall c \geq 0$.
- (A.4) $\operatorname{conv}\{\mathcal{V}\}$ is a nonempty closed subset in $\mathbb{R}^m$ containing the null vector in its interior: $0_m \in \operatorname{Int}\{\operatorname{conv}\{\mathcal{V}\}\}$.
- (A.5) There exists $\rho \in [0, 1)$ such that $\mathcal{P}(d_{\max}) \subset \operatorname{conv}\{\rho\mathcal{V}\}$.

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From the point of view of hybrid systems, system (1) with control laws restricted to $\mathcal{V}$ may be seen as a particular class of *switched affine systems* [9], [12], [22]. Note that for $d = 0$ system (1) may be re-expressed as

$$\dot{x} = \tilde{A}_\sigma x + \tilde{b}_\sigma, \quad \text{with} \quad \tilde{A}_i = A, \quad \tilde{b}_i = Bv_i, \quad i \in \mathcal{I}_N$$

where the function $\sigma : \mathbb{R}^n \to \mathcal{I}_N$ (the switching law) is such that $\sigma(x) = i \Leftrightarrow u(x) = v_i$. Therefore the synthesis of a control law u may be seen as a switching law design problem for a particular class of switched affine systems. However, we do not require the existence of a Hurwitz convex combination (the matrix A may have eigenvalues with positive real part). For local stabilization we only require that (A.1) holds.

For the particular case when the set of vectors $\mathcal{V}$ form a simplex in $\mathbb{R}^m$ ($N = m + 1$, every subset of m vectors in $\mathcal{V}$ are linearly independent and there exists $m + 1$ positive scalars $\nu_i, i \in \mathcal{I}_{m+1}$ such that $\sum_{i=1}^{m+1} \nu_i v_i = 0$, $\sum_{i=1}^{m+1} \nu_i = 1$), the design of a control u with values constrained to the set $\mathcal{V}$ is a *simplex*-type variable structure control problem (see [3], [5] and the references with).

In this technical note we are interested in the design of control laws $u : \mathbb{R}^n \to \mathcal{V}$ of the form

$$u(x) \in \arg\min_{v \in \mathcal{V}} x^T \Gamma v \tag{2}$$

where $\Gamma \in \mathbb{R}^{n \times m}$ is a matrix to be determined. Note that for the case when the input u is a scalar constraint to the set $\mathcal{V} = \{-v, v\}$, with $v > 0$ a given constant, $u(x) = v$ whenever $x^T \Gamma v \leq x^T \Gamma(-v)$, i.e., for $x^T \Gamma \leq 0$. Similarly, $u(x) = -v$ whenever $x^T \Gamma \geq 0$. Then, for $\mathcal{V} = \{-v, v\}$, with $v > 0$, the control law (2) is reduced to the classical relay control $u(x) \in -v\text{sign}(\Gamma^T x)$.

Since the values of the input are restricted to a finite set, the closed loop system (1), (2) has a discontinuous right-hand side. Solutions are considered in the sense of Filippov [15]. We recall that for a differential equation $\dot{x} = f(x)$, with f a locally bounded discontinuous vector field, a *Filippov solution* of the system on the interval $I = [t_a, t_b] \subset [0, \infty)$ is an absolutely continuous map $\phi : [t_a, t_b] \to \mathbb{R}^n$ such that the differential inclusion $\dot{\phi}(t) \in F(\phi(t))$ is satisfied for almost every $t \in [t_a, t_b]$, with $F(x) = \cap_{\epsilon > 0} \cap_{\mu(N)=0} \text{conv}\{f(\bar{x}) : \bar{x} \in \mathcal{B}(x, \epsilon) \setminus N\}$ where μ represents the usual Lebesgue measure. For the case of the closed-loop system (1), (2) the differential inclusion used for defining Filippov solutions takes the particular form [15]

$$F(x) = \text{conv}\ \Big\{Ax + B(\tilde{u} + d), d \in \mathcal{P}(d_{\max}), \\ \tilde{u} \in \arg\min\nolimits_{v \in \mathcal{V}} x^T \Gamma v\Big\}. \tag{3}$$

The existence of a least one solution starting from each initial condition is guaranteed if for every $x \in \mathbb{R}^n$, $F(x)$ is locally bounded, upper semi-continuous and takes non-empty, compact and convex values, which is the case for (3). For stability definitions in the context of discontinuous systems see [2], [15]. The goal of the technical note is to provide criteria for the synthesis of a relay control law (2) that ensures *local stability* of Filippov solutions associated to the closed-loop system (1), (2). We provide optimization methods for control design while enlarging the domain of attraction. Finite time reachability properties to sliding manifolds and the robustness with respect to matched perturbations, time-varying uncertainties or sampled-data implementations will be discussed.

