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SAMPLED-DATA DISTRIBUTED H_∞ CONTROL OF TRANSPORT REACTION SYSTEMS*

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Abstract. We develop, for the first time, sampled-data distributed H_∞ control of a class of parabolic systems. These systems are governed by one-dimensional semilinear transport reaction equations with additive disturbances. A network of stationary sensing devices provides spatially averaged state measurements over the N sampling spatial intervals. We suggest a sampled-data controller design, where the sampling intervals in time and in space are bounded. Our sampled-data static output feedback enters the equation through N shape functions (which are localized in the space) multiplied by the corresponding state measurements. Sufficient conditions for the internal exponential stability and for L_2-gain analysis of the closed-loop system are derived via direct Lyapunov method in terms of linear matrix inequalities (LMIs). By solving these LMIs, upper bounds on the sampling intervals that preserve the internal stability and the resulting L_2-gain can be found. Numerical examples illustrate the efficiency of the method.

Key words. sampled-data control, H_∞ control, distributed parameter systems, Lyapunov method, linear matrix inequalities

AMS subject classifications. 93C57, 93C20, 93B36, 93D15, 93D30

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1. Introduction.

Many important plants, such as chemical reactors and heat transfer processes, are governed by partial differential equations and are often described by uncertain models. The existing results [4, 9, 19, 25] on robust control of uncertain distributed parameter systems extend the state space or the frequency domain H_∞ approach and are confined to the linear case. It is thus of interest to develop consistent methods that are capable of providing the desired performance of distributed parameter systems in spite of significant model uncertainties. The linear matrix inequalities (LMI) approach [5] is definitely among such methods. In the recent papers [16, 17] an LMI approach was introduced for the stability analysis of linear heat and wave equations and for the boundary (continuous-time) H_∞ control of uncertain distributed parameter systems.

We develop H_∞ sampled-data controllers for parabolic systems governed by semilinear convection-diffusion equations with distributