Systems & Control Letters 140 (2020) 104698

Contents lists available at ScienceDirect

Systems & Control Letters

journal homepage: www.elsevier.com/locate/sysconle

Constrained control of 1-D parabolic PDEs using sampled in space sensing and actuation[☆]

Wen Kang [a,b,*], Emilia Fridman [b]

[a] School of Automation and Electrical Engineering, University of Science and Technology Beijing, China
[b] Department of Electrical Engineering-Systems, Tel Aviv University, Israel

ARTICLE INFO

Article history:
Received 27 September 2019
Received in revised form 9 April 2020
Accepted 15 April 2020
Available online xxxx

Keywords:
Distributed parameter systems
Parabolic systems
Burgers' equation
Point actuation
Constrained control

ABSTRACT

We consider distributed control of a class of 1-D parabolic PDEs under distributed in-domain point actuation and measurements in the presence of control constraints. This class includes unstable diffusion-reaction equations as well as stable Burgers' equations, where we aim to locally improve the convergence. We suggest an observer-based control law that employs the averaged values of the observer state. This allows to regionally stabilize the system. We derive linear matrix inequalities (LMIs) conditions that provide an estimate on the set of initial conditions starting from which the state trajectories of the system are exponentially converging to zero. A numerical example validates the efficiency of the method.

1. Preface

The authors are honored to be able to contribute this article to this special issue, dedicated to the memory of professor Ruth F. Curtain. The second author wants to share some memories about this great woman who was a leader in infinite-dimensional systems.

I met Ruth on ECC 1993 in Groningen. It was my first conference out of Russia. With my background in time-delay systems, I wanted to study H_∞ control for time-delay systems. I approached Prof. Curtain, and she gave me references on the topic and later on sent me a thesis of her Ph.D. student Bert van Keulen. Her encouragement and help were very important for me.

We met on numerous conferences, and had interesting discussions on various topics. I remember her nice plenary lectures. This bright and elegant woman was an inspiring model for me. Ruth explained me the importance of special sessions on distributed parameter systems. Now I greatly appreciate this opportunity not to be lost on huge conferences.

In conclusion, I would like to cite [1]: "There is no doubt that the flourishing state of infinite-dimensional systems today is greatly due to her long-term leadership. Many of her colleagues continued her efforts, and the field remains on the firm ground".

2. Introduction

In many cases, the constraints on the control input should be taken into account for practical application of control laws. There have been some important works on constrained distributed control of PDEs (see e.g. [2–5]). In [2], the internal feedbacks with input constraints of quasi-linear heat equation were designed and the domains of attractions were found via the Galerkin method. In [3,4], global stabilization by distributed saturated control of 1-D Korteweg–de Vries and wave equations was studied. Global stabilization of linear or semilinear system in the Hilbert space by using constrained control was presented in [5]. In [6,7], regional boundary stabilization of coupled linear ODE-heat system and nonlinear Schrödinger equation in the presence of actuator saturation was presented respectively. The results in [6,7] were based on the backstepping method [8] and on direct Lyapunov method for finding domains of attraction of the resulting target systems.

In [9,10] point in-domain control of unstable diffusion equation under the collocated point state measurements was studied. In the absence of disturbances in the equation, a linear static output feedback may globally stabilize the system. However, in the presence of control input constraints, it is not clear if such a controller can achieve at least regional stability. Finding domain of attraction seems to be not possible here. In the present paper,

☆ This work was supported by Israel Science Foundation (Grant No. 673/19, 1128/14), National Natural Science Foundation of China (Grant No. 61803026), Beijing Science Foundation for the Excellent Youth Scholars (Grant No. 2018000020124G067), Fundamental Research Funds for the Central Universities (Grant No. FRF-TP-18-032A1), China Postdoctoral Science Foundation (Grant No. 2018M640065, 2019T120047).

* Corresponding author at: School of Automation and Electrical Engineering, University of Science and Technology Beijing, China.
E-mail addresses: kangwen@amss.ac.cn (W. Kang), emilia@eng.tau.ac.il (E. Fridman).

https://doi.org/10.1016/j.sysconle.2020.104698
0167-6911/

for stabilization of 1-D parabolic PDEs under point actuation and measurements, we suggest an observer-based control law that employs averaged values of the observer. This allows to regionally stabilize the system and to give an estimate on the domain of attraction via Lyapunov method. Moreover, sensors and actuators are not supposed to be collocated.

We consider a class of parabolic PDEs that includes unstable heat equations as well as stable Burgers' equations, where we aim at locally improving the convergence. As a by-product, we design a novel observer for Burgers' equation by using sampled in space measurements. Note that Burgers' equation is used in traffic problems modeling the hydrodynamic mode of car clustering (see e.g. [11]). Burgers' equation describes also models in fluid mechanics, nonlinear acoustics and gas dynamics. Boundary stabilization of Burgers' equation has been extensively studied (see e.g. [8,12–15] and the references therein). In [16], some stability results were provided for distributed and boundary stabilization of Burgers' equation.

The article is organized as follows. In Section 3, the problem statement is presented and useful lemmas are introduced. Section 4 is devoted to construction of an observer by using point measurements. Then based on the observer, a point controller is designed and LMI conditions are presented for the stability analysis of the closed-loop system. In Section 5, we design a constrained controller via LMIs. We find an estimate on the set of initial conditions starting from which the state trajectories of the system are exponentially attracted to a bounded set. A numerical example is presented in Section 6. Section 7 contains some concluding remarks and possible further research lines. Some preliminary results (with skipped proofs) for stabilization of Burgers' equation will be presented in Kang and Fridman [17].

Notation. Throughout the paper, $L^2(0,1)$ stands for the Hilbert space of square integrable scalar functions $z(x)$ on $(0,1)$ with the corresponding norm $\|z\|^2_{L^2(0,1)} = \int_0^1 |z(x)|^2 dx$. $H_0^1(0,1)$ is the closure in $H^1(0,1)$ of the set of smooth functions that are vanishing at $x=0$ and $x=1$. It is equipped with the norm $\|z\|^2_{H_0^1(0,1)} = \int_0^1 |z'(x)|^2 dx$.

3. Problem formulation and useful lemma

We consider the following 1-D parabolic PDEs:

$$\begin{cases} z_t(x,t) = \gamma z_{xx}(x,t) - \alpha z(x,t) z_x(x,t) + \lambda z(x,t) \\ \qquad + \sum_{j=0}^{N-1} \delta(x-\bar{x}_j) u_j(t), \\ z(x,0) = z_0(x), \end{cases} \tag{3.1}$$

under the Dirichlet boundary conditions

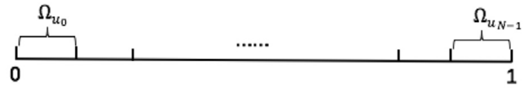
$$z(0,t) = z(1,t) = 0, \tag{3.2}$$

where $\gamma > 0$ is viscosity, $\lambda > 0$ denotes a constant coefficient, $\alpha \geq 0$, $z(x,t)$ is the state of parabolic PDEs, and $u_j(t)$ $(j = 0,1,\ldots,N-1)$ are the control inputs. The coefficient $\alpha \geq 0$ is supposed to be positive for the case of Burgers' equation or zero for the case of unstable diffusion-reaction equation. If $\lambda < \gamma\pi^2$, the open-loop system (with $u_j(t) \equiv 0$) is exponentially stable. For $\lambda > \gamma\pi^2$, the open-loop system (with $u_j(t) \equiv 0$) may become unstable.

We assume that $\{\Omega_{u_j}\}_{j=0}^{N-1}$ is a partition of $[0,1]$. The intervals Ω_{u_j} are upper bounded by Δ_u:

$$0 < |\Omega_{u_j}| \leq \Delta_u,$$

where Δ_u is the maximum subdomain length $\max_j |\Omega_{u_j}|$.



$$\Delta_u = \max_{0 \leq j \leq N-1} |\Omega_{u_j}|$$

The control inputs $u_j(t)$ enter (3.1) through the Dirac delta function at some points $\bar{x}_j \in \Omega_{u_j}$.[1]

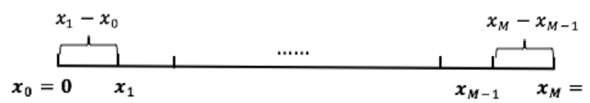
Sensors are supposed to be uncollocated with actuators providing point measurements of the state

$$y_k(t) = z(x_k,t), \quad k = 0,\ldots,M,$$

where $0 = x_0 < x_1 < \cdots < x_M = 1$.

Moreover,

$$x_{k+1} - x_k \leq \Delta_y, \quad k = 0,\ldots,M-1.$$



$$\Delta_y = \max_{0 \leq k \leq M-1} |x_{k+1} - x_k|$$

It is well-known that global stabilization by constrained control can be achieved for linear finite-dimensional systems that have no eigenvalues in the right-half plane. Since linear system (3.1), (3.2) with $\alpha = 0$ and $u_j = 0$ has exponentially growing solutions for $\lambda > \gamma\pi^2$, our objective in this paper is to obtain regional stabilization with a bound on domain of attraction. In our case of point actuation and measurements, the control inputs and measurements are multiplied by unbounded operator, whereas the convergence of the closed-loop system can be proved only in L^2-norm (see e.g. [18]). To obtain a bound on the initial conditions, differently from [18], we suggest a controller by employing averaged measurements of the observer state that leads to regional exponential stability of the closed-loop system.

