DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS SERIES S
Volume **2**, Number **3**, September **2009**

doi:10.3934/dcdss.2009.2.559

pp. **559–581**

STABILITY OF THE HEAT AND OF THE WAVE EQUATIONS WITH BOUNDARY TIME-VARYING DELAYS

Serge Nicaise and Julie Valein

Université de Valenciennes et du Hainaut Cambrésis
LAMAV, FR CNRS 2956
Institut des Sciences et Techniques of Valenciennes
F-59313 - Valenciennes Cedex 9, France

Emilia Fridman

School of Electrical Engineering
Tel Aviv University
Tel Aviv, 69978, Israel

Abstract. Exponential stability analysis via Lyapunov method is extended to the one-dimensional heat and wave equations with time-varying delay in the boundary conditions. The delay function is admitted to be time-varying with an *a priori* given upper bound on its derivative, which is less than 1. Sufficient and explicit conditions are derived that guarantee the exponential stability. Moreover the decay rate can be explicitly computed if the data are given.

1. Introduction.

Time-delay often appears in many biological, electrical engineering systems and mechanical applications, and in many cases, delay is a source of instability [5]. In the case of distributed parameter systems, even arbitrarily small delays in the feedback may destabilize the system (see e.g. [3, 9, 15, 10]). The stability issue of systems with delay is, therefore, of theoretical and practical importance.

There are only a few works on Lyapunov-based technique for Partial Differential Equations (PDEs) with delay. Most of these works analyze the case of *constant delays*. Thus, stability conditions and exponential bounds were derived for some scalar heat and wave equations with constant delays and with Dirichlet boundary conditions without delay in [16, 17]. Stability and instability conditions for the wave equations with constant delay can be found in [10, 12]. The stability of linear parabolic systems with constant coefficients and internal constant delays has been studied in [6] in the frequency domain.

Recently the stability of PDEs with *time-varying delays* was analyzed in [2, 4, 13] via Lyapunov method. In the case of linear systems in the Hilbert space, the conditions of [2, 4, 13] assume that the operator acting on the delayed state is bounded, which means that this condition can not be applied to boundary delays. These conditions were applied to PDEs without delays in the boundary conditions (to $2D$ Navier-Stokes and to a scalar heat equations in [2], to a scalar heat and to a scalar wave equations in [4, 13]).

2000 *Mathematics Subject Classification.* Primary: 93D15; Secondary: 35L05.
Key words and phrases. Heat equation, wave equation, time-varying delay, stability, Lyapunov functional.

In the present paper we analyze exponential stability of the heat and wave equations with time-varying boundary delay. Our main novel contribution is an extension of previous results from [10, 12] to time-varying delays. This extension is not straightforward due to the loss of translation-invariance. In the constant delay case the exponential stability was proved in [10, 12] by using the observability inequality which can not be applicable in the time-varying case (since the system is not invariant by translation). Hence we introduce new Lyapunov functionals with exponential terms and an additional term for the wave equation, which take into account the dependence of the delay with respect to time. For the treatment of other problems with Lyapunov technique see [4, 13, 11]. Note further that to the best of our knowledge the heat equation with boundary delay has not been treated in the literature. Contrary to [10, 12], the existence results do not follow from standard semi-group theory because the spatial operator depends on time due to the time-varying delay. Therefore we use the variable norm technique of Kato [7, 8]. Finally for each problem we give explicit sufficient conditions that guarantee the exponential decay and for the first time we characterize the optimal decay rate that can be explicitly computed once the data are given.

The paper is mainly decomposed in two parts treating the heat equation (section 2) and the wave equation (section 3). In the first subsection, we set the problem under consideration and prove existence results by using semigroup theory. In the second subsection we find sufficient conditions for the strict decay of the energy and finally in the last subsection we show that these conditions yield an exponential decay.

2. **Exponential stability of the delayed heat equation.**

First, we consider the system described by

$$\left\{\begin{array}{cc} u_t(x,\,t) - a u_{xx}(x,\,t) = 0, & 0 < x < \pi,\, t > 0, \\ u(0,\,t) = 0, & t > 0, \\ u_x(\pi,\,t) = -\mu_0 u(\pi,\,t) - \mu_1 u(\pi,\,t-\tau(t)), & t > 0, \\ u(x,\,0) = u^0(x), & 0 < x < \pi, \\ u(\pi,\,t-\tau(0)) = f^0(t-\tau(0)), & 0 < t < \tau(0), \end{array}\right. \tag{1}$$

with the constant parameter $a > 0$ and where $\mu_0,\,\mu_1 \geq 0$ are fixed nonnegative real numbers, the time-varying delay $\tau(t)$ satisfies

$$\dot{\tau}(t) < 1,\, \forall t > 0, \tag{2}$$

and

$$\exists M > 0 :\, 0 < \tau_0 \leq \tau(t) \leq M,\, \forall t > 0. \tag{3}$$

Moreover, we assume that

$$\tau \in W^{2,\infty}([0,\,T]),\, \forall T > 0. \tag{4}$$

The boundary-value problem (1) describes the propagation of heat in a homogeneous one-dimensional rod with a fixed temperature at the left end. Here a stands for the heat conduction coefficient, $u(x,t)$ is the value of the temperature field of the plant at time moment t and location x along the rod. In the sequel, the state dependence on time t and spatial variable x is suppressed whenever possible.

2.1. **Well-posedness of the problem.** We aim to show that problem (1) is well-posed. For that purpose, we use semi-group theory and adapt the ideas from [10].

We introduce the Hilbert space

$$V = \{\phi \in H^1(0,\,\pi)\,:\,\phi(0) = 0\}.$$

We transform our system (1) as follows. Let us introduce the auxiliary variable $z(\rho, t) = u(\pi,\, t - \tau(t)\rho)$ for $\rho \in (0,\,1)$ and $t > 0$. Note that z verifies the transport equation for $0 < \rho < 1$ and $t > 0$

$$\begin{cases} \tau(t) z_t(\rho,\, t) + (1 - \dot{\tau}(t)\rho) z_\rho(\rho,\, t) = 0, \\ z(0,\, t) = u(\pi,\, t), \\ z(\rho,\, 0) = f^0(-\tau(0)\rho). \end{cases} \tag{5}$$

Therefore, the problem (1) is equivalent to

$$\begin{cases} u_t(x,\, t) - a u_{xx}(x,\, t) = 0, & 0 < x < \pi,\, t > 0, \\ \tau(t) z_t(\rho,\, t) + (1 - \dot{\tau}(t)\rho) z_\rho(\rho,\, t) = 0, & 0 < \rho < 1,\, t > 0, \\ u(0,\, t) = 0,\, u_x(\pi,\, t) = -\mu_0 u(\pi,\, t) - \mu_1 z(1,\, t), & t > 0, \\ z(0,\, t) = u(\pi,\, t), & t > 0, \\ u(x,\, 0) = u^0(x), & 0 < x < \pi, \\ z(\rho,\, 0) = f^0(-\tau(0)\rho), & 0 < \rho < 1. \end{cases} \tag{6}$$

If we introduce

$$U := (u,\, z)^\top,$$

then U satisfies

$$U_t = (u_t,\, z_t)^\top = (a u_{xx},\, \frac{\dot{\tau}(t)\rho - 1}{\tau(t)} z_\rho)^\top.$$

Consequently the problem (1) may be rewritten as the first order evolution equation

$$\begin{cases} U_t = \mathcal{A}(t) U \\ U(0) = (u^0,\, f^0(-\tau(0)))^\top = U_0, \end{cases} \tag{7}$$

where the time dependent operator $\mathcal{A}(t)$ is defined by

$$\mathcal{A}(t) \begin{pmatrix} u \\ z \end{pmatrix} = \begin{pmatrix} a u_{xx} \\ \frac{\dot{\tau}(t)\rho - 1}{\tau(t)} z_\rho \end{pmatrix},$$

with domain

$$\mathcal{D}(\mathcal{A}(t)) := \{(u,\, z) \in (V \cap H^2(0,\, \pi)) \times H^1(0,\, 1)\, : \\ z(0) = u(\pi),\, u_x(\pi) = -\mu_0 u(\pi) - \mu_1 z(1)\}.$$

Notice that the domain of the operator $\mathcal{A}(t)$ is independent of the time t, i.e.

$$\mathcal{D}(\mathcal{A}(t)) = \mathcal{D}(\mathcal{A}(0)),\, \forall t > 0. \tag{8}$$

Now, we introduce the Hilbert space

$$H = L^2(0,\, \pi) \times L^2(0,\, 1)$$

equipped with the usual inner product

$$\left\langle \begin{pmatrix} u \\ z \end{pmatrix}, \begin{pmatrix} \tilde{u} \\ \tilde{z} \end{pmatrix} \right\rangle = \int_0^\pi u \tilde{u} dx + \int_0^1 z(\rho) \tilde{z}(\rho) d\rho.$$

A general theory for equations of type (7) has been developed using semigroup theory [7, 8, 14]. The simplest way to prove existence and uniqueness results is to show that the triplet $\{\mathcal{A}, H, Y\}$, with $\mathcal{A} = \{\mathcal{A}(t) : t \in [0,\, T]\}$, for some fixed $T > 0$ and $Y = \mathcal{D}(\mathcal{A}(0))$, forms a CD-system (or constant domain system, see [7, 8]). More precisely, the following theorem gives the existence and uniqueness results and is proved in Theorem 1.9 of [7] (see also Theorem 2.13 of [8] or [1])

Theorem 2.1. *Assume that*

(i) $Y = \mathcal{D}(\mathcal{A}(0))$ is a dense subset of H,

(ii) (8) holds,

(iii) for all $t \in [0, T]$, $\mathcal{A}(t)$ generates a strongly continuous semigroup on H and the family $\mathcal{A} = \{\mathcal{A}(t) : t \in [0, T]\}$ is stable with stability constants C and m independent of t (i.e. the semigroup $(S_t(s))_{s\geq 0}$ generated by $\mathcal{A}(t)$ satisfies $\|S_t(s)u\|_H \leq Ce^{ms}\|u\|_H$, for all $u \in H$ and $s \geq 0$),

(iv) $\partial_t\mathcal{A}$ belongs to $L^\infty_*([0, T], B(Y, H))$, the space of equivalent classes of essentially bounded, strongly measure functions from $[0, T]$ into the set $B(Y, H)$ of bounded operators from Y into H.

