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Matrix inequality-based observer design for a class of distributed transport-reaction systems[‡]

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SUMMARY

The problem of designing a globally exponentially convergent observer for a class of (linear in transport and nonlinear in generation) semi-linear parabolic distributed systems is addressed within a matrix inequality framework, yielding (i) sufficient convergence conditions with physical meaning and (ii) the weight of a Lyapunov functional as design degree of freedom. The proposed approach is illustrated and tested with a representative case example in chemical reaction engineering. Copyright © 2013 John Wiley & Sons, Ltd.

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1. INTRODUCTION

This paper is concerned with the design of an infinite-dimensional observer for a class of semi-linear parabolic systems with linear convective-diffusive transport and nonlinear generation. The system class includes a diversity of important processes in science and engineering [1–3]. In the literature, this problem has been addressed, basically, with two approaches: (i) finite-dimensional approximation of the distributed system followed by finite-dimensional observer design and functioning assessment (also called early lumping) [4–7] and (ii) direct infinite-dimensional convergent observer design for the distributed system [8–11] followed by finite-dimensional (numerical package-based or tailored) implementation [12, 13].

In the finite-dimensional approximation approach [4–7], the observation problem is addressed with the tools available in the more developed field of finite-dimensional systems, and the convergence assessment is with respect to the actual distributed system. The infinite-dimensional designs have been performed according with inertial manifolds [2], backstepping-like integral transformations [11], variable structure estimation schemes [9], nonlinear evolution equations [14], open-loop observers [15], and absolute stability [8].

Motivated by a finite-dimensional nonlinear system observer design [18, 19] within an adjustable-weight Lyapunov function approach, in this paper, the problem of designing an observer for a class of infinite-dimensional semi-linear-distributed systems with linear diffusion-convection transport,

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‡Preliminary results were presented at 2008 IFAC World Congress, Seoul, South Korea [16], and at 2009 IFAC Symposium on Robust Control Design, Haifa, Israel [17].

nonlinear reaction, and boundary measurements is considered. The problem is addressed within a weighted Lyapunov function framework, where (i) the weight function is regarded as a design degree of freedom and (ii) the associated algebraic inequality convergence conditions are handled with a linear matrix inequality (LMI) procedure. The result is an infinite-dimensional Luenberger-like observer with (i) a simple structure and construction-tuning procedure, (ii) global convergence conditions with physical meaning, and (iii) the Lyapunov weight as an effective design degree of freedom.

The proposed approach is illustrated and tested with a representative tubular reactor case example with non-monotonic kinetics, which includes an important class of single-state profile chemical, biochemical, and physiological isothermal-distributed systems. Methodologically speaking, the present study can be seen as an inductive step toward the consideration of isothermal and non-isothermal single and multi-state transport-reaction systems with boundary and/or domain measurements.

The content of this paper puts together and completes preliminary reported material on the subject: in [16], the infinite-dimensional observer design problem was addressed with an adjustable-weight Lyapunov function, and in [17], the related convergence inequalities were handled with an LMI approach recalled from optimal boundary control design [20].

The paper is organized as follows. The observation problem is stated in Section 2. Space-dependent LMI conditions for observer exponential convergence are derived in Section 3. The approach is applied to the tubular reactor example in Section 4. Finally, some conclusions are drawn in Section 5.

Notation

The space of n-times continuously differentiable functions is denoted by $\mathcal{C}^n$, $z_\xi(\xi)$ (or $z_{\xi\xi}(\xi)$) denotes the first (or second) derivative of the function $z(\xi) \in \mathcal{C}^2$, $z_{\min} = \min_{\xi\in[0,1]} z(\xi)$, and $z_{\max} = \max_{\xi\in[0,1]} z(\xi)$, and $L^2([0,1])$ denotes the space of square Lebesgue integrable functions, that is, functions with finite norm

$$||z(x,t)|| = \left(\int_0^1 z^2(x,t)\mathrm{d}x\right)^{1/2}, \tag{1}$$

and $H^1([0,1]) = W^{1,2}([a,b],\mathbb{R})$ is the Sobolev space of absolutely continuous functions $h : [a,b] \to \mathbb{R}$ with square Lebesgue integrable derivative h_x, and norm $||h||^2_{W^{1,2}} = \int_a^b h_x^2(\xi)d\xi$. For later use, recall the following Lemma.

Lemma 1 (Wirtinger's inequality [21]*)*
For $h \in W^{1,2}([a,b],\mathbb{R})$, the following holds:

(a) If $h(a) = 0$, the following inequality holds with $\gamma = 4$

$$\int_a^b h^2(\xi)d\xi \leqslant \frac{\gamma(b-a)^2}{\pi^2}\int_a^b h_\xi^2(\xi)d\xi. \tag{2}$$

(b) If $h(a) = h(b) = 0$, inequality (2) holds with $\gamma = 1$.

2. THE OBSERVATION PROBLEM

In this section, the observation problem for the class of semi-linear parabolic systems is formulated, and sufficient conditions for well-posedness are presented.

2.1. Problem statement

Consider the class of transport-reaction parabolic semi-linear systems

$$z_t(x,t) = \delta(x)z_{xx}(x,t) - \kappa(x)z_x(x,t) - \sigma[x,t,z(x,t)] + \gamma_d(x)u_d(x,t) \tag{3}$$

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with Danckwerts [3] boundary and initial conditions

$$\begin{aligned}\delta(0)z_x(0,t) &= \kappa(0)[z(0,t) - u_0(t)] \\ \delta(1)z_x(1,t) &= \gamma_1 u_1(t) \\ z(x,0) &= z^0(x) \in \mathcal{C}^1\end{aligned} \tag{4}$$

and two boundary measurements

$$y(t) = [y_0(t), y_1(t)]^T = [z(0,t),\, z(1,t)]^T\ , \tag{5}$$

where t is the time, x is the dimensionless space in the interval $[0, 1]$, z is the state profile at time t, z^0 is the initial profile, $\delta \in \mathcal{C}^1$ (or $\kappa \in \mathcal{C}^1$) is the diffusion (or convection) space-dependent function, σ is a nonlinear source function, u_d is the exogenous input function on $(0, 1)$ with gain function γ_d, u_0 (or u_1) is the exogenous input at the Robin (or Neumann) boundary $x = 0$ (or $x = 1$), and y_0 (or y_1) is the measurement at the boundary $x = 0$ (or $x = 1$). Throughout the developments, the space and time dependencies will be denoted explicitly only when it is required for clarity.

The linear operator δz_{xx} (or κz_x) corresponds to the diffusive (or convective) transport, σ is the nonlinear generation, $\gamma_d u_d \in \mathcal{C}^1([0,1] \times \mathbb{R}_+)$ is the linear-distributed domain input, and $(u_0, u_1) \in \mathcal{C}^2(\mathbb{R}_+) \times \mathcal{C}^2(\mathbb{R}_+)$ is the linear boundary injection pair. In an important class of practical situations, the diffusion-convection term (δ, κ) is constant, and σ depends only on z. The dependence of (δ, κ, σ) on (x, t) in system (3)–(4) is meant for methodological generality purposes, in the understanding that space-dependent diffusion coefficients can be found in problems for tubes with diameter varying in space ([22]).

It is assumed that $\delta(x) > 0$, $\sigma(x,t,z) \in \mathcal{C}^1$ with respect to $z\ \forall\, x, t$, and $\exists\, s_l \in \mathbb{R}$ such that

$$\sigma_z(x,t,z) \geqslant s_l, \quad \forall\, x \in (0,1),\ t \geqslant 0,\ z \in \mathbb{R}, \tag{6}$$

where s_l is a lower bound for the slope of the generation function σ, with $s_l \geqslant 0$ (or < 0) when σ depends monotonically (or non-monotonically) on z.

The system (3)–(4) has a local unique strong solution for any initial profile $z^0(x) \in H^1([0,1])$ that satisfies the boundary conditions (4), if the given conditions on the exogenous inputs (u_d, u_0, u_1), the diffusion, convection and drift coefficients (δ, κ, and γ_d, respectively), and the condition (6) for the nonlinear source σ are satisfied. The details of this well-posedness result are presented in Subsection 2.2.