We aim to design a constrained controller that regionally stabilizes the system for $\alpha = 0$ or improves its convergence for $\alpha > 0$. The following useful lemmas will be employed in the proof of our main results:

Lemma 3.1 (*Wirtinger's Inequality [19]*). *For $a < b$, let $g \in H^1(a,b)$ be a scalar function with $g(a) = 0$ or $g(b) = 0$. Then*

$$\int_a^b g^2(x)dx \leq \frac{4(b-a)^2}{\pi^2} \int_a^b \left[\frac{dg}{dx}(x)\right]^2 dx.$$

Moreover, if $g \in H_0^1(a,b)$, then

$$\int_a^b g^2(x)dx \leq \frac{(b-a)^2}{\pi^2} \int_a^b \left[\frac{dg}{dx}(x)\right]^2 dx.$$

[1] The Dirac delta function is a distribution (a generalized function, such as a probability distribution) that is also a measure (i.e. it assigns a value to a function) - terms that come from probability and set theory. Dirac delta function is defined indirectly by specifying its effect on a continuous test function $\varphi(\xi)$ as

$$\langle \delta(\xi - x), \varphi(\xi) \rangle = \varphi(x).$$

Lemma 3.2 (*Agmon's Inequality, See p. 20 in [8]*). *For any $z \in H^1(0,1)$, the following inequalities hold:*

$$\max_{x\in[0,1]} |z(x,t)|^2 \le z^2(0) + 2\|z(t)\|_{L^2(0,1)}\|z_x(t)\|_{L^2(0,1)},$$
$$\max_{x\in[0,1]} |z(x,t)|^2 \le z^2(1) + 2\|z(t)\|_{L^2(0,1)}\|z_x(t)\|_{L^2(0,1)}.$$

4. Observer and feedback for regional stabilization of system

Based on the point measurements we construct the following observer for system (3.1), (3.2) to estimate the value of the state $z(x,t)$:

$$\begin{cases} \hat z_t(x,t) = \gamma \hat z_{xx}(x,t) - \alpha \hat z(x,t)\hat z_x(x,t) + \lambda \hat z(x,t) \\ +\displaystyle\sum_{j=0}^{N-1}\delta(x-\bar x_j)u_j(t) - L\sum_{k=0}^{M-1} b_k(x)[\hat z(x_k,t) - z(x_k,t)], \\ \hat z(0,t) = \hat z(1,t) = 0, \\ \hat z(x,0) = 0, \end{cases} \tag{4.1}$$

where $L > 0$ will be chosen later.

As in [20], here the shape functions are given by

$$b_k(x) = \begin{cases} 1, & x \in \Gamma_k \triangleq [x_k, x_{k+1}), \\ 0, & otherwise, \end{cases} \quad k = 0,\dots,M-1. \tag{4.2}$$

Let $e(x,t) = \hat z(x,t) - z(x,t)$ be the estimation error. It is easy to see that e is governed by

$$\begin{cases} e_t(x,t) = \gamma e_{xx}(x,t) - \alpha e(x,t)\hat z_x(x,t) - \alpha \hat z(x,t)e_x(x,t) \\ +\alpha e(x,t)e_x(x,t) + \lambda e(x,t) - L\displaystyle\sum_{k=0}^{M-1} b_k(x)e(x_k,t), \\ e(0,t) = e(1,t) = 0, \\ e(x,0) = -z_0(x). \end{cases} \tag{4.3}$$

We propose the following observer-based feedback controller to stabilize the system (3.1), (3.2):

$$u_j(t) = -K\int_{\Omega_{u_j}} \hat z(\xi,t)d\xi, \tag{4.4}$$

where K is a positive constant to be determined later.

The closed-loop system (4.1), (4.3) corresponding to controller (4.4) becomes

$$\begin{cases} \hat z_t(x,t) = \gamma \hat z_{xx}(x,t) - \alpha \hat z(x,t)\hat z_x(x,t) + \lambda \hat z(x,t) \\ -L\displaystyle\sum_{k=0}^{M-1} b_k(x)e(x_k,t) - \sum_{j=0}^{N-1}\delta(x-\bar x_j)K\int_{\Omega_{u_j}}\hat z(\xi,t)d\xi \\ \hat z(0,t) = \hat z(1,t) = 0, \\ e_t(x,t) = \gamma e_{xx}(x,t) - \alpha e(x,t)\hat z_x(x,t) - \alpha \hat z(x,t)e_x(x,t) \\ +\alpha e(x,t)e_x(x,t) + \lambda e(x,t) - L\displaystyle\sum_{k=0}^{M-1} b_k(x)e(x_k,t), \\ e(0,t) = e(1,t) = 0, \\ \hat z(x,0) = 0, \ \ e(x,0) = -z_0(x). \end{cases} \tag{4.5}$$

Now we study the well-posedness of (4.5). We investigate the coupled system (4.5) in the energy state space

$\mathcal{H} = L^2(0,1)\times L^2(0,1)$

with the norm $\|(f,g)\|^2_{\mathcal{H}} = \|f\|^2_{L^2(0,1)} + \|g\|^2_{L^2(0,1)}$.

Let

$\mathcal{H}_1 = H^1_0(0,1)\times H^1_0(0,1)$

be the Hilbert space with the norm:

$\|(f,g)\|^2_{\mathcal{H}_1} = \|f'\|^2_{L^2(0,1)} + \|g'\|^2_{L^2(0,1)}$.

Following [10] (see Definition 1 in [10]), we give the following definition of the solution to system (4.5):

Definition 4.1. For any $T > 0$, a function $(\hat z(\cdot,t), e(\cdot,t)) \in C([0,T];\mathcal{H}_1)$ such that $(\hat z_t(\cdot,t), e_t(\cdot,t)) \in L^\infty([0,T];\mathcal{H})\cap L^2([0,T];\mathcal{H}_1)$, is said to be a solution of the boundary value problem (4.5) initialized by $z_0 \in H^2(0,1)\cap H^1_0(0,1)$ if for every $(\phi(\xi),\varphi(\xi)) \in \mathcal{H}_1$, the functions $\int_0^1 \hat z(\xi,t)\phi(\xi)d\xi$ and $\int_0^1 e(\xi,t)\varphi(\xi)d\xi$ are absolutely continuous on $[0,T]$ and relation

$$\begin{cases} \frac{d}{dt}\int_0^1 \hat z(\xi,t)\phi(\xi)d\xi + \gamma\int_0^1 \hat z_\xi(\xi,t)\phi_\xi(\xi)d\xi \\ = -\alpha\int_0^1 \hat z(\xi,t)\hat z_\xi(\xi,t)\phi(\xi)d\xi + \lambda\int_0^1 \hat z(\xi,t)\phi(\xi)d\xi \\ -L\displaystyle\sum_{k=0}^{M-1} e(x_k,t)\int_{\Gamma_k}\phi(\xi)d\xi \\ -K\displaystyle\sum_{j=0}^{N-1}\phi(\bar x_j)\int_{\Omega_{u_j}}\hat z(\xi,t)d\xi, \\ \frac{d}{dt}\int_0^1 e(\xi,t)\varphi(\xi)d\xi + \gamma\int_0^1 e_\xi(\xi,t)\varphi_\xi(\xi)d\xi \\ = -\alpha\int_0^1 [e(\xi,t)\hat z_\xi(\xi,t) + \hat z(\xi,t)e_\xi(\xi,t)]\varphi(\xi)d\xi \\ +\alpha\int_0^1 e(\xi,t)e_\xi(\xi,t)\varphi(\xi)d\xi + \lambda\int_0^1 e(\xi,t)\varphi(\xi)d\xi \\ -L\displaystyle\sum_{k=0}^{M-1} e(x_k,t)\int_{\Gamma_k}\varphi(\xi)d\xi \end{cases} \tag{4.6}$$

holds for almost all $t \in [0,T]$.

The weak solution concept (4.6) is based on the integration-by-parts property

$$\int_0^1 \hat z_{\xi\xi}(\xi,t)\phi(\xi)d\xi = -\int_0^1 \hat z_\xi(\xi,t)\phi_\xi(\xi)d\xi,$$
$$\int_0^1 e_{\xi\xi}(\xi,t)\varphi(\xi)d\xi = -\int_0^1 e_\xi(\xi,t)\varphi_\xi(\xi)d\xi$$

of the Sobolev derivatives of $H^1_0(0,1)$-valued functions for any test function $(\phi(\xi),\varphi(\xi)) \in \mathcal{H}_1$.

Now we are in a position to formulate the conditions that guarantee regional stability and well-posedness of the closed-loop system:

Theorem 4.1. *Consider the system (3.1), (3.2) under the observer-based controller (4.4), where $\hat z$ is governed by (4.1). For $\alpha > 0$, given positive scalars Δ_u, Δ_y, δ and tuning parameters $0 < \beta < 1$, $C > 0$, assume that there exist positive scalars K, L, $\mu_i (i = 1,2)$ and nonnegative scalars $\mu_i (i = 3,4)$ such that the following LMIs hold:*

$$-\gamma + \mu_1 + \mu_3 + \frac{C}{2} \le 0, \tag{4.7}$$

$$-\gamma + \mu_2 + \mu_4 + \frac{C}{2} \le 0, \tag{4.8}$$

$$\Theta_1 = \begin{bmatrix} \theta_1 & -\frac{K}{2} & -\frac{\beta L}{2} \\ * & -\mu_1\frac{\pi^2}{4\Delta_u^2} & 0 \\ * & * & \theta_2 \end{bmatrix} \le 0, \tag{4.9}$$

$$\Theta_2 = \begin{bmatrix} \theta_3 & -\frac{L}{2} & -\frac{(1-\beta)L}{2} \\ * & -\mu_2\frac{\pi^2}{4\Delta_y^2} & -\frac{L}{2} \\ * & * & \theta_4 \end{bmatrix} \le 0, \tag{4.10}$$

where

$$\begin{aligned}&\theta_1=\beta[-K-\mu_3\pi^2+\lambda+\delta],\\&\theta_2=\beta[-L-\mu_4\pi^2+\lambda+\delta],\\&\theta_3=(1-\beta)[-K-\mu_3\pi^2+\lambda+\delta],\\&\theta_4=(1-\beta)[-L-\mu_4\pi^2+\lambda+\delta].\end{aligned}\tag{4.11}$$

Then for any initial function in (4.5) $z_0\in H^2(0,1)\cap H_0^1(0,1)$ satisfying $\|z_0\|_{L^2(0,1)}<\alpha^{-1}C$, the following holds:
(i) [Well-posedness] *A solution of the system (4.5) exists in the sense that*

$$\begin{aligned}&(\hat z,e)\in C([0,T];\mathcal{H}_1),\\&(\hat z_t,e_t)\in L^\infty([0,T];\mathcal{H})\cap L^2([0,T];\mathcal{H}_1).\end{aligned}\tag{4.12}$$

holds for all $T>0$.
(ii) [Regional stability] *The solution of the closed-loop system (3.1), (3.2), (4.4) satisfies*

$$\|z(\cdot,t)\|_{L^2(0,1)}\le\sqrt{2}e^{-\delta t}\|z_0\|_{L^2(0,1)},\ \forall t\ge0.\tag{4.13}$$

Furthermore, if the strict LMIs (4.9), (4.10) are feasible for $\delta=0$ and (4.7), (4.8) are satisfied, then the closed-loop system is exponentially stable with a small enough decay rate.

