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Sliding mode control of Schrödinger equation-ODE in the presence of unmatched disturbances[☆]

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ABSTRACT

In this paper, we consider boundary stabilization for a cascade of Schrödinger equation-ODE system with both, matched and unmatched disturbances. The backstepping method is first applied to transform the system into an equivalent target system where the target system is input-to-state stable. To reject the matched disturbance, the sliding mode control (SMC) law is designed for the target system. The well-posedness of the closed-loop system is proved, and the reachability of the sliding manifold in finite time is justified by infinite-dimensional system theory. It is shown that the resulting closed-loop system is input-to-state stable. A Numerical example illustrates the efficiency of the sliding mode design that reduces the ultimate bound of the closed-loop system by rejecting the matched disturbance.

1. Introduction

In the present paper, we consider stabilization of the Schrödinger equation-ODE cascade system with matched and unmatched disturbances. The main contribution of this paper is the design of a state feedback controller that practically stabilizes the coupled system in the presence of small unmatched disturbance by rejecting the matched disturbance. The control problems for unperturbed Schrödinger equations have been well studied and many nice results have been obtained. For instance, E. Machtyngier [1] discussed the exact controllability of Schrödinger equation in bounded domains with Dirichlet boundary condition. E. Machtyngier and E. Zuazua in [2] further considered the stabilization problem of the Schrödinger equation. By introducing multiplier techniques and constructing energy functionals, they have proved the exponential stabilization of the system. M. Krstic developed backstepping approach to deal with the problem of stabilization of Schrödinger equation in [3–5].

During the last decade, a considerable amount of attention has been paid to stability and control of systems described by partial differential equations (PDEs) subject to external disturbances. In [6,7], a stabilizing controller is designed for vibrating system with uncertainty by the Lyapunov functional approach. Input-to-state stability of the wave equation with a boundary disturbance is studied in [8]. Stabilization for a wave equation with distributed control and uncertainty by variable structure control is considered in [9]. Direct output feedback stabilization for a heat equation by the Lyapunov function method is discussed in [9]. More recently, the sliding mode boundary control is designed for a one-dimensional unstable heat equation in [10]. The sliding mode control is also applied to deal with stabilization for one dimensional wave equation, Euler–Bernoulli equation, Schrödinger equation, and cascaded heat partial differential equation system, where the control channel is subject to external disturbance, in [11,12], [13] and [14] respectively. SMC of finite-dimensional systems in the presence of unmatched disturbances is considered in [15]. In [16], SMC is designed to guarantee minimization of unmatched disturbance effects on system motions in a sliding mode. However, the problem of feasible controller design for coupled ODE–PDE systems as well as for coupled PDE–PDE systems is far from being complete, and this problem is rather challenging.

In the present paper, to the best of our knowledge, the backstepping-based sliding mode controller is designed for PDEs in the presence of both, unmatched and matched disturbances. Moreover, boundary backstepping-based SMC is extended to a new class of PDEs: cascade of ODE-Schrödinger equation. We consider

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the following cascade with disturbances:

$$\begin{cases} \dot{X}(t) = AX(t) + Bu(0,t) + B_1 d_1(t), \ t > 0, \\ u_t(x,t) = -\mathrm{i}u_{xx}(x,t), \ 0 < x < 1, \ t > 0, \\ u_x(0,t) = CX(t), \ t > 0, \\ u_x(1,t) = U(t) + d_2(t), \ t > 0, \end{cases} \tag{1.1}$$

where $A \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{n\times 1}$, $C \in \mathbb{C}^{1\times n}$, $X(t) \in \mathbb{C}^{n\times 1}$ is the state of ordinary differential equation, $u(x,t) \in \mathbb{C}$ is the displacement of Schrödinger equation, and $U(t) \in \mathbb{C}$ is the control actuation. The unmatched disturbance d_1 and the matched one d_2 are assumed to be measurable and bounded functions: $|d_1(t)| \le \Delta$ and $|d_2(t)| \le M$, where $\Delta > 0$ and $M > 0$ are known upper bounds.

Our main objective is state-feedback practical stabilization of the coupled system in the presence of small unmatched disturbance $d_1(t)$ and matched bounded disturbance $d_2(t)$. We design a SMC to reject the matched disturbance. We further establish the reachability of the sliding manifold in finite time and the existence and uniqueness of the solution. Finally, input-to-state stability (ISS) of the target closed-loop system is analyzed.

The structure of the paper is as follows. In the next section, we transform system (1.1) into the equivalent target system by the backstepping method. Section 3 is devoted to the matched disturbance rejection by SMC approach. We design a sliding mode control and prove the existence and uniqueness of solution of the closed-loop system. The reachability of the sliding manifold in finite time is presented. In Section 4, the Lyapunov method is used to show that the closed-loop system on the sliding mode surface is input-to-state stable. An example with numerical simulation is presented in Section 5 for illustration of the effectiveness of the method. Concluding remarks are presented in Section 6.

Notation. The Sobolev space $W^{k,p}(\Omega)$ is defined as $W^{k,p}(\Omega) = \{u : D^\alpha u \in L^p(\Omega), \ \textit{for all} \ 0 \le |\alpha| \le k\}$ with norm $\|u\|_{W^{k,p}} = \left\{\sum_{0\le|\alpha|\le k} \|D^\alpha u\|^p_{L^p}\right\}^{\frac{1}{p}}$. $W^{k,2}(\Omega) = H^k(\Omega)$ is the Sobolev space of absolutely continuous scalar functions on Ω with square integrable derivatives of the order $k \ge 1$.

2. Backstepping transformation

First, following (M. Krstic, A. Smyshlyaev [4]), we introduce a transformation for $[X, u] \to [X, w]$ in the form

$$\begin{cases} X(t) = X(t), \\ w(x,t) = u(x,t) - \displaystyle\int_0^x q(x,y)u(y,t)dy - \gamma(x)X(t) \end{cases} \tag{2.1}$$

where

$$q(x,y) = \int_0^{x-y} \mathrm{i}\gamma(\sigma)Bd\sigma, \tag{2.2}$$

$$\gamma(x) = \begin{bmatrix} K & C & \mathrm{i}KA \end{bmatrix} e^{\begin{bmatrix} 0 & 0 & -\mathrm{i}BC \\ I & 0 & \mathrm{i}A \\ 0 & I & 0 \end{bmatrix} x} \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix}. \tag{2.3}$$

The transformations (2.1) transform the system (1.1) into the intermediate system of ODE-Schrödinger cascades of the following form:

$$\begin{cases} \dot{X}(t) = (A+BK)X(t) + Bw(0,t) + B_1 d_1(t), \\ w_t(x,t) = -\mathrm{i}w_{xx}(x,t), \\ w_x(0,t) = 0, \\ w_x(1,t) = W(t) + d_2(t), \end{cases} \tag{2.4}$$

where $w(x,t) \in \mathbb{C}$. Assume that (A, B) is stabilizable and $K \in \mathbb{C}^{1\times n}$ is chosen such that $A + BK$ is Hurwitz. Here $W(t)$ is intermediate system controller of the form:

$$W(t) = U(t) - q(1,1)u(1,t) - \int_0^1 q_x(1,y)u(y,t)dy - \gamma'(1)X(t). \tag{2.5}$$