Then, problem (7) has a unique solution $U \in C([0, T], Y) \cap C^1([0, T], H)$ for any initial datum in Y.

Lemma 2.2. *$D(\mathcal{A}(0))$ is dense in H.*

Proof. Let $(f, h)^\top \in H$ be orthogonal to all elements of $D(\mathcal{A}(0))$, namely

$$0 = \left\langle \left(\begin{array}{c} u \\ z \end{array} \right), \left(\begin{array}{c} f \\ h \end{array} \right) \right\rangle = \int_0^\pi uf dx + \int_0^1 z(\rho)h(\rho)d\rho,$$

for all $(u, z)^\top \in D(\mathcal{A}(0))$.

We first take $u = 0$ and $z \in \mathcal{D}(0, 1)$. As $(0, z) \in D(\mathcal{A}(0))$, we get

$$\int_0^1 z(\rho)h(\rho)d\rho = 0.$$

Since $\mathcal{D}(0, 1)$ is dense in $L^2(0, 1)$, we deduce that $h = 0$.

In the same manner, by taking $z = 0$ and $u \in \mathcal{D}(0, \pi)$ we see that $f = 0$. □

Let us suppose now that the speed of the delay satisfies

$$\dot{\tau}(t) \leq d < 1, \forall t > 0 \tag{9}$$

and that μ_0, μ_1 satisfy

$$\mu_1^2 \leq (1 - d)\mu_0^2. \tag{10}$$

Under these conditions, we will show that the operator $\mathcal{A}(t)$ generates a C_0-semigroup in H and using the variable norm technique of Kato from [7], that problem (6) (and then (1)) has a unique solution.

For that purpose, we introduce the following time-dependent inner product on H

$$\left\langle \left(\begin{array}{c} u \\ z \end{array} \right), \left(\begin{array}{c} \tilde{u} \\ \tilde{z} \end{array} \right) \right\rangle_t = \int_0^\pi u\tilde{u} dx + q\tau(t)\int_0^1 z(\rho)\tilde{z}(\rho)d\rho,$$

where q is a positive constant chosen later on, with associated norm denoted by $\|.\|_t$.

Theorem 2.3. *For an initial datum $U_0 \in H$, there exists a unique solution $U \in C([0, +\infty), H)$ to problem (7). Moreover, if $U_0 \in D(\mathcal{A}(0))$, then*

$$U \in C([0, +\infty), D(\mathcal{A}(0))) \cap C^1([0, +\infty), H).$$

Proof. We first prove that

$$\frac{\|\phi\|_t}{\|\phi\|_s} \leq e^{\frac{c}{2\tau_0}|t-s|},\ \forall t,\ s \in [0,\ T] \tag{11}$$

where $\phi = (u,\ z)^\top$ and c is a positive constant. For all $s,\ t \in [0,\ T]$, we have

$$\|\phi\|_t^2 - \|\phi\|_s^2\, e^{\frac{c}{\tau_0}|t-s|} = \left(1 - e^{\frac{c}{\tau_0}|t-s|}\right)\int_0^\pi u^2 dx + q\left(\tau(t) - \tau(s)e^{\frac{c}{\tau_0}|t-s|}\right)\int_0^1 z(\rho)^2 d\rho.$$

We notice that $1 - e^{\frac{c}{\tau_0}|t-s|} \leq 0$. Moreover $\tau(t) - \tau(s)e^{\frac{c}{\tau_0}|t-s|} \leq 0$ for some $c > 0$. Indeed,

$$\tau(t) = \tau(s) + \dot{\tau}(a)(t-s), \quad \text{where } a \in (s,\ t),$$

and thus,

$$\frac{\tau(t)}{\tau(s)} \leq 1 + \frac{|\dot{\tau}(a)|}{\tau(s)}\,|t-s|\,.$$

By (4), $\dot{\tau}$ is bounded and therefore,

$$\frac{\tau(t)}{\tau(s)} \leq 1 + \frac{c}{\tau_0}\,|t-s| \leq e^{\frac{c}{\tau_0}|t-s|},$$

by (3), which proves (11).

Now we calculate $\langle \mathcal{A}(t)U,\ U\rangle_t$ for a t fixed. Take $U = (u,\ z)^\top \in D(\mathcal{A}(t))$. Then

$$\begin{aligned}
\langle \mathcal{A}(t)U,\ U\rangle_t &= \left\langle \begin{pmatrix} a u_{xx} \\ \frac{\dot{\tau}(t)\rho - 1}{\tau(t)} z_\rho \end{pmatrix},\ \begin{pmatrix} u \\ z \end{pmatrix} \right\rangle_t \\
&= a\int_0^\pi u_{xx} u\, dx - q\int_0^1 z_\rho(\rho) z(\rho)(1 - \dot{\tau}(t)\rho)\, d\rho.
\end{aligned}$$

By integrating by parts in space in the first term of this right hand side, we have

$$\begin{aligned}
\langle \mathcal{A}(t)U,\ U\rangle_t &= -a\int_0^\pi u_x^2 dx + a[u u_x]_0^\pi - q\int_0^1 z_\rho(\rho) z(\rho)(1 - \dot{\tau}(t)\rho)\, d\rho \\
&= -a\int_0^\pi u_x^2 dx - a\mu_0 u(\pi,\ t)^2 - a\mu_1 u(\pi,\ t) u(\pi,\ t - \tau(t)) \\
&\quad -q\int_0^1 z_\rho(\rho) z(\rho)(1 - \dot{\tau}(t)\rho)\, d\rho.
\end{aligned}$$

Moreover, we have by integrating by parts in ρ:

$$\begin{aligned}
\int_0^1 z_\rho(\rho) z(\rho)(1 - \dot{\tau}(t)\rho)\, d\rho &= \int_0^1 \frac{1}{2}\frac{\partial}{\partial\rho}(z(\rho)^2)(1 - \dot{\tau}(t)\rho)\, d\rho \\
&= \frac{\dot{\tau}(t)}{2}\int_0^1 z(\rho)^2 d\rho + \frac{1}{2} u^2(\pi,\ t - \tau(t))(1 - \dot{\tau}(t)) \\
&\quad -\frac{1}{2} u^2(\pi,\ t).
\end{aligned}$$

Therefore

$$\begin{aligned}\langle \mathcal{A}(t)U, U\rangle_t &= -a\int_0^\pi u_x^2dx - \frac{q\dot\tau(t)}{2}\int_0^1 z(\rho)^2d\rho - a\mu_0 u(\pi,t)^2\\ &\quad -a\mu_1 u(\pi,t)u(\pi,t-\tau(t)) - \frac{q}{2}u(\pi,t-\tau(t))^2(1-\dot\tau(t))\\ &\quad +\frac{q}{2}u(\pi,t)^2\\ &\le -a\int_0^\pi u_x^2dx + (\frac{q}{2}-a\mu_0)u^2(\pi,t) - a\mu_1 u(\pi,t)u(\pi,t-\tau(t))\\ &\quad -\frac{q}{2}u(\pi,t-\tau(t))^2(1-d) + \frac{q\,|\dot\tau(t)|}{2\tau(t)}\tau(t)\int_0^1 z(\rho)^2d\rho.\end{aligned}$$

We can see that this inequality can be written

$$\langle \mathcal{A}(t)U, U\rangle_t \le -a\int_0^\pi u_x^2dx + (u(\pi,t),\, u(\pi,t-\tau(t)))\Psi_q(u(\pi,t),\, u(\pi,t-\tau(t)))^\top + \kappa(t)\langle U, U\rangle_t\,,$$

where

$$\kappa(t) = \frac{(\dot\tau(t)^2+1)^{\frac12}}{2\tau(t)} \tag{12}$$

and where Ψ_q is the 2×2 matrix defined by

$$\Psi_q = \frac12\begin{pmatrix} q-2a\mu_0 & -a\mu_1\\ -a\mu_1 & -q(1-d)\end{pmatrix}. \tag{13}$$

As $-q(1-d)<0$, we notice that the matrix Ψ_q is negative (in the sense that $X\Psi_qX^\top \le 0$, for all $X=(x_1,x_2)\in\mathbb{R}^2$) if and only if

$$q^2 - 2a\mu_0 q + \frac{a^2\mu_1^2}{1-d} \le 0. \tag{14}$$

The discriminant of this second order polynomial (in q) is

$$\Delta = 4a^2\left(\mu_0^2 - \frac{\mu_1^2}{1-d}\right),$$

which is non negative if and only if (10) holds. Therefore, the matrix Ψ_q is negative for some $q>0$ if and only if (10) is satisfied. Hence, we choose q satisfying (14) or equivalently such that

$$a\mu_0 - a\sqrt{\mu_0^2-\frac{\mu_1^2}{1-d}} \le q \le a\mu_0 + a\sqrt{\mu_0^2-\frac{\mu_1^2}{1-d}}.$$

Such a choice of q yields

$$\langle \mathcal{A}(t)U, U\rangle_t - \kappa(t)\langle U, U\rangle_t \le 0, \tag{15}$$

which proves the dissipativeness of $\tilde{\mathcal{A}}(t) = \mathcal{A}(t) - \kappa(t)I$ for the inner product $\langle\cdot,\cdot\rangle_t$.

Moreover $\kappa'(t) = \frac{\ddot\tau(t)\dot\tau(t)}{2\tau(t)(\dot\tau(t)+1)^{\frac12}} - \frac{\dot\tau(t)(\dot\tau(t)^2+1)^{\frac12}}{2\tau(t)^2}$ is bounded on $[0,T]$ for all $T>0$ (by (4)) and we have

$$\frac{d}{dt}\mathcal{A}(t)U = \begin{pmatrix} 0\\ \frac{\ddot\tau(t)\tau(t)\rho - \dot\tau(t)(\dot\tau(t)\rho-1)}{\tau(t)^2}z_\rho\end{pmatrix}$$

with $\frac{\ddot{\tau}(t)\tau(t)\rho-\dot{\tau}(t)(\dot{\tau}(t)\rho-1)}{\tau(t)^2}$ bounded on $[0, T]$ by (4). Thus

$$\frac{d}{dt}\tilde{\mathcal{A}}(t) \in L_*^\infty([0, T], B(D(\mathcal{A}(0)), H)), \tag{16}$$

the space of equivalence classes of essentially bounded, strongly measurable functions from $[0, T]$ into $B(D(\mathcal{A}(0)), H)$.