Proof. (i) Based on the Galerkin approximation method, the well-posedness result will be proved in Appendix.
(ii) We construct the Lyapunov function $V(t)$ as follows:

$$V(t)=\frac12\int_0^1[\hat z^2(x,t)+e^2(x,t)]dx.\tag{4.14}$$

As in [1] (see the proof of Theorem 2.1, pp.115), let us multiply the first equation of (4.5) by $\hat z$, multiply the third equation of (4.5) by e and integrate in [0, 1] to obtain

$$\begin{aligned}\dot V(t)=&-\gamma\int_0^1[\hat z_x^2(x,t)+e_x^2(x,t)]dx+\lambda\int_0^1[\hat z^2(x,t)+e^2(x,t)]dx\\&-L\sum_{k=0}^{M-1}\int_{\Gamma_k}[\hat z(x,t)+e(x,t)]e(x_k,t)dx\\&-K\sum_{j=0}^{N-1}\int_{\Omega_{u_j}}\hat z(\bar x_j,t)\hat z(x,t)dx\\&+\alpha\int_0^1 e(x,t)[-e(x,t)\hat z_x(x,t)-\hat z(x,t)e_x(x,t)]dx.\end{aligned}\tag{4.15}$$

Denote $g_k(x,t)\triangleq\int_x^{x_k}e_\xi(\xi,t)d\xi$, $f_j(x,t)\triangleq\int_x^{\bar x_j}\hat z_\xi(\xi,t)d\xi$.

Then the L-term and K-term of (4.15) can be presented in the form

$$\begin{aligned}&-L\sum_{k=0}^{M-1}\int_{\Gamma_k}[\hat z(x,t)+e(x,t)]e(x_k,t)dx\\&=-L\sum_{k=0}^{M-1}\int_{\Gamma_k}[\hat z(x,t)+e(x,t)][e(x,t)+g_k(x,t)]dx\\&=-L\int_0^1\hat z(x,t)e(x,t)dx-L\int_0^1e^2(x,t)dx\\&-L\sum_{k=0}^{M-1}\int_{\Gamma_k}[\hat z(x,t)+e(x,t)]g_k(x,t)dx,\end{aligned}\tag{4.16}$$

$$\begin{aligned}&-K\sum_{j=0}^{N-1}\int_{\Omega_{u_j}}\hat z(\bar x_j,t)\hat z(x,t)dx\\&=-K\sum_{j=0}^{N-1}\int_{\Omega_{u_j}}[f_j(x,t)+\hat z(x,t)]\hat z(x,t)dx\\&=-K\int_0^1\hat z^2(x,t)dx-K\sum_{j=0}^{N-1}\int_{\Omega_{u_j}}\hat z(x,t)f_j(x,t)dx.\end{aligned}\tag{4.17}$$

Integration of the term "$-e(x,t)e_x(x,t)\hat z(x,t)$" from 0 to 1 in x by parts and substitution of the boundary conditions $e(0,t)=e(1,t)=0$ lead to

$$-\int_0^1e(x,t)e_x(x,t)\hat z(x,t)dx=\frac12\int_0^1e^2(x,t)\hat z_x(x,t)dx.\tag{4.18}$$

The Agmon's inequality (Lemma 3.2) yields

$$\max_{x\in[0,1]}|e(x,t)|^2\le2\|e(\cdot,t)\|_{L^2(0,1)}\cdot\|e_x(\cdot,t)\|_{L^2(0,1)}.\tag{4.19}$$

By using (4.18), (4.19), Young's and Cauchy-Schwarz's inequalities we obtain

$$\begin{aligned}&\int_0^1e(x,t)[-e(x,t)\hat z_x(x,t)-\hat z(x,t)e_x(x,t)]dx\\&\overset{(4.18)}{=}-\frac12\int_0^1e^2(x,t)\hat z_x(x,t)dx\\&\le\frac12\max_{x\in[0,1]}|e(x,t)|^2\|\hat z_x(\cdot,t)\|_{L^1(0,1)}\\&\overset{(4.19)}{\le}\|e(\cdot,t)\|_{L^2(0,1)}\|e_x(\cdot,t)\|_{L^2(0,1)}\|\hat z_x(\cdot,t)\|_{L^1(0,1)}\\&\overset{\text{Cauchy's}}{\le}\|e(\cdot,t)\|_{L^2(0,1)}\|e_x(\cdot,t)\|_{L^2(0,1)}\|\hat z_x(\cdot,t)\|_{L^2(0,1)}\\&\overset{\text{Young's}}{\le}\|e(\cdot,t)\|_{L^2(0,1)}\left[\frac12\|e_x(\cdot,t)\|^2_{L^2(0,1)}+\frac12\|\hat z_x(\cdot,t)\|^2_{L^2(0,1)}\right].\end{aligned}\tag{4.20}$$

Due to $g_k(x_k)=f_j(\bar x_j)=0$, we further apply Wirtinger's inequality (Lemma 3.1). For any positive constants $\mu_i (i=1,2)$ we have

$$0\le\mu_1\sum_{j=0}^{N-1}\int_{\Omega_{u_j}}\left[\hat z_x^2(x,t)-\frac{\pi^2}{4\Delta_u^2}f_j^2(x,t)\right]dx,\tag{4.21}$$

$$0\le\mu_2\sum_{k=0}^{M-1}\int_{\Gamma_k}\left[e_x^2(x,t)-\frac{\pi^2}{4\Delta_y^2}g_k^2(x,t)\right]dx.\tag{4.22}$$

Moreover, under the homogeneous Dirichlet boundary conditions (since $\hat z(0,t)=\hat z(1,t)=e(0,t)=e(1,t)=0$)

$$0\le\mu_3\left[\|\hat z_x(\cdot,t)\|^2_{L^2(0,1)}-\pi^2\|\hat z(\cdot,t)\|^2_{L^2(0,1)}\right],\tag{4.23}$$

$$0\le\mu_4\left[\|e_x(\cdot,t)\|^2_{L^2(0,1)}-\pi^2\|e(\cdot,t)\|^2_{L^2(0,1)}\right]\tag{4.24}$$

hold, where $\mu_3\ge0$ and $\mu_4\ge0$.

Set

$$\begin{aligned}&\eta_j=\operatorname{col}\left\{\hat z(x,t),f_j(x,t),e(x,t)\right\},\\&\sigma_k=\operatorname{col}\left\{\hat z(x,t),g_k(x,t),e(x,t)\right\}.\end{aligned}$$

Substituting (4.16), (4.17), (4.20) into the right-hand side of (4.15), and adding (4.21)–(4.24) to (4.15), for any $\beta\in(0,1)$ we obtain

$$\begin{aligned}&\dot V(t)+2\delta V(t)\\&\le\sum_{j=0}^{N-1}\int_{\Omega_{u_j}}\eta_j^\top\Theta_1\eta_jdx+\sum_{k=0}^{M-1}\int_{\Gamma_k}\sigma_k^\top\Theta_2\sigma_kdx\\&-\left(\gamma-\mu_1-\mu_3-\frac{\alpha}{2}\|e\|_{L^2(0,1)}\right)\int_0^1\hat z_x^2(x,t)dx\\&-\left(\gamma-\mu_2-\mu_4-\frac{\alpha}{2}\|e\|_{L^2(0,1)}\right)\int_0^1e_x^2(x,t)dx,\end{aligned}\tag{4.25}$$

where Θ_1 and Θ_2 are given by (4.9), (4.10) respectively.

We next prove that (4.13) is satisfied. Similar to [21], we first assume that

$$\|e(\cdot,t)\|_{L^2(0,1)}<\alpha^{-1}C,\ \forall t\ge0.\tag{4.26}$$

Then from (4.25), it follows that

$$\dot{V}(t) + 2\delta V(t) \leq 0 \tag{4.27}$$

if $\Theta_1 \leq 0$, $\Theta_2 \leq 0$, and (4.7), (4.8) hold.

Moreover, (4.27) implies

$$\|(\hat{z}(\cdot,t), e(\cdot,t))\|^2_{\mathcal{H}} \leq e^{-2\delta t}\|(\hat{z}(\cdot,0), e(\cdot,0))\|^2_{\mathcal{H}}, \tag{4.28}$$

that together with Minkowski's inequality leads to

$$\begin{aligned} &\|z(\cdot,t)\|_{L^2(0,1)} = \|\hat{z}(\cdot,t) - e(\cdot,t)\|_{L^2(0,1)} \\ &\leq \sqrt{2}\|(\hat{z}(\cdot,t), e(\cdot,t))\|_{\mathcal{H}} \leq \sqrt{2}e^{-\delta t}\|(\hat{z}(\cdot,0), e(\cdot,0))\|_{\mathcal{H}} \\ &\leq \sqrt{2}e^{-\delta t}\|z_0\|_{L^2(0,1)}. \end{aligned}$$

Now we prove (4.26). Due to $V(t) = \frac{1}{2}\|(\hat{z}(\cdot,t), e(\cdot,t))\|^2_{\mathcal{H}}$, it is sufficient to show that

$$V(t) < \frac{1}{2}(\alpha^{-1}C)^2, \quad \forall t \geq 0. \tag{4.29}$$

Indeed, for $t = 0$, the inequality (4.29) holds. Let (4.29) be false for some $t_1 > 0$. Then $V(t_1) \geq \frac{1}{2}(\alpha^{-1}C)^2 > V(0)$. Since V is continuous in time, there must exist $t^* \in (0, t_1]$ such that

$$V(t) < \frac{1}{2}(\alpha^{-1}C)^2 \ \ \forall t \in [0, t^*) \ \text{ and } \ V(t^*) = \frac{1}{2}(\alpha^{-1}C)^2. \tag{4.30}$$

The first relation of (4.30), together with the feasibility of $\Theta_1 \leq 0$, $\Theta_2 \leq 0$ and (4.7), (4.8), guarantees that $\dot{V}(t) + 2\delta V(t) \leq 0$ on $[0, t^*)$. Therefore, $V(t^*) \leq V(0) < \frac{1}{2}(\alpha^{-1}C)^2$. This contradicts the second relation of (4.30). Thus, (4.29) and consequently, (4.26), (4.27) are true, which implies (4.13) provided that $\|z_0\|_{L^2(0,1)} < \alpha^{-1}C$.