III. Main Results

From the point of view of Lyapunov stability theory, Assumption (A.1) is equivalent [10] with

$$(A + BK)^T P + P(A + BK) \prec -2\delta P. \tag{4}$$

For $\gamma, \beta > 0$ let $\mathcal{E}(P, \gamma) := \{x \in \mathbb{R}^n : x^T P x < \gamma\}$ denote the γ level set of the function $V(x) = x^T P x$ and $\mathcal{C}_{\beta\mathcal{V}}(K)$ the subset of the state space for which Kx belongs to the convex hull of $\beta\mathcal{V}$, $\mathcal{C}_{\beta\mathcal{V}}(K) := \{x \in \mathbb{R}^n : Kx \in \text{conv}\{\beta\mathcal{V}\}\}$. Consider a set of co-vectors $h_i \in \mathbb{R}^{1 \times m}, i \in \mathcal{I}_{n_h}$ describing the dual representation [11], [39] of the polytope $\text{conv}\{\mathcal{V}\}$

$$\text{conv}\{\mathcal{V}\} = \left\{y \in \mathbb{R}^m : \ h_i y \leq 1, i \in \mathcal{I}_{n_h}\right\}. \tag{5}$$

The following theorem provides design conditions for the control law (2).

Theorem 3.1: Consider Assumptions (A.2)–(A.6) and the closed-loop system (1), (2) with $\Gamma = PB$. Then for any

$$\gamma \leq \min_{i \in \mathcal{I}_{n_h}} (1 - \rho)^2 \left(h_i K P^{-1} K^T h_i^T\right)^{-1} \tag{6}$$

a) the origin $x = 0$ of the closed-loop system is *locally exponentially stable* in $\Omega_0 := \mathcal{E}(P, \gamma)$;

b) if $rank(B) = m \leq n$ then, for $s = B^T P x$ the surface $s = 0$ is *finite time reachable* whenever $x(0) \in \mathcal{E}(P, \gamma)$, i.e., exists $t_f \in [0, \infty)$ such that $s(t) = 0$ for all $t \geq t_f$.

Furthermore, if for some P satisfying (4), $A^T P + PA$ is negative semi-definite then

c) the origin of the closed-loop system is *globally asymptotically stable*.

Proof: a) Consider the function $V(x) = x^T P x$ with P satisfying Assumption (A.6). Then

$$\nabla V^T(x)\,(Ax + B(u + d)) = M_1(\rho, u, d) + M_2(\rho, u) \tag{7}$$

where $M_1(\rho, u, d) = 2x^T PB(\rho u + d)$ and $M_2(\rho, u) = 2x^T P(Ax + B(1 - \rho)u)$. Consider $\rho \in [0, 1)$ such that (A.5) is satisfied. Since $\mathcal{P}(d_{\max}) \subset \text{conv}\{\rho\mathcal{V}\}$ and the minimum of a linear function over a convex polytope is reached in the set of vertices [39], $\min_{v \in \rho\mathcal{V}} x^T PBv = \min_{v \in \text{conv}\{\rho\mathcal{V}\}} x^T PBv \leq \min_{v \in \text{vert}\{\mathcal{P}(d_{\max})\}} x^T PBv$. Denote the ith column of B as b_i. Remark that $\min_{v \in \text{vert}\{\mathcal{P}(d_{\max})\}} x^T PBv = \sum_{i=1}^m (-d_{\max} x^T P b_i \text{sign}(x^T P b_i))$. Let d_i denotes the ith component of the vector d. Then for $u \in \arg\min_{v \in \mathcal{V}} x^T PBv$, we obtain that

$$M_1(\rho, u, d) \leq 2x^T P \left(\sum_{i=1}^m b_i \left(-d_{\max}\text{sign}(x^T P b_i) + d_i\right)\right) \leq 0. \tag{8}$$

Let v_i be one of the control vectors in $\mathcal{V}$ for some $i \in \mathcal{I}_N$. From (2), with $\Gamma = PB$, $u(x)$ may take the value v_i only when v_i minimizes the expression $x^T PBv, v \in \mathcal{V}$, that is $x^T PB(v_j - v_i) \geq 0, \forall j \in \mathcal{I}_N$. Furthermore, since $\rho \in [0, 1)$