The transformation (2.1) is invertible,

$$\begin{cases} X(t) = X(t), \\ u(x,t) = w(x,t) + \displaystyle\int_0^x l(x,y)w(y,t)dy + \psi(x)X(t), \end{cases} \tag{2.6}$$

where

$$l(x,y) = \int_0^{x-y} \mathrm{i}\psi(\sigma)Bd\sigma, \tag{2.7}$$

$$\psi(x) = \begin{bmatrix} K & C \end{bmatrix} e^{\begin{bmatrix} 0 & \mathrm{i}(A+BK) \\ I & 0 \end{bmatrix} x} \begin{bmatrix} I \\ 0 \end{bmatrix}. \tag{2.8}$$

Next, a further transformation from $[X, w] \to [X, z]$ is given by

$$\begin{cases} X(t) = X(t), \\ z(x,t) = w(x,t) - \displaystyle\int_0^x k(x,y)w(y,t)dy, \end{cases} \tag{2.9}$$

where

$$k(x,y) = -cix\frac{I_1\left(\sqrt{ci(x^2-y^2)}\right)}{\sqrt{ci(x^2-y^2)}}, \tag{2.10}$$

and I_1 is the modified Bessel function,

$$I_1(x) = \sum_{n=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{2n+1}}{n!(n+1)!}. \tag{2.11}$$

Hence, we obtain the target system:

$$\begin{cases} \dot{X}(t) = (A+BK)X(t) + Bz(0,t) + B_1 d_1(t), \\ z_t(x,t) = -\mathrm{i}z_{xx}(x,t) - cz(x,t), \\ z_x(0,t) = 0, \\ z_x(1,t) = Z(t) + d_2(t), \end{cases} \tag{2.12}$$

where $c > 0$ and

$$Z(t) = W(t) - k(1,1)w(1,t) - \int_0^1 k_x(1,y)w(y,t)dy. \tag{2.13}$$

Then,

$$\begin{aligned} Z(t) &= U(t) - q(1,1)u(1,t) - \int_0^1 q_x(1,y)u(y,t)dy - \gamma'(1)X(t) \\ &- k(1,1)\left[u(1,t) - \int_0^1 q(1,y)u(y,t)dy - \gamma(1)X(t)\right] \\ &- \int_0^1 k_x(1,y)\left[u(y,t) - \int_0^y q(y,\tau)u(\tau,t)d\tau - \gamma(y)X(t)\right]dy. \end{aligned} \tag{2.14}$$

The inverse of the transformation (2.9) can be found as follows

$$\begin{cases} X(t) = X(t), \\ w(x,t) = z(x,t) + \displaystyle\int_0^x p(x,y)z(y,t)dy, \end{cases} \tag{2.15}$$

where

$$p(x,y) = -cix\frac{J_1(\sqrt{ci(x^2-y^2)})}{\sqrt{ci(x^2-y^2)}}, \tag{2.16}$$

and J_1 is the Bessel function of first kind.

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3. Sliding mode controller design

In order to choose a sliding mode surface, we consider the system (2.12) in the energy space $\mathcal{H}_1 = \mathbb{C}^n \times H^1(0,1)$. The inner product induced norm of $\mathcal{H}_1$ is given by

$$\langle [X,f],[Y,g]\rangle = X^T\overline{Y} + \int_0^1 f(x)\overline{g(x)}dx + \int_0^1 f'(x)\overline{g'(x)},\ \forall [X,f],[Y,g]\in\mathcal{H}_1. \tag{3.1}$$

Design the sliding mode surface $\mathcal{S}$ for system (2.12) as a closed-subspace of $\mathcal{H}_1$:

$$\mathcal{S} = \left\{[X,f]\in\mathcal{H}_1 \middle| \int_0^1 f(x)dx = 0\right\}. \tag{3.2}$$

The corresponding sliding mode function for system (2.12) is then given by

$$\begin{aligned} S(t) &= \int_0^1 z(x,t)dx \\ &= \int_0^1\left[w(x,t) - \int_0^x k(x,y)w(y,t)dy\right]dx \end{aligned} \tag{3.3}$$

or, as a function of u and X, as

$$\begin{aligned} S(t) = &\int_0^1 u(x,t)dx - \int_0^1\int_0^x q(x,y)u(y,t)dydx - \int_0^1 \gamma(x)X(t)dx \\ &- \int_0^1\int_0^x k(x,y)u(y,t)dydx \\ &+ \int_0^1\int_0^x k(x,y)\int_0^y q(y,\tau)u(\tau,t)d\tau dydx \\ &+ \int_0^1\int_0^x k(x,y)\gamma(y)X(t)dydx. \end{aligned} \tag{3.4}$$

Next we seek a "reachability of the sliding manifold in finite time" by designing a sliding mode feedback. Differentiating formally the sliding surface function we have

$$\begin{aligned} \dot{S}(t) &= \int_0^1 z_t(x,t)dx = \int_0^1[-\mathrm{i}z_{xx}(x,t) - cz(x,t)]dx \\ &= -\mathrm{i}z_x(1,t) - cS(t) \\ &= -\mathrm{i}Z(t) - \mathrm{i}d_2(t) - cS(t). \end{aligned} \tag{3.5}$$

We design the feedback controller as

$$Z(t) = -\mathrm{i}(M+\eta)\mathrm{sign}(S(t)). \tag{3.6}$$

Then the dynamics on the sliding surface is governed by the following ODE:

$$\dot{S}(t) = -cS(t) - (M+\eta)\mathrm{sign}(S(t)) - \mathrm{i}d_2(t). \tag{3.7}$$

Hence,

$$\begin{aligned} \frac{d}{dt}|S(t)|^2 &= 2\mathrm{Re}\left\{\overline{S(t)}\dot{S}(t)\right\} \\ &= -2c|S(t)|^2 - 2(M+\eta)|S(t)| - 2\mathrm{Re}\left\{\overline{S(t)}\mathrm{i}d_2(t)\right\} \\ &\le -2\eta|S(t)|. \end{aligned} \tag{3.8}$$

It is well-known that (3.8) guarantees reachability of the sliding manifold in finite time [17].