Let us prove that $\mathcal{A}(t)$ is maximal, i.e., that $\lambda I - \mathcal{A}(t)$ is surjective for some $\lambda > 0$ and $t > 0$.

Let $(f, h)^T \in H$. We look for $U = (u, z)^T \in D(\mathcal{A}(t))$ solution of

$$(\lambda I - \mathcal{A}(t)) \begin{pmatrix} u \\ z \end{pmatrix} = \begin{pmatrix} f \\ h \end{pmatrix}$$

or equivalently

$$\begin{cases} \lambda u - a u_{xx} = f \\ \lambda z + \frac{1-\dot{\tau}(t)\rho}{\tau(t)} z_\rho = h. \end{cases} \tag{17}$$

Suppose that we have found u with the appropriate regularity. We can then determine z, indeed z satisfies the differential equation

$$\lambda z + \frac{1-\dot{\tau}(t)\rho}{\tau(t)} z_\rho = h$$

and the boundary condition $z(0) = u(\pi)$. Therefore z is explicitly given by

$$z(\rho) = u(\pi)e^{-\lambda\tau(t)\rho} + \tau(t)e^{-\lambda\tau(t)\rho}\int_0^\rho e^{\lambda\tau(t)\sigma}h(\sigma)d\sigma,$$

if $\dot{\tau}(t) = 0$, and

$$z(\rho) = u(\pi)e^{\lambda\frac{\tau(t)}{\dot{\tau}(t)}\ln(1-\dot{\tau}(t)\rho)} + e^{\lambda\frac{\tau(t)}{\dot{\tau}(t)}\ln(1-\dot{\tau}(t)\rho)}\int_0^\rho \frac{h(\sigma)\tau(t)}{1-\dot{\tau}(t)\sigma}e^{-\lambda\frac{\tau(t)}{\dot{\tau}(t)}\ln(1-\dot{\tau}(t)\sigma)}d\sigma,$$

otherwise. This means that once u is found with the appropriate properties, we can find z. In particular, we have if $\dot{\tau}(t) = 0$,

$$z(1) = u(\pi)e^{-\lambda\tau(t)} + \tau(t)e^{-\lambda\tau(t)}\int_0^1 e^{\lambda\tau(t)\sigma}h(\sigma)d\sigma = u(\pi)e^{-\lambda\tau(t)} + z^0,$$

where $z^0 = \tau(t)e^{-\lambda\tau(t)}\int_0^1 e^{\lambda\tau(t)\sigma}h(\sigma)d\sigma$ is a fixed real number depending only on h, and if $\dot{\tau}(t) \neq 0$

$$\begin{aligned} z(1) &= u(\pi)e^{\lambda\frac{\tau(t)}{\dot{\tau}(t)}\ln(1-\dot{\tau}(t))} + e^{\lambda\frac{\tau(t)}{\dot{\tau}(t)}\ln(1-\dot{\tau}(t))}\int_0^1 \frac{h(\sigma)\tau(t)}{1-\dot{\tau}(t)\sigma}e^{-\lambda\frac{\tau(t)}{\dot{\tau}(t)}\ln(1-\dot{\tau}(t)\sigma)}d\sigma \\ &= u(\pi)e^{\lambda\frac{\tau(t)}{\dot{\tau}(t)}\ln(1-\dot{\tau}(t))} + z^0, \end{aligned}$$

where $z^0 = e^{\lambda\frac{\tau(t)}{\dot{\tau}(t)}\ln(1-\dot{\tau}(t))}\int_0^1 \frac{h(\sigma)\tau(t)}{1-\dot{\tau}(t)\sigma}e^{-\lambda\frac{\tau(t)}{\dot{\tau}(t)}\ln(1-\dot{\tau}(t)\sigma)}d\sigma$ depends only on h.

It remains to find u. By (17), u must satisfy

$$\lambda u - a u_{xx} = f.$$

Multiplying this identity by a test function ϕ, integrating in space and using integration by parts, we obtain

$$\int_0^\pi (\lambda u - a u_{xx})\phi dx = \int_0^\pi (\lambda u\phi + a u_x\phi_x)dx - a u_x(\pi)\phi(\pi) + a u_x(0)\phi(0).$$

But using the fact that $(u, z)^\top$ must belong to $D(\mathcal{A}(t))$, we have

$$\int_0^\pi (\lambda u - a u_{xx})\phi dx = \int_0^\pi (\lambda u\phi + a u_x \phi_x) dx + a\mu_0 u(\pi)\phi(\pi) + a\mu_1 z(1)\phi(\pi).$$

Therefore

$$\int_0^\pi (\lambda u\phi + a u_x \phi_x) dx + a\mu_0 u(\pi)\phi(\pi) + a\mu_1 z(1)\phi(\pi) = \int_0^\pi f\phi dx.$$

Using the above expression for $z(1)$, we arrive at the problem

$$\int_0^\pi (\lambda u\phi + a u_x \phi_x) dx + a(\mu_0 + \mu_1 e^{-\lambda\tau(t)}) u(\pi)\phi(\pi) \\ = \int_0^\pi f\phi dx - a\mu_1 z^0 \phi(\pi),\ \forall \phi \in V \quad (18)$$

if $\dot{\tau}(t) = 0$, or otherwise

$$\int_0^\pi (\lambda u\phi + a u_x \phi_x) dx + a(\mu_0 + \mu_1 e^{\lambda \frac{\tau(t)}{\dot{\tau}(t)} \ln(1-\dot{\tau}(t))}) u(\pi)\phi(\pi) \\ = \int_0^\pi f\phi dx - a\mu_1 z^0 \phi(\pi),\ \forall \phi \in V. \quad (19)$$

These problems have a unique solution $u \in V$ by Lax-Milgram's lemma, because the left-hand side of (18) or (19) is coercive on V.

If we consider $\phi \in \mathcal{D}(0, \pi) \subset V$, then u satisfies

$$\lambda u - a u_{xx} = f \text{ in } \mathcal{D}'(0, \pi).$$

This directly implies that $u \in H^2(0, \pi)$ and then $u \in V \cap H^2(0, \pi)$. Coming back to (18) and by integrating by parts, we find, for $\dot{\tau}(t) = 0$,

$$a[u_x(\pi) + (\mu_0 + \mu_1 e^{-\lambda\tau(t)}) u(\pi)]\phi(\pi) = -a\mu_1 z^0 \phi(\pi),$$

and then

$$\begin{aligned} u_x(\pi) &= -(\mu_0 + \mu_1 e^{-\lambda\tau(t)}) u(\pi) - \mu_1 z^0 \\ &= -\mu_0 u(\pi) - \mu_1 (e^{-\lambda\tau(t)} u(\pi) + z^0) \\ &= -\mu_0 u(\pi) - \mu_1 z(1). \end{aligned}$$

We find the same result if $\dot{\tau}(t) \neq 0$.

In summary we have found $(u, z)^\top \in D(\mathcal{A}(t))$ satisfying (17) and thus $\lambda I - \mathcal{A}(t)$ is surjective for some $\lambda > 0$ and $t > 0$. Since $\kappa(t) > 0$, we directly deduce that

$$\lambda I - \tilde{\mathcal{A}}(t) = (\lambda + \kappa(t)) I - \mathcal{A}(t) \text{ is surjective} \quad (20)$$

for some $\lambda > 0$ and $t > 0$.

Then, (11), (15) and (20) imply that the family $\tilde{\mathcal{A}} = \{\tilde{\mathcal{A}}(t) : t \in [0, T]\}$ is a stable family of generators in H with stability constants independent of t, by Proposition 1.1 from [7]. Therefore, the assumptions (i)-(iv) of Theorem 2.1 are verified by (8), (11), (15), (16), (20) and Lemma 2.2, and thus, the problem

$$\begin{cases} \tilde{U}_t = \tilde{\mathcal{A}}(t)\tilde{U} \\ \tilde{U}(0) = U_0. \end{cases}$$

has a unique solution $\tilde{U} \in C([0, +\infty), H)$ and, if $U_0 \in D(\mathcal{A}(0))$,

$$\tilde{U} \in C([0, +\infty), D(\mathcal{A}(0))) \cap C^1([0, +\infty), H).$$

Setting

$$U(t) = e^{\beta(t)} \tilde{U}(t)$$

with $\beta(t)=\int_0^t \kappa(s)ds$, we remark that it is a solution of (7) because

$$\begin{aligned} U_t(t) &= \kappa(t)e^{\beta(t)}\tilde{U}(t)+e^{\beta(t)}\tilde{U}_t(t)\\ &= \kappa(t)e^{\beta(t)}\tilde{U}(t)+e^{\beta(t)}\tilde{\mathcal{A}}(t)\tilde{U}(t)\\ &= e^{\beta(t)}(\kappa(t)\tilde{U}(t)+\tilde{\mathcal{A}}(t)\tilde{U}(t))\\ &= e^{\beta(t)}\mathcal{A}(t)\tilde{U}(t)=\mathcal{A}(t)e^{\beta(t)}\tilde{U}(t)\\ &= \mathcal{A}(t)U(t), \end{aligned}$$

which concludes the proof. □

2.2. **The decay of the energy.** We here suppose that

$$\mu_1^2<(1-d)\mu_0^2. \tag{21}$$

Let us choose the following energy

$$E(t)=\frac{1}{2}\int_0^\pi u^2(x,\,t)dx+\frac{q\tau(t)}{2}\int_0^1 u^2(\pi,\,t-\tau(t)\rho)d\rho, \tag{22}$$

where q is a positive constant chosen later.

Proposition 1. *Let (9) and (21) be satisfied. Then for all regular solution of problem (1), the energy is decreasing and satisfies*

$$E'(t)\le -a\int_0^\pi u_x^2(x,\,t)dx+(u(\pi,\,t),\,u(\pi,\,t-\tau(t)))\,\Psi_q\,(u(\pi,\,t),\,u(\pi,\,t-\tau(t)))^\top<0, \tag{23}$$

where Ψ_q is the matrix defined in (13).