Note that the feasibility of LMIs (4.9), (4.10) with $\delta = 0$ implies its feasibility with a small enough $\delta > 0$. Therefore, if LMIs (4.9), (4.10) hold for $\delta = 0$ and (4.7), (4.8) are satisfied, then the closed-loop system is exponentially stable with a small decay rate.

The proof of (i) and (ii) are completed. □

Remark 4.1. If the LMIs of Theorem 4.1 are feasible for $\alpha = 0$, then the results are global: for any initial function $z_0 \in H^2(0,1) \cap H^1_0(0,1)$, the system (3.1), (3.2) under the observer-based controller (4.4) governed by (4.5) is exponentially stable meaning that the solution of the closed-loop system satisfies (4.13).

Remark 4.2. For the case of the Neumann boundary conditions:

$$z_x(0,t) = z_x(1,t) = 0, \tag{4.31}$$

Wirtinger's inequality (Lemma 3.1) is not applicable. Hence, (4.23) and (4.24) hold iff $\mu_3 = \mu_4 = 0$. Therefore, diffusion-reaction equation (3.1) with $\alpha = 0$ and the Neumann boundary conditions (4.31) under the observer-based controller (4.4) with (4.5) is exponentially stable if the LMIs of Theorem 4.1 hold with $\mu_3 = \mu_4 = 0$. Note that due to additional terms in Agmon's inequality under the Neumann boundary conditions, (4.20) cannot be obtained, and hence, it is not clear how to extend Theorem 4.1 for $\alpha \neq 0$.

Remark 4.3. Given any desirable decay rate $\delta > 0$, $L > \lambda + \delta$, and a bound $C < 2\gamma$ on the initial conditions $\|z_0\|_{L^2(0,1)} < \alpha^{-1}C$, the LMIs in Theorem 4.1 are always feasible for large enough K and small enough Δ_u and Δ_y (i.e. the number of the sensors and actuators is large enough). Indeed, consider Θ_1 and Θ_2 given by (4.9) and (4.10) respectively. Choose $\mu_1 = \mu_2$, $\mu_3 = \mu_4 = \gamma - \mu_1 - \frac{C}{2}$ that satisfy (4.7) and (4.8). Then for $\beta = 0.5$, LMIs (4.9) and (4.10) are feasible for large enough $K > \lambda + \delta$ and small enough Δ_u and Δ_y.

Remark 4.4. From the LMI conditions of Theorem 4.1, it can be seen that

(i) If γ becomes larger, for the same estimate on the set of initial conditions (inside the domain of attraction), a larger decay rate δ can be achieved.

(ii) If γ becomes larger, for the same decay rate δ, a larger domain of attraction can be obtained.

(iii) If α becomes smaller, for the same decay rate δ, a larger domain of attraction can be obtained.

5. Constrained control: regional stabilization

In this section, we consider (3.1), (3.2) with the point control law which is subject to the following amplitude constraint:

$$|u_j(t)| \leq \bar{u}, \ \ (j = 0, \ldots, N-1). \tag{5.1}$$

We design the observer-based feedback controller in the form:

$$u_j^{sat}(t) = \mathrm{sat}(u_j(t), \bar{u}), \ \ j = 0, \ldots, N-1, \tag{5.2}$$

where the saturation function is defined by

$$\mathrm{sat}(u_j, \bar{u}) = \mathrm{sign}(u_j)\min(|u_j|, \bar{u}),$$

and $u_j(t)$ is given by (4.4).

We will find domain of attraction for the closed-loop system (4.1), (4.3) subject to (5.2). Denoting the state trajectory of closed-loop system (4.1), (4.3) subject to (5.2) with the initial condition $(0, -z_0)$ by $(\hat{z}(x,t;0), e(x,t;-z_0))$, the domain of attraction of the closed-loop system is then the set

$$\begin{aligned} \mathcal{S} = \{&(0, -z_0) \in (H^2(0,1) \cap H^1_0(0,1))^2 : \\ &\lim_{t\to\infty} \|(\hat{z}(x,t;0), e(x,t;-z_0))\|_{\mathcal{H}} = 0\}. \end{aligned}$$

For $\alpha > 0$, we will obtain an estimate $\tilde{\mathcal{X}}_C \subset \mathcal{S}$ on the domain of attraction, where

$$\tilde{\mathcal{X}}_C = \{(0, -z_0) \in (H^2(0,1) \cap H^1_0(0,1))^2 : \|z_0\|_{L^2(0,1)} < \alpha^{-1}C\},$$

and C is a scalar that will be maximized in the sequel.

Theorem 5.1. *Consider the system (3.1), (3.2) under the observer-based constrained controller (5.2) governed by (4.1), (4.3). For $\alpha > 0$, given positive scalars K, L, Δ_u, Δ_y, δ, $\bar{u}$ and tuning parameters $0 < \beta < 1$, $C > 0$, assume that there exist positive scalars $\mu_i (i = 1, 2)$ and nonnegative scalars $\mu_i (i = 3, 4)$ such that (4.7)–(4.10) and*

$$\bar{u} \geq K(\Delta_u)^{\frac{1}{2}}\alpha^{-1}C \tag{5.3}$$

hold. Then for any initial condition z_0 from the set

$$\mathcal{X}_C = \{z_0 \in H^2(0,1) \cap H^1_0(0,1) : \|z_0\|_{L^2(0,1)} < \alpha^{-1}C\}, \tag{5.4}$$

a unique solution of the closed-loop system exists. Moreover, the closed-loop system initialized with $z_0 \in \mathcal{X}_C$ is exponentially stable:

$$\|z(\cdot,t)\|_{L^2(0,1)} \leq \sqrt{2}e^{-\delta t}\|z_0\|_{L^2(0,1)}, \ \ \forall t \geq 0. \tag{5.5}$$

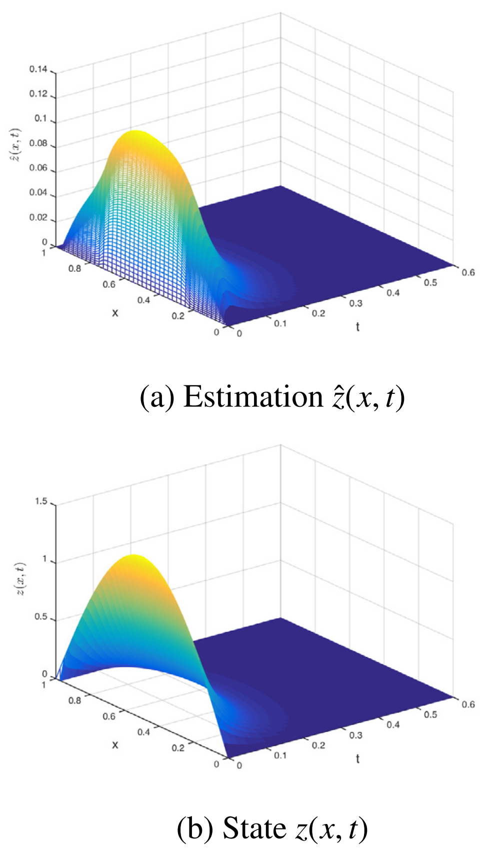
Proof. From (4.4), the Cauchy–Schwarz inequality yields

$$\begin{aligned} |u_j(t)| \ &= K\left|\int_{\Omega_{u_j}} \hat{z}(\xi,t)d\xi\right| \leq K|\Omega_{u_j}|^{\frac{1}{2}} \cdot \|\hat{z}\|_{L^2(\Omega_{u_j})} \\ &\leq K(\Delta_u)^{\frac{1}{2}}\|\hat{z}\|_{L^2(\Omega_{u_j})} \leq K(\Delta_u)^{\frac{1}{2}}\|(\hat{z}, e)\|_{\mathcal{H}}. \end{aligned} \tag{5.6}$$

Given $\bar{u} > 0$, we define the following set:

$$\mathcal{L}(K, \bar{u}) = \{(\hat{z}, e) \in \mathcal{H} : K(\Delta_u)^{\frac{1}{2}}\|(\hat{z}, e)\|_{\mathcal{H}} \leq \bar{u}\}. \tag{5.7}$$

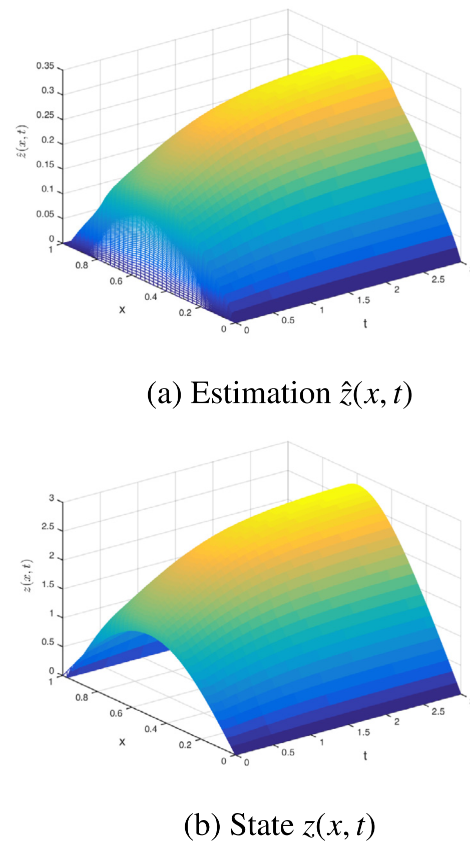
Then from the inequality (5.6) and the definition above, we can obtain the following implication: if $(\hat{z}, e) \in \mathcal{L}(K, \bar{u})$, then $|u_j(t)| \leq \bar{u}$, $(j = 0, \ldots, N-1)$, and the saturation is avoided. Thus, the



(a) Estimation $\hat{z}(x,t)$

(b) State $z(x,t)$

Fig. 1. Closed-loop system (with observer-based constrained controller) with $\lambda = 10.7$ under point actuation and measurements.



(a) Estimation $\hat{z}(x,t)$

(b) State $z(x,t)$

Fig. 2. Closed-loop system (with observer-based constrained controller) with $\lambda = 26$ under point actuation and measurements.

closed-loop system (4.1), (4.3) subject to (5.2) admits the linear representation (4.5).