The observation problem consists in exponentially (in the sense of the L_2 norm) estimating the state variable $z(x,t)$ on the basis of the (Luenberger-like) nonlinear observer

$$\begin{aligned}\hat{z}_t &= \delta(x)\hat{z}_{xx} - \kappa(x)\hat{z}_x - \sigma(x,t,\hat{z}) + \gamma_d u_d - l_d^T(x)(\hat{y} - y) \\ \hat{y} &= [\hat{z}(0,t), \hat{z}(1,t)]^T, \qquad\qquad l_d^T(x) = [l_{d0}(x), l_{d1}(x)] \\ \delta(0)\hat{z}_x(0,t) &= \kappa(0)[\hat{z}(0,t) - u_0(t)] - l_0^T(\hat{y} - y)\ , \quad l_0 = [l_{00}\ ,\ l_{01}]^T, \\ \delta(1)\hat{z}_x(1,t) &= \gamma_1 u_1(t) - l_1^T(\hat{y} - y)\ , \qquad l_1 = [l_{10}\ ,\ l_{11}]^T \\ \hat{z}(x,0) &= \hat{z}^0(x)\end{aligned} \tag{7}$$

where $\hat{z}(x,t)$ is the state estimate, $\hat{z}^0(x)$ is its initial estimate, $l_d^T(\hat{y} - y)$ with domain measurement injection vector $l_d(x) \in \mathcal{C}^1([0,1]) \times \mathcal{C}^1([0,1])$, $l_0^T(\hat{y} - y)$ $\left[\text{ or } l_1^T(\hat{y} - y)\right]$ with constant l_0 (or l_1) is the boundary measurement injection at $x = 0$ (or $x = 1$). The observer (7) has two Robin boundary conditions and is made by the actual system plus measurement injections. As the injections vanish, the observer system (7) becomes the actual system (3)–(4) with Neumann–Robin boundary condition pair. The observer dynamics (7) has a local unique strong solution for any initial function $\hat{z}^0(x) \in H^1([0,1])$ satisfying the boundary conditions, if the exogenous inputs (u_d, u_0, u_1), transport coefficients (δ, κ), drift amplitude $(\gamma_d(x))$, nonlinear source (σ), and observer domain injection gain (l_d) satisfy the conditions stated previously. The details of this well-posedness result are presented in Subsection 2.2.

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The gain function pair $(l_{d0}, l_{d1})(x)$ and the constant gain quartet $(l_{00}, l_{01}, l_{10}, l_{11})$ are degrees of freedom in the estimator design task.

The subtraction of system (3)–(4) from its observer (7) yields the well-posed estimation error dynamics

$$\begin{aligned} e_t &= \delta e_{xx} - \kappa e_x - \varphi(x,t,z;e) - l_d^T(x)\varepsilon(t), \\ \delta_0 e_x(0,t) &= \kappa_0 e(0,t) - l_0^T \varepsilon, \\ \delta_1 e_x(1,t) &= -l_1^T \varepsilon, \\ e(x,0) &= e^0(x) \end{aligned} \tag{8}$$

with Robin boundary conditions, where

$$e(x,t) = \hat{z}(x,t) - z(x,t), \quad \varepsilon = \hat{y} - y \tag{9}$$

$$\varphi(x,t,z;e) := \sigma(x,t,z+e) - \sigma(x,t,z), \quad \varphi(x,t,z;0) = 0, \tag{10}$$

$e(x,t)$ is the estimation error profile at time t and ε is the measurement error that drives the observer dynamics. By virtue of the mean value theorem for derivatives and the low slope bound expression (6), φ is bounded as follows

$$\varphi(x,t,z;e)e = \sigma_z(x,t,z+\eta e)e^2 \geqslant s_l e^2 \;\; \forall\, x \in (0,1),\, t \geqslant 0,\, z \in \mathbb{R}, \quad \text{with } \eta \in (0,1). \tag{11}$$

Given that the system and observer dynamics have a local unique strong solution, the estimation error dynamics (8) has a local unique strong solution for any initial function $e^0(x) \in H^1([0,1])$ satisfying the boundary conditions, if the conditions on the exogenous inputs (u_d, u_0, u_1), transport coefficients (δ, κ), drift amplitude $(\gamma_d(x))$, nonlinear source (σ), and observer domain injection gain (l_d) are satisfied.

Thus, the estimation problem amounts to designing the gain function pair (l_{d0}, l_{d1}) and the gain constant quartet $(l_{00}, l_{01}, l_{10}, l_{11})$ so that the state estimate $\hat{z}(x,t)$ exponentially converges, in the sense of the L^2 norm (1), to the system state $z(x,t)$, or equivalently, there are positive constants a and λ so that the solutions $e(x,t)$ of the estimation error dynamics (8) exponentially vanish according to

$$\forall\, e_0 \in L^2([0,1],\mathbb{R}) \;\Rightarrow\; ||e(\cdot,t)|| \leqslant a \left|\left|e^0\right|\right| e^{-\lambda t}, \tag{12}$$

where $||\cdot||$ is defined in (1).

From an industrial implementation perspective, this design problem in continuous space-time formulation corresponds to a basic feasibility assessment, in the understanding that, in a second stage, a suitable finite-dimensional approximation scheme [12, 13] should be applied in such a way that an adequate compromise between state reconstruction, robustness, and complexity (in terms of number of ODEs or difference equations) is obtained.

2.2. *Well-posedness*

The system dynamics (3)–(4) and the observer dynamics (7) can be written in the generalized form:

$$\zeta_t(x,t) = \delta(x)\zeta_{xx}(x,t) - \kappa(x)\zeta_x(x,t) + f[x,t,\zeta(x,t)] \tag{13}$$

$$\begin{aligned} \delta(0)\zeta_x(0,t) - \kappa_0\zeta(0,t) = f_0(t), \quad \delta(1)\zeta_x(1,t) - \kappa_1\zeta(1,t) = f_1(t), \\ \zeta(x,0) = \zeta^0(x) \end{aligned} \tag{14}$$

with

$$\begin{aligned} &\zeta = z(\text{or } \hat{z}), \quad f(x,t,\zeta) = \sigma(x,t,\zeta) + \gamma_d(x)u_d(t) + l_d(x)[\hat{y}(t) - y(t)], \\ &\kappa_0 = \kappa(0) - l_0, \quad f_0(t) = \kappa(0)u_0(t) - l_0 y(t), \quad \kappa_1 = -l_1, \quad f_1(t) = \gamma_1 u_1(t) - l_1 y(t). \end{aligned} \tag{15}$$

where $l_d, l_0, l_1 = 0$ in (3)–(4). From the analysis of system (13)–(14), the next result on its well-posedness follows. Because its rigorous proof goes beyond the scope of the present paper, in Appendix A is provided a sketch of the proof.

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Lemma 2 (Proof sketch in Appendix A)
Consider the PDE system (13)–(14) with $\delta(x) > 0, \kappa(x) \in \mathcal{C}^1([0,1])$, $f \in \mathcal{C}^1([0,1] \times \mathbb{R}_+ \times L_2([0,1]))$, and $f_0, f_1 \in \mathcal{C}^2(\mathbb{R}_+)$. Then a unique strong solution exists for (13)–(14) for all initial conditions $\zeta^0(x) \in H^1([0,1])$ satisfying the boundary conditions (14).

The proof of the preceding lemma (see sketch in Appendix A) is based on a state transformation $\zeta \to \omega$ with homogeneous boundary conditions for the transformed variable ω. For the homogeneous problem, the steps presented in [20] are followed exploiting the fact that the linear differential operator generates an analytic (smoothening) semi-group and the nonlinear operator is Lipschitz with respect to all its arguments. The existence of a strong solution is essential for the application of the Lyapunov approach employed in the next section.

3. CONVERGENCE ASSESSMENT

In this section, the estimation design problem is addressed within a weighted Lyapunov framework, with the weight function as design degree of freedom. The analysis of the corresponding dissipation mechanism leads to an LMI convergence condition, which depends on the spatial coordinate, the observer gains, and the Lyapunov weight function.