Proof. Differentiating (22) and by (1), we obtain

$$\begin{aligned} E'(t) &= \int_0^\pi uu_tdx+\frac{q\dot{\tau}(t)}{2}\int_0^1 u^2(\pi,\,t-\tau(t)\rho)d\rho\\ &\quad +q\tau(t)\int_0^1 u(\pi,\,t-\tau(t)\rho)u_t(\pi,\,t-\tau(t)\rho)(1-\dot{\tau}(t)\rho)d\rho\\ &= a\int_0^\pi uu_{xx}dx+\frac{q\dot{\tau}(t)}{2}\int_0^1 u^2(\pi,\,t-\tau(t)\rho)d\rho\\ &\quad +q\tau(t)\int_0^1 u(\pi,\,t-\tau(t)\rho)u_t(\pi,\,t-\tau(t)\rho)(1-\dot{\tau}(t)\rho)d\rho. \end{aligned}$$

By integrating by parts in space, we find

$$\begin{aligned} a\int_0^\pi uu_{xx}dx &= -a\int_0^\pi u_x^2dx+a[u(x,\,t)u_x(x,\,t)]_0^\pi\\ &= -a\int_0^\pi u_x^2dx-a\mu_0u^2(\pi,\,t)-a\mu_1u(\pi,\,t)u(\pi,\,t-\tau(t)). \end{aligned}$$

Setting $z(\rho,t) = u(\pi,\, t-\tau(t)\rho)$, we see that $z_\rho(\rho,t) = -\tau(t)u_t(\pi,\, t-\tau(t)\rho)$, and by integrating by parts in ρ, we get

$$\begin{aligned}
&\int_0^1 u(\pi,\, t-\tau(t)\rho)u_t(\pi,\, t-\tau(t)\rho)(1-\dot\tau(t)\rho)d\rho\\
&= -\frac{1}{\tau(t)}\int_0^1 z(\rho,\, t)z_\rho(\rho,\, t)(1-\dot\tau(t)\rho)d\rho\\
&= -\frac{1}{2\tau(t)}\int_0^1 \partial_\rho(z(\rho,\, t)^2)(1-\dot\tau(t)\rho)d\rho\\
&= \frac{1}{2\tau(t)}\int_0^1 z(\rho,\, t)^2(-\dot\tau(t))d\rho - \frac{1}{2\tau(t)}[z^2(\rho,\, t)(1-\dot\tau(t)\rho)]_0^1\\
&= -\frac{\dot\tau(t)}{2\tau(t)}\int_0^1 u^2(\pi,\, t-\tau(t)\rho)d\rho - \frac{1-\dot\tau(t)}{2\tau(t)}u^2(\pi,\, t-\tau(t)) + \frac{1}{2\tau(t)}u^2(\pi,\, t).
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
E'(t) = -a\int_0^\pi u_x^2dx - (a\mu_0 - \frac{q}{2})u^2(\pi,\, t) - a\mu_1 u(\pi,\, t)u(\pi,\, t-\tau(t))\\
- \frac{q}{2}(1-\dot\tau(t))u^2(\pi,\, t-\tau(t)).
\end{aligned}$$

We can see that this inequality can be written as

$$E'(t) \le -a\int_0^\pi u_x^2dx + (u(\pi,\, t),\, u(\pi,\, t-\tau(t)))\Psi_q(u(\pi,\, t),\, u(\pi,\, t-\tau(t)))^\top.$$

As $-q(1-d) < 0$, Ψ_q is negative definite if and only if

$$q^2 - 2a\mu_0 q + \frac{a^2\mu_1^2}{1-d} < 0. \tag{24}$$

The discriminant of this second order polynomial is $\Delta = 4a^2\left(\mu_0^2 - \frac{\mu_1^2}{1-d}\right)$, which is positive if and only if (21) holds. Therefore, the matrix Ψ_q is negative definite for some $q > 0$ if and only if (21) is satisfied, and in that case, we choose q such that

$$a\mu_0 - a\sqrt{\mu_0^2 - \frac{\mu_1^2}{1-d}} < q < a\mu_0 + a\sqrt{\mu_0^2 - \frac{\mu_1^2}{1-d}}, \tag{25}$$

which concludes the proof. □

2.3. **Exponential stability.** In this section, we prove the exponential stability of the heat equation (1) by using the following Lyapunov functional

$$\mathcal{E}(t) = E(t) + \gamma\mathcal{E}_2(t), \tag{26}$$

where $\gamma > 0$ is a parameter that will be fixed small enough later on, E is the standard energy defined by (22) and $\mathcal{E}_2$ is defined by

$$\mathcal{E}_2(t) = q\int_{t-\tau(t)}^t e^{2\delta(s-t)}u^2(\pi,\, s)ds = q\tau(t)\int_0^1 e^{-2\delta\tau(t)\rho}u^2(\pi,\, t-\tau(t)\rho)d\rho, \tag{27}$$

where $\delta > 0$ is a fixed positive real number.

Remark 1. Let us notice that the energies E and $\mathcal{E}$ are equivalent, since

$$E(t) \le \mathcal{E}(t) \le (2\gamma+1)E(t).$$

The result about the decay of the energy E is the following one:

Theorem 2.4. *Let (9) and (21) be satisfied. Then the energy E decays exponentially, more precisely there exist two positive constants α and C such that*

$$E(t) \leq Ce^{-\alpha t}E(0), \forall t > 0.$$

Proof. First, we differentiate $\mathcal{E}_2$ to have

$$\frac{d}{dt}\mathcal{E}_2(t) = \frac{\dot{\tau}(t)}{\tau(t)}\mathcal{E}_2(t) + q\tau(t)\int_0^1 (-2\delta\dot{\tau}(t)\rho)e^{-2\delta\tau(t)\rho}u^2(\pi,\, t-\tau(t)\rho)d\rho + J,$$

where

$$J = 2q\tau(t)\int_0^1 e^{-2\delta\tau(t)\rho}u(\pi,\, t-\tau(t)\rho)u_t(\pi,\, t-\tau(t)\rho)(1-\dot{\tau}(t)\rho)d\rho.$$

Moreover, by noticing one more time that $z(\rho,\, t) = u(\pi,\, t-\tau(t)\rho)$ and by integrating by parts in ρ, we have

$$\begin{aligned} J &= -q\int_0^1 e^{-2\delta\tau(t)\rho}\frac{\partial}{\partial\rho}(z(\rho,\, t)^2)(1-\dot{\tau}(t)\rho)d\rho \\ &= q\int_0^1 z^2(\rho,\, t)(-2\delta\tau(t)(1-\dot{\tau}(t)\rho)-\dot{\tau}(t))e^{-2\delta\tau(t)\rho}d\rho \\ &\quad -qe^{-2\delta\tau(t)}z^2(1,\, t)(1-\dot{\tau}(t)) + qz^2(0,\, t) \\ &= q\int_0^1 (-2\delta\tau(t)(1-\dot{\tau}(t)\rho)-\dot{\tau}(t))u^2(\pi,\, t-\tau(t)\rho)e^{-2\delta\tau(t)\rho}d\rho \\ &\quad -qe^{-2\delta\tau(t)}u^2(\pi,\, t-\tau(t))(1-\dot{\tau}(t)) + qu^2(\pi,\, t). \end{aligned}$$

Therefore, we have

$$\begin{aligned} \frac{d}{dt}\mathcal{E}_2(t) &= \frac{\dot{\tau}(t)}{\tau(t)}\mathcal{E}_2(t) + q\int_0^1 (-2\delta\tau(t)-\dot{\tau}(t))u^2(\pi,\, t-\tau(t)\rho)e^{-2\delta\tau(t)\rho}d\rho \\ &\quad -qe^{-2\delta\tau(t)}u^2(\pi,\, t-\tau(t))(1-\dot{\tau}(t)) + qu^2(\pi,\, t) \\ &= -2\delta\mathcal{E}_2(t) - qe^{-2\delta\tau(t)}u^2(\pi,\, t-\tau(t))(1-\dot{\tau}(t)) + qu^2(\pi,\, t). \end{aligned}$$

As $\dot{\tau}(t) < 1$ (see (2)), we obtain

$$\frac{d}{dt}\mathcal{E}_2(t) \leq -2\delta\mathcal{E}_2(t) + qu^2(\pi,\, t). \tag{28}$$

Consequently, gathering (23), (26) and (28), we obtain

$$\begin{aligned} \frac{d}{dt}\mathcal{E}(t) &\leq -(a\mu_0 - \frac{q}{2} - q\gamma)u^2(\pi,\, t) - \frac{q}{2}(1-d)u^2(\pi,\, t-\tau(t)) \\ &\quad -a\mu_1 u(\pi,\, t)u(\pi,\, t-\tau(t)) - a\int_0^\pi u_x^2(x,\, t)dx - 2\gamma\delta\mathcal{E}_2(t), \end{aligned}$$

or equivalently

$$\begin{aligned} \frac{d}{dt}\mathcal{E}(t) \leq (u(\pi,\, t),\, u(\pi,\, t-\tau(t)))\tilde{\Psi}_q(u(\pi,\, t),\, u(\pi,\, t-\tau(t)))^\top \\ - a\int_0^\pi u_x^2(x,\, t)dx - 2\gamma\delta\mathcal{E}_2(t), \end{aligned} \tag{29}$$

where $\tilde{\Psi}_q$ is the 2×2 matrix defined by

$$\tilde{\Psi}_q = \frac{1}{2}\begin{pmatrix} q(1+2\gamma) - 2a\mu_0 & -a\mu_1 \\ -a\mu_1 & -q(1-d) \end{pmatrix} = \Psi_q + \gamma\begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}.$$

Now fix $q > 0$ such that Ψ_q is negative definite (consequence of the assumption (21)). By a perturbation argument, we deduce that for $\gamma > 0$ small enough, $\tilde{\Psi}_q$ is negative. More precisely, we take $\gamma = \frac{-\lambda}{q}$, when λ is the greatest negative eigenvalue of Ψ_q, or equivalently $X\Psi_q X^\top \leq \lambda |X|^2$, for all $X \in \mathbb{R}^2$. We can easily check that

$$\lambda = \frac{1}{4}\left(-2a\mu_0 + dq + \sqrt{4a^2(\mu_0^2+\mu_1^2) + 4a(d-2)\mu_0 q + (d-2)^2q^2}\right) < 0. \quad (30)$$

Therefore, for $\gamma = \frac{-\lambda}{q}$, we find

$$\frac{d}{dt}\mathcal{E}(t) \leq -2\delta\gamma\mathcal{E}_2(t) - a\int_0^\pi u_x^2(x,\,t)dx. \quad (31)$$