From Theorem 4.1, we find that if there exists $\delta > 0$ such that the strict LMIs (4.9), (4.10) are feasible and (4.7), (4.8) hold, then (4.28) is satisfied. Hence, the trajectories $(\hat{z}(x,t;0), e(x,t;-z_0))$ starting from initial function $(0,-z_0) \in \tilde{\mathcal{X}}_C$ (i.e. $z_0 \in \mathcal{X}_C$) remain within

$$\mathcal{X} = \{(\hat{z}, e) \in \mathcal{H} : \|(\hat{z}, e)\|_{\mathcal{H}} < \alpha^{-1}C\}.$$

The "ball" $\mathcal{X}$ is contained in $\mathcal{L}(K, \bar{u})$, if the following implication holds

$$\|(\hat{z}, e)\|_{\mathcal{H}} < \alpha^{-1}C \Longrightarrow K(\Delta_u)^{\frac{1}{2}}\|(\hat{z}, e)\|_{\mathcal{H}} < \bar{u}$$

for all $(\hat{z}, e) \in \mathcal{H}$, i.e. if

$$K(\Delta_u)^{\frac{1}{2}}\|(\hat{z}, e)\|_{\mathcal{H}} \le (\alpha^{-1}C)^{-1}\bar{u}\|(\hat{z}, e)\|_{\mathcal{H}}.$$

The latter inequality is guaranteed if (5.3) is satisfied. Therefore, the inequality (5.3) guarantees the saturation avoidance, and together with Theorem 4.1 imply that

$$\lim_{t\to\infty} \|(\hat{z}(x,t;0), e(x,t;-z_0))\|_{\mathcal{H}} = 0.$$

Hence, (5.5) holds. The proof is completed. □

Consider the system (3.1), (3.2) with $\alpha = 0$. Then for the heat equation, we obtain the following result:

Corollary 5.1. *Consider the system (3.1), (3.2) with $\alpha = 0$ under the observer-based constrained controller (5.2) governed by (4.1), (4.3). Given positive scalars Δ_u, Δ_y, δ, $\bar{u}$ and tuning parameters $0 < \beta < 1$, $C_1 > 0$, assume that there exist positive scalars K, L, $\mu_i (i = 1, 2)$ and nonnegative scalars $\mu_i (i = 3, 4)$ such that (4.7)–(4.10) and $\bar{u} \ge K(\Delta_u)^{\frac{1}{2}} C_1$ hold. Then for any initial condition z_0 from the set*

$$\mathcal{X}_{C_1} = \{z_0 \in H^2(0,1) \cap H_0^1(0,1) : \|z_0\|_{L^2(0,1)} < C_1\},$$

a unique solution of the closed-loop system exists. Moreover, the closed-loop system initialized with $z_0 \in \mathcal{X}_{C_1}$ is exponentially stable.

Remark 5.1. From Remark 4.2 and Corollary 5.1, we obtain the following result directly:

Diffusion-reaction equation (3.1) with $\alpha = 0$ and the Neumann boundary conditions (4.31) under the observer-based controller (5.2) with (4.1), (4.3) is exponentially stable for any initial value $z_0 \in \mathcal{X}_{C_1}$ if the LMIs of Corollary 5.1 hold with $\mu_3 = \mu_4 = 0$.

6. Example

Consider the system (3.1), (3.2) with parameters $\gamma = \alpha = 1$, $\lambda = 10.7 > \gamma\pi^2$ under the point measurements. The open-loop system is unstable. For the observer-based constrained control law (5.2) governed by (4.1), (4.3) with $K = 21$ and $\bar{u} = 7.5$, by using Yalmip we verify LMI conditions of Theorem 5.1 with $\beta = 0.5$, $\Delta_u = 0.125$, $\Delta_y = 1/6$, $L = 15$, $\delta = 0.1$. We obtain that $\max C = 1$, and find that the closed-loop system (3.1), (3.2), (4.1), (4.3), (5.2) preserves the exponential stability for any initial values satisfying $\|z_0\|_{L^2(0,1)} < 1$.

Next a finite difference method is applied to compute the state of the closed-loop system (3.1), (3.2) under the observer-based constrained controller (5.2) governed by (4.1), (4.3). We choose the same values of parameters and the initial condition $z_0(x) = 1.4\sin(\pi x)$, $0 \le x \le 1$. Hence, $\|z_0\|_{L^2(0,1)} < 1$.

The steps of space and time are taken as 0.025 and 0.0003, respectively. Assume that there are 7 in-domain sensors transmitting point measurements at $x_0 = 0, x_1 = 1/6, x_2 = 1/3, x_3 = 1/2, x_4 = 2/3, x_5 = 5/6$ and $x_6 = 1$. Here $\Delta_y = 1/6$. Simulation of solutions under the controller $u_j(t) = -21\int_{\Omega_{u_j}} \hat{z}(\xi, t)d\xi$ with $\Omega_{u_j} = \left[\frac{j}{8}, \frac{j+1}{8}\right)$ $(j = 0, \dots, 7)$, and $\Delta_u = 0.125$, where the spatial domain is divided into eight sub-domains, shows that the closed-loop system is exponentially stable (see Fig. 1). By verifying the LMI conditions of Theorem 5.1, we obtain the maximum value of $\lambda = 10.7$ that preserves the exponential stability. By simulation of the solution to the closed-loop system starting from the same initial condition, we find that stability is preserved for essentially larger values of λ till approximately $\lambda = 25$. However, for $\lambda = 26$ the solution becomes unbounded (see Fig. 2). The simulations of the solutions confirm the theoretical results and illustrate their conservatism.

7. Conclusion

In this paper, an observer for Burgers equation by using the point measurements was constructed. This allowed to achieve regional stabilization under the point in-domain constrained controller that employs the averaged values of the observer. An estimate on the domain of attraction was found by using LMIs. One of the directions for the future research may be extension of the obtained results to the observer-based boundary control of coupled ODE-PDE system.

CRediT authorship contribution statement

Wen Kang: Investigation, Methodology, Visualization, Writing-original draft, Writing-review & editing. **Emilia Fridman:** Conceptualization, Methodology, Supervision, Writing-review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Appendix. Proof of Theorem 4.1(i)

As in [22,23], the proof is based on the Galerkin approximation method. Given $T > 0$. Suppose that $\{\phi_n\}_{n=1}^{\infty}$ is an orthonormal basis for $H^2(0,1) \cap H_0^1(0,1)$ with the norm $\|f\|_{H^2(0,1)\cap H_0^1(0,1)} = \|f''\|_{L^2(0,1)} + \|f'\|_{L^2(0,1)}$. For any $N \in \mathbb{Z}^+$, define a finite-dimensional subspace of $H_0^1(0,1)$ by $V_N = \mathrm{span}\{\phi_1, \phi_2, \dots, \phi_N\}$. A Galerkin approximation solution to (4.5) is constructed as follows:

$$\left(\hat{z}^N(x,t), e^N(x,t)\right) = \left(\sum_{n=1}^{N} a_n^N(t)\phi_n(x), \sum_{n=1}^{N} g_n^N(t)\phi_n(x)\right)$$

which satisfies

$$\begin{cases} \langle \hat{z}_t^N(\cdot,t), \phi\rangle + \alpha\langle \hat{z}^N(\cdot,t)\hat{z}_x^N(\cdot,t), \phi\rangle + \gamma\langle \hat{z}_x^N(\cdot,t), \phi'\rangle \\ -\lambda\langle \hat{z}^N(\cdot,t), \phi\rangle = -L\sum_{k=0}^{M-1} e^N(x_k,t)\int_{\Gamma_k}\phi(\xi)d\xi \\ -K\sum_{j=0}^{N-1}\phi(\bar{x}_j)\int_{\Omega_{u_j}}\hat{z}^N(\xi,t)d\xi, \\ \langle e_t^N(\cdot,t), \varphi\rangle + \alpha\langle e^N(\cdot,t)\hat{z}_x^N(\cdot,t), \varphi\rangle + \alpha\langle \hat{z}^N(\cdot,t)e_x^N(\cdot,t), \varphi\rangle \\ -\alpha\langle e^N(\cdot,t)e_x^N(\cdot,t), \varphi\rangle + \gamma\langle e_x^N(\cdot,t), \varphi'\rangle - \lambda\langle e^N(\cdot,t), \varphi\rangle \\ = -L\sum_{k=0}^{M-1} e^N(x_k,t)\int_{\Gamma_k}\varphi(\xi)d\xi, \qquad \forall\, \phi, \varphi \in V_N, \\ a_n^N(0) = 0,\ g_n^N(0) = -\langle z_0, \phi_n\rangle,\ n = 1,\dots,N. \end{cases} \tag{A.1}$$

Set $X_1(t) = (a_1^N(t), \dots, a_N^N(t))^T, X_2(t) = (g_1^N(t), \dots, g_N^N(t))^T$ and $X(t) = (X_1(t), X_2(t))^T$. From (A.1), substituting $\phi = \varphi = \phi_n$, $n = 1, \dots, N$ we obtain that $X(t)$ satisfies a nonlinear ODE system:

$$\dot{X}(t) = (A+B)X(t) + F(X_1(t), X_2(t)), \tag{A.2}$$

with

$$A = \left[\begin{array}{c:c} \Phi & 0 \\ \hdashline 0 & \Phi \end{array}\right],\quad F = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix},$$

$$B = \left[\begin{array}{c:c} -K\sum_{j=0}^{N-1}\int_{\Omega_{u_j}}\phi_1(\xi)d\xi\cdot c_2 & -L\sum_{k=0}^{M-1}\int_{\Gamma_k}\phi_1(\xi)d\xi\cdot c_1 \\ \vdots & \vdots \\ -K\sum_{j=0}^{N-1}\int_{\Omega_{u_j}}\phi_N(\xi)d\xi\cdot c_2 & -L\sum_{k=0}^{M-1}\int_{\Gamma_k}\phi_N(\xi)d\xi\cdot c_1 \\ \hdashline & -L\sum_{k=0}^{M-1}\int_{\Gamma_k}\phi_1(\xi)d\xi\cdot c_1 \\ 0 & \vdots \\ & -L\sum_{k=0}^{M-1}\int_{\Gamma_k}\phi_N(\xi)d\xi\cdot c_1 \end{array}\right],$$

where

$$\begin{aligned} \Phi &= -\gamma(\{\langle\phi_i', \phi_j'\rangle\}_{i,j=1}^N)^T + \lambda I_N, \\ F_1 &= -\alpha\left(\langle X_1^T(\{\phi_i\phi_j'\}_{i,j=1}^N)X_1, \phi_k\rangle_{k=1}^N\right)^T, \\ F_2 &= -\alpha\left(\langle X_1^T(\{\phi_i\phi_j'\}_{i,j=1}^N)X_2, \phi_k\rangle_{k=1}^N\right)^T \\ &\quad -\alpha\left(\langle X_2^T(\{\phi_i\phi_j'\}_{i,j=1}^N)X_1, \phi_k\rangle_{k=1}^N\right)^T \\ &\quad +\alpha\left(\langle X_2^T(\{\phi_i\phi_j'\}_{i,j=1}^N)X_2, \phi_k\rangle_{k=1}^N\right)^T, \\ c_1 &= (\phi_1(x_k), \phi_2(x_k), \dots, \phi_N(x_k)), \\ c_2 &= (\phi_1(\bar{x}_j), \phi_2(\bar{x}_j), \dots, \phi_N(\bar{x}_j)). \end{aligned}$$

Hence, there exist functions $a_1^N(t), \dots, a_N^N(t), g_1^N(t), \dots, g_N^N(t)$ on some interval $[0, T_N)$. We will show next that these functions can be extended for all $t \ge 0$.