$$(1 - \rho)2x^T PB(v_j - v_i) \geq 0, \forall j \in \mathcal{I}_N. \tag{9}$$

Since for any $\rho \in [0, 1)$ the set $\mathcal{C}_{(1-\rho)\mathcal{V}}(K)$ is a non-empty subset of $\mathbb{R}^m$ containing the origin and Kx is a continuous application, there exists $\gamma > 0$ such that $\mathcal{E}(P, \gamma) \subset \mathcal{C}_{(1-\rho)\mathcal{V}}(K)$. For this relation to hold, it is necessary and sufficient that none of the hyperplanes $h_i K x = (1 - \rho)$, $i \in \mathcal{I}_{n_h}$, crosses the ellipsoid $\mathcal{E}(P, \gamma)$, that is γ should be smaller than the minimum of the quadratic function V along any of the hyperplanes $h_i K x = (1 - \rho)$, $\gamma \leq \min_{h_i K x = (1-\rho), i \in \mathcal{I}_{n_h}} x^T P x$, i.e., $(1 - \rho)^2 (h_i K P^{-1} K^T h_i^T)^{-1}$. Using standard arguments this leads to condition (6) (see also Chapter 5 in [10]). Then, for any $x \in \mathcal{E}(P, \gamma)$, there exists N scalars $\alpha_j(x) \geq 0$, $\forall j \in \mathcal{I}_N$ with $\sum_{j=1}^N \alpha_j(x) = 1$ such that $Kx = \sum_{j=1}^N \alpha_j(x)(1 - \rho)v_j$. Multiplication of (9) by the appropriate coefficients $\alpha_j(x)$ and summing leads to $2x^T PB(Kx - (1 - \rho)v_i) \geq 0, \forall j \in \mathcal{I}_N$ whenever $u(x) = v_i$. To show that $M_2(\rho, u) < -2\delta V(x)$ when $u(x) = v_i, \forall i \in \mathcal{I}_N$ and $x \in$

$\mathcal{E}(P,\gamma)\setminus\{0\}$, it is sufficient that $M_2(\rho,v_i)+2x^TPB(Kx-(1-\rho)v_i)<-2\delta V(x)$, for all $x\in\mathcal{E}(P,\gamma)\setminus\{0\}$ which holds true since from (4) and the definition of $M_2(\rho,u)$ we have

$$M_2(\rho,v_i)+2x^TPB\left(Kx-(1-\rho)v_i\right) = 2x^TP(A+BK)x<-2\delta V(x) \quad (10)$$

for all $i\in\mathcal{I}_N$ and $x\in\mathcal{E}(P,\gamma)\setminus\{0\}$. From (7), (8) and (10) we have that $\nabla V^T(x)(Ax+B(\tilde u+d))<-2\delta V(x)$ for all $x\in\mathcal{E}(P,\gamma)\setminus\{0\}$, $\tilde u\in\arg\min_{v\in\mathcal{V}}x^TPBv$, that is $\max_{y\in F(x)}\nabla V^T(x)y<-2\delta V(x)$, $\forall x\in\mathcal{E}(P,\gamma)\setminus\{0\}$, with $F(x)$ given in (3).

b) The dynamic of s is given by

$$\dot s = B^TP\left(Ax+B(u+d)\right) \quad (11)$$

where u from (2) can be re-expressed as

$$u\in\arg\min_{v\in\mathcal{V}} s^Tv. \quad (12)$$

Since $P\succ 0$ and B is full rank, then $M=B^TPB=M^T\succ 0$ and there exists $L=M^{-1}=L^T\succ 0$. Let $\psi(s)=s^TLs$. Note that

$$\nabla\psi^T(s)B^TP\left(Ax+B(u+d)\right)=2s^T(u+\omega) \quad (13)$$

where $\omega=LB^TPAx+d$. Let $c^*=\max c$ such that $\mathcal{P}(c)\subset\mathrm{conv}\{\mathcal{V}\}$. Then

$$\min_{v\in\mathcal{V}}s^Tv=\min_{v\in\mathrm{conv}\{\mathcal{V}\}}s^Tv\le\min_{v\in\mathrm{vert}\{\mathcal{P}(c^*)\}}s^Tv=-c^*\mathrm{sign}(s). \quad (14)$$

From (12)–(14), we obtain that

$$\nabla\psi^T(s)B^TP\left(Ax+B(u+d)\right)\le 2s^T\left(-c^*\mathrm{sign}(s)+\omega\right).$$

Then, for any $|\omega|_\infty<c^*$ there exists $\epsilon>0$ such that

$$\nabla\psi^T(s)B^TP\left(Ax+B(u+d)\right)\le-\epsilon\sqrt{\psi(s)} \quad (15)$$

whenever $s\neq 0$. Inequality (15) is sufficient to guarantee the (local) finite-time stability of (11), (12) (see [6], [13]) and therefore the finite time reachability of the surface $s=0$ for any initial condition in an invariant level set of V with

$$|LB^TPAx|_\infty<c^*-|d|_\infty. \quad (16)$$

Let $R=LB^TPA$ and let r_i denote the i^{th} row of R, $i=1,\dots,m$. The set in (16) can be described by

$$\mathcal{D}(c^*,d_{\max})=\left\{x\in\mathbb{R}:|r_ix|<c^*-d_{\max},\ i\in\mathcal{I}_m\right\}. \quad (17)$$