As $u(0,\,t) = 0$ for all $t > 0$, by the min-max principle, we have

$$\int_0^\pi u^2(x,\,t)dx \leq 4\int_0^\pi u_x^2(x,\,t)dx,$$

because the first eigenvalue of the Laplace operator with Dirichlet boundary condition at 0 and Neumann boundary condition at π is $-\frac{1}{4}$. Therefore

$$-a\int_0^\pi u_x^2(x,\,t)dx \leq -\frac{a}{4}\int_0^\pi u^2(x,\,t)dx.$$

This estimate in (31) and by the definition (27) of $\mathcal{E}_2$, we obtain

$$\frac{d}{dt}\mathcal{E}(t) \leq -\frac{a}{4}\int_0^\pi u^2(x,\,t)dx - 2q\delta\gamma\tau(t)\int_0^1 e^{-2\delta\tau(t)\rho}u^2(\pi,\,t-\tau(t)\rho)d\rho.$$

Since $\tau(t) \leq M$ (see (3)) and in view of the definition (22) of $E(t)$, there exists a constant $\gamma' > 0$ (depending on γ and δ, namely $\gamma' \leq \min(\frac{a}{2},\, 4\delta\gamma e^{-2\delta M})$) such that

$$\frac{d}{dt}\mathcal{E}(t) \leq -\gamma' E(t).$$

By applying Remark 1, we obtain

$$\frac{d}{dt}\mathcal{E}(t) \leq -\frac{\gamma'}{2\gamma+1}\mathcal{E}(t).$$

This implies that

$$\mathcal{E}(t) \leq \mathcal{E}(0)e^{-\alpha t},$$

with

$$\alpha = \frac{\gamma'}{2\gamma+1} \leq \frac{1}{q-2\lambda}\min\left(\frac{aq}{2},\, -4\lambda\delta e^{-2\delta M}\right).$$

Remark 1 leads to

$$E(t) \leq \mathcal{E}(t) \leq \mathcal{E}(0)e^{-\alpha t} \leq (2\gamma+1)E(0)e^{-\alpha t}.$$

□

Remark 2. In the proof of Theorem 2.4, we notice that we have explicitely calculated the decay rate of the energy, given by

$$\alpha = \frac{1}{q-2\lambda}\min\left(\frac{aq}{2},\, -4\lambda\delta e^{-2\delta M}\right),$$

where λ is given by (30), q by (25) and δ is a positive real number. Therefore, we can choose δ so that the decay of the energy is as quick as possible. For that purpose, we notice that the function $\delta \to -4\lambda\delta e^{-2\delta M}$ admits a maximum at $\delta = \frac{1}{2M}$ and that this maximum is $\frac{-2\lambda}{Me}$. Thus the larger decay rate of the energy is given by

$$\alpha_{max} = \frac{1}{q-2\lambda}\min\left(\frac{aq}{2},\, \frac{-2\lambda}{Me}\right).$$

Obviously, this quantity can be calculated if the data μ_0, μ_1 and τ are given.

3. **Exponential stability of the delayed wave equation.** We now consider the system described by

$$\begin{cases} u_{tt}(x,\,t) - a u_{xx}(x,\,t) = 0, & 0 < x < \pi,\, t > 0, \\ u(0,\,t) = 0, & t > 0, \\ u_x(\pi,\,t) = -\mu_0 u_t(\pi,\,t) - \mu_1 u_t(\pi,\,t-\tau(t)), & t > 0, \\ u(x,\,0) = u^0(x),\, u_t(x,\,0) = u^1(x), & 0 < x < \pi, \\ u_t(\pi,\,t-\tau(0)) = f^0(t-\tau(0)), & 0 < t < \tau(0), \end{cases} \tag{32}$$

whith the constant parameter $a > 0$ and where μ_0, $\mu_1 \geq 0$ are fixed nonnegative real numbers, the time-varying delay $\tau(t)$ still satisfies (2), (3) and (4).

The boundary-value problem (32) describes the oscillations of a homogeneous string fixed at 0 and with a feedback law at π.

3.1. **Well-posedness of the problem.** We aim to show that problem (32) is well-posed. For that purpose, we use the same ideas than before.

We transform our system (32) as follows. Let us introduce the auxiliary variable $z(\rho,t) = u_t(\pi,\,t-\tau(t)\rho)$ for $\rho \in (0,\,1)$ and $t > 0$. Note that z verifies the transport equation for $0 < \rho < 1$ and $t > 0$ (compare with (5))

$$\begin{cases} \tau(t) z_t(\rho,\,t) + (1-\dot{\tau}(t)\rho) z_\rho(\rho,\,t) = 0, \\ z(0,\,t) = u_t(\pi,\,t), \\ z(\rho,\,0) = f^0(-\tau(0)\rho). \end{cases} \tag{33}$$

Therefore, the problem (32) is equivalent to

$$\begin{cases} u_{tt}(x,\,t) - a u_{xx}(x,\,t) = 0, & 0 < x < \pi,\, t > 0, \\ \tau(t) z_t(\rho,\,t) + (1-\dot{\tau}(t)\rho) z_\rho(\rho,\,t) = 0, & 0 < \rho < 1,\, t > 0, \\ u(0,\,t) = 0,\, u_x(\pi,\,t) = -\mu_0 u_t(\pi,\,t) - \mu_1 z(1,\,t), & t > 0, \\ z(0,\,t) = u_t(\pi,\,t), & t > 0, \\ u(x,\,0) = u^0(x),\, u_t(x,\,0) = u^1(x), & 0 < x < \pi, \\ z(\rho,\,0) = f^0(-\tau(0)\rho), & 0 < \rho < 1. \end{cases} \tag{34}$$

If we introduce

$$U := (u,\,u_t,\,z)^\top,$$

then U satisfies

$$U_t = (u_t,\,u_{tt},\,z_t)^\top = (u_t,\,a u_{xx},\,\frac{\dot{\tau}(t)\rho - 1}{\tau(t)} z_\rho)^\top.$$

Consequently the problem (32) may be rewritten as the first order evolution equation

$$\begin{cases} U_t = \mathcal{A}(t) U \\ U(0) = (u^0,\,u^1,\,f^0(-\tau(0).))^\top = U_0, \end{cases} \tag{35}$$

where the time dependent operator $\mathcal{A}(t)$ is defined by

$$\mathcal{A}(t) \begin{pmatrix} u \\ \omega \\ z \end{pmatrix} = \begin{pmatrix} \omega \\ a u_{xx} \\ \frac{\dot{\tau}(t)\rho - 1}{\tau(t)} z_\rho \end{pmatrix},$$

with domain

$$\mathcal{D}(\mathcal{A}(t)) := \{(u,\,\omega,\,z) \in (V \cap H^2(0,\,\pi)) \times V \times H^1(0,\,1) : \\ z(0) = \omega(\pi),\, u_x(\pi) = -\mu_0 \omega(\pi) - \mu_1 z(1)\},$$

where we recall that

$$V = \{\phi \in H^1(0,\,\pi) \,:\, \phi(0) = 0\}.$$

Again, we notice that the domain of the operator $\mathcal{A}(t)$ is independent of the time t, i.e.

$$\mathcal{D}(\mathcal{A}(t)) = \mathcal{D}(\mathcal{A}(0)),\ \forall t > 0. \tag{36}$$

Now, we introduce the Hilbert space

$$H = V \times L^2(0,\,\pi) \times L^2(0,\,1)$$

equipped with the usual inner product

$$\left\langle \begin{pmatrix} u \\ \omega \\ z \end{pmatrix}, \begin{pmatrix} \tilde{u} \\ \tilde{\omega} \\ \tilde{z} \end{pmatrix} \right\rangle = \int_0^\pi (a u_x \tilde{u}_x + \omega\tilde{\omega})dx + \int_0^1 z(\rho)\tilde{z}(\rho)d\rho.$$

Lemma 3.1. *$D(\mathcal{A}(0))$ is dense in H.*

Proof. The proof is the same as the one of Lemma 2.1 of [12], we give it for the sake of completeness. Let $(f,\,g,\,h)^\top \in H$ be orthogonal to all elements of $D(\mathcal{A}(0))$, namely

$$0 = \left\langle \begin{pmatrix} u \\ \omega \\ z \end{pmatrix}, \begin{pmatrix} f \\ g \\ h \end{pmatrix} \right\rangle = \int_0^\pi (a u_x f_x + \omega g)dx + \int_0^1 z(\rho)h(\rho)d\rho,$$

for all $(u,\,\omega,\,z)^\top \in D(\mathcal{A}(0))$.

We first take $u = 0$ and $w = 0$ and $z \in \mathcal{D}(0,\,1)$. As $(0,\,0,\,z) \in D(\mathcal{A}(0))$, we get

$$\int_0^1 z(\rho)h(\rho)d\rho = 0.$$

Since $\mathcal{D}(0,\,1)$ is dense in $L^2(0,\,1)$, we deduce that $h = 0$.

In the same manner, by taking $u = 0$, $z = 0$ and $\omega \in \mathcal{D}(0,\,\pi)$ we see that $g = 0$.

The above orthogonality condition is then reduced to

$$0 = a\int_0^\pi u_x f_x dx,\ \forall (u,\,\omega,\,z) \in D(\mathcal{A}(0)).$$

By restricting ourselves to $\omega = 0$ and $z = 0$, we obtain

$$\int_0^1 u_x f_x dx = 0,\ \forall (u,\,0,\,0) \in D(\mathcal{A}(0)).$$

But we easily check that $(u,\,0,\,0) \in D(\mathcal{A}(0))$ if and only if $u \in D(\Delta) = \{v \in H^2(0,\pi) \,:\, v(0) = 0, v'(1) = 0\}$, the domain of the Laplace operator with mixed boundary conditions. Since it is well known that $D(\Delta)$ is dense in V (equipped with the inner product $< .,\, . >_V$), we conclude that $f = 0$. □

As before we suppose that the speed of the delay satisfies (9) and (10). Under these conditions, we will show that the operator $\mathcal{A}(t)$ generates a C_0-semigroup in H and the unique solvability of problem (35).

For that purpose, we introduce the following time-dependent inner product on H

$$\left\langle \begin{pmatrix} u \\ \omega \\ z \end{pmatrix}, \begin{pmatrix} \tilde{u} \\ \tilde{\omega} \\ \tilde{z} \end{pmatrix} \right\rangle_t = \int_0^\pi (a u_x \tilde{u}_x + \omega\tilde{\omega})dx + q\tau(t)\int_0^1 z(\rho)\tilde{z}(\rho)d\rho,$$

where q is a positive constant chosen such that Ψ_q is negative (guaranteed by the assumptions (9) and (10)), with associated norm denoted by $\|.\|_t$.