Lemma A.1. *For any $t \ge 0$, the following inequality holds:*

$$\sup_{t\ge 0}\sup_{N}\Big[\|\hat{z}^N(\cdot,t)\|_{L^2}^2 + \|e^N(\cdot,t)\|_{L^2}^2 + \int_0^t[\|\hat{z}_x^N(\cdot,s)\|_{L^2}^2 + \|e_x^N(\cdot,s)\|_{L^2}^2]ds\Big] < \infty. \tag{A.3}$$

Proof. For $t \geq 0$, substitute $(\phi, \varphi) = (\hat{z}^N(x,t), e^N(x,t))$ into (A.1). Then from (4.25), the feasibility of LMIs (4.7)–(4.10) lead to

$$\begin{aligned}
&\frac{d}{dt}\|(\hat{z}^N(\cdot,t), e^N(\cdot,t))\|^2_{\mathcal{H}} + 2\delta\|(\hat{z}^N(\cdot,t), e^N(\cdot,t))\|^2_{\mathcal{H}}\\
&\leq -\left(\gamma - \mu_1 - \mu_3 - \frac{\alpha}{2}\|e^N(\cdot,t)\|_{L^2}\right)\|\hat{z}^N_x(\cdot,t)\|^2_{L^2}\\
&-\left(\gamma - \mu_2 - \mu_4 - \frac{\alpha}{2}\|e^N(\cdot,t)\|_{L^2}\right)\|e^N_x(\cdot,t)\|^2_{L^2}\\
&\leq 0,
\end{aligned}$$

which implies (A.3). □

In order to pass to the limits as $N \to \infty$, we need N-independent a-priori-estimates via several lemmas.

Lemma A.2. *For any $z_0 \in H^2(0,1) \cap H^1_0(0,1)$, the following inequality holds:*

$$\sup_N \left[\|\hat{z}^N_t(\cdot,0)\|_{L^2} + \|e^N_t(\cdot,0)\|_{L^2}\right] < \infty. \tag{A.4}$$

Proof. Set $t = 0$ in (A.1). Then

$$\begin{cases}
\langle \hat{z}^N_t(\cdot,0), \phi\rangle + \alpha\langle \hat{z}^N(\cdot,0)\hat{z}^N_x(\cdot,0), \phi\rangle + \gamma\langle \hat{z}^N_x(\cdot,0), \phi_x\rangle\\
-\lambda\langle \hat{z}^N(\cdot,0), \phi\rangle = -L\sum_{k=0}^{M-1} e^N(x_k,0)\int_{\Gamma_k}\phi(\xi)d\xi\\
-K\sum_{j=0}^{N-1}\phi(\bar{x}_j)\int_{\Omega_{u_j}}\hat{z}^N(\xi,0)d\xi,\\
\langle e^N_t(\cdot,0), \varphi\rangle + \alpha\langle e^N(\cdot,0)\hat{z}^N_x(\cdot,0), \varphi\rangle + \alpha\langle \hat{z}^N(\cdot,0)e^N_x(\cdot,0), \varphi\rangle\\
-\alpha\langle e^N(\cdot,0)e^N_x(\cdot,0), \varphi\rangle + \gamma\langle e^N_x(\cdot,0), \varphi_x\rangle - \lambda\langle e^N(\cdot,0), \varphi\rangle\\
= -L\sum_{k=0}^{M-1} e^N(x_k,0)\int_{\Gamma_k}\varphi(\xi)d\xi.
\end{cases} \tag{A.5}$$

Due to $\hat{z}^N(\cdot,0) = \sum_{n=1}^N a^N_n(t)\phi_n(0) = 0$, substituting $\phi = \hat{z}^N_t(x,0)$ and $\varphi = e^N_t(x,0)$ into (A.5), we obtain

$$\begin{aligned}
\|\hat{z}^N_t(\cdot,0)\|^2_{L^2} &= -L\sum_{k=0}^{M-1}\langle e^N(x_k,0), b_k(\cdot)\hat{z}^N_t(\cdot,0)\rangle,\\
\|e^N_t(\cdot,0)\|^2_{L^2} &= \langle \gamma e^N_{xx}(\cdot,0) + \lambda e^N(\cdot,0) - \alpha e^N(\cdot,0)\hat{z}^N_x(\cdot,0)\\
&\quad -\alpha\hat{z}^N(\cdot,0)e^N_x(\cdot,0) + \alpha e^N(\cdot,0)e^N_x(\cdot,0), e^N_t(\cdot,0)\rangle\\
&\quad -L\sum_{k=0}^{M-1}\langle e^N(x_k,0), b_k(\cdot)e^N_t(\cdot,0)\rangle.
\end{aligned} \tag{A.6}$$

Due to $e^N(\cdot,0) = \sum_{n=1}^N\langle -z_0(\cdot), \phi_n\rangle\phi_n$, from (A.6), the Minkowski and Cauchy-Schwarz inequalities lead to

$$\begin{aligned}
\|e^N_t(\cdot,0)\|_{L^2} &\leq \gamma\|e^N_{xx}(\cdot,0)\|_{L^2} + \lambda\|e^N(\cdot,0)\|_{L^2}\\
&+\alpha\|e^N(\cdot,0)\hat{z}^N_x(\cdot,0)\|_{L^2} + \alpha\|\hat{z}^N(\cdot,0)e^N_x(\cdot,0)\|_{L^2}\\
&+\alpha\|e^N(\cdot,0)e^N_x(\cdot,0)\|_{L^2} + L\max_{0\leq x\leq 1}|e^N(\cdot,0)|\\
&\leq K_1\|e^N(\cdot,0)\|_{H^2} + \alpha\|e^N(\cdot,0)\|_{L^\infty}\|e^N_x(\cdot,0)\|_{L^2}\\
&+L\|e^N(\cdot,0)\|_{L^\infty}
\end{aligned} \tag{A.7}$$

for some positive constant $K_1 > 0$.

By using Sobolev inequality, we obtain

$$\|e^N(\cdot,0)\|_{L^\infty} \leq \|e^N(\cdot,0)\|_{H^1}. \tag{A.8}$$

From (A.7) and (A.8) it follows that $\sup_N\|e^N_t(\cdot,0)\|_{L^2} < \infty$. Together with (A.6), it implies (A.4). □

Lemma A.3. *For any $z_0 \in H^2(0,1) \cap H^1_0(0,1)$ and $t \geq 0$, the following inequality holds:*

$$\begin{aligned}
&\sup_N \Big[\|\hat{z}^N_t(\cdot,t)\|^2_{L^2} + \|e^N_t(\cdot,t)\|^2_{L^2}\\
&+\int_0^t\left(\|\hat{z}^N_{xt}(\cdot,s)\|^2_{L^2} + \|e^N_{xt}(\cdot,s)\|^2_{L^2}\right)ds\Big] < \infty.
\end{aligned} \tag{A.9}$$

Proof. For $t \geq 0$, differentiating the first and second equation of (A.1) with respect to t, we have

$$\begin{cases}
\langle \hat{z}^N_{tt}(\cdot,t), \phi\rangle + \alpha\langle \hat{z}^N_t(\cdot,t)\hat{z}^N_x(\cdot,t), \phi\rangle + \alpha\langle \hat{z}^N(\cdot,t)\hat{z}^N_{xt}(\cdot,t), \phi\rangle\\
+\gamma\langle \hat{z}^N_{xt}(\cdot,t), \phi_x\rangle - \lambda\langle \hat{z}^N_t(\cdot,t), \phi\rangle\\
= -L\sum_{k=0}^{M-1} e^N_t(x_k,t)\int_{\Gamma_k}\phi(\xi)d\xi\\
-K\sum_{j=0}^{N-1}\phi(\bar{x}_j)\int_{\Omega_{u_j}}\hat{z}^N_t(\xi,t)d\xi,\\
\langle e^N_{tt}(\cdot,t), \varphi\rangle + \alpha\langle e^N_t(\cdot,t)\hat{z}^N_x(\cdot,t), \varphi\rangle + \alpha\langle e^N(\cdot,t)\hat{z}^N_{xt}(\cdot,t), \varphi\rangle\\
+\alpha\langle \hat{z}_t(\cdot,t)e^N_x(\cdot,t), \varphi\rangle + \alpha\langle \hat{z}^N(\cdot,t)e^N_{xt}(\cdot,t), \varphi\rangle\\
-\alpha\langle e^N_t(\cdot,t)e^N_x(\cdot,t), \varphi\rangle - \alpha\langle e^N(\cdot,t)e^N_{xt}(\cdot,t), \varphi\rangle\\
+\gamma\langle e^N_{xt}(\cdot,t), \varphi_x\rangle - \lambda\langle e^N_t(\cdot,t), \varphi\rangle\\
= -L\sum_{k=0}^{M-1} e^N_t(x_k,t)\int_{\Gamma_k}\varphi(\xi)d\xi.
\end{cases} \tag{A.10}$$