The closed-loop target system (2.12) under the state feedback control (3.6) becomes

$$\begin{cases} \dot{X}(t) = (A+BK)X(t) + Bz(0,t) + B_1d_1(t), \\ z_t(x,t) = -\mathrm{i}z_{xx}(x,t) - cz(x,t), \\ z_x(0,t) = 0, \\ z_x(1,t) = -\mathrm{i}(M+\eta)\mathrm{sign}(S(t)) + d_2(t) \triangleq \tilde{d}(t). \end{cases} \tag{3.9}$$

3.1. Existence of solution to (3.9)

Next, the Riesz basis approach is adopted to prove the well-posedness of system (3.9). Define the operator $\mathbb{A}$ as

$$\begin{cases} \mathbb{A}(X,f) = \left((A+BK)X + Bf(0), -\mathrm{i}f''(x) - cf(x)\right), \\ D(\mathbb{A}) = \{(X,f)\in\mathbb{C}^n\times H^2(0,1) | f'(0) = f'(1) = 0\}. \end{cases} \tag{3.10}$$

With the operator $\mathbb{A}$ at hand, we can present system (3.9) as an evolutionary equation in $\mathcal{H}_1$:

$$\frac{d}{dt}\begin{pmatrix} X(t) \\ z(\cdot,t)\end{pmatrix} = \mathbb{A}\begin{pmatrix} X(t) \\ z(\cdot,t)\end{pmatrix} + \begin{pmatrix} B_1d_1(t) \\ 0\end{pmatrix} + \mathbb{B}\tilde{d}(t), \tag{3.11}$$

where

$$\mathbb{B} = \begin{pmatrix} 0 \\ \mathrm{i}\delta(x-1)\end{pmatrix}. \tag{3.12}$$

Now we show that $\mathbb{A}$ generates C_0-semigroup $e^{\mathbb{A}t}$ and $\mathbb{B}$ is admissible for $e^{\mathbb{A}t}$(G. Weiss, 1989). To this purpose, it suffices to show that $\mathbb{A}^*$ generates C_0-semigroup $e^{\mathbb{A}^*t}$. Indeed, a straightforward computation gives

$$\begin{cases} \mathbb{A}^*(\Psi,\phi)^\top = \left(\overline{(A+BK)}^\top\Psi, \mathrm{i}\phi'' - c\phi\right)^\top, \\ \quad \forall(\Psi,\phi)^\top\in D(\mathbb{A}^*), \\ D(\mathbb{A}^*) = \{(\Psi,\phi)^\top\in\mathbb{C}^n\times H^2(0,1) | \phi'(0) = \mathrm{i}\overline{B}^\top\Psi, \phi'(1) = 0\}. \end{cases} \tag{3.13}$$

The dual system to (3.11) is

$$\begin{cases} \dot{\Psi}(t) = \overline{(A+BK)}^\top\Psi(t),\ t>0 \\ \phi_t(x,t) = \mathrm{i}\phi_{xx}(x,t) - c\phi(x,t),\ x\in(0,1),\ t>0, \\ \phi_x(0,t) = \mathrm{i}\overline{B}^\top\Psi,\ \phi_x(1,t) = 0,\ t>0, \\ Y_1(t) = \mathbb{B}^*(\Psi,\phi) = \mathrm{i}\phi(1,t), t>0. \end{cases} \tag{3.14}$$

Lemma 3.1. *Let $\mathbb{A}^*$ be given by (3.13), let s_k be the eigenvalue of $\overline{(A+BK)}^\top$ with the corresponding eigenvector Ψ_k and assume that $s_k \notin \{\lambda_l = -c - \mathrm{i}(l^2\pi^2)\}$. Then we have the following assertion:*

The eigenvalues of $\mathbb{A}^*$ are given by

$$\{s_k, \lambda_l, k = 1,\ldots,n, l = 0,1,2,\ldots\} \tag{3.15}$$

and the corresponding to s_k and λ_l eigenfunctions are given by

$$W_k = (\Psi_k,\phi_k)^\top, W_l = (0,\phi_l)^\top, \tag{3.16}$$

where

$$\begin{aligned} \phi_l &= \frac{\cos(l\pi x)}{l\pi}, \\ \phi_k &= \frac{\sqrt{2}(1-\mathrm{i})}{2\sqrt{s_k+c}}\overline{B}^\top\Psi_k\frac{\cosh\left((1-\mathrm{i})\sqrt{\frac{s_k+c}{2}}(1-x)\right)}{\sinh\left((1-\mathrm{i})\sqrt{\frac{s_k+c}{2}}\right)}. \end{aligned} \tag{3.17}$$

Moreover, $\{W_l, W_k, l = 0,1,\ldots,k = 1,2,\ldots,n\}$ forms a Riesz basis for $\mathcal{H}_1$. $\mathbb{A}^*$ generates a C_0-semigroup $e^{\mathbb{A}^*t}$ on $\mathcal{H}_1$.

Proof. First, we compute the eigenvalues and the corresponding eigenfunctions of $\mathbb{A}^*$. Let $\mathbb{A}^*W = \lambda W$, where $\lambda\in\sigma(\mathbb{A}^*)$ and $W = (\Psi,\phi)\in D(\mathbb{A}^*)$. Then we have that Ψ and ϕ satisfy the eigenvalue problem

$$\begin{cases} \lambda\Psi = \overline{(A+BK)}^\top\Psi, \\ \lambda\phi = \mathrm{i}\phi'' - c\phi, \\ \phi'(0) = \mathrm{i}\overline{B}^\top\Psi, \phi'(1) = 0. \end{cases} \tag{3.18}$$

There are two cases:
Case 1: $\Psi = 0$. Here (3.18) becomes

$$\begin{cases} \lambda\phi = \mathrm{i}\phi'' - c\phi, \\ \phi'(0) = \phi'(1) = 0, \end{cases} \tag{3.19}$$

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which has nontrivial solutions $(\lambda_l, \phi_l(x))$, $l \in \mathbb{N}$, where $\lambda_l = -c - \mathrm{i}(l^2\pi^2)$, $\phi_l(x) = \frac{\cos(l\pi x)}{l\pi}$, $l \in \mathbb{N}$. Therefore, we obtain that the eigenvalues λ_l and the corresponding eigenfunctions $W_l(x) = \left(0, \frac{\cos(l\pi x)}{l\pi}\right)^\top$.

Case 2: $\Psi \neq 0$. Here $s_k\Psi_k = \overline{(A+BK)}^\top \Psi_k$, $k = 1, 2, \ldots, n$. Then ϕ_k satisfy the following equation:

$$\begin{cases} s_k\phi_k = \mathrm{i}\phi_k'' - c\phi_k, \\ \phi_k'(0) = \mathrm{i}\overline{B}^\top \Psi_k, \ \phi_k'(1) = 0. \end{cases} \tag{3.20}$$

Let

$$\phi_k(x) = c_1 \cosh\left(\sqrt{-\mathrm{i}(s_k+c)}x\right) + c_2 \sinh\left(\sqrt{-\mathrm{i}(s_k+c)}x\right), \tag{3.21}$$

where c_1, c_2 are constants and $\sqrt{-\mathrm{i}} = \frac{1-\mathrm{i}}{\sqrt{2}}$. Substituting (3.21) into the boundary conditions of (3.20), we obtain

$$\begin{cases} \sqrt{-\mathrm{i}(s_k+c)}c_2 = \mathrm{i}\overline{B}^\top\Psi_k, \\ \sqrt{-\mathrm{i}(s_k+c)}\Big[c_1 \sinh(\sqrt{-\mathrm{i}(s_k+c)}) \\ \quad + c_2\cosh(\sqrt{-\mathrm{i}(s_k+c)})\Big] = 0. \end{cases} \tag{3.22}$$

Hence,

$$\begin{cases} c_1 = \dfrac{\sqrt{2}(1-\mathrm{i})}{2\sqrt{s_k+c}}\overline{B}^\top\Psi_k \dfrac{\cosh(\sqrt{-\mathrm{i}(s_k+c)})}{\sinh(\sqrt{-\mathrm{i}(s_k+c)})}, \\ c_2 = -\dfrac{\sqrt{2}(1-\mathrm{i})}{2\sqrt{s_k+c}}\overline{B}^\top\Psi_k. \end{cases} \tag{3.23}$$

Therefore, the solution of (3.20) has the form

$$\phi_k(x) = \frac{\sqrt{2}(1-\mathrm{i})}{2\sqrt{s_k+c}}\overline{B}^\top\Psi_k \frac{\cosh\left((1-\mathrm{i})\sqrt{\frac{s_k+c}{2}}(1-x)\right)}{\sinh\left((1-\mathrm{i})\sqrt{\frac{s_k+c}{2}}\right)}. \tag{3.24}$$

This implies (3.17).