Theorem 3.2. *For an initial datum $U_0 \in H$, there exists a unique solution $U \in C([0, +\infty), H)$ to problem (35). Moreover, if $U_0 \in D(\mathcal{A}(0))$, then*

$$U \in C([0, +\infty), D(\mathcal{A}(0))) \cap C^1([0, +\infty), H).$$

Proof. We first notice that

$$\frac{\|\phi\|_t}{\|\phi\|_s} \leq e^{\frac{c}{2\tau_0}|t-s|}, \forall t, s \in [0, T] \tag{37}$$

where $\phi = (u, \omega, z)^\top$ and c is a positive constant. Indeed, for all $s, t \in [0, T]$, we have

$$\|\phi\|_t^2 - \|\phi\|_s^2 e^{\frac{c}{\tau_0}|t-s|} = \left(1 - e^{\frac{c}{\tau_0}|t-s|}\right) \int_0^\pi (au_x^2 + \omega^2)dx + q\left(\tau(t) - \tau(s)e^{\frac{c}{\tau_0}|t-s|}\right) \int_0^1 z(\rho)^2 d\rho,$$

and we conclude as in the proof of Theorem 2.3.

Now we calculate $\langle \mathcal{A}(t)U, U\rangle_t$ for a $t > 0$ fixed. For an arbitrary $U = (u, \omega, z)^\top \in D(\mathcal{A}(t))$, we have

$$\begin{aligned}\langle \mathcal{A}(t)U, U\rangle_t &= \left\langle \begin{pmatrix} \omega \\ au_{xx} \\ \frac{\dot\tau(t)\rho - 1}{\tau(t)} z_\rho \end{pmatrix}, \begin{pmatrix} u \\ \omega \\ z \end{pmatrix} \right\rangle_t \\ &= \int_0^\pi (a\omega_x u_x + au_{xx}\omega)dx - q\int_0^1 z_\rho(\rho)z(\rho)(1 - \dot\tau(t)\rho)d\rho.\end{aligned}$$

By integrating by parts in space, we have

$$\begin{aligned}\langle \mathcal{A}(t)U, U\rangle_t &= a[\omega u_x]_0^\pi - q\int_0^1 z_\rho(\rho)z(\rho)(1 - \dot\tau(t)\rho)d\rho \\ &= -a\mu_0 z(0)^2 - a\mu_1 z(0)z(1) - q\int_0^1 z_\rho(\rho)z(\rho)(1 - \dot\tau(t)\rho)d\rho.\end{aligned}$$

Moreover, we have by integrating by parts in ρ:

$$\begin{aligned}\int_0^1 z_\rho(\rho)z(\rho)(1 - \dot\tau(t)\rho)d\rho &= \int_0^1 \frac{1}{2}\frac{\partial}{\partial\rho}(z(\rho)^2)(1 - \dot\tau(t)\rho)d\rho \\ &= \frac{\dot\tau(t)}{2}\int_0^1 z(\rho)^2 d\rho + \frac{1}{2}z(1)^2(1 - \dot\tau(t)) - \frac{1}{2}z^2(0).\end{aligned}$$

These two identities yield

$$\langle \mathcal{A}(t)U, U\rangle_t = -a\mu_0 z(0)^2 - a\mu_1 z(0)z(1) - \frac{q}{2}z(1)^2(1 - \dot\tau(t)) + \frac{q}{2}z^2(0) - \frac{q\dot\tau(t)}{2}\int_0^1 z(\rho)^2 d\rho.$$

We can see, that this identity implies that

$$\langle \mathcal{A}(t)U, U\rangle_t \leq (z(0), z(1))\Psi_q(z(0), z(1))^\top + \kappa(t)\langle U, U\rangle_t,$$

where Ψ_q is the matrix defined by (13) and $\kappa(t)$ is given by (12). As we have chosen q such that the matrix Ψ_q is negative, we have

$$\langle \mathcal{A}(t)U, U\rangle_t - \kappa(t)\langle U, U\rangle_t \leq 0, \tag{38}$$

which proves the dissipativeness of $\tilde{\mathcal{A}}(t) = \mathcal{A}(t) - \kappa(t)I$ for the inner product $\langle \cdot, \cdot\rangle_t$.

As in the proof of Theorem 2.3, we see that (4) implies that

$$\frac{d}{dt}\tilde{\mathcal{A}}(t) \in L^\infty_*([0,\,T],\,B(D(\mathcal{A}(0)),\,H)). \tag{39}$$

Let us finally prove that $\mathcal{A}(t)$ is maximal, i.e., that $\lambda I - \mathcal{A}(t)$ is surjective for some $\lambda > 0$ and $t > 0$.

Let $(f,\,g,\,h)^T \in H$. We look for $U = (u,\,\omega,\,z)^T \in D(\mathcal{A}(t))$ solution of

$$(\lambda I - \mathcal{A}(t)) \begin{pmatrix} u \\ \omega \\ z \end{pmatrix} = \begin{pmatrix} f \\ g \\ h \end{pmatrix}$$

or equivalently

$$\begin{cases} \lambda u - \omega = f \\ \lambda \omega - a u_{xx} = g \\ \lambda z + \frac{1-\dot{\tau}(t)\rho}{\tau(t)} z_\rho = h. \end{cases} \tag{40}$$

Suppose that we have found u with the appropriate regularity. Then, we have

$$\omega = -f + \lambda u \in V.$$

We can then determine z, indeed z satisfies the differential equation

$$\lambda z + \frac{1-\dot{\tau}(t)\rho}{\tau(t)} z_\rho = h$$

and the boundary condition $z(0) = \omega(\pi) = -f(\pi) + \lambda u(\pi)$. Therefore z is explicitly given by

$$z(\rho) = \lambda u(\pi) e^{-\lambda\tau(t)\rho} - f(\pi) e^{-\lambda\tau(t)\rho} + \tau(t) e^{-\lambda\tau(t)\rho} \int_0^\rho e^{\lambda\tau(t)\sigma} h(\sigma)\, d\sigma,$$

if $\dot{\tau}(t) = 0$, and

$$\begin{aligned} z(\rho) &= \lambda u(\pi) e^{\lambda \frac{\tau(t)}{\dot{\tau}(t)} \ln(1-\dot{\tau}(t)\rho)} - f(\pi) e^{\lambda \frac{\tau(t)}{\dot{\tau}(t)} \ln(1-\dot{\tau}(t)\rho)} \\ &\quad + e^{\lambda \frac{\tau(t)}{\dot{\tau}(t)} \ln(1-\dot{\tau}(t)\rho)} \int_0^\rho \frac{h(\sigma)\tau(t)}{1-\dot{\tau}(t)\sigma} e^{-\lambda \frac{\tau(t)}{\dot{\tau}(t)} \ln(1-\dot{\tau}(t)\sigma)} d\sigma, \end{aligned}$$

otherwise. This means that once u is found with the appropriate properties, we can find z and ω. In particular, we have if $\dot{\tau}(t) = 0$,

$$z(1) = \lambda u(\pi) e^{-\lambda\tau(t)} - f(\pi) e^{-\lambda\tau(t)} + \tau(t) e^{-\lambda\tau(t)} \int_0^1 e^{\lambda\tau(t)\sigma} h(\sigma)\, d\sigma = \lambda u(\pi) e^{-\lambda\tau(t)} + z^0,$$

where $z^0 = -f(\pi)e^{-\lambda\tau(t)} + \tau(t)e^{-\lambda\tau(t)} \int_0^1 e^{\lambda\tau(t)\sigma} h(\sigma)\,d\sigma$ is a fixed real number depending only on f and h, and if $\dot{\tau}(t) \neq 0$

$$\begin{aligned} z(1) &= \lambda u(\pi) e^{\lambda \frac{\tau(t)}{\dot{\tau}(t)} \ln(1-\dot{\tau}(t))} - f(\pi) e^{\lambda \frac{\tau(t)}{\dot{\tau}(t)} \ln(1-\dot{\tau}(t))} \\ &\quad + e^{\lambda \frac{\tau(t)}{\dot{\tau}(t)} \ln(1-\dot{\tau}(t))} \int_0^1 \frac{h(\sigma)\tau(t)}{1-\dot{\tau}(t)\sigma} e^{-\lambda \frac{\tau(t)}{\dot{\tau}(t)} \ln(1-\dot{\tau}(t)\sigma)} d\sigma \\ &= \lambda u(\pi) e^{\lambda \frac{\tau(t)}{\dot{\tau}(t)} \ln(1-\dot{\tau}(t))} + z^0, \end{aligned}$$

where

$$z^0 = -f(\pi) e^{\lambda \frac{\tau(t)}{\dot{\tau}(t)} \ln(1-\dot{\tau}(t))} + e^{\lambda \frac{\tau(t)}{\dot{\tau}(t)} \ln(1-\dot{\tau}(t))} \int_0^1 \frac{h(\sigma)\tau(t)}{1-\dot{\tau}(t)\sigma} e^{-\lambda \frac{\tau(t)}{\dot{\tau}(t)} \ln(1-\dot{\tau}(t)\sigma)} d\sigma$$

depends only on f and h.

It remains to find u. By (40), u must satisfy

$$\lambda^2 u - a u_{xx} = g + \lambda f.$$

Multiplying this identity by a test function ϕ, integrating in space and using integration by parts, we obtain

$$\int_0^\pi (\lambda^2 u - a u_{xx})\phi dx = \int_0^\pi (\lambda^2 u\phi + a u_x \phi_x) dx - a u_x(\pi)\phi(\pi) + a u_x(0)\phi(0).$$

But using the fact that $(u,\, \omega,\, z)^\top$ must belong to $D(\mathcal{A}(t))$, we have

$$\int_0^\pi (\lambda^2 u - a u_{xx})\phi dx = \int_0^\pi (\lambda^2 u\phi + a u_x \phi_x) dx + a\mu_0 \omega(\pi)\phi(\pi) + a\mu_1 z(1)\phi(\pi).$$

Therefore

$$\int_0^\pi (\lambda^2 u\phi + a u_x \phi_x) dx + a\mu_0 \omega(\pi)\phi(\pi) + a\mu_1 z(1)\phi(\pi) = \int_0^\pi (g + \lambda f)\phi dx.$$

Using the above expression for $z(1)$ and $\omega = \lambda u - f$, we arrive at the problem

$$\int_0^\pi (\lambda^2 u\phi + a u_x \phi_x) dx + a(\mu_0 + \mu_1 e^{-\lambda\tau(t)})\lambda u(\pi)\phi(\pi) \\ = \int_0^\pi (g + \lambda f)\phi dx + a(\mu_0 f(\pi) - \mu_1 z^0)\phi(\pi),\ \forall \phi \in V \quad (41)$$

if $\dot{\tau}(t) = 0$, or otherwise

$$\int_0^\pi (\lambda^2 u\phi + a u_x \phi_x) dx + a(\mu_0 + \mu_1 e^{\lambda \frac{\tau(t)}{\dot{\tau}(t)} \ln(1-\dot{\tau}(t)\rho)})\lambda u(\pi)\phi(\pi) \\ = \int_0^\pi (g + \lambda f)\phi dx + a(\mu_0 f(\pi) - \mu_1 z^0)\phi(\pi),\ \forall \phi \in V. \quad (42)$$

These problems have a unique solution $u \in V$ by Lax-Milgram's lemma, because the left-hand side of (41) or (42) is coercive on V.