Substituting $(\phi, \varphi) = (\hat{z}^N_t(\cdot,t), e^N_t(\cdot,t))$ into (A.10) we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left[\|\hat{z}^N_t(\cdot,t)\|^2_{L^2} + \|e^N_t(\cdot,t)\|^2_{L^2}\right]\\
&\leq -\gamma\left[\|\hat{z}^N_{xt}(\cdot,t)\|^2_{L^2} + \|e^N_{xt}(\cdot,t)\|^2_{L^2}\right]\\
&+\lambda\left[\|\hat{z}^N_t(\cdot,t)\|^2_{L^2} + \|e^N_t(\cdot,t)\|^2_{L^2}\right]\\
&+\alpha\left[\langle \hat{z}^N(\cdot,t)\hat{z}^N_{xt}(\cdot,t), \hat{z}^N_t(\cdot,t)\rangle - \langle e^N(\cdot,t)e^N_{xt}(\cdot,t), e^N_t(\cdot,t)\rangle\right]\\
&+\alpha\left[\langle e^N_t(\cdot,t)\hat{z}^N_x(\cdot,t), e^N_t(\cdot,t)\rangle + \langle e^N(\cdot,t)e^N_t(\cdot,t), \hat{z}^N_{xt}(\cdot,t)\rangle\right]\\
&+\alpha\left[\langle \hat{z}^N_t(\cdot,t)e^N_x(\cdot,t), e^N_t(\cdot,t)\rangle + \langle \hat{z}^N(\cdot,t)e^N_{xt}(\cdot,t), e^N_t(\cdot,t)\rangle\right]\\
&+L\|e^N_t(\cdot,t)\|_{L^\infty}\|\hat{z}^N_t(\cdot,t)\|_{L^2} + L\|e^N_t(\cdot,t)\|_{L^\infty}\|e^N_t(\cdot,t)\|_{L^2}\\
&+K\|\hat{z}^N_t(\cdot,t)\|_{L^\infty}\|\hat{z}^N_t(\cdot,t)\|_{L^2}.
\end{aligned} \tag{A.11}$$

By using Sobolev inequality, we have

$$\begin{aligned}
&\|\hat{z}^N(\cdot,t)\|_{L^\infty} \leq \|\hat{z}^N(\cdot,t)\|_{H^1},\ \|e^N(\cdot,t)\|_{L^\infty} \leq \|e^N(\cdot,t)\|_{H^1},\\
&\|\hat{z}^N_t(\cdot,t)\|_{L^\infty} \leq \|\hat{z}^N_t(\cdot,t)\|_{H^1},\ \|e^N_t(\cdot,t)\|_{L^\infty} \leq \|e^N_t(\cdot,t)\|_{H^1}.
\end{aligned}$$

Then, the Young inequality leads to

$$\begin{aligned}
&L\|e^N_t(\cdot,t)\|_{L^\infty}\|\hat{z}^N_t(\cdot,t)\|_{L^2}\\
&\leq \frac{L}{2}[\epsilon_0\|e^N_{xt}(\cdot,t)\|^2_{L^2} + \frac{1}{\epsilon_0}\|\hat{z}^N_t(\cdot,t)\|^2_{L^2}],\\
&L\|e^N_t(\cdot,t)\|_{L^\infty}\|e^N_t(\cdot,t)\|_{L^2}\\
&\leq \frac{L}{2}[\epsilon_0\|e^N_{xt}(\cdot,t)\|^2_{L^2} + \frac{1}{\epsilon_0}\|e^N_t(\cdot,t)\|^2_{L^2}],\\
&K\|\hat{z}^N_t(\cdot,t)\|_{L^\infty}\|\hat{z}^N_t(\cdot,t)\|_{L^2}\\
&\leq \frac{K}{2}[\epsilon_0\|\hat{z}^N_{xt}(\cdot,t)\|^2_{L^2} + \frac{1}{\epsilon_0}\|\hat{z}^N_t(\cdot,t)\|^2_{L^2}],\ \forall\epsilon_0 > 0.
\end{aligned}$$

From Lemma A.1, together with the latter inequality and Young's inequality we have

$$\begin{aligned} &\langle \hat{z}^N(\cdot,t)\hat{z}^N_{xt}(\cdot,t), \hat{z}^N_t(\cdot,t)\rangle - \langle e^N(\cdot,t)e^N_{xt}(\cdot,t), e^N_t(\cdot,t)\rangle \\ &+\langle e^N(\cdot,t)e^N_t(\cdot,t), \hat{z}^N_{xt}(\cdot,t)\rangle + \langle \hat{z}^N(\cdot,t)e^N_{xt}(\cdot,t), e^N_t(\cdot,t)\rangle \\ &\le K_2\left[\|\hat{z}^N_{xt}(\cdot,t)\|_{L^2}\|\hat{z}^N_t(\cdot,t)\|_{L^2} + \|e^N_{xt}(\cdot,t)\|_{L^2}\|e^N_t(\cdot,t)\|_{L^2}\right] \\ &+K_2\left[\|\hat{z}^N_{xt}(\cdot,t)\|_{L^2}\|e^N_t(\cdot,t)\|_{L^2} + \|e^N_{xt}(\cdot,t)\|_{L^2}\|e^N_t(\cdot,t)\|_{L^2}\right] \\ &\le K_2\left[2\epsilon_1\|\hat{z}^N_{xt}(\cdot,t)\|^2_{L^2} + \frac{1}{\epsilon_1}\|\hat{z}^N_t(\cdot,t)\|^2_{L^2} + 2\epsilon_1\|e^N_{xt}(\cdot,t)\|^2_{L^2}\right. \\ &\left.+\frac{3}{\epsilon_1}\|e^N_t(\cdot,t)\|^2_{L^2}\right] \ \forall \epsilon_1 > 0 \end{aligned} \tag{A.12}$$

for some constant $K_2 > 0$.

Similarly,

$$\begin{aligned} &\langle e^N_t(\cdot,t)\hat{z}^N_x(\cdot,t), e^N_t(\cdot,t)\rangle + \langle \hat{z}^N_t(\cdot,t)e^N_x(\cdot,t), e^N_t(\cdot,t)\rangle \\ &\le K_3\left[\epsilon_2\|e^N_{xt}(\cdot,t)\|^2_{L^2} + \epsilon_2\|\hat{z}^N_{xt}(\cdot,t)\|^2_{L^2} + \frac{2}{\epsilon_2}\|e^N_t(\cdot,t)\|^2_{L^2}\right] \end{aligned} \tag{A.13}$$

holds for any $\epsilon_2 > 0$ for some constant $K_3 > 0$.

Choose $\epsilon_i > 0 (i = 0, 1, 2)$ such that $\gamma > (L + \frac{K}{2})\epsilon_0 + (2K_2\epsilon_1 + K_3\epsilon_2)\alpha$. Then from (A.11)–(A.13), we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left[\|\hat{z}^N_t(\cdot,t)\|^2_{L^2} + \|e^N_t(\cdot,t)\|^2_{L^2}\right] \\ &\le -\left(\gamma - L\epsilon_0 - \frac{K\epsilon_0}{2} - 2\alpha K_2\epsilon_1 - \alpha K_3\epsilon_2\right) \\ &\times\left[\|\hat{z}^N_{xt}(\cdot,t)\|^2_{L^2} + \|e^N_{xt}(\cdot,t)\|^2_{L^2}\right] \\ &+\left(\lambda + \frac{L}{\epsilon_0} + \frac{K}{2\epsilon_0} + \frac{3K_2}{\epsilon_1}\alpha + \frac{2K_3}{\epsilon_2}\alpha\right) \\ &\times\left[\|\hat{z}^N_t(\cdot,t)\|^2_{L^2} + \|e^N_t(\cdot,t)\|^2_{L^2}\right] \end{aligned}$$

Application of the Gronwall inequality and Lemma A.2 yield (A.9).

Continuation of proof of Theorem 4.1(i):

Given $T > 0$. From Lemmas A.1–A.3, and Corollary 4.19 and Proposition 4.17 of [24] (see pp. 104, 106), we can extract a subsequence, which is still denoted by N, such that

$$\begin{cases} (\hat{z}^N, e^N) \rightharpoonup (\hat{z}, e) \ \text{in} \ L^\infty(0,T;\mathcal{H}_1) \ \text{weak star}, \\ (\hat{z}^N_t, e^N_t) \rightharpoonup (\hat{z}_t, e_t) \ \text{in} \ L^\infty(0,T;\mathcal{H}) \ \text{weak star}, \\ (\hat{z}^N_t, e^N_t) \rightharpoonup (\hat{z}_t, e_t) \ \text{in} \ L^2(0,T;\mathcal{H}_1) \ \text{weak} \end{cases} \tag{A.14}$$

Indeed, let us first recall the following results from [24]:
Corollary 4.19 of [24] (Reflexive weak compactness): Let X be a reflexive Banach space and x_n a bounded sequence in X. Then x_n has a subsequence that converges weakly in X.
Proposition 4.17 of [24]: Weak convergence implies weak−∗ convergence.

It should be noticed that in Corollary 4.19 of [24], the subsequence converges weakly in the original space. From Lemma A.1, it follows that $(\hat{z}^N, e^N)$ is bounded sequence in $\mathcal{H}_1$. Then, from Corollary 4.19 and Proposition 4.17 of [24], we have

$$(\hat{z}^N, e^N) \rightharpoonup (\hat{z}, e) \ \text{in} \ L^\infty(0,T;\mathcal{H}_1) \ \text{weak star},$$

Similarly, from Corollary 4.19 and Proposition 4.17 of [24], and Lemmas A.2–A.3, we have

$$\begin{cases} (\hat{z}^N_t, e^N_t) \rightharpoonup (\hat{z}_t, e_t) \ \text{in} \ L^\infty(0,T;\mathcal{H}) \ \text{weak star}, \\ (\hat{z}^N_t, e^N_t) \rightharpoonup (\hat{z}_t, e_t) \ \text{in} \ L^2(0,T;\mathcal{H}_1) \ \text{weak} \end{cases}$$