Motivated by the idea of setting the Lyapunov energy weight as a design degree of freedom in a finite-dimensional system observer design [18], let us set the positive-definite weighted candidate Lyapunov functional for the observation error dynamics (8),

$$V : L^2([0,1], \mathbb{R}) \to \mathbb{R}_+, \quad V(e) = \int_0^1 w(x) e^2(x,t) \mathrm{d}x \,, \tag{16}$$

with adjustable weighting function $0 < w(x) \in \mathcal{C}^1$.

From the comparison principle [23], the estimation error $e(x,t)$ (8) is exponentially convergent (12) with amplitude constant

$$a = \sqrt{w_{\max}/w_{\min}}, \quad w_{\max} = \max w(x), \quad w_{\min} = \min w(x). \tag{17}$$

if, along the trajectories of the estimation error dynamics (8), the dissipation inequality

$$\dot{V}(e) + 2\lambda V(e) \leqslant 0 \tag{18}$$

is met with a positive constant $\lambda > 0$.

3.1. Dissipation inequality

The time derivation of the Lyapunov function (16), along the error dynamics (8), followed by integration by parts, substitution of the slope lower bound expression (6), and application of Wirtinger's inequality (Lemma 1) leads to the linear dissipation inequality (derivation in Appendix A) in the variables $e(x,t)$ and $\varepsilon(t)$ defined in (9)

$$\dot{V} + 2\lambda V \leqslant \int_0^1 \left\{ \Upsilon_{11}(x) e^2(x,t) + 2e(x,t) \Upsilon_{12}(x)^T \varepsilon(t) + \varepsilon(t)^T \Upsilon_{22} \varepsilon(t) \right\} \mathrm{d}x, \tag{19}$$

with quadratic integral error dependence, where the scalar function $\Upsilon_{11}(x)$ is given by

$$\Upsilon_{11}(x) = \frac{(\tilde{\delta}_x)^2}{2\tilde{\delta}(x)}(x) - 2\pi^2 \tilde{\delta}(x) + \tilde{\kappa}_x(x) + 2w(x)(\lambda - s_l), \tag{20}$$

$$\tilde{\delta}(x) = \delta(x) w(x), \; \tilde{\kappa}(x) = \kappa(x) w(x), \tag{21}$$

the 1×2 row vector Υ_{12} is

$$\Upsilon_{12}(x) = [\Upsilon_{12,0}, \; \Upsilon_{12,1}] \tag{22}$$

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$$\Upsilon_{12,0}(x) = -w(x)l_{d,0}(x) + 2\pi^2\tilde{\delta}^{1/2}(x)\tilde{\delta}^{1/2}(0)\chi_{[0,0.5]}(x)$$
$$\Upsilon_{12,1}(x) = -w(x)l_{d,1}(x) + 2\pi^2\tilde{\delta}^{1/2}(x)\tilde{\delta}^{1/2}(1)\chi_{[0.5,1]}(x), \quad (23)$$

where $\chi_{[a,b]}(x)$ is the characteristic function of the interval $[a, b]$, that is,

$$\chi_{[a,b]}(x) = \begin{cases} 1, & \text{if } x \in [a, b] \\ 0, & \text{otherwise.} \end{cases}$$

The constant and symmetric 2×2 matrix Υ_{22} is defined as

$$\Upsilon_{22} = \begin{bmatrix} \Upsilon_{22,0} & \Upsilon_{22,01} \\ \Upsilon_{22,01} & \Upsilon_{22,1} \end{bmatrix} \quad (24)$$

$$\begin{aligned} \Upsilon_{22,0} &= 2w(0)l_{00} - \tilde{\kappa}(0) - 2\pi^2\tilde{\delta}(0) \\ \Upsilon_{22,01} &= w(0)l_{01} - w(1)l_{10} \\ \Upsilon_{22,1} &= -2w(1)l_{11} - \tilde{\kappa}(1) - 2\pi^2\tilde{\delta}(1). \end{aligned} \quad (25)$$

In matrix form, inequality (19) is written as follows:

$$\dot{V} + 2\lambda V \leqslant \int_0^1 \begin{bmatrix} e(x,t) \\ \varepsilon(t) \end{bmatrix}^T \begin{bmatrix} \Upsilon_{11}(x) & \Upsilon_{12}(x) \\ \Upsilon_{12}^T(x) & \Upsilon_{22} \end{bmatrix} \begin{bmatrix} e(x,t) \\ \varepsilon(t) \end{bmatrix} \mathrm{d}x, \quad (26)$$

with $\Upsilon_{11}, \Upsilon_{12}$, and Υ_{22} given by (20), (22), and (24), respectively. Inequality (26) shows that the derivative of the Lyapunov functional can be bounded by the space integral of a quadratic form in the estimation error variables $e(x,t)$ and $\varepsilon(t)$, along the space and on the boundaries. This fact is fundamental for the results of the next subsection.

3.2. Convergence condition

On the basis of the preceding developments, the following theorem, which is the main result of this paper, provides *sufficient LMI-based* exponential convergence conditions for observer (7), involving the parameters of the plant, the output injection gains, and the weight function $w(x)$ of the candidate Lyapunov functional (16). To state the theorem, let us introduce the matrix-valued function

$$\Upsilon(x) = \begin{bmatrix} \Upsilon_{11}(x) & \Upsilon_{12}(x) \\ \Upsilon_{12}^T(x) & \Upsilon_{22} \end{bmatrix}, \; x \in [0, 1], \quad (27)$$

with the entries $\Upsilon_{11}, \Upsilon_{12}$, and Υ_{22} given by (20), (22), and (24), respectively.

Theorem 1 (Proof in Appendix B)
Let $w(x) > 0$ be a given $\mathcal{C}^1$ weight function. The estimation error $e(x,t)$, associated with observer (7), globally exponentially converges to zero with amplitude $a = \sqrt{w_{\max}/w_{\min}}$, defined in (17) according to inequality (12), if there exist a constant $\lambda > 0$, constant vectors l_0, $l_1 \in \mathbb{R}^2$, and a function $l_d(x) = [l_{d0}(x), l_{d1}(x)]^T : [0, 1] \to \mathbb{R}^2$ such that the LMI

$$\Upsilon(x) \leqslant 0 \,, \; \forall\, x \in [0, 1] \,, \quad (28)$$

holds with $\Upsilon(x)$ given in (27). ◇

According to Theorem 3, the observer can be designed as follows: for a given weight function $w(x)$, solve the corresponding MI (28) with respect to the set $[\lambda \,,\; l_0 \,,\; l_1 \,,\; l_d(x)]$ (which is linear in the design parameters).

Because the LMI (28) is parameterized by the space variable x, that is, it consists of an uncountable number of LMIs, it is more difficult to solve than the simpler control counterpart drawn with unit weight [20]. However, as stated in the following corollary, the feasibility of the three-dimensional LMI (28) is equivalent to the existence of a positive λ such that the scalar differential inequality (29) is met (which does not depend on the observer gains $[\, l_0 \,,\, l_1 \,,\, l_d]$).