Given P, c^*, $d_{\max}$ and γ such that $\mathcal{E}(P,\gamma)\subset\mathcal{C}_{(1-\rho)\mathcal{V}}(K)$, a level γ_s for which $\mathcal{E}(P,\gamma_s)\subset\mathcal{D}(c^*,d_{\max})$ may be obtained using the description (17) and standard arguments (see Chapter 5 in [10]): $\gamma_s<\min_{i\in\mathcal{I}_m,r_iP^{-1}r_i^T\neq 0}(c^*-d_{\max})^2(r_iP^{-1}r_i^T)^{-1}$. If $\gamma\le\gamma_s$, then for any $x(0)\in\mathcal{E}(P,\gamma)$, $x(t)\in\mathcal{E}(P,\gamma)\subseteq\mathcal{E}(P,\gamma_s)\subset\mathcal{D}(c^*,d_{\max}),\forall t>0$. Then (15) holds, and the system reaches the surface $B^TPx=0$ in finite time. For the case $\gamma>\gamma_s$, remark that $V(x(t))<e^{-\delta t}V(x(0)),\forall x(0)\in\mathcal{E}(P,\gamma)$. Then there exists $t_s>0$ such that $x(t)\in\mathcal{E}(P,\gamma_s)$ for all $t>t_s$. Therefore, $\forall t>t_s$, relation (15) holds, i.e., the surface $B^TPx=0$ is reachable in finite time.

c) From (4) we have that

$$x^T(A^TP+PA)x<-2\delta x^TPx \text{ whenever } B^TPx=0,\quad x\neq 0. \quad (18)$$

Since $x^T(A^TP+PA)x\le 0$, $\forall x\in\mathbb{R}^n$

$$\nabla V^T(x)\left(Ax+B(\tilde u+d)\right)\le\nabla V^T(x)B(\tilde u+d) \quad (19)$$

$\forall\tilde u\in\arg\min_{v\in\mathcal{V}}x^TPBv$ where $V(x)=x^TPx$. Consider the maximum $c^*>0$ such that $\mathcal{P}(c^*)\subset\mathrm{conv}\{\mathcal{V}\}$. From (A.3)–(A.6), $d_{\max}<c^*$. Following similar arguments as in the proof of points a), b) one may show that

$$\nabla V^T(x)B(\tilde u+d)\le-(c^*-d_{\max})s^T\mathrm{sign}(s)<0 \quad (20)$$

for all $\tilde u\in\arg\min_{v\in\mathcal{V}}x^TPBv$, and all x such that $B^TPx\neq 0$. From (18), (19), (20) we have that $\nabla V^T(x)(Ax+B(\tilde u+d))<0,\forall x\neq 0$, $\tilde u\in\arg\min_{v\in\mathcal{V}}x^TPBv$ that is $\max_{y\in F(x)}\nabla V^T(x)y<0,\forall x\neq 0$, with $F(x)$ given in (3). ■

IV. LMI Criteria for Relay Control Design

In this section LMI design methods are provided for a control of the form (2). We treat the more general case with $A(\mu(t))\in\mathcal{A}:=\mathrm{conv}\{A_1,A_2,\dots,A_{n_v}\}$ where $\mu(t)=[\mu_1(t)\ \mu_2(t)\ \dots\ \mu_{n_v}(t)]^T$ are the barycentric coordinates of A in $\mathcal{A}$.

Corollary 4.1: Consider the system

$$\dot x=A(\mu)x+B(u+d) \quad (21)$$

where $\mu(\cdot)$ is measurable, Assumptions (A.2)–(A.5) and the dual representation of the polytope $\mathrm{conv}\{\mathcal{V}\}$ in (5). Given $\delta>0,\gamma>0$, assume that there exists (Q,λ,ϵ) solution to the set of linear matrix inequalities

$$Q=Q^T\succ 0,\quad \lambda>0,$$
$$A_jQ+QA_j^T-\lambda BB^T\prec-2\delta Q,\quad\forall j\in\mathcal{I}_{n_v}, \quad (22)$$
$$\begin{bmatrix}\epsilon I & I\\ * & Q\gamma\end{bmatrix}\succ 0, \quad (23)$$
$$\begin{bmatrix}1 & \frac{\lambda}{2(1-\rho)}h_iB^T\gamma\\ * & Q\gamma\end{bmatrix}\succ 0,\quad i\in\mathcal{I}_{n_h} \quad (24)$$

Then the origin $x=0$ of the closed-loop system (21),(2) with $\Gamma=Q^{-1}B$ is locally asymptotically stable in the ellipsoid $\mathcal{E}(Q^{-1},\gamma)$ containing the ball $\mathcal{B}(0,c_B)$ with $c_B=1/\sqrt{\epsilon}$. Furthermore, if $rank(B)=m\le n$, the surface $s=B^TQ^{-1}x=0$ is finite time reachable for any $x(0)\in\mathcal{E}(Q^{-1},\gamma$.