For any $\psi \in C_0^\infty(0,T)$ and $(\phi, \varphi) \in \mathcal{H}_1$

$$\begin{cases} \int_0^T \langle \hat{z}^N_t(\cdot,t), \phi\rangle\psi(t)dt + \alpha\int_0^T \langle \hat{z}^N(\cdot,t)\hat{z}^N_x(\cdot,t), \phi\rangle\psi(t)dt \\ +\gamma\int_0^T \langle \hat{z}^N_x(\cdot,t), \phi_x\rangle\psi(t)dt - \lambda\int_0^T \langle \hat{z}^N(\cdot,t), \phi\rangle\psi(t)dt \\ +L\sum_{k=0}^{M-1}\int_0^T e^N(x_k,t)\int_{\Gamma_k}\phi(\xi)d\xi\,\psi(t)dt \\ +K\sum_{j=0}^{N-1}\int_0^T \phi(\bar{x}_j)\int_{\Omega_{u_j}}\hat{z}^N(\xi,t)d\xi\,\psi(t)dt = 0, \\ \int_0^T \langle e^N_t(\cdot,t), \varphi\rangle\psi(t)dt + \alpha\int_0^T \langle e^N(\cdot,t)\hat{z}^N_x(\cdot,t), \varphi\rangle\psi(t)dt \\ +\alpha\int_0^T \langle \hat{z}^N(\cdot,t)e^N_x(\cdot,t), \varphi\rangle\psi(t)dt \\ -\alpha\int_0^T \langle e^N(\cdot,t)e^N_x(\cdot,t), \varphi\rangle\psi(t)dt \\ +\gamma\int_0^T \langle e^N_x(\cdot,t), \varphi_x\rangle\psi(t)dt - \lambda\int_0^T \langle e^N(\cdot,t), \varphi\rangle\psi(t)dt \\ +L\sum_{k=0}^{M-1}\int_0^T e^N(x_k,t)\int_{\Gamma_k}\varphi(\xi)d\xi\,\psi(t)dt = 0. \end{cases} \tag{A.15}$$

From (A.14), we have

$$(\hat{z}^N, e^N) \rightharpoonup (\hat{z}, e) \ \text{in} \ L^\infty(0,T;\mathcal{H}_1) \ \text{weak star},$$

Thus, the convergence of $\hat{z}^N_x$ and e^N_x can be obtained in L^2-norm in the sense that

$$(\hat{z}^N_x, e^N_x) \rightharpoonup (\hat{z}_x, e_x) \ \text{in} \ L^\infty(0,T;\mathcal{H}) \ \text{weak star},$$

Note that

$$\begin{aligned} \|\hat{z}^N\hat{z}^N_x - \hat{z}\hat{z}_x\| &= \|\hat{z}^N\hat{z}^N_x - \hat{z}\hat{z}^N_x + \hat{z}\hat{z}^N_x - \hat{z}\hat{z}_x\| \\ &\le \|\hat{z}^N\hat{z}^N_x - \hat{z}\hat{z}^N_x\| + \|\hat{z}\hat{z}^N_x - \hat{z}\hat{z}_x\| \\ &\le \|\hat{z}^N - \hat{z}\|\cdot\|\hat{z}^N_x\| + \|\hat{z}^N_x - \hat{z}_x\|\cdot\|\hat{z}\|. \end{aligned}$$

Passing to the limits as $N \to \infty$ in the above inequality, we obtain

$$\hat{z}^N\hat{z}^N_x \rightharpoonup \hat{z}\hat{z}_x,$$

Similarly, we have

$$e^N e^N_x \rightharpoonup ee_x,$$

Thus, passing to the limits as $N \to \infty$ in (A.15), it follows that

$$\begin{cases} \langle \hat{z}_t, \phi\rangle + \alpha\langle \hat{z}\hat{z}_x, \phi\rangle + \gamma\langle \hat{z}_x, \phi_x\rangle - \lambda\langle \hat{z}, \phi\rangle \\ = -L\sum_{k=0}^{M-1} e(x_k,t)\int_{\Gamma_k}\phi(\xi)d\xi - K\sum_{j=0}^{N-1}\phi(\bar{x}_j)\int_{\Omega_{u_j}}\hat{z}(\xi,t)d\xi, \\ \langle e_t, \varphi\rangle + \alpha\langle e\hat{z}_x + \hat{z}e_x - ee_x, \varphi\rangle + \gamma\langle e_x, \varphi_x\rangle - \lambda\langle e, \varphi\rangle \\ = -L\sum_{k=0}^{M-1} e(x_k,t)\int_{\Gamma_k}\varphi(\xi)d\xi, \ t \in [0,T] \text{ a.e.} \end{cases} \tag{A.16}$$

for any $(\phi, \varphi) \in \mathcal{H}_1$.

Hence,

$$\begin{cases} \langle \hat{z}_t, \phi\rangle + \alpha\langle \hat{z}\hat{z}_x, \phi\rangle + \gamma\langle \hat{z}_x, \phi_x\rangle - \lambda\langle \hat{z}, \phi\rangle \\ = -L\sum_{k=0}^{M-1} e(x_k,t)\langle b_k, \phi\rangle \\ -\sum_{j=0}^{N-1} K\langle \delta(x-\bar{x}_j), \phi\rangle\int_{\Omega_{u_j}}\hat{z}(\xi,t)d\xi, \\ \langle e_t, \varphi\rangle + \alpha\langle e\hat{z}_x + \hat{z}e_x - ee_x, \varphi\rangle + \gamma\langle e_x, \varphi_x\rangle - \lambda\langle e, \varphi\rangle \\ = -L\sum_{k=0}^{M-1} e(x_k,t)\langle b_k, \varphi\rangle, \ t \in [0,T] \text{ a.e.} \end{cases} \tag{A.17}$$

Taking $\phi, \varphi \in C_0^\infty(0,1)$ in (A.17), we obtain the generalized derivatives $\hat{z}_{xx}$ and e_{xx} exist, and

$$\begin{cases} \hat{z}_t(x,t) = \gamma \hat{z}_{xx}(x,t) - \alpha \hat{z}(x,t)\hat{z}_x(x,t) + \lambda \hat{z}(x,t) \\ -L \sum_{k=0}^{M-1} b_k(x) e(x_k,t) - \sum_{j=0}^{N-1} \delta(x-\bar{x}_j) K \int_{\Omega_{u_j}} \hat{z}(\xi,t) d\xi, \\ e_t(x,t) = \gamma e_{xx}(x,t) - \alpha e(x,t)\hat{z}_x(x,t) - \alpha \hat{z}(x,t) e_x(x,t) \\ +\alpha e(x,t) e_x(x,t) + \lambda e(x,t) - L \sum_{k=0}^{M-1} b_k(x) e(x_k,t), \end{cases}$$

for almost all $t \in [0,T]$. Therefore, for any $T > 0$, there exists a solution to the system (4.5) for all $t \in [0,T]$ in the sense that (4.12) holds. □

References

[1] M. Wakaiki, Y. Yamamoto, Stability analysis of perturbed infinite-dimensional sampled-data systems, Systems Control Lett. 138 (2020) 104652.

[2] N.H. El-Farra, A. Armaou, P.D. Christofides, Analysis and control of parabolic PDE systems with input constraints, Automatica 39 (2003) 715–725.

[3] S. Marx, E. Cerpa, V. Prieur, Stabilization of a linear Korteweg–de Vries equation with a saturated internal control, in: Proc. European Control Conference, Vol. 41, Linz, Austria, 2015, pp. 867–872.

[4] C. Prieur, S. Tarbouriech, J.M.G. da Silva, Well-posedness and stability of 1D wave equation with saturating distributed input, in: Proc. IEEE Conf. on Decision and Control, California, USA, 2014, pp. 2846–2851.

[5] M. Slemrod, Feedback stabilization of a linear control system in Hilbert space with an a priori bounded control, Math. Control Signals Systems 2 (1989) 265–285.

[6] W. Kang, E. Fridman, Boundary control of delayed ODE-Heat cascade under actuator saturation, Automatica 83 (2017) 252–261.

[7] W. Kang, E. Fridman, Boundary constrained control of delayed nonlinear Schrödinger equation, IEEE Trans. Automat. Control 63 (2018) 3873–3880.

[8] A. Balogh, M. Krstic, Burgers' equation with nonlinear boundary feedback: H^1 stability, well-posedness and simulation, Math. Probl. Eng. 6 (2000) 189–200.

[9] A. Azouani, E.S. Titi, Feedback control of nonlinear dissipative systems by finite determining parameters - A reaction–diffusion Paradigm, Evol. Equ. Control Theory 3 (2014) 579–594.

[10] A. Pisano, Y. Orlov, On the ISS properties of a class of parabolic DPS' with discontinuous control using sampled-in-space sensing and actuation, Automatica 81 (2017) 447–454.

[11] T. Nagatani, H. Emmerich, K. Nakanishi, Burgers equation for kinetic clustering in traffic flow, Physica A 255 (1998) 158–162.

[12] C.I. Byrnes, D.S. Gilliam, V.I. Shubov, On the global dynamics of a controlled viscous Burgers' equation, J. Dynam. Control Syst. 4 (1998) 457–519.

[13] M. Krstic, On global stabilization of Burgers' equation by boundary control, Systems Control Lett. 37 (1999) 123–141.

[14] M. Krstic, L. Magnis, R. Vazquez, Nonlinear stabilization of shock-like unstable equilibria in the viscous Burgers PDE, IEEE Trans. Automat. Control 53 (2008) 1678–1683.

[15] W.-J. Liu, M. Krstic, Backstepping boundary control of Burgers' equation with actuator dynamics, Systems Control Lett. (2000) 291–303.

[16] H.V. Ly, K.D. Mease, E.S. Titi, Distributed and boundary control of the viscous Burgers' equation, Numer. Funct. Anal. Optim. 18 (1997) 143–188.

[17] W. Kang, E. Fridman, Stabilization of Burgers' equation by constrained control, in: Proc. IFAC World Congress, Germany, 2020.

[18] A. Selivanov, E. Fridman, Delayed point control of a reaction–diffusion PDE under discrete-time point measurements, Automatica 96 (2018) 224–233.

[19] G.H. Hardy, J.E. Littlewood, G. Pólya, Inequalities, Mathematical Library, Cambridge, 1988.

[20] E. Fridman, A. Blighovsky, Robust sampled-data control of a class of semilinear parabolic systems, Automatica 48 (2012) 826–836.

[21] A. Selivanov, E. Fridman, Sampled-data relay control of diffusion PDEs, Automatica 82 (2017) 59–68.

[22] F.F. Jin, B.Z. Guo, Lyapunov approach to output feedback stabilization for the Euler–Bernoulli beam equation with boundary input disturbance, Automatica 52 (2015) 95–102.

[23] N.A. Larkin, Korteweg–de Vries and Kuramoto–Sivashinsky equations in bounded domains, J. Math. Anal. Appl. 297 (2004) 169–185.

[24] J.C. Robinson, Infinite-Dimensional Dynamical Systems: An Introduction to Dissipative Parabolic PDEs and the Theory of Global Attractors, Cambridge University Press, 2001.