Corollary 1
Let $w(x) > 0$ be a given $\mathcal{C}^1$ weight function. Suppose there exists a constant $\lambda > 0$ such that

$$\Upsilon_{11}(x) = \frac{(\tilde{\delta}_x)^2(x)}{2\tilde{\delta}(x)} - 2\pi^2\tilde{\delta}(x) + \tilde{\kappa}_x(x) + 2w(x)(\lambda - s_l) \leqslant 0\,,\ \forall x \in [0,1]\,. \tag{29}$$

Then, the choice

$$\begin{aligned} l_{d0}(x) &= \frac{2\pi^2}{w(x)}\tilde{\delta}^{1/2}(0)\tilde{\delta}^{1/2}(x)\chi_{[0,0.5]}(x)\,,\ l_{d1}(x) = \frac{2\pi^2}{w(x)}\tilde{\delta}^{1/2}(1)\tilde{\delta}^{1/2}(x)\chi_{[0.5,1]}(x), \\ l_{00} &\leqslant \frac{\tilde{\kappa}(0) + 2\pi^2\tilde{\delta}(0)}{2w(0)} = \frac{1}{2}[\kappa(0) + 2\pi^2\delta(0)],\ l_{11} \geqslant -\frac{\tilde{\kappa}(1) + 2\pi^2\tilde{\delta}(1)}{2w(1)} = -\frac{1}{2}[\kappa(1) + 2\pi^2\delta(1)], \\ l_{01} &= l_{10} = 0 \end{aligned} \tag{30}$$

provides a (particular) solution of (28), so that (7) globally exponentially converges according to the inequality (12) with amplitude $a = \sqrt{w_{\max}/w_{\min}}$, defined in (17). ◇

3.3. Dissipation adjustment

Hitherto, the observer convergence has been ensured for a given Lyapunov weight function $w(x) \in \mathcal{C}^2$. In principle, the problem of jointly finding the weight function $w(x)$ and the gains $[l_d, l_o, l_1]$ can be addressed by maximizing the dissipation bound (i.e., the right-hand side term in inequality (26)). The consideration of this problem goes beyond the scope of the present study, and here, it suffices to state that such methodological possibility exists and to circumscribe ourselves to address a simpler version of the problem.

For this aim, recall the dissipation inequality (19) and observe that, although the second (Υ_{12}) and third (Υ_{22}) coefficient functions of the dissipation bound depend on the observer gains, the first term (Υ_{11}) does not. Consequently, a solution for the Lyapunov weight function $\omega(x)$ of inequality (29) can be found by maximizing the smallest dissipation rate, that is, the right-hand side of the the inequality

$$0 < \lambda \leqslant \frac{1}{2w(x)}\left[\frac{(\tilde{\delta}_x)^2(x)}{2\tilde{\delta}(x)} - 2\pi^2\tilde{\delta}(x) + \tilde{\kappa}_x(x) + 2w(x)(\lambda - s_l)\right] \tag{31}$$

It must be pointed out that, when the weight $w(x)$ is constant, (i) the feasibility region over the parameter space of inequality (31) is strongly restricted in comparison to the case when $w(x)$ is a degree of freedom, and (ii) the amplitude a (17) is equal to one, meaning that the estimator response (12) does not have overshooting. In other words, the choice of a constant weight $w(x) = 1$ restricts the observer design. Thus, finding a suitable weight endows the observer design with an interesting degree of freedom, at the cost (reasonable as we shall see) of the complexity introduced by a space-dependent weight function.

3.4. Discussion

The proposed design method is rather simple, as the observer design amounts to calculating the output injection gains that render feasible the MI (28). It is rather flexible, as semi-linear systems with constant or varying coefficients, different boundary conditions, and boundary and/or distributed injection can be used. Point or interval domain measurements (see e.g., [4]), and flux measurements (z_x) can be handled. Even though in Theorem 3, a nonlinear sink σ with from below-bounded slope was assumed, the case with nonlinear source σ with from above-bounded slope (Lipschitz constant) can be addressed with the same approach.

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Because the selection of the distributed injection gain $l_d(x)$ is an important design degree of freedom, different functions have been considered in the literature, among them are eigenfunction combinations in the linear (or linearized) case (e.g., [24]), particular decaying functions generated by integral backstepping-like transformations [11], point injections according to further measurements in the domain (e.g., [4]), and interval (step-like) injections (e.g., [6]). The proposed method does not impose any *a priori* restriction on $l_d(x)$, so that, in general, different alternatives can be explored.

4. TUBULAR REACTOR APPLICATION EXAMPLE

In this section, the proposed observer design methodology is illustrated and tested with a representative case example of the semi-linear parabolic system class (3): an isothermal tubular reactor with non-monotonic reaction rate (which underlies an important class of catalytic, bioprocesses, and physiology reactions [1,3,5,25]). The purpose is threefold: (i) the illustration of the theoretical developments of the previous section, (ii) the identification of the dependence of the MI (28) on the Lyapunov weight function $w(x)$, and (iii) the interpretation with physical meaning of the resulting solvability conditions.

Consider an isothermal tubular reactor [1] with area A, length L, pure reactant concentration C_r, and volumetric feed flow q (at $\zeta = 0$) at reactant concentration $C_e \leqslant C_r$, where the reactant is converted into product through the reaction with rate function $R(C)$ along the axial interval $0 \leqslant \zeta \leqslant L$, constant diffusion-convection pair (D, v), Danckwerts' boundary conditions [3], and non-monotonic (Haldane [26] or Langmuir-Hinshelwood [27] type) reaction rate, and actual time t_a. The reactant concentration profile over the axial interval is denoted by $C(\zeta, t)$. In terms of dimensionless variables (x, t and c), and Peclet (P_e) and Damköhler (ϕ) numbers,

$$\begin{gathered} 0 \leqslant x = \zeta/L \leqslant 1, \quad t = t_a v/L, \quad v = q/A, \quad 0 \leqslant c = C/C_r \leqslant 1, \\ P_e = vL/D, \quad \phi = R_r L/v, \quad R_r = R(C_r) \end{gathered} \tag{32}$$

the reactor dynamics are modeled by the distributed system of the form (3)

$$\begin{gathered} c_t(x,t) = \frac{1}{P_e} c_{xx}(x,t) - c_x(x,t) - \phi r(c(x,t)), \\ \frac{1}{P_e} c_x(0,t) = c\,(0,t) - c_e(t), \ \frac{1}{P_e} c_x(1,t) = 0\,, \\ y(t) = [y_0(t),\ y_1(t)]^T = [c(0,t)\,,\ c\,(1,t)]^T \\ c(x,0) = c_0(x)\,, \end{gathered} \tag{33}$$

where

$$r(c) = R(cC_r)/(R_r C_r) = \frac{(1+k_I)^2 c}{(1+k_I c)^2}, \quad \phi = \frac{\phi_k}{(1+k_I)^2}, \quad \phi_k = \frac{kL}{v}$$

C_r (or R_r) is a reference concentration (or reaction) value, $r(c)$ is the reaction rate with linear mass action growth (kc) and quadratic inhibition $(1 + k_I c)^2$ (k and k_I are constant), t is the dimensionless time with respect to convection characteristic time (t_v), $c(x,t)$ is the time-varying concentration profile, and ϕ_k is the Damköhler number associated with the mass action kinetics constant k. The Peclet (or Damköhler) number P_e (or ϕ) represents the ratio of diffusion t_D (or mass action reaction rate t_R)-to-convective t_v characteristic time

$$P_e = t_D/t_v, \quad \phi = t_R/t_v, \quad t_D = L^2/D, \quad t_v = L/v, \quad t_R = R_r^{-1}. \tag{34}$$

As the diffusive transport becomes considerably larger than the convective one, the Peclet vanishes ($P_e \to 0$), and the behavior of the tubular reactor approaches the one of a (perfectly mixed) continuous stirred tank reactor [25], with one or more critical points, depending on (ϕ, k_I). Finally, y_0 (or y_1) denotes the measurement of the inlet (or exit) concentration.

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The corresponding Luenberger observer (7) for reactor (33) is given by

$$\begin{aligned}
\hat{c}_t(x,t) &= \frac{1}{P_e}\hat{c}_{xx}(x,t) - \hat{c}_x(x,t) - \phi r(\hat{c}(x,t)) + l_d(x)\epsilon(t), \\
\epsilon(t) &= [y_0(t) - \hat{c}(0,t),\ y_1(t) - c(1,t)]^T \\
\frac{1}{P_e}\hat{c}_x(0,t) &= \hat{c}(0,t) - c_e(t) - l_0\epsilon(t), \\
\frac{1}{P_e}c_x(1,t) &= -l_1\epsilon(t) \\
\hat{c}(x,0) &= \hat{c}_0(x)\ ,
\end{aligned} \tag{35}$$

where $l_d(x)$ is the distributed injection gain function and l_0 (or l_1) is the constant vector injection gain at the left (or right) boundary. To analyze the related convergence sufficient condition inequality set (29), (30) associated with Corollary 1, let us introduce the family of weight functions

$$\Omega = \{w(x) = e^{\theta x},\ 0 \leqslant x \leqslant 1 | \theta \in \mathbb{R}\} \tag{36}$$

that includes the unit (w_u) and Sturm–Liouville (w_s) weight functions,

$$\begin{aligned}
\theta_u = 0 &\Rightarrow w_u(x) = 1 \\
\theta_s = -P_e &\Rightarrow w_s(x) = e^{-P_e x}.
\end{aligned} \tag{37}$$

Observe that, as particular case, the weight w_s is the integration factor that makes self-adjoint the spatial transport operator of the PDE (33). In the absence of convection ($P_e = 0$), the Sturm–Liouville weight w_s becomes the unit one w_u.

In the notation of Corollary 1, stated in terms of the semi-linear parabolic system employed in the derivation of the LMI (28) and inequality (29), the functions $\delta, \kappa, \tilde{\delta}, \tilde{\kappa}$ of the reactor case example reactor (33) are given by

$$\delta(x) = \frac{1}{P_e}\ , \quad \kappa(x) = 1\ , \quad \tilde{\delta}(x) = \frac{1}{P_e}e^{\theta x}\ , \quad \tilde{\kappa}(x) = e^{\theta x}. \tag{38}$$

The corresponding entries Υ_{11}, Υ_{12}, and Υ_{22} of matrix Υ (27), related to the statement of Theorem 1 and Corollary 1, are given by

$$\begin{aligned}
\Upsilon_{11}(x) &= e^{\theta x}\left\{2(\lambda - s_l) - \frac{2\pi^2}{P_e} + \theta\left(1 + \frac{\theta}{2P_e}\right)\right\}, \quad s_l = -\phi_k/27 \\
\Upsilon_{12}(x) &= -e^{\theta x} l_d^T(x) + \frac{2\pi^2}{P_e}e^{\theta x/2}\left[\chi_{[0,0.5]}(x)\ , \quad \chi_{[0.5,1]}(x)e^{\theta/2}\right] \\
\Upsilon_{22} &= \begin{bmatrix} 2l_{00} - 1 - 2\frac{\pi^2}{P_e}\ , & l_{01} - \frac{1}{P_e}e^{\theta}l_{10} \\ l_{01} - \frac{1}{P_e}e^{\theta}l_{10} & \left(-2l_{11} - 1 - 2\frac{\pi^2}{P_e}\right)e^{\theta} \end{bmatrix}.
\end{aligned} \tag{39}$$

where s_l is the smallest slope of the reaction rate r over the concentration interval [0, 1] [31].

The application of inequality condition (29) and the convergence amplitude formula (17) to the reactor case yields inequality (40) and equation (41):

$$\Upsilon_{11}(x) = 2e^{\theta x}[\mu(P_e, \phi, \theta) - \lambda] \leqslant 0 \tag{40}$$

$$a = e^{|\theta|/2} \tag{41}$$

where

$$\mu(P_e, \phi, \theta) = \frac{\pi^2}{P_e} - \phi_k/27 - \frac{\theta}{2}\left(1 + \frac{\theta}{2P_e}\right) > 0, \tag{42}$$

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λ (or a) is the observer convergence rate (or amplitude). Because Condition (40) ensures the existence of gains for observer convergence (Corollary 1), let us rewrite inequality (40) in the following form

$$\begin{aligned} &\lambda_{P_e} - (\lambda_\phi + \lambda_{P_e}^\theta) \geqslant \lambda > 0 \\ &\qquad \lambda_{P_e} = \frac{\pi^2}{P_e},\ \lambda_\phi = \phi_k/27,\ \lambda_{P_e}^\theta = \frac{\theta}{2}\left(1 + \frac{\theta}{2P_e}\right) \end{aligned} \tag{43}$$

where λ is the observer convergence rate, λ_{P_e} is the stabilizing contribution of the diffusion-convection mechanism, and λ_ϕ (or $\lambda_{P_e}^\theta$) is the destabilizing contribution due to the reaction (or diffusion)-convection, with $\lambda_{P_e}^\theta$ depending quadratically on the weight parameter θ. Thus, Condition (43) and equation (41) display, with physical meaning, the way in which the estimator functioning is underline by a suitable interplay between the system diffusive-convective transport and reaction mechanisms (P_e, ϕ), the convergence rate λ and amplitude a, and the weight function parameter θ.

In principle, the choices of equipment as well as of monitoring and control schemes should be performed simultaneously [28]. From this perspective, the reactor length (L) and area (A) are design degrees of freedom that should be decided in the light of Condition (43) for robustly convergent profile estimation. In particular, when the Peclet (P_e) and Damköhler (ϕ) numbers have been specified, the parameter θ of the weight function is a design degree of freedom. According to equations (43) and (41), (i) the destabilizing term $\left(\lambda_{P_e}^\theta\right)$ of the convergence rate λ (43) depends quadratically on the parameter θ of the weight function $w(x)$ (36), depending on the value (P_e, ϕ), and (ii) the convergence amplitude a (41) grows isotonically with θ. The value of θ that maximizes the upper bound of the convergence rate λ (43) is given by

$$\theta_* := \text{argmin}_\theta \left[\lambda_{P_e}^\theta(P_e, \theta)\right] = -P_e, \tag{44}$$

implying that the Sturm–Liouville weight $w_s(x)$ $(\theta = -P_e)$ (37) maximizes the reconstruction rate (λ_s) of the concentration profile (43) with the amplitude (41),

$$\lambda_s = \frac{\pi^2}{P_e} + \frac{P_e}{4} - \frac{\phi}{27} > 0, \quad a_s = e^{P_e/2}. \tag{45}$$

For comparison purposes, let us write the convergence rate-amplitude pair for the unit-weight case $(\theta = 0)$ (37):

$$\lambda_u = \frac{\pi^2}{P_e} - \frac{\phi}{27} = \lambda_s - \frac{P_e}{4} > 0, \quad a_u = 1. \tag{46}$$

For illustration purposes, in Figure 1 are plotted, in the (P_e, ϕ)-parameter space, the values of the dissipation bounds λ_s $(\theta = -P_e)$, λ_u $(\theta = 0)$ and an intermediate one $(\theta = -3)$. The dissipation bound λ_s (45) of the Sturm–Liouville weight w_s (37) is greater than the one λ_u (46) of the unit weight w_s (37), especially at large P_e values. The amplitude of the Sturm–Liouville weight is larger than the one of the unit weight, with a difference that grows exponentially with P_e. Because large amplitudes signify large error overshoots in the transient error response, the unit weight yields the best transient error response.

Thus, depending on the reactor Peclet–Damköhler pair (P_e, ϕ), the intensity of the measurement noise and the characteristics of the unmodeled parasitic (high-frequency) dynamics, the weight parameter (θ), and the observer gains (l_d, l_0, l_1) must be chosen such that the observer functioning is underline by a suitable compromise between convergence rate (λ), transient error overshoot (proportional to a), and asymptotic offset (inversely proportional to λ).

According to the preceding results, for reactors with low Peclet numbers ($P_e < 5$ to 6), or equivalently, with highly dispersive regime, the Sturm–Liouville weight tends to the unit weight, implying that this weight should be used. This is in agreement with the well-known fact that, as the Peclet number vanishes, the distributed tubular reactor becomes a perfectly mixed continuous-time

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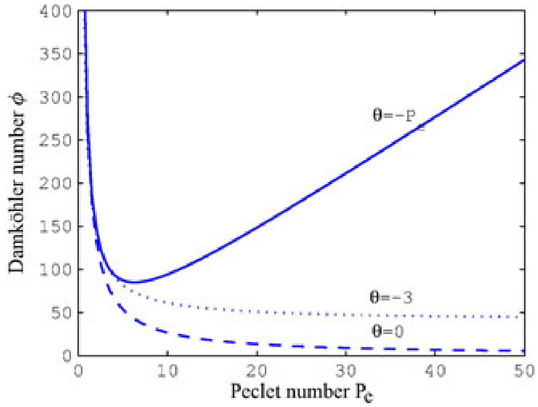

Figure 1. Graph, in the (P_e, ϕ) parameter space, of the dissipation bounds λ_s of the Sturm–Liouville weight function w_s (37) (continuous line), λ_u of the unit weight w_u (37) (dashed line), and the one for the intermediate weight function $w(x) = e^{-3x}$ (dotted line).

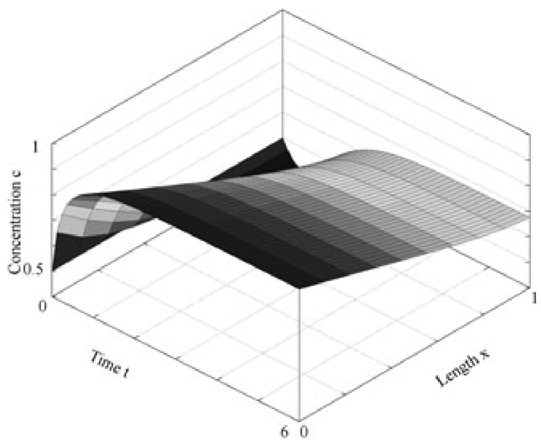

Figure 2. Tubular reactor response for $(P_e, \phi, k_I) = (5, 4, 3)$ and constant initial concentration profile $c_0(x) = 0.5$.

(lumped) reactor with constant spatial profile [25]. Differently, for large P_e numbers, where the reactor behaves like a plug-flow reactor, the Sturm-Liouville weight is the preferred choice. This is in agreement with the fact that the observer convergence rate is dominated by the stabilizing convective flow [25]. Intermediate Peclet cases should be analyzed in the light of the particular reaction intensity (measured by ϕ). In general, monotonic reactions favor the convergence rate.

Observer functioning

In this subsection, numerical simulation results are presented to illustrate the observer functioning in comparison to its open-loop version (i.e., with gains $l_0, l_1, l_d = 0$). The numerical simulations have been performed using the numerical package *Octave* on the basis of a finite-difference spatial approximation with 10 discretization points and a backward Euler algorithm for solving the associated system of stiff coupled ODEs in time.

To illustrate the observer functioning under conditions, which resemble to ones assessed in previous tubular reactor control studies, let us consider the parameter set

$$(P_e, \phi, k_I) = (5, 4, 3) \tag{47}$$

for a tubular reactor with a balanced reaction-to-convection mechanism, comparable diffusion with respect to convection, and appreciable reaction inhibition at play.

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In Figure 2 is presented the evolution of the concentration profile when the 'actual' reactor is started at an initial constant profile $c_0(x) = 0.5$, showing that the reactor has an overall settling time of about three to four dimensionless time units. In Figure 3 are presented the estimation (a) and error (b) evolution of the open-loop version (with zero gains) of the proposed observer (35), starting from a deviated initial profile $\hat{c}_0(x) = 1$, showing that: (i) as expected, the overall profile error vanishes in about three to four natural settling times, (ii) due to the convective transport and reaction intensity, the convergence is faster in the entrance than in the exit, and (iii) due to the intensity of reaction inhibition after the reactor entrance, the observer exhibits a comparatively slower convergence.

In Figure 4 are presented the estimation (a) and error (b) responses of the observer with boundary measurements, for the intermediate Peclet weight function and the gains

$$w(x) = e^{-3x}, \quad l_{d0}(x) = \frac{2\pi^2}{w(x)} \left(\frac{w(0)}{P_e}\right)^{1/2} \left(\frac{w(x)}{P_e}\right)^{1/2} \chi_{[0,0.5]}(x),$$
$$l_{d1}(x) = \frac{2\pi^2}{w(x)} \left(\frac{w(1)}{P_e}\right)^{1/2} \left(\frac{w(x)}{P_e}\right)^{1/2} \chi_{[0.5,1]}(x),\ l_{00} = -15, l_{11} = 15,\ l_{01} = l_{10} = 0 \tag{48}$$

chosen according to the LMI condition (30). The related feasibility region is presented in Figure 1, in the understanding that this choice of weight function and gains corresponds to a suitable compromise between convergence speed (λ) and error overshoot (a).

The comparison of the results of Figure 4 with the ones of Figure 3 evidences that (i) the estimator convergence is about two times faster than the natural process dynamics, or equivalently, than the

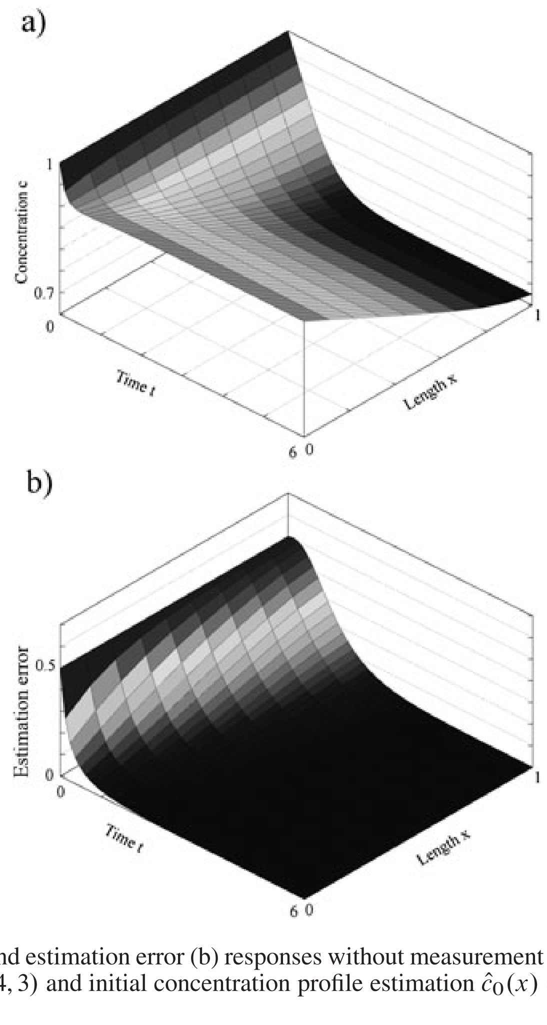

Figure 3. Estimation (a) and estimation error (b) responses without measurement injection for $(P_e, \phi, k_I) = (5, 4, 3)$ and initial concentration profile estimation $\hat{c}_0(x) = 1$.

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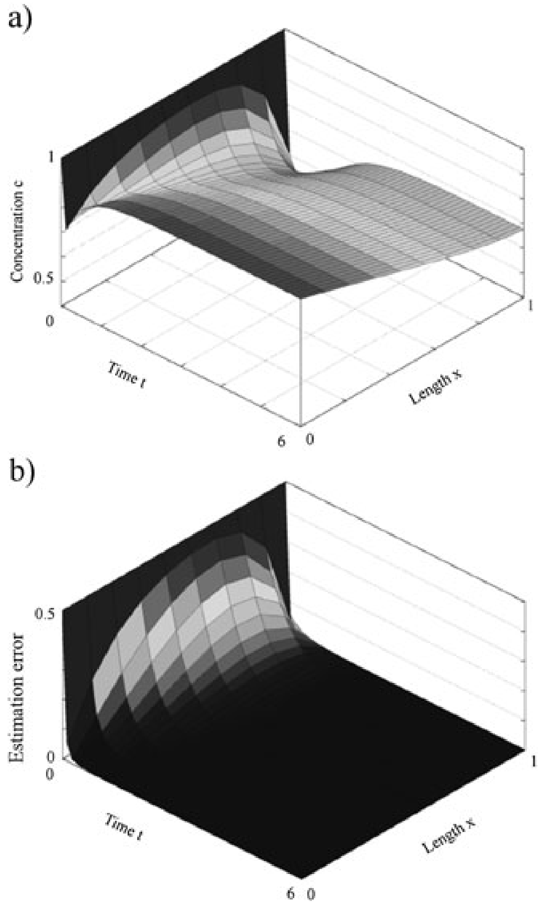

Figure 4. Estimation (a) and estimation error (b) responses with measurement injection for $(P_e, \phi, k_I) = (5, 4, 3)$, initial concentration profile estimate $\hat{c}_0(x) = 1$, $w(x) = \exp(-3x)$, and gains l_d, l_0, l_1 given in (48).

observer without measurements, (ii) as expected, the convergence in the boundaries is comparatively faster, and (iii) there is a wave-like information propagation from the reactor boundaries toward the interior of the reactor.