We show next that $\{W_l, W_k, l = 0, 1, \ldots, k = 1, 2, \ldots, n\}$ forms a Riesz basis in $\mathcal{H}_1$. Since $\{\Psi_k, k = 1, 2, \ldots, n\}$ is a basis in $\mathbb{C}^n$ and $\left\{\phi_l = \frac{\cos(l\pi x)}{l\pi}, l = 0, 1, 2, \ldots\right\}$ forms an orthogonal basis in $H^1(0,1)$, we obtain that $\{F_l, F_k, l = 0, 1, \ldots, k = 1, 2, \ldots, n\}$ forms an orthogonal basis in $\mathcal{H}_1$, where $F_l = (0, \phi_l)$ and $F_k = (\Psi_k, 0)$. It follows from (3.17) that

$$\sum_{k=1}^n \|W_k - F_k\|^2 + \sum_{l=0}^\infty \|W_l - F_l\|^2 = \sum_{k=1}^n \|\phi_k\|_{H^1(0,1)} < \infty. \tag{3.25}$$

Then, by classical Bari's theorem, $\{W_l, W_k, l = 0, 1, \ldots, k = 1, 2, \ldots, n\}$ forms a Riesz basis for $\mathcal{H}_1$. This implies $\mathbb{A}^*$ generates a C_0-semigroup $e^{\mathbb{A}^*t}$ on $\mathcal{H}_1$. □

Lemma 3.2. *Let $\mathbb{A}$ be defined by (3.10). Then $\mathbb{A}$ generates a C_0-semigroup $e^{\mathbb{A}t}$ and $\mathbb{B}$ is admissible for $e^{\mathbb{A}t}$ (see [18]).*

Proof. It is shown that $\mathbb{A}^*$ generates a C_0-semigroup $e^{\mathbb{A}^*t}$ on $\mathcal{H}_1$ in Lemma 3.1. Then by [19, Proposition 2.8.5, Proposition 2.8.1], we obtain that $\mathbb{A}$ generates a C_0-semigroup $e^{\mathbb{A}t}$. For any $W_0(\cdot,0) = (\Psi(0), \phi(\cdot,0))^\top \in \mathcal{H}_1$, suppose that

$$W_0(x,0) = \sum_{l=0}^\infty a_l W_l(x) + \sum_{k=1}^n b_k W_k(x), \tag{3.26}$$

$$\|W_0\| \asymp \sum_{l=0}^\infty |a_l|^2 + \sum_{k=1}^n |b_k|^2. \tag{3.27}$$

We thus obtain the solution of (3.14):

$$\begin{aligned} W(x,t) &= (\Psi(t), \phi(x,t))^\top \\ &= \sum_{l=0}^\infty a_l e^{\lambda_l t} W_l(x) + \sum_{k=1}^n b_k e^{s_k t} W_k(x), \end{aligned} \tag{3.28}$$

where

$$\begin{aligned} \Psi(t) &= \sum_{k=1}^n b_k e^{s_k t}\Psi_k, \ \phi(x,t) \\ &= \sum_{l=0}^\infty a_l e^{\lambda_l t}\phi_l(x) + \sum_{k=1}^n b_k e^{s_k t}\phi_k(x). \end{aligned} \tag{3.29}$$

It then follows from (3.17) that

$$\begin{aligned} \phi_l(1) &= \frac{\cos(l\pi)}{l\pi}, \ \phi_k(1) \\ &= \frac{\sqrt{2}(1-\mathrm{i})}{2\sqrt{s_k+c}}\overline{B}^\top\Psi_k \frac{1}{\sinh\left((1-\mathrm{i})\sqrt{\frac{s_k+c}{2}}\right)}, \end{aligned} \tag{3.30}$$

$$\begin{aligned} Y_1(t) &= \mathrm{i}\phi(1,t) = \mathrm{i}\left[\sum_{l=0}^\infty a_l e^{\lambda_l t}\phi_l(1) + \sum_{k=1}^n b_k e^{s_k t}\phi_k(1)\right] \\ &= \mathrm{i}\Bigg[\sum_{l=0}^\infty \frac{(-1)^l}{l\pi} a_l e^{\lambda_l t} + \sum_{k=1}^n b_k e^{s_k t}\frac{\sqrt{2}(1-\mathrm{i})}{2\sqrt{s_k+c}}\overline{B}^\top \\ &\quad \times \Psi_k \frac{1}{\sinh\left((1-\mathrm{i})\sqrt{\frac{s_k+c}{2}}\right)}\Bigg]. \end{aligned} \tag{3.31}$$

By Ingham's inequality ([20, Theorem 9.1, 173]), there exists $T > 0$ such that

$$\int_0^T |Y_1(t)|^2 dt \le K_T \sum_{l=0}^\infty |a_l|^2 + L_T \sum_{k=1}^n |b_k|^2 \le M_T\|W_0\|^2 \tag{3.32}$$

for some constants K_T, L_T, and M_T, which depend only on T. This shows that $\mathbb{B}^*$ is admissible for $e^{\mathbb{A}^*t}$. Therefore, $\mathbb{B}$ is admissible for $e^{\mathbb{A}t}$. □

Due to [18], given $d_1(t)$ and $\tilde{d}(t)$ for any initial value $(X(0), z(\cdot,0))^\top \in \mathcal{H}_1$ there exists a unique solution to (3.11) such that $(X(t), z(\cdot,t))^\top \in C(0,\infty;\mathcal{H}_1)$. Now, we apply the Banach fixed-point theorem to prove the following result:

Theorem 3.1. *Suppose that $|d_2(t)| \le M$ for all $t \ge 0$, and $S(t)$ be defined by (3.3). Then for any $(X(\cdot,0), z(\cdot,0)) \in \mathcal{H}_1$ and $S(t) \neq 0$, there exists a $t_{\max} > 0$ such that (3.9) admits a unique solution $(X,z) \in C(0, t_{max}; \mathcal{H}_1)$ and $S(t) = 0$ for $t \ge t_{\max}$.*

Proof. For any $T > 0$ and $W_0 = (X(0), z(\cdot,0)) \in \mathcal{H}_1$, if $S \in C[0,T]$, $S(t) \neq 0$ for all $t \in [0,T)$, then by the admissibility of $\mathbb{B}$ claimed by Lemma 3.2, there exists a unique solution $W = (X(t), z(\cdot,t)) \in C(0,T;\mathcal{H}_1)$. Let $(f,g) = (0,1) \in D(\mathbb{A}^*)$. By the admissibility of $\mathbb{B}$, we have from (3.11), that

$$\begin{aligned} \frac{d}{dt}\langle W(t), (f,g)\rangle &= \langle W(t), \mathbb{A}^*(f,g)\rangle + \mathbb{B}^*(f,g)\tilde{d}(t) \\ &= \langle W(t), \mathbb{A}^*(f,g)\rangle - \mathrm{i}\tilde{d}(t), \end{aligned} \tag{3.33}$$

where $W(t) = (X(t), z(\cdot,t))$.