Proof: Using convexity arguments, condition (22), implies the existence of $K=-\lambda/2B^TQ^{-1}$ and $V(x)=x^TQ^{-1}x$ such that $\nabla V^T(x)(A(\mu)+BK)x<-2\delta V(x),\forall A(\mu)\in\mathcal{A},x\neq 0$. Condition (23) is equivalent (by Schur complement) with $x^T(1/c_B^2I-(Q\gamma)^{-1})x>0$. Then $1/c_B^2x^Tx<1$ implies $x^TQ^{-1}\gamma^{-1}x<1$ which guarantees that $\mathcal{B}(0,c_B)\subset\mathcal{E}(Q^{-1},\gamma)$. Applying the Schur complement lemma, and using the notation $K=-\lambda/2B^TQ^{-1}$, condition (24) leads to $\gamma<(1-\rho)^2(h_iKQK^Th_i^T)^{-1},\forall\ i\in\mathcal{I}_{n_h}$, which implies that $\mathcal{E}(Q^{-1},\gamma)\subset\mathcal{C}_{(1-\rho)\mathcal{V}}(K)$. Local exponential stability in $\mathcal{E}(Q^{-1},\gamma)$ may be proven using the same steps as in the proof of Theorem 3.1, a), with $P=Q^{-1}$.

Considering $\psi=s^TLs$ with $L=(B^TQ^{-1}B)^{-1}$, $\omega=LB^TPA(\mu)x+d$, and $c^*=\max\ c$ such that $\mathcal{P}(c)\subset\mathrm{conv}\{\mathcal{V}\}$, the inequality $\nabla\psi^T(s)B^TP(A(\mu)x+B(u+d))\le 2s^T(-c^*\mathrm{sign}(s)+\omega)$ holds for all $s\neq 0$, $A(\mu)\in\mathcal{A}$. Since $|LB^TPA(\mu)x|_\infty\le\sum_{j=1}^{n_v}\mu_j|LB^TPA_jx|_\infty$, there exists $\epsilon>0$ such that

$$\nabla\psi^T(s)B^TP\left(A(\mu)x+B(u+d)\right)\le-\epsilon\sqrt{\psi(s)} \quad (25)$$

whenever $|LB^TPA_jx|_\infty\le c^*-|d|_\infty,\forall\ j\in\mathcal{I}_{n_v}$. Considering $R_j=LB^TPA_j$, $r_{i,j}$, the i^{th} row of R_j, $j\in\mathcal{I}_{n_v}$, and (25), the finite time convergence of ψ is guaranteed from any invariant level set of V that may be placed in $\mathcal{D}(c^*,d_{\max})=\{x\in\mathbb{R}^n:|r_{i,j}x|<c^*-d_{\max},(i,j)\in\mathcal{I}_m\times\mathcal{I}_{n_v}\}$. The rest of the proof follows the arguments used for Theorem 3.1, b). ■

Remark 1: The existence of a solution (Q, λ, ϵ) to the LMI optimization problem $\inf\ \epsilon$ under the constraints (22)–(24), guarantees that any Filippov solution of the closed-loop system (1), (2) (with $\Gamma = Q^{-1}B$), originating from $\mathcal{E}(Q^{-1}, \gamma)$ is exponentially converging to the origin. By minimizing ϵ, the size of the invariant ellipsoid is maximized. Note that without any loss of generality we may always consider $\gamma = 1$. If the LMIs (22)–(24) are satisfied for $(Q_0, \lambda_0, \epsilon_0)$, then they are also satisfied for $\gamma = 1$ with $(Q_0\gamma_0, \lambda_0\gamma_0, \epsilon_0)$. Given $d_{\max}$, the minimum ρ s.t. $\mathcal{P}(d_{\max}) \subset \mathrm{conv}\{\rho\mathcal{V}\}$ can be computed from the standard optimization problem

$$\inf \rho \text{ s.t. } h_i y \leq \rho, \quad \forall\, y \in \mathrm{vert}\left\{\mathcal{P}(d_{\max})\right\}, \quad i \in \mathcal{I}_{n_h} \tag{26}$$

and (27) and (28), as shown at the bottom of the page.

V. Sampled-Data Implementation

Consider the sequence of sampling times $\{t_k\}_{k\in\mathbb{N}}$, with

- (A.7) $t_0 = 0, \quad t_k < t_{k+1}, \quad \forall \quad k \in \mathbb{N}, \ \lim_{k\to\infty} t_k = \infty$ and $T_k := t_{k+1} - t_k \in (0, \overline{T}]$, where $\overline{T}$ is a known bound.

Denote $x_k = x(t_k)$, and consider a sampled-data implementation of the relay control (2)

$$u(x_k) \in \arg\min_{v\in\mathcal{V}} x_k^T \Gamma v. \tag{29}$$

We will study the practical stabilization of the system

$$\dot{x}(t) = A(\mu(t))x(t) + B(u(x_k) + d(t)) \tag{30}$$

$\forall t \in [t_k, t_{k+1})$, with $A(\mu(t)) \in \mathcal{A}$ under the assumption

- (A.8) $\exists\, P_0 \succ 0, K \in \mathbb{R}^{n\times m}, \delta_0 > 0$ such that

$$(A_i + BK)^T P_0 + P_0(A_i + BK) \prec -2\delta_0 P_0, \forall i \in \mathcal{I}_{n_v}.$$

Theorem 5.1: Consider system (30), (29) and Assumptions (A.2)–(A.5), (A.7), (A.8). Denote $\tilde{A}_j = A_j + BK, j \in \mathcal{I}_{n_v}$. Given tuning parameter δ, γ, let there exist $P, U \succ 0$ matrices P_2, P_3 and $\beta > 0$, $\beta_d > 0$ such that:

a)

$$\begin{bmatrix} I & h_i K/(1-\rho) \\ * & P\gamma^{-1} \end{bmatrix} \succ 0, \forall i \in \mathcal{I}_{n_h} \tag{31}$$

b)$\beta + \beta_d \cdot m \cdot d_{\max}^2 < 2\gamma\delta\overline{T}^{-1}$;

c)the LMIs (27), (28) are satisfied for all $v \in \mathcal{V}, j \in \mathcal{I}_{n_v}$.