5. CONCLUSIONS

A globally exponentially convergent observer design for a class of semi-linear parabolic distributed transport-reaction systems has been developed. In addition to the observer gains, the weight function of a Lyapunov functional was regarded as a design degree of freedom. The consideration of the problem within a Lyapunov framework led to a convergence condition in terms of LMIs. The conditions capture the fundamental interplay between system characteristics, convergence (error amplitude and state reconstruction rate) features, functional weight, and observer gains in the light of a specific estimation objective. The proposed approach was illustrated and tested with a representative tubular reactor case example. The convergence conditions were interpreted in terms of dimensionless parameters with physical meaning, which describe the relative importance of the (linear) transport and (nonlinear) generation mechanisms.

The necessary convergence conditions exhibited the interplay between estimator (rate and amplitude) convergence features, the weight function parameter, and the Peclet–Damköhler dimensionless number pair associated with the reactor transport and reaction mechanisms. Because these dimensionless numbers depend on the reactor area and length, the observer design can be part of a joint process-monitoring-control design.

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From an industrial implementation viewpoint, the proposed design corresponds to the basic feasibility step and constitutes the point of departure to perform the second implementation stage, by applying a suitable finite-dimensional approximation scheme ([12, 13]) so that there is an adequate trade-off between state reconstruction speed and accuracy, robustness, and algorithm complexity in the light of the specific estimation objective.

In principle, the proposed one-state one-space dimensional observer design for semi-linear parabolic distributed parameter systems with boundary measurements can be extended to (i) address the problem of choosing the number of sensors and their locations, (ii) include more general weight functions with respect to certain estimation objectives, such as inference of states at the effluent or at the region of maximum reaction rate, (iii) design output feedback controllers on the basis of existing state feedback-distributed control approaches ([20]), and (iv) address the case of two-state (concentration–temperature) exothermic tubular reactors with temperature boundary and/or domain measurements in particular (cf. [10]), or of multi-state transport-reaction systems in general.

APPENDIX A: WELL-POSEDNESS

Introduce the state transformation

$$w(x,t) = \zeta(x,t) + \alpha(x) f_0(t) + \beta(x) f_1(t) \tag{A.1}$$

with $\alpha, \beta \in \mathcal{C}^1[0,1]$ being any function pair satisfying

$$\begin{aligned} &\alpha(0) = \frac{1}{\kappa_0}, \alpha'(0) = 0, \quad \alpha(1) = \alpha'(1) = 0 \\ &\beta(0) = \beta'(0) = 0, \quad \beta(1) = \frac{1}{\kappa_1}, \beta'(1) = 0. \end{aligned} \tag{A.2}$$

Particular functions satisfying these conditions are sigmoidal increasing (β) and decreasing (α) ones. For w given by (A.1), the associated dynamics read

$$\begin{aligned} &w_t(x,t) = \delta(x) w_{xx}(x,t) - \kappa(x) w_x(x,t) + \Phi(t,x,w(x,t)) \\ &\delta(0) w_x(0,t) - \kappa_0 w(0,t) = 0, \quad \delta(1) w_x(1,t) - \kappa_1 w(1,t) = 0. \end{aligned} \tag{A.3}$$

where

$$\Phi(t,x,w(x,t)) = f[x,t,\zeta(x,t) + \alpha(x) f_0(t) + \beta(x) f_1(t)] + \alpha(x)\dot{f}_0(t) + \beta(x)\dot{f}_1(t). \tag{A.4}$$

In the form of an abstract differential equation, the preceding dynamics is written as

$$v_t(t) - Av(t) = F[t, v(t)], \quad v \in D(A) \tag{A.5}$$

in the Hilbert space $L_2([0,1])$ where the operator

$$A = \delta(x)\frac{\partial^2}{\partial x^2} - \kappa(x)\frac{\partial}{\partial x} \tag{A.6}$$

has the dense domain

$$D(A) = \left\{ v \in H^2([0,1]) \mid \mathcal{B}v = 0 \right\} \tag{A.7}$$

with $\mathcal{B}$ being the boundary operator

$$\begin{aligned} &\mathcal{B} = \chi(x)\left[\delta(0)\frac{\partial}{\partial x}(0) - \kappa_0\right] + (1-\chi(x))\left[\delta(1)\frac{\partial}{\partial x}(1) - \kappa_1\right], \\ &\chi = \begin{cases} 1, & x = 0 \\ 0, & x = 1 \end{cases}, \quad x \in \{0,1\}. \end{aligned} \tag{A.8}$$

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Given that $\delta(x) > 0$, the parabolic operator A is the infinitesimal generator of an analytic semi-group $T(t)$. From this fact, followed by the application of Hille–Yosida's Theorem for $C0$ semi-groups [29], the semi-group $T(t)$ satisfies the following inequality

$$||T(t)|| \leqslant m \exp(-\lambda t) \tag{A.9}$$

for some positive constants pair (m, λ), and $||T(t)||$ being the norm induced by A [29]. Let H be a Hilbert space, then the domain $D(A) = A^{-1}H$ of the operator A forms another Hilbert space with the graph inner product $(x, y)_{D(A)} = \langle Ax, Ay\rangle, x, y \in D(A)$. The domain $D(A)$ of A is thus continuously embedded into H, that is, $D(A) \subset H$, dense in H, and the inequality $|x| \leqslant \omega|Ax|$ holds for all $x \in D(A)$ and some constant $\omega > 0$.

Apart from this, the square root $\sqrt{A}$ of the operator A is rigorously introduced on $D(A)$ as a positive-definite solution X of the algebraic operator equation $X^2 = A$. By continuity, this operator is well-posed on the domain

$$D(\sqrt{A}) = \left\{v \in H^1([0,1]) | \mathcal{B}v = 0\right\} \tag{A.10}$$

and continuously embedded into H, whereas $D(A)$ is continuously embedded into $D(\sqrt{A})$. Hence, it holds that $D(A) \subset D(\sqrt{A}) \subset H$, and the following inequalities apply for the operator pair $(A, \sqrt{A})$

$$|z| \leqslant \omega|\sqrt{A}z| \quad \forall z \in D(\sqrt{A}), \quad |\sqrt{A}z| \leqslant \omega|Ax|, \quad \forall z \in D(A)$$

with a generic constant $\omega > 0$.

Given that $f_0, f_1 \in \mathcal{C}^2(\mathbb{R}_+)$, $\gamma_d u_d \in \mathcal{C}^1([0,1] \times \mathbb{R}_+)$, and l_d are Lipschitz continuous, the nonlinear function Φ (A.4) defines a nonlinear operator $F : \mathbb{R} \times L_2([0,1]) \to L_2([0,1])$ that satisfies the following inequality

$$||F(t_1, z_1) - F(t_2, z_2)|| \leqslant L\left(|t_1 - t_2| + ||\sqrt{A}(z_1 - z_2)||_{L_2}\right) \tag{A.11}$$

with some constant L.