Substitution of $\mathbb{A}^*(f(x), g(x)) = (0, -c)$ into (3.33) gives

$$\dot{S}(t) = -cS(t) - \mathrm{i}\tilde{d}(t), \forall\, t \ge 0 \quad \text{a.e.}, \tag{3.34}$$

where $\tilde{d}(t)$ is defined in (3.9).

Now we may assume without loss of generality that $S(t_0) = S_0 \neq 0$. In this case, it follows from (3.34) that

$$S(t) = e^{-c(t-t_0)}S_0 - \int_{t_0}^{t} e^{-c(t-t_0-\tau)}[(M+\eta)\mathrm{sign}(S(\tau)) + \mathrm{i}d_2(\tau)]d\tau, \ \forall t \geq t_0. \tag{3.35}$$

Define a closed subspace of $C\left[t_0, t_0 + \frac{|S_0|}{4(2M+\eta)}\right]$ by

$$\Omega = \left\{ S \in C\left[t_0, t_0 + \frac{|S_0|}{4(2M+\eta)}\right] \middle| S(t_0) = S_0, \ |S(t)| \geq \frac{3|S_0|}{4}e^{-ct}, \ \forall t \in \left[t_0, t_0 + \frac{|S_0|}{4(2M+\eta)}\right] \right\}, \tag{3.36}$$

and a mapping $\mathcal{F}$ on Ω by

$$(\mathcal{F}S)(t) = e^{-c(t-t_0)}S_0 - \int_{t_0}^{t} e^{-c(t-t_0-\tau)}[(M+\eta)\mathrm{sign}(S(\tau)) + \mathrm{i}d_2(\tau)]d\tau. \tag{3.37}$$

Then for any $S \in \Omega$, we have

$$|(\mathcal{F}S)(t)| \geq e^{-c(t-t_0)}\left[|S_0| - (t-t_0)(2M+\eta)\right] \geq \frac{3|S_0|}{4}e^{-ct}. \tag{3.38}$$

The above inequality shows that $\mathcal{F}\Omega \subset \Omega$. Furthermore,

$$\begin{aligned} &|\mathcal{F}S_1(t) - \mathcal{F}S_2(t)| \\ &\leq (M+\eta)\int_{t_0}^{t} e^{-c(t-t_0-\tau)}\left|\mathrm{sign}(S_1) - \mathrm{sign}(S_2)\right| d\tau \\ &\leq (M+\eta)\|S_1 - S_2\|_{C\left[t_0, t_0+\frac{|S_0|}{4(2M+\eta)}\right]} \cdot \frac{4}{3|S_0|} \cdot \frac{|S_0|}{4(2M+\eta)} \\ &\leq \frac{(M+\eta)}{3(2M+\eta)}\|S_1 - S_2\|_{C\left[t_0, t_0+\frac{|S_0|}{4(2M+\eta)}\right]}. \end{aligned} \tag{3.39}$$

This shows that the mapping $\mathcal{F}$ defined by (3.37) is a contraction mapping on Ω. According to the Banach fixed-point theorem [21], there exists a unique, nonzero solution S in $C\left[t_0, t_0 + \frac{|S_0|}{4(2M+\eta)}\right]$.

We conclude that when $S(t_0) \neq 0$, there exists a positive value $t_{\max}$ such that there exists a unique solution to (3.34) and such that $S(t_{\max}) = 0$. It then follows from (3.8) that $|S(t)|$ must be decreasing in $[0, t_{\max})$ and $|S(t)| > 0$ for all $t \in [0, t_{\max})$. Since $S(t)$ is continuous, the reachability of sliding manifold in finite time (3.8) implies that $S(t) \equiv 0$ for all $t \geq t_{\max}$. □

4. ISS (Input-to-state stability) analysis of the system on the sliding surface

On the sliding surface $\mathcal{S}(t) = 0$, the system (3.9) becomes

$$\begin{cases} \dot{X}(t) = (A+BK)X(t) + Bz(0,t) + B_1 d_1(t), \\ z_t(x,t) = -\mathrm{i}z_{xx}(x,t) - cz(x,t), \\ z_x(0,t) = 0, \\ \int_0^1 z(x,t)dx = 0. \end{cases} \tag{4.1}$$

Now we are in a position to consider the input-to-state stability for the system (4.1). By integrating from 0 to 1 in x the second equation of system (4.1), we can obtain that

$$\begin{aligned} \int_0^1 z_t(x,t)dx &= -\mathrm{i}\int_0^1 z_{xx}(x,t)dx - c\int_0^1 z(x,t)dx \\ &= -\mathrm{i}z_x(1,t) + \mathrm{i}z_x(0,t) - c\int_0^1 z(x,t)dx. \end{aligned} \tag{4.2}$$

The boundary conditions of the system (4.1):

$$z_x(0,t) = \int_0^1 z(x,t)dx = 0 \tag{4.3}$$

and the equality (4.2) guarantee $z_x(1,t) = 0$.

Consider the Lyapunov function below:

$$E(t) = \frac{1}{2}\int_0^1 |z(x,t)|^2 dx + \frac{1}{2}\int_0^1 |z_x(x,t)|^2 dx. \tag{4.4}$$

Now by taking a derivative of the function along the solution of system (4.1), we obtain that

$$\begin{aligned} \dot{E}(t) &= \frac{1}{2}\left[\int_0^1 z\overline{z_t}dx + \int_0^1 z_t\bar{z}dx + \int_0^1 z_x\overline{z_{xt}}dx + \int_0^1 z_{xt}\overline{z_x}dx\right] \\ &= -c\int_0^1 |z(x,t)|^2 dx - c\int_0^1 |z_x(x,t)|^2 dx \\ &= -2cE(t). \end{aligned} \tag{4.5}$$

Hence,

$$E(t) \leq e^{-2ct}E(0),$$

i.e.

$$\|z(\cdot,t)\|^2 + \|z_x(\cdot,t)\|^2 \leq e^{-2ct}\left[\|z(\cdot,0)\|^2 + \|z_x(\cdot,0)\|^2\right]. \tag{4.6}$$

Using the Poincaré's and Agmon's inequalities, we find

$$|z(0,t)|^2 \leq \|z(x,t)\|^2 + \|z_x(x,t)\|^2, \tag{4.7}$$

implying

$$|z(0,t)|^2 \leq e^{-2ct}\left[\|z(\cdot,0)\|^2 + \|z_x(\cdot,0)\|^2\right]. \tag{4.8}$$

In order to prove the input-to state stability of 'X' system in the following:

$$\dot{X}(t) = (A+BK)X(t) + Bz(0,t) + B_1 d_1(t), \tag{4.9}$$

we will use the following lemma.