Then for $\Gamma = PB$, any solution $x(t)$ of the system with $x(0) \in \Omega_0 = \mathcal{E}(P, \gamma)$ converges exponentially to $\Omega_\infty = \mathcal{E}(P, C)$ as $t \to \infty$, with $C = (2\delta)^{-1}(\beta + \beta_d \cdot m \cdot d_{\max}^2)\overline{T}$.

Proof: Condition a) implies that $\mathcal{E}(P, \gamma) \subset \mathcal{C}_{(1-\rho)\mathcal{V}}(K)$. Condition b) guarantees that $\Omega_\infty = \mathcal{E}(P, C) \subset \Omega_0 = \mathcal{E}(P, \gamma)$. For $\tau(t) = t - t_k$, condition c) implies the existence of functions $V(x) = x^T P x$ and

$$W(t) = (t_{k+1} - t_k - \tau(t)) \int_{t_k}^{t} e^{2\delta(s-t)} \dot{x}^T(s) U \dot{x}(s) ds$$

such that

$$\dot{V}(x(t)) + \dot{W}(t) + 2\delta(V(x(t)) + W(t)) - \\ - \left(\beta + \beta_d \cdot d^T(t)d(t)\right)\overline{T} < 0 \tag{32}$$

for all $x(t) \in \mathcal{E}(P, \gamma)$, $t \in [t_k, t_{k+1})$. This will be shown as follows. Note that $u(x_k) = v_i$, for some $i \in \mathcal{I}_N$, if

$$x_k^T PB(v_j - v_i) \geq 0, \forall j \in \mathcal{I}_N.$$

Then for $u(x_k) \in \arg\min_{v\in\mathcal{V}} x_k^T PBv$

$$2(1-\rho)x_k^T PB(v_i - u(x_k)) \geq 0, \forall i \in \mathcal{I}_N. \tag{33}$$

Furthermore, using condition a), for all $x(t) \in \Omega_0$, there exists N scalars $\alpha_j(x(t)) \geq 0$, $\forall j \in \mathcal{I}_N$ with $\sum_{j=1}^N \alpha_j(x(t)) = 1$ such that

$$Kx(t) = \sum_{j=1}^{N} \alpha_j\left(x(t)\right)(1-\rho)v_j.$$

Multiplying (33) by the barycentric coordinates of $Kx(t)$ in $\mathrm{conv}\{(1-\rho)\mathcal{V}\}$, and summing, we have

$$2\left(x(t) - \tau(t)\eta(t)\right)^T PB\left(Kx(t) - (1-\rho)u(x_k)\right) \geq 0 \tag{34}$$

for all $t \in [t_k, t_{k+1})$, where $\eta(t) = (x(t) - x_k) \cdot \tau(t)^{-1}$. Moreover, using $\mathcal{P}(d_{\max}) \subset \mathrm{conv}\{\rho\mathcal{V}\}$ in the same line of the proof of Theorem 3.1, $2x_k^T PB(\rho u(x_k) + d(t)) \leq 0$. Then

$$\begin{aligned} &2x(t)^T PB\left(\rho u(x_k) + d(t)\right) \\ &\quad = 2\left(x_k + \tau(t)\eta(t)\right)^T PB\left(\rho u(x_k) + d(t)\right) \\ &\quad \leq 2\tau(t)\eta^T(t) PB\left(\rho u(x_k) + d(t)\right) \end{aligned} \tag{35}$$

for $t \in [t_k, t_{k+1})$. Note that

$$\begin{aligned} \dot{V}\left(x(t)\right) &= 2x^T(t)P\left(A(\mu(t))x(t) + B(1-\rho)u(x_k)\right) \\ &\quad + 2x(t)^T PB\left(\rho u(x_k) + d(t)\right). \end{aligned} \tag{36}$$

Adding (34)to (36) and using the inequality (35) leads to

$$\begin{aligned} \dot{V}\left(x(t)\right) &\leq 2x^T(t)P\left(A(\mu(t)) + BK\right)x(t) \\ &\quad + 2\tau(t)\eta^T(t)PB\left(u(x_k) + d(t) - Kx(t)\right) \end{aligned} \tag{37}$$

$$\Theta_j^0(v) = \begin{bmatrix} (U - P_3^T - P_3)\overline{T} & (P_3^T A_j - P_2)\overline{T} & \overline{T}P_3^T Bv & \overline{T}P_3^T B \\ * & \tilde{A}_j^T P + P\tilde{A}_j + 2\delta P + (A_j^T P_2 + P_2^T A_j)\overline{T} & \overline{T}P_2^T Bv & \overline{T}P_2^T B \\ * & * & -\beta\overline{T}I & 0 \\ * & * & * & -\beta_d\overline{T}I \end{bmatrix} \prec 0 \tag{27}$$

$$\Theta_j^{\overline{T}}(v) = \begin{bmatrix} -(P_3^T + P_3)\overline{T} & (P_3^T A_j - P_2)\overline{T} & \overline{T}P_3^T Bv & \overline{T}P_3^T B & 0 \\ * & \tilde{A}_j^T P + P\tilde{A}_j + 2\delta P + \left(A_j^T P_2 + P_2^T A_j\right)\overline{T} & \overline{T}P_2^T Bv & \overline{T}P_2^T B & -(PBK)^T\overline{T} \\ * & * & -\beta\overline{T}I & 0 & (PBv)^T\overline{T} \\ * & * & * & -\beta_d\overline{T}I & (PB)^T\overline{T} \\ * & * & * & * & -\overline{T}Ue^{-2\delta\overline{T}} \end{bmatrix} \prec 0 \tag{28}$$

for all $x(t) \in \mathcal{E}(P,\gamma)$ and $t \in [t_k, t_{k+1})$. Using Jensen's inequality [11]

$$\dot{W}(t) + 2\delta W(t) \leq (T_k - \tau(t))\dot{x}^T(t)U\dot{x}(t) - \tau(t)\eta^T(t)U\eta(t)e^{-2\delta\overline{T}}.$$

From (37), using the descriptor form [16]

$$2\overline{T}(P_2 x + P_3\dot{x}))^T\left(-\dot{x} + A(\mu) + B\left(u(x_k) + d\right)\right) = 0$$

a sufficient condition for (32) to hold is

$$\begin{aligned} &2x^T P\left(A(\mu) + BK\right)x + 2\delta x^T P x \\ &\quad + 2\overline{T}(P_2 x + P_3\dot{x}))^T\left(-\dot{x} + A(\mu) + B(v + d)\right) \\ &\quad + 2\tau\eta^T PB(v + d - Kx) + (\overline{T} - \tau)\dot{x}^T U\dot{x} \\ &\quad - \tau\eta^T U\eta e^{-2\delta\overline{T}} - (\beta + \beta_d \cdot d^T d)\overline{T} < 0 \end{aligned} \tag{38}$$

for all $\tau \in [0, \overline{T}]$, $v \in \mathcal{V}$, and μ in the unit simplex, with $x = x(t)$, $\dot{x} = \dot{x}(t)$, $\tau = \tau(t)$, $d = d(t)$, $\mu = \mu(t)$, $\eta = \eta(t)$. Consider the vector $z = [\dot{x}^T\ x^T\ 1\ d^T]^T$ and the definitions of $\Theta_j^0(v)$ and $\Theta_j^{\overline{T}}(v)$ in (27), (28). One may remark that for any $u(x_k) \in \mathcal{V}$, the left side of the inequality (38) can be expressed as a convex combination of $z^T\Theta_j^0(u(x_k))z$, $[z^T\ \ \eta^T]\Theta_j^{\overline{T}}(u(x_k))[z^T\ \eta^T]^T$, with $j \in \mathcal{I}_{n_v}$. Using convexity arguments, the set of LMIs (27), (28) are sufficient for (38) to hold. To end the proof, note that for $|d|_\infty \leq d_{\max}$, (32) leads to $\dot{V}(x(t)) + \dot{W}(t) + 2\delta(V(x(t)) + W(t)) - (\beta + \beta_d \cdot m \cdot d_{\max}^2)\overline{T} < 0$, for all $x(t) \in \mathcal{E}(P,\gamma)$, $t \in [t_k, t_{k+1})$. Using the comparison principle $V(x(t)) < e^{-2\delta t}V(x(0)) + C$, $\forall t > 0$, which means that $x(t)$ converges exponentially to the attractive ellipsoid $\Omega_\infty = \mathcal{E}(P, C)$. ■

Example 1: Consider a system (1) described by

$$A = \begin{bmatrix} a & -1 \\ 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \mathcal{V} = \left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix}\right\}$$

with $a = 1$, $|d|_\infty < d_{\max} = 0.01$. The set $\mathrm{conv}\{\mathcal{V}\}$ in (5) is characterized by $h_1 = [-1\ 1]$, $h_2 = [1\ \ 1]$, $h_3 = [0\ \ -1/2]$. Addressing the optimization problems (26) and $\inf\ \epsilon$ under the constraints (22)–(24) with $\gamma = 1$, $\delta = 0.25$, leads to a control law (2) with $\Gamma = PB$ and

$$P = \begin{bmatrix} 3.25 & 0 \\ 0 & 3.25 \end{bmatrix}$$

which ensures the local (robust) stabilization in $\Omega_0 = \mathcal{E}(P, 1)$, containing the ball with the radius $c_B = 0.55$. For this example $s = B^T Px$ corresponds to the origin. Then the equilibrium point is finite-time reachable. Let us remark that the boundary of the domain of attraction is not far from the unstable equilibrium points of the closed-loop system: $-A^{-1}Bv_2 = [1.5\ \ -0.5]^T$, $-A^{-1}Bv_3 = [-0.5\ 1.5]^T$. Furthermore, for $x(0) = [0.501\ \ -0.501]^T$, simulations with constant sampling interval $t_{k+1} - t_k = 10^{-5}, \forall k \in \mathbb{N}$ and $d_{\max} = 0$, illustrate an unstable system behavior. Note that $|x(0)| = 0.708$, to be compared with $c_B < 0.55$ for which local stabilization is ensured. This gives an idea about the accuracy of the ellipsoidal estimation of the domain of attraction.

Let us remarks that for the system under study the matrix A is unstable. Therefore it is impossible to apply the classical global stabilization control design techniques based on the existence of a stable convex combination [9], [12]. The example shows that there are classes of switched affine systems that can be locally stabilized even if not stable convex combination exists.

Next we consider a sampled-data implementation of the control law of the form (29) with the obtained Lyapunov matrix P for a time-varying sampling interval upper-bounded by $\overline{T} = 10^{-3}$. Using

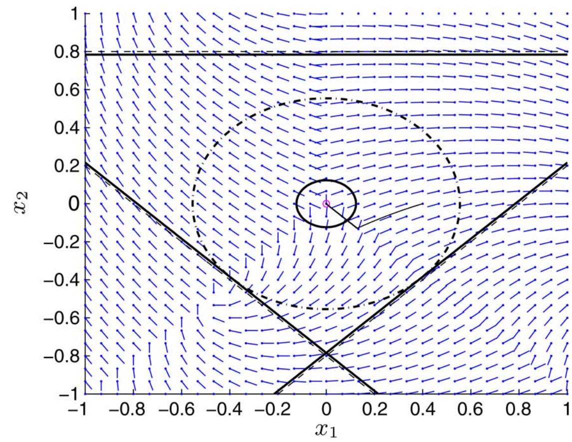

Fig. 1. Phase space for the closed-loop system in Example 1 with $|d|_\infty \leq 0.01$ and a sampled-data implementation of the control with $\overline{T} = 10^{-3}$. Ellipsoid in dotted-dashed line—Ω_0. Ellipsoid in solid line—Ω_∞. Solid black lines—limiting hyperplanes of $\mathcal{C}_{(1-\rho)\mathcal{V}}(K)$.

this Lyapunov matrix, the gain $K = -\lambda/2B^T P$, and minimizing the quantity $\beta + \beta_d \cdot m \cdot d_{\max}^2$ for which there exists U, P_2 and P_3 satisfying the conditions of Theorem 5.1 with $\gamma = 1$, $\delta = 0.22$, we may show that any system solution in the ellipsoid $\Omega_0 = \mathcal{E}(P, 1)$ converges exponentially to the ellipsoid $\Omega_\infty = \mathcal{E}(P, 0.05)$. An illustration of this sets is provided in Fig. 1. A simulation from the initial condition $x(0) = [0.4\ 0]^T$ is presented under arbitrary variations of the matched perturbation and of the sampling interval.

Assume now that the parameter a is time-varying in [0.97, 1.03]. Let us consider a continuous-time control design based on Corollary 4.1 for $|d|_\infty < d_{\max} = 0.01$. For $\gamma = 1$, solving the LMI problem (22)–(24) (for the two vertex of the A matrix) while minimizing ϵ, leads to a control law of the form (2) with $\Gamma = PB$ and

$$P = \begin{bmatrix} 0.33 & 0 \\ 0 & 0.33 \end{bmatrix}$$

which ensures local stabilization of the continuous-time systems in $\Omega_0 = \mathcal{E}(P, 1)$ for any $|d|_\infty < d_{\max} = 0.01$ and any $a(t) \in [0, 97, 1.03]$. Using Theorem 5.1 with $\gamma = 1$, $\delta = 0.22$, $K = -\lambda/2B^T P$, and minimizing the quantity $\beta + \beta_d \cdot m \cdot d_{\max}^2$ for which there exists U, P_2 and P_3, we may show that the closed-loop system (with a sampled-data control) converges to the ellipsoid $\Omega_\infty = \mathcal{E}(P, 0.17)$ for any time-varying sampling interval upper-bounded by $\overline{T} = 10^{-3}$.

VI. Conclusion

This article presents a new convex optimization approach for the design of relay feedback control. Simple LMI-based criteria are proposed for the local stabilization of linear systems in the presence of matched perturbations and time-varying uncertainties in the state matrix. The approach is used for studying practical stability of the sampled-data control implementation. Sampling jitters may be easily taken into account. Several extensions are possible for including useful performance and robustness specifications.

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