If we consider $\phi \in \mathcal{D}(0,\, \pi) \subset V$, then u satisfies

$$\lambda^2 u - a u_{xx} = g + \lambda f \text{ in } \mathcal{D}'(0,\, \pi).$$

This directly implies that $u \in H^2(0,\, \pi)$ and then $u \in V \cap H^2(0,\, \pi)$. Coming back to (41) and by integrating by parts, we find, for $\dot{\tau}(t) = 0$,

$$a[u_x(\pi) + (\mu_0 + \mu_1 e^{-\lambda\tau(t)})\lambda u(\pi)]\phi(\pi) = a(\mu_0 f(\pi) - \mu_1 z^0)\phi(\pi),$$

and then

$$\begin{aligned} u_x(\pi) &= -(\mu_0 + \mu_1 e^{-\lambda\tau(t)})\lambda u(\pi) - (\mu_1 z^0 - \mu_0 f(\pi)) \\ &= -\mu_0(\lambda u(\pi) - f(\pi)) - \mu_1(e^{-\lambda\tau(t)}\lambda u(\pi) + z^0) \\ &= -\mu_0\omega(\pi) - \mu_1 z(1). \end{aligned}$$

We find the same result if $\dot{\tau}(t) \neq 0$.

In summary we have found $(u,\, \omega,\, z)^\top \in D(\mathcal{A}(t))$ satisfying (40) and thus $\lambda I - \mathcal{A}(t)$ is surjective for some $\lambda > 0$ and $t > 0$. Again as $\kappa(t) > 0$, this proves that

$$\lambda I - \tilde{\mathcal{A}}(t) = (\lambda + \kappa(t))I - \mathcal{A}(t) \text{ is surjective} \quad (43)$$

for some $\lambda > 0$ and $t > 0$.

Then, (37), (38) and (43) imply that the family $\tilde{\mathcal{A}} = \{\tilde{\mathcal{A}}(t) : t \in [0,\, T]\}$ is a stable family of generators in H with stability constants independent of t, by Proposition 1.1 from [7]. Therefore, the assumptions (i)-(iv) of Theorem 2.1 are verified by (36), (37), (38), (39), (43) and Lemma 3.1, and thus, the problem

$$\begin{cases} \tilde{U}_t = \tilde{\mathcal{A}}(t)\tilde{U} \\ \tilde{U}(0) = U_0. \end{cases}$$

has a unique solution $\tilde{U} \in C([0, +\infty), H)$ and, if $U_0 \in D(\mathcal{A}(0))$,

$$\tilde{U} \in C([0, +\infty), D(\mathcal{A}(0))) \cap C^1([0, +\infty), H).$$

As before, the requested solution of (35) is given by

$$U(t) = e^{\beta(t)}\tilde{U}(t)$$

with $\beta(t) = \int_0^t \kappa(s)ds$. □

3.2. **The decay of the energy.** As for the heat equation, we restrict the hypothesis (10) to obtain the decay of the energy. Namely we suppose that (21) holds.

Let us choose the following energy (which corresponds to the time-dependent inner product in H)

$$E(t) = \frac{1}{2}\int_0^\pi \left(u_t^2(x, t) + au_x^2(x, t)\right) dx + \frac{q\tau(t)}{2}\int_0^1 u_t^2(\pi, t - \tau(t)\rho)d\rho, \tag{44}$$

where q is a positive constant chosen such that Ψ_q is negative definite (possible if (9) and (21) hold).

Proposition 2. *Let the assumptions (9) and (21) be satisfied. Then for all regular solution of problem (32), the energy is decreasing and verifies*

$$E'(t) \leq (u_t(\pi, t), u_t(\pi, t - \tau(t)))\, \Psi_q\, (u_t(\pi, t), u_t(\pi, t - \tau(t)))^\top < 0, \tag{45}$$

where Ψ_q is the matrix defined in (13).

Remark 3. In the case where the delay is constant in time, i.e. $\tau(t) = \tau$ for all $t > 0$ and thus $d = 0$, we recover the results from [10, 12]. Indeed in [10, 12], the energy is decreasing under the assumption $\mu_1 < \mu_0$.

Proof. Differentiating (44) and integrating by parts in space, we obtain

$$\begin{aligned} E'(t) &= \int_0^\pi (u_t u_{tt} + au_x u_{xt})dx + \frac{q\dot{\tau}(t)}{2}\int_0^1 u_t^2(\pi, t - \tau(t)\rho)d\rho \\ &\quad + q\tau(t)\int_0^1 u_t(\pi, t - \tau(t)\rho)u_{tt}(\pi, t - \tau(t)\rho)(1 - \dot{\tau}(t)\rho)d\rho \\ &= \int_0^\pi u_t(u_{tt} - au_{xx})dx + a[u_x u_t]_0^\pi + \frac{q\dot{\tau}(t)}{2}\int_0^1 u_t^2(\pi, t - \tau(t)\rho)d\rho \\ &\quad + q\tau(t)\int_0^1 u_t(\pi, t - \tau(t)\rho)u_{tt}(\pi, t - \tau(t)\rho)(1 - \dot{\tau}(t)\rho)d\rho \\ &= au_x(\pi, t)u_t(\pi, t) + \frac{q\dot{\tau}(t)}{2}\int_0^1 u_t^2(\pi, t - \tau(t)\rho)d\rho \\ &\quad + q\tau(t)\int_0^1 u_t(\pi, t - \tau(t)\rho)u_{tt}(\pi, t - \tau(t)\rho)(1 - \dot{\tau}(t)\rho)d\rho. \end{aligned}$$

Recalling that $z(\rho, t) = u_t(\pi, t - \tau(t)\rho)$, we see that

$$\begin{aligned} \int_0^1 u_t(\pi, t - \tau(t)\rho)u_{tt}(\pi, t - \tau(t)\rho)(1 - \dot{\tau}(t)\rho)d\rho &= -\frac{\dot{\tau}(t)}{2\tau(t)}\int_0^1 u_t^2(\pi, t - \tau(t)\rho)d\rho \\ &\quad - \frac{1 - \dot{\tau}(t)}{2\tau(t)}u_t^2(\pi, t - \tau(t)) + \frac{1}{2\tau(t)}u_t^2(\pi, t). \end{aligned}$$

Therefore, we obtain

$$
\begin{aligned}
E'(t) &= a u_x(\pi, t) u_t(\pi, t) + \frac{q\dot{\tau}(t)}{2}\int_0^1 u_t^2(\pi, t-\tau(t)\rho)d\rho \\
&\quad -\frac{q\dot{\tau}(t)}{2}\int_0^1 u_t^2(\pi, t-\tau(t)\rho)d\rho - q\frac{1-\dot{\tau}(t)}{2}u_t^2(\pi, t-\tau(t)) + \frac{q}{2}u_t^2(\pi, t),
\end{aligned}
$$

which implies

$$
\begin{aligned}
E'(t) = -a\mu_0 u_t^2(\pi, t) - a\mu_1 u_t(\pi, t-\tau(t))u_t(\pi, t) - \frac{q}{2}(1-\dot{\tau}(t))u_t^2(\pi, t-\tau(t)) \\
+ \frac{q}{2}u_t^2(\pi, t).
\end{aligned}
$$

By the condition (9) we can see that this identity yields

$$
E'(t) \leq (u_t(\pi, t), u_t(\pi, t-\tau(t)))\, \Psi_q \, (u_t(\pi, t), u_t(\pi, t-\tau(t)))^\top .
$$

This concludes the proof as Ψ_q is negative definite. □

3.3. **Exponential stability.** In this section, under the assumptions (9) and (21), we prove the exponential stability of the wave equation (32) by using the following Lyapunov functional

$$
\mathcal{E}(t) = E(t) + \gamma\left(2\int_0^\pi x u_t u_x dx + \mathcal{E}_2(t)\right), \tag{46}
$$

where $\gamma > 0$ is a parameter that will be fixed small enough later on, E is the standard energy defined by (44) with q a positive constant fixed such that Ψ_q is negative definite and $\mathcal{E}_2$ is defined by

$$
\mathcal{E}_2(t) = q\int_{t-\tau(t)}^t e^{2\delta(s-t)}u_t^2(\pi, s)ds = q\tau(t)\int_0^1 e^{-2\delta\tau(t)\rho}u_t^2(\pi, t-\tau(t)\rho)d\rho, \tag{47}
$$

where $\delta > 0$ is a fixed positive real number.

The Lyapunov functional $E(t) + 2\gamma\int_0^\pi x u_t u_x dx$ is standard in problems with boundary conditions with memory (see for instance [11]). We have added the two terms to the standard energy $E(t)$ in order to take into account the dependence of τ with respect to t.

First we notice that the energies E and $\mathcal{E}$ are equivalent.