Lemma 4.1 ([22]). *Let $V : [0,\infty) \to \mathbb{R}^+$ be an absolutely continuous function. If there exist $\alpha > 0$ and $\gamma > 0$ such that the derivative of V satisfies almost everywhere the inequality*

$$\frac{d}{dt}V(t) + 2\alpha V(t) - \gamma|d(t)|^2 \leq 0$$

then it follows that for all $|d(t)| \leq \Delta$,

$$V(t) \leq e^{-2\alpha(t-t_0)}V(t_0) + \frac{\gamma}{2\alpha}\Delta^2, \ t \geq t_0.$$

Proposition 4.1. *Consider (4.1). The solution of this system satisfies the bound (4.7). Moreover, given scalars $0 < \alpha < c$ and $\gamma_1 > 0$, if there exists a matrix $P_1 > 0$ that satisfies the LMI*

$$\Theta = \begin{bmatrix} P_1(A+BK) + \overline{A+BK}^{\top}P_1 + 2\alpha P_1 & P_1B_1 \\ * & -\gamma_1 I \end{bmatrix} < 0, \tag{4.10}$$

then for all $|d_1(t)| \leq \Delta$ the following holds:

$$\limsup_{t\to\infty} \overline{X}^{\top}P_1X \leq \frac{\gamma_1}{2\alpha}\Delta^2. \tag{4.11}$$

Proof. Consider the following Lyapunov function for the ISS analysis of (4.9):

$$V(t) = \overline{X}^{\top}(t)P_1X(t), \quad P_1 > 0. \tag{4.12}$$

Differentiating V along the trajectories of the system (4.9) yields

$$\begin{aligned}\frac{d}{dt}V(t)+2\alpha V(t)-\gamma|z(0,t)|^2-\gamma_1|d_1(t)|^2\\ \le 2\overline{X}^\top(t)P_1\dot{X}(t)+2\alpha\overline{X}^\top(t)P_1X(t)-\gamma|z(0,t)|^2-\gamma_1|d_1(t)|^2\\ \le 2\overline{X}^\top(t)P_1[(A+BK)X(t)+Bz(0,t)+B_1d_1(t)]\\ +2\alpha\overline{X}^\top(t)P_1X(t)-\gamma|z(0,t)|^2-\gamma_1|d_1(t)|^2.\end{aligned} \tag{4.13}$$

Setting $\eta(t)=\mathrm{col}\{X(t),z(0,t),d_1(t)\}$ we obtain that

$$\begin{aligned}\frac{d}{dt}V(t)+2\alpha V(t)-\gamma|z(0,t)|^2-\gamma_1|d_1(t)|^2 \le \overline{\eta(t)}^\top\Theta_1\eta(t)\\ \le 0\end{aligned} \tag{4.14}$$

if

$$\Theta_1=\begin{bmatrix}P_1(A+BK)+\overline{A+BK}^\top P_1+2\alpha P_1 & P_1B & P_1B_1\\ * & -\gamma I & 0\\ * & * & -\gamma_1 I\end{bmatrix} < 0. \tag{4.15}$$

The feasibility of (4.10) implies $\Theta_1<0$ for large enough γ. From (4.14) by comparison principle we obtain

$$\begin{aligned}V(t)\le e^{-2\alpha t}V(0)+\int_0^t e^{-2\alpha(t-s)}[\gamma|z(0,s)|^2\\ +\gamma_1|d_1(s)|^2]ds,\ t\ge 0.\end{aligned} \tag{4.16}$$

Then the exponential bound (4.8) on $|z(0,t)|$ and $|d_1(t)|\le\Delta$ imply (4.11). □

Remark 4.1. Since $A+BK$ is Hurwitz, the $(1,1)$-term of Θ in (4.10) is negative for small enough $0<\alpha<c$. Then by Schur complement, the LMI (4.10) is feasible for large enough $\gamma_1>0$.

Remark 4.2. If an unmatched disturbance appears in the right-hand side of Schrödinger equation, the system becomes

$$\begin{cases}\dot{X}(t)=AX(t)+Bu(0,t),\ t>0,\\ u_t(x,t)=-\mathrm{i}u_{xx}(x,t)+B_1d_1(t),\ 0<x<1,\ t>0,\\ u_x(0,t)=CX(t),\ t>0,\\ u_x(1,t)=U(t)+d_2(t),\ t>0.\end{cases} \tag{4.17}$$

By the same backstepping transformations (2.1) and (2.9), we obtain the following target system:

$$\begin{cases}\dot{X}(t)=(A+BK)X(t)+Bz(0,t),\\ z_t(x,t)=-\mathrm{i}z_{xx}(x,t)-cz(x,t)+\bar{B}_1(x)d_1(t),\\ z_x(0,t)=0,\\ z_x(1,t)=Z(t)+d_2(t),\end{cases} \tag{4.18}$$

where

$$\begin{aligned}\bar{B}_1(x)\triangleq B_1\Bigg[1-\int_0^x(q(x,y)+k(x,y))dy\\ +\int_0^x\int_0^y k(x,y)q(y,s)dsdy\Bigg].\end{aligned} \tag{4.19}$$

From the backstepping transformations (2.1) and (2.9), it can be shown that

$$\begin{aligned}\|\bar{B}_1\|_{C[0,1]}=\max_{0\le x\le 1}|\bar{B}_1(x)|\le[1+\|q\|+\|k\|\\ +\|q\|\cdot\|k\|]\cdot|B_1|.\end{aligned} \tag{4.20}$$

SMC may not be efficient in the case of perturbed Schrödinger equation, because in the case of the disturbance d_1 in the right-hand side of Schrödinger equation, this disturbance may become dominant after the backstepping, where the norm of $\bar{B}_1$ may become large. So, compensation of the matched disturbances only by using SMC may not be efficient.

We will show next that the simple Lyapunov function that has been used in the present paper is not appropriate to ISS analysis in this case. If we choose the same sliding mode surface (3.2), the boundary conditions of the system (4.18) on the sliding mode surface (3.2) become

$$z_x(0,t)=\int_0^1 z(x,t)dx=0. \tag{4.21}$$

By integrating from 0 to 1 in x the second equation of system (4.18), we find

$$\begin{aligned}\int_0^1 z_t(x,t)dx=-\mathrm{i}\int_0^1 z_{xx}(x,t)dx-c\int_0^1 z(x,t)dx\\ +\int_0^1\bar{B}_1(x)dx\cdot d_1(t).\end{aligned} \tag{4.22}$$

The boundary conditions (4.21) and the equality (4.22) imply

$$z_x(1,t)=-\mathrm{i}\int_0^1\bar{B}_1(x)dx\cdot d_1(t). \tag{4.23}$$

Consider the following Lyapunov function:

$$V_1(t)=\frac{1}{2}\int_0^1|z(x,t)|^2dx. \tag{4.24}$$

Differentiating V_1 along the system (4.18) subject to (4.21), we find

$$\begin{aligned}\dot{V}_1(t)+2\alpha V_1(t)-\gamma_1|d_1(t)|^2\\ \le -2(c-\alpha)V_1(t)+\|\bar{B}_1\|_{C[0,1]}\cdot|d_1(t)|\cdot[|z(1,t)|+\|z\|]\\ -\gamma_1|d_1(t)|^2.\end{aligned} \tag{4.25}$$

Due to the term $|z(1,t)|$ in the right-hand side of (4.25), the latter cannot be non-positive for all d_1 and all admissible initial conditions $z(x,0)$. Therefore, the simple Lyapunov function that has been used in the present paper is not appropriate to ISS analysis in this case.