In consequence, Theorem 3.3.3 of [30] implies the (local) existence of a unique strong solution of (A.5) with initial condition $v_0 \in D(\sqrt{A})$ (A.10) and thus for a local strong solution of (A.3) for any $w_0(x) \in D(\sqrt{A})$ (A.10). This implies the existence of a unique strong solution $z(x,t)$ of (13) for any initial condition $z_0(x) \in H^1([0,1])$ satisfying the boundary conditions (14). **QED.**

APPENDIX B: LYAPUNOV DISSIPATION INEQUALITY (28)

The time-derivation of (16) along (8) yields

$$\begin{aligned} \dot{V} + 2\lambda V &= 2\int_0^1 \left[wee_t + \lambda we^2\right] \mathrm{d}x \\ &= 2\int_0^1 \left[w\delta ee_{xx} - w\kappa ee_x - we\varphi(x,t,z;e) - wel_d^T(x)\varepsilon(t) + \lambda we^2\right] \mathrm{d}x \\ &\leqslant 2\int_0^1 \left[\tilde{\delta}ee_{xx} - \tilde{\kappa}ee_x - wel_d^T(x)\varepsilon(t) + w(\lambda - s_l)e^2\right] \mathrm{d}x\,, \end{aligned} \tag{D}$$

where the last inequality follows from (11), with $\tilde{\delta} = \delta w$ and $\tilde{\kappa} = \kappa w$ (20). After integration by parts, one obtains

$$-2\int_0^1 \tilde{\kappa}ee_x \mathrm{d}x = -\int_0^1 \tilde{\kappa}\frac{d(e^2)}{\mathrm{d}x}\mathrm{d}x = -\tilde{\kappa}e^2\Big|_0^1 + \int_0^1 \tilde{\kappa}_x e^2 \mathrm{d}x \tag{B.1}$$

$$2\int_0^1 \tilde{\delta}ee_{xx}\mathrm{d}x = 2\,\tilde{\delta}ee_x\Big|_0^1 - 2\int_0^1 \left(\tilde{\delta}e\right)_x e_x \mathrm{d}x\,. \tag{B.2}$$

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On the other hand, the substitution of the equation

$$\begin{aligned}\left(\tilde{\delta}e\right)_x e_x &= \left(\tilde{\delta}_x e + \tilde{\delta}e_x\right)e_x = \left\{\left[\left(\tilde{\delta}^{1/2}\right)^2\right]_x e + \tilde{\delta}e_x\right\}e_x = \left[2\tilde{\delta}^{1/2}\left(\tilde{\delta}^{1/2}\right)_x e + \tilde{\delta}e_x\right]e_x\\
&= \left[\left(\tilde{\delta}^{1/2}\right)_x e + \tilde{\delta}^{1/2}e_x\right]^2 - \left[\left(\tilde{\delta}^{1/2}\right)_x e\right]^2 = \left[\left(\tilde{\delta}^{1/2}e\right)_x\right]^2 - \left[\left(\tilde{\delta}^{1/2}\right)_x e\right]^2\\
&= \left[\left(\tilde{\delta}^{1/2}e\right)_x\right]^2 - \left[\frac{\tilde{\delta}_x}{2\tilde{\delta}^{1/2}}\right]^2 e^2 = \left[\left(\tilde{\delta}^{1/2}e\right)_x\right]^2 - \frac{\left(\tilde{\delta}_x\right)^2}{4\tilde{\delta}}e^2\end{aligned}$$

into equation (B.2) yields

$$2\int_0^1 \tilde{\delta}ee_{xx}\mathrm{d}x = 2\,\tilde{\delta}ee_x\Big|_0^1 - 2\int_0^1\left[\left(\tilde{\delta}^{1/2}e\right)_x\right]^2\mathrm{d}x + \int_0^1 \frac{\left(\tilde{\delta}_x\right)^2}{2\tilde{\delta}}e^2\mathrm{d}x.$$

By virtue of Wirtinger's inequality (Lemma 1 (a)), we have that

$$\begin{aligned}-2\int_0^1\left[\left(\tilde{\delta}^{1/2}e\right)_x\right]^2\mathrm{d}x &= -2\int_0^{0.5}\left[\left(\tilde{\delta}^{1/2}e\right)_x\right]^2\mathrm{d}x - 2\int_{0.5}^1\left[\left(\tilde{\delta}^{1/2}e\right)_x\right]^2\mathrm{d}x\\
&\leqslant -2\pi^2\int_0^{0.5}\left[\tilde{\delta}^{1/2}(x)e(x,t) - \tilde{\delta}^{1/2}(0)e(0,t)\right]^2\mathrm{d}x\\
&\quad -2\pi^2\int_{0.5}^1\left[\tilde{\delta}^{1/2}(x)e(x,t) - \tilde{\delta}^{1/2}(1)e(1,t)\right]^2\mathrm{d}x\\
&= -2\pi^2\int_0^1\tilde{\delta}(x)e^2(x,t)\mathrm{d}x - 2\pi^2\left[\tilde{\delta}(0)e^2(0,t) + \tilde{\delta}(1)e^2(1,t)\right]\\
&\quad +4\pi^2\int_0^1\tilde{\delta}^{1/2}(x)e(x,t)\left[\tilde{\delta}^{1/2}(0)e(0,t)\chi_{[0,0.5]}(x) + \tilde{\delta}^{1/2}(1)e(1,t)\chi_{[0.5,1]}(x)\right]\mathrm{d}x.\end{aligned}$$

The substitution of this inequality into the dissipation inequality (D) yields

$$\begin{aligned}\dot{V} + 2\lambda V &\leqslant \left(2\tilde{\delta}ee_x - \tilde{\kappa}e^2\right)\Big|_0^1 - 2\pi^2\int_0^1\tilde{\delta}(x)e^2(x,t)\mathrm{d}x - 2\pi^2\left[\tilde{\delta}(0)e^2(0,t) + \tilde{\delta}(1)e^2(1,t)\right]\\
&+4\pi^2\int_0^1\tilde{\delta}^{1/2}(x)e(x,t)\left[\tilde{\delta}^{1/2}(0)e(0,t)\chi_{[0,0.5]}(x) + \tilde{\delta}^{1/2}(1)e(1,t)\chi_{[0.5,1]}(x)\right]\mathrm{d}x\\
&+\int_0^1\left\{\left[\frac{\left(\tilde{\delta}_x\right)^2}{2\tilde{\delta}} + \tilde{\kappa}_x + 2w(\lambda - s_l)\right]e^2 - 2wel_d^T(x)\varepsilon(t)\right\}\mathrm{d}x\,. \qquad \text{(E)}\end{aligned}$$

From the substitution of the boundary conditions (8), in the first term of the right side of the preceding inequality, we have that

$$\begin{aligned}\left(2\tilde{\delta}ee_x-\tilde{\kappa}e^2\right)\Big|_0^1 &= 2\tilde{\delta}(1)e(1,t)e_x(1,t)-\tilde{\kappa}(1)e^2(1,t)-2\tilde{\delta}(0)e(0,t)e_x(0,t)+\tilde{\kappa}(0)e^2(0,t)\\ &= 2\tilde{\delta}(1)e(1,t)\left[-\frac{1}{\delta(1)}[l_{10}e(0,t)+l_{11}e(1,t)]\right]-\tilde{\kappa}(1)e^2(1,t)\\ &\quad -2\tilde{\delta}(0)e(0,t)\left[\frac{\kappa(0)}{\delta(0)}e(0,t)-\frac{1}{\delta(0)}[l_{00}e(0,t)+l_{01}e(1,t)]\right]+\tilde{\kappa}(0)e^2(0,t)\\ &= \left[2\frac{\tilde{\delta}(0)}{\delta(0)}(l_{00}-\kappa(0))+\tilde{\kappa}(0)\right]e^2(0,t)+2\left[\frac{\tilde{\delta}(0)}{\delta(0)}l_{01}-\frac{\tilde{\delta}(1)}{\delta(1)}l_{10}\right]e(0,t)e(1,t)\\ &\quad +\left[-2\frac{\tilde{\delta}(1)}{\delta(1)}l_{11}-\tilde{\kappa}(1)\right]e^2(1,t)\\ &= [2w(0)l_{00}-\tilde{\kappa}(0))]e^2(0,t)+2[w(0)l_{01}-w(1)l_{10}]e(0,t)e(1,t)\\ &\quad +[-2w(1)l_{11}-\tilde{\kappa}(1)]e^2(1,t)\end{aligned}$$

The substitution of this equation in (E) followed by rearrangement yields the dissipation LMI

$$\dot{V}+2\lambda V \leqslant \int_0^1 [e(x,t)\ ,\ \varepsilon(t)]\Upsilon(x)[e(x,t)\ ,\ \varepsilon(t)]^T\,\mathrm{d}x \leqslant 0$$

with $\Upsilon(x)$ (27) satisfying (28). Finally, from the application of the comparison lemma, it follows that

$$w_{\min}\,||e||^2 \leqslant V(e) \leqslant V(e_0)e^{-2\lambda t} \leqslant w_{\max}\,||e_0||^2\,e^{-2\lambda t}$$

or equivalently, that (12) is proven. **QED.**

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