Lemma 3.3. *For γ small enough, there exist two positive constants $C_1(\gamma)$ and $C_2(\gamma)$ such that*

$$
C_1(\gamma)E(t) \leq \mathcal{E}(t) \leq C_2(\gamma)E(t).
$$

Proof. We have

$$
2\gamma\int_0^\pi x u_x u_t dx \leq \gamma\pi\int_0^\pi (u_x^2 + u_t^2)dx \leq \gamma\pi c\int_0^\pi (a u_x^2 + u_t^2)dx,
$$

where $c = \max(1, \frac{1}{a})$ and

$$
\gamma q\tau(t)\int_0^1 e^{-2\delta\tau(t)\rho}u_t^2(\pi, t-\tau(t)\rho)d\rho \leq \gamma q\tau(t)\int_0^1 u_t^2(\pi, t-\tau(t)\rho)d\rho.
$$

As $c \geq 1$, these estimates yield

$$
\mathcal{E}(t) \leq (1 + 2\gamma c\pi)E(t).
$$

Moreover, by definition we have

$$
\gamma\mathcal{E}_2(t) \geq 0
$$

and by Cauchy-Schwarz's inequality

$$2\gamma\int_0^\pi xu_xu_tdx \geq -\gamma\pi\int_0^\pi (u_x^2+u_t^2)dx.$$

Then

$$\mathcal{E}(t) \geq E(t) - c\gamma\pi\int_0^\pi (au_x^2+u_t^2)dx,$$

and therefore, for γ small enough ($\gamma < \frac{1}{2c\pi}$), we obtain

$$\mathcal{E}(t) \geq (1-2c\pi\gamma)E(t).$$

□

We are ready to state our result about the decay of the energy E:

Theorem 3.4. *Let (9) and (21) be satisfied. Then the energy E decays exponentially, more precisely there exist two positive constants α and C such that*

$$E(t) \leq Ce^{-\alpha t}E(0),\ \forall t > 0.$$

Proof. First we remark that

$$\begin{aligned}\frac{d}{dt}\left(2\int_0^\pi xu_tu_xdx\right) &= 2\int_0^\pi xu_{tt}u_xdx + 2\int_0^\pi xu_tu_{xt}dx \\ &= 2a\int_0^\pi xu_{xx}u_xdx + 2\int_0^\pi xu_tu_{xt}dx \\ &= a\int_0^\pi x\partial_x(u_x^2)dx + 2\int_0^\pi xu_tu_{xt}dx \\ &= -a\int_0^\pi u_x^2dx + a\pi u_x^2(\pi,\, t) + 2\int_0^\pi xu_tu_{xt}dx.\end{aligned}$$

But by integrating by parts in space and by (32), we have

$$\int_0^\pi xu_tu_{xt}dx = -\int_0^\pi xu_{xt}u_tdx - \int_0^\pi u_t^2dx + \pi u_t^2(\pi,\, t),$$

that is to say

$$2\int_0^\pi xu_tu_{xt}dx = -\int_0^\pi u_t^2dx + \pi u_t^2(\pi,\, t).$$

Thus

$$\frac{d}{dt}\left(2\int_0^\pi xu_tu_xdx\right) = -\int_0^\pi (u_t^2+au_x^2)dx + \pi u_t^2(\pi,\, t) + a\pi u_x^2(\pi,\, t).$$

By the boundary conditions in (32) and Cauchy-Schwarz's inequality, we finally find

$$\begin{aligned}\frac{d}{dt}\left(2\int_0^\pi xu_tu_xdx\right) \leq -\int_0^\pi (u_t^2+au_x^2)dx + \pi(1+2a\mu_0^2)u_t^2(\pi,\, t) \\ + 2a\pi\mu_1^2u_t^2(\pi,\, t-\tau(t)).\end{aligned} \quad (48)$$

Then, we differentiate $\mathcal{E}_2$ to have

$$\frac{d}{dt}\mathcal{E}_2(t) = \frac{\dot\tau(t)}{\tau(t)}\mathcal{E}_2(t) + q\tau(t)\int_0^1 (-2\delta\dot\tau(t)\rho)e^{-2\delta\tau(t)\rho}u_t^2(\pi,\, t-\tau(t)\rho)d\rho + J_w,$$

where

$$J_w = 2q\tau(t)\int_0^1 e^{-2\delta\tau(t)\rho}u_t(\pi,\, t-\tau(t)\rho)u_{tt}(\pi,\, t-\tau(t)\rho)(1-\dot\tau(t)\rho)d\rho.$$

As in the proof of Theorem 2.4, by integration by parts, we have

$$\begin{aligned} J_w &= q\int_0^1 (-2\delta\tau(t)(1-\dot\tau(t)\rho)-\dot\tau(t))u_t(\pi,\, t-\tau(t)\rho)e^{-2\delta\tau(t)\rho}d\rho \\ &\quad -qe^{-2\delta\tau(t)}u_t^2(\pi,\, t-\tau(t))(1-\dot\tau(t))+qu_t^2(\pi,\, t). \end{aligned}$$

These identities yield

$$\begin{aligned} \frac{d}{dt}\mathcal{E}_2(t) &= \frac{\dot\tau(t)}{\tau(t)}\mathcal{E}_2(t)+q\int_0^1(-2\delta\tau(t)-\dot\tau(t))u_t^2(\pi,\, t-\tau(t)\rho)e^{-2\delta\tau(t)\rho}d\rho \\ &\quad -qe^{-2\delta\tau(t)}u_t^2(\pi,\, t-\tau(t))(1-\dot\tau(t))+qu_t^2(\pi,\, t) \\ &= \frac{\dot\tau(t)}{\tau(t)}\mathcal{E}_2(t)+\frac{-2\delta\tau(t)-\dot\tau(t)}{\tau(t)}\mathcal{E}_2(t)-qe^{-2\delta\tau(t)}u_t^2(\pi,\, t-\tau(t))(1-\dot\tau(t)) \\ &\quad +qu_t^2(\pi,\, t). \end{aligned}$$

As $\dot\tau(t)<1$ (see (2)), we obtain

$$\frac{d}{dt}\mathcal{E}_2(t)\le -2\delta\mathcal{E}_2(t)+qu_t^2(\pi,\, t). \tag{49}$$

Consequently, gathering (46), (48) and (49), we obtain

$$\begin{aligned} &\frac{d}{dt}\mathcal{E}(t)\le -\gamma\int_0^\pi(u_t^2+au_x^2)dx-2\gamma\delta\mathcal{E}_2(t) \\ &+(u_t(\pi,\, t),\, u_t(\pi,\, t-\tau(t)))\,\tilde\Phi_q\,(u_t(\pi,\, t),\, u_t(\pi,\, t-\tau(t)))^\top, \end{aligned}$$

where $\tilde\Phi_q$ is the matrix defined by

$$\begin{aligned} \tilde\Phi_q &= \tfrac12\begin{pmatrix} q(1+2\gamma)-2a\mu_0+2\gamma\pi(1+2a\mu_0^2) & -a\mu_1 \\ -a\mu_1 & 4a\gamma\pi\mu_1^2-q(1-d) \end{pmatrix} \\ &= \Psi_q+\gamma\begin{pmatrix} q+\pi(1+2a\mu_0^2) & 0 \\ 0 & 2a\pi\mu_1^2 \end{pmatrix}. \end{aligned}$$

Noticing that $\max\left(q+\pi(1+2a\mu_0^2),\, 2a\pi\mu_1^2\right)=q+\pi(1+2a\mu_0^2)$ by (21), for γ sufficiently small, i.e. $\gamma\le\frac{-\lambda}{q+\pi(1+2a\mu_0^2)}$, where λ is the greatest negative eigenvalue of Ψ_q given by (30), $\tilde\Phi_q$ is negative and therefore

$$\frac{d}{dt}\mathcal{E}(t)\le-\gamma\int_0^\pi(u_t^2+au_x^2)dx-2\delta\gamma\mathcal{E}_2(t).$$

By the definition (47) of $\mathcal{E}_2$, this estimate becomes

$$\frac{d}{dt}\mathcal{E}(t)\le-\gamma\int_0^\pi(u_t^2+au_x^2)dx-2\delta\gamma q\tau(t)e^{-2\delta\tau(t)}\int_0^1 u_t^2(\pi,\, t-\tau(t)\rho)d\rho.$$

Since $\tau(t)\le M$ (see (3)), in view of the definition of E, there exists a constant $\gamma'>0$ (depending on γ and δ: $\gamma'\le 2\gamma\min\left(1,\, 2\delta e^{-2\delta M}\right)$) such that

$$\frac{d}{dt}\mathcal{E}(t)\le-\gamma' E(t).$$

By applying Lemma 3.3, we arrive at

$$\frac{d}{dt}\mathcal{E}(t)\le-\alpha\mathcal{E}(t),$$

where α is explicitely given by $\alpha=\frac{\gamma'}{1+2\gamma c\pi}$. Therefore

$$\mathcal{E}(t)\le\mathcal{E}(0)e^{-\alpha t},$$

and Lemma 3.3 allows to conclude the proof:

$$E(t) \leq \frac{1}{1-2c\pi\gamma}\mathcal{E}(t) \leq \frac{1}{1-2c\pi\gamma}\mathcal{E}(0)e^{-\alpha t} \leq \frac{1+2\gamma c\pi}{1-2c\pi\gamma}E(0)e^{-\alpha t}.$$

□

Remark 4. In the case where the delay is constant in time and $a = 1$, we recover some results from [10, 12]. Moreover, in [10, 12], the energy is defined by

$$E(t) = \frac{1}{2}\int_0^\pi (u_t^2(x,\, t) + u_x^2(x,\, t))dx + \frac{\xi}{2}\int_0^1 u_t^2(\pi,\, t-\tau\rho)d\rho,$$

where ξ is a positive constant satisfying

$$\tau\mu_1 \leq \xi \leq \tau(2\mu_0 - \mu_1),$$

under the condition (21), which corresponds to the definition (44) of E with $q = \frac{\xi}{\tau}$.

Remark 5. In the proof of Theorem 3.4, we notice that we can explicitely calculate the decay rate α of the energy, given by

$$\alpha = \frac{2\gamma}{1+2\gamma c\pi}\min\left(1,\, 2\delta e^{-2\delta M}\right),$$

with

$$\gamma < \frac{1}{2c\pi} \text{ and } \gamma \leq \frac{-\lambda}{q+\pi(1+2a\mu_0^2)}$$

(by Lemma 3.3 and Theorem 3.4) and $c = \max\left(1,\, \frac{1}{a}\right)$ where λ is given by (30), q by (25) and δ is a positive real number. Therefore, we can choose δ such that the decay of the energy is as quick as possible. By Remark 2, we get that the larger decay rate of the energy is given by

$$\alpha_{max} = \frac{2\gamma}{1+2\gamma c\pi}\min\left(1,\, \frac{1}{Me}\right).$$

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Received April 2008; revised September 2008.

E-mail address: `Serge.Nicaise@univ-valenciennes.fr`
E-mail address: `Julie.Valein@univ-valenciennes.fr`
E-mail address: `emilia@eng.tau.ac.il`