5. Example

Consider the following simple system with unknown (bounded) perturbations $d_1(t)$ and $d_2(t)$:

$$\begin{cases}\dot{X}(t)=\begin{bmatrix}0 & 2\\ 6 & -4\end{bmatrix}X(t)+\begin{bmatrix}1\\ 1\end{bmatrix}u(0,t)+\begin{bmatrix}1\\ 0\end{bmatrix}d_1(t),\\ u_t(x,t)=-\mathrm{i}u_{xx}(x,t),\\ u_x(0,t)=\begin{bmatrix}1 & 0\end{bmatrix}X(t),\\ u_x(1,t)=U(t)+d_2(t),\end{cases} \tag{5.1}$$

where

$$\begin{aligned}X(t)=\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix},A=\begin{bmatrix}0 & 2\\ 6 & -4\end{bmatrix},B=\begin{bmatrix}1\\ 1\end{bmatrix},\\ B_1=\begin{bmatrix}1\\ 0\end{bmatrix},C=\begin{bmatrix}1 & 0\end{bmatrix}.\end{aligned} \tag{5.2}$$

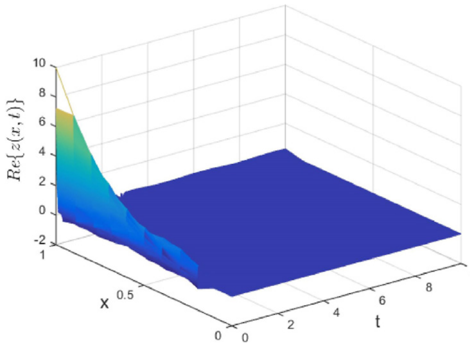
We choose $K=\begin{bmatrix}-7 & 0\end{bmatrix}$ such that $A+BK$ is Hurwitz, where

$$A+BK=\begin{bmatrix}-7 & 2\\ -1 & -4\end{bmatrix} \tag{5.3}$$

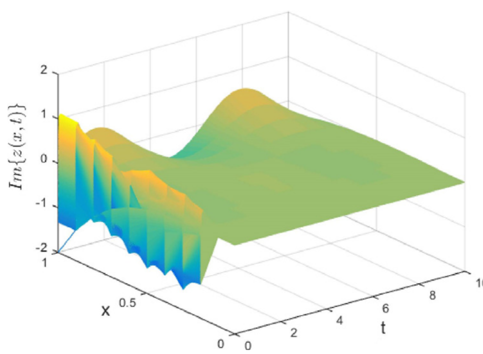
and the eigenvalues of $A+BK$ are $\lambda_1=-5,\lambda_2=-6$.

Then by backstepping transformation (2.1), we get the intermediate system:

$$\begin{cases}\dot{X}(t)=\begin{bmatrix}-7 & 2\\ -1 & -4\end{bmatrix}X(t)+\begin{bmatrix}1\\ 1\end{bmatrix}w(0,t)+\begin{bmatrix}1\\ 0\end{bmatrix}d_1(t),\\ w_t(x,t)=-\mathrm{i}w_{xx}(x,t),\\ w_x(0,t)=0,\\ w_x(1,t)=W(t)+d_2(t).\end{cases} \tag{5.4}$$

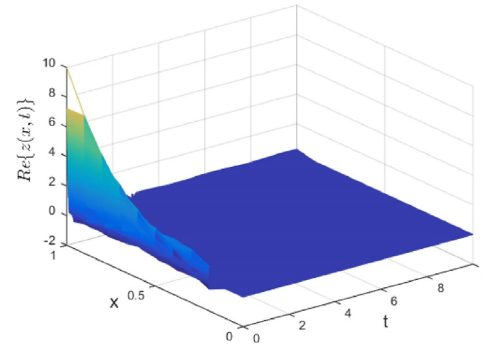


(a) Displacement $\mathrm{Re}\{z(x,t)\}$.

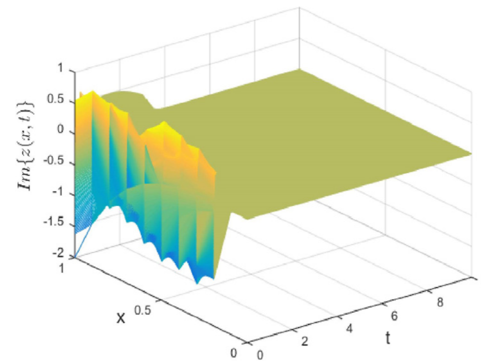


(b) Displacement $\mathrm{Im}\{z(x,t)\}$.

Fig. 1. Displacement $z(x,t)$ of the target system without SMC.



(a) Displacement $\mathrm{Re}\{z(x,t)\}$.



(b) Displacement $\mathrm{Im}\{z(x,t)\}$.

Fig. 2. Displacement $z(x,t)$ of the target system with SMC.

By a further backstepping transformation (2.9), we choose $c = 1$ and obtain the target system:

$$\begin{cases} \dot{X}(t) = \begin{bmatrix} -7 & 2 \\ -1 & -4 \end{bmatrix} X(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} z(0,t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} d_1(t), \\ z_t(x,t) = -\mathrm{i} z_{xx}(x,t) - z(x,t), \\ z_x(0,t) = 0, \\ z_x(1,t) = Z(t) + d_2(t). \end{cases} \tag{5.5}$$

Under the sliding mode control (3.6), the closed-loop system (5.5) becomes

$$\begin{cases} \dot{X}(t) = \begin{bmatrix} -7 & 2 \\ -1 & -4 \end{bmatrix} X(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} z(0,t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} d_1(t), \\ z_t(x,t) = -\mathrm{i} z_{xx}(x,t) - z(x,t), \\ z_x(0,t) = 0, \\ z_x(1,t) = -\mathrm{i}(M+\eta)\mathrm{sign}(S(t)) + d_2(t). \end{cases} \tag{5.6}$$

On the sliding mode surface, the system (5.6) is given by

$$\begin{cases} \dot{X}(t) = \begin{bmatrix} -7 & 2 \\ -1 & -4 \end{bmatrix} X(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} z(0,t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} d_1(t), \\ z_t(x,t) = -\mathrm{i} z_{xx}(x,t) - z(x,t), \\ z_x(0,t) = 0, \\ \int_0^1 z(x,t)dx = 0. \end{cases} \tag{5.7}$$

Applying Proposition 4.1, we fix $\alpha = 0.3$, and minimize γ_1. Then we obtain that the solution of LMI is feasible with $\gamma_1 = 0.1764$.

5.1. Numerical simulation

We give now some simulation results to illustrate the effects of the SMC. Consider the original system (5.1) and target system (5.5) with the initial value:

$$X(0) = \begin{bmatrix} 3 \\ -1+\mathrm{i} \end{bmatrix}, z(x,0) = 10x^3 - 2\mathrm{i}x^4, \tag{5.8}$$

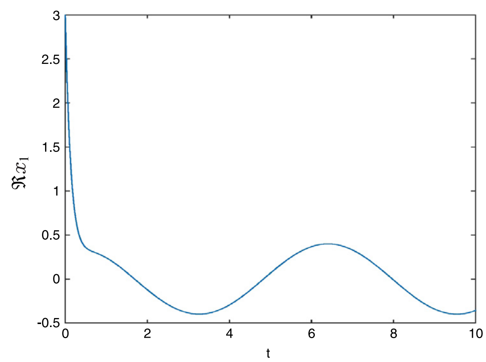
$$u(x,0) = z(x,0) + \int_0^x p(x,y)z(y,0)dy + \int_0^x l(x,y)\left[z(y,0) + \int_0^y p(y,s)z(s,0)ds\right]dy + \psi(x)X(0). \tag{5.9}$$

The disturbances are described as $d_1(t) = \cos t$ and $d_2(t) = 2\mathrm{i}\sin t$. Note that both $d_1(t)$ and $d_2(t)$ are uniformly bounded.

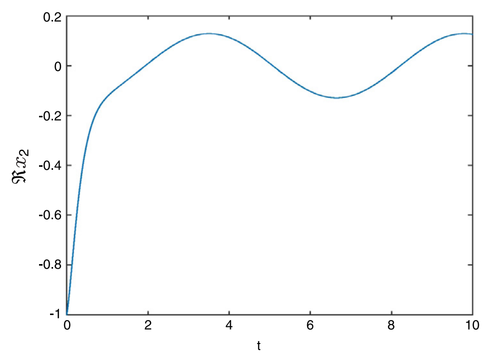
A finite difference method is applied to compute the displacements of the system. Fig. 1(a, b) displays the real and imaginary parts of displacement of target system (5.5) without SMC respectively. Here the steps of space and time are chosen as 0.1 and 0.0001, respectively. Fig. 5 show the displacement of original system (5.1) respectively without SMC.

Further the proposed sliding mode controller (3.6) is applied with the design parameters $M = 2$ and $\eta = 1$. Fig. 2(a, b) demonstrates the real and imaginary parts of displacement of target system (5.6) with the same space and time step sizes used. Compared with Fig. 1(a, b), we see the advantage of SMC approach. It is seen that the displacement is obviously convergent. Fig. 6 shows the displacement of original system (5.1) with SMC respectively.

Fig. 3 displays the real part of $X(t)$ without SMC. Fig. 4 shows the real part of $X(t)$ with SMC. Due to unmatched disturbance, it is seen that $X(t)$ is ultimately bounded. According to Figs. 3 and 4, it is clear that the ultimate bound of $|X(t)|$ is larger if there is no SMC. This is consistent with the theoretical results.

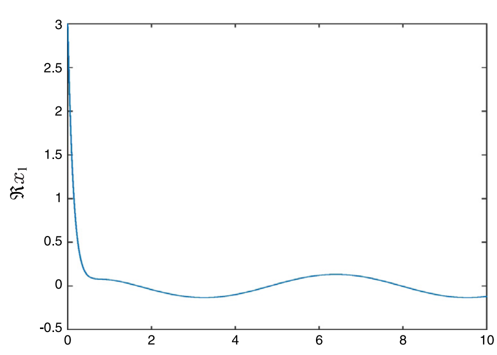


(a) Displacement Re{$x_1(t)$}.

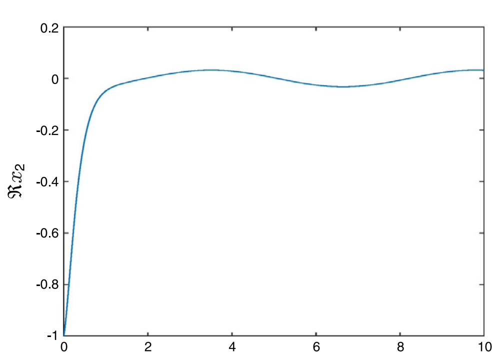


(b) Displacement Re{$x_2(t)$}.

Fig. 3. Real part of displacement $X(t)$ (without SMC).

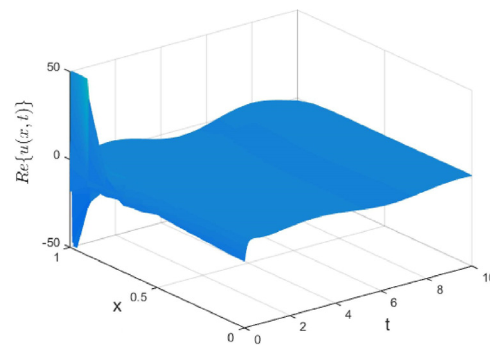


(a) Displacement Re{$x_1(t)$}.

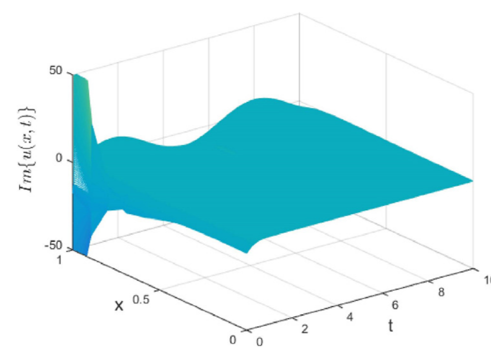


(b) Displacement Re{$x_2(t)$}.

Fig. 4. Real part of displacement $X(t)$ (with SMC).

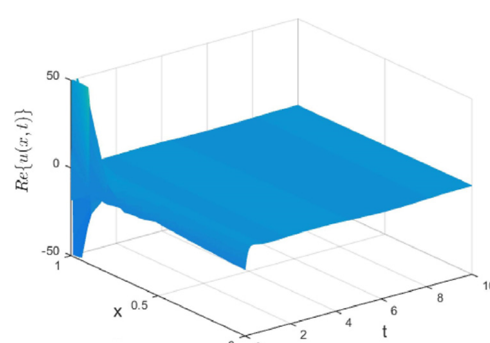


(a) Displacement Re{$u(x, t)$}.

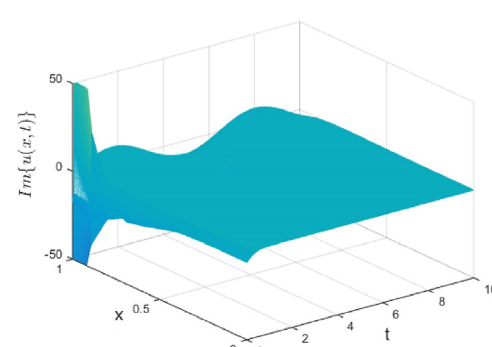


(b) Displacement Im{$u(x, t)$}.

Fig. 5. Displacement $u(x, t)$ of original system without SMC.



(a) Displacement Re{$u(x, t)$}.



(b) Displacement Im{$u(x, t)$}.

Fig. 6. Displacement $u(x, t)$ of original system with SMC.

6. Conclusion

In this paper, to the best of our knowledge, sliding mode boundary control is developed for systems with unmatched and matched disturbances. The backstepping method is applied to transform the perturbed system into the target system. Sliding mode controller is designed to reject the matched disturbance. The closed-loop system is shown to have a unique solution and can reach the sliding surface in finite time. The closed-loop system on the sliding surface is shown to be ISS. A numerical example illustrates the effectiveness of the proposed design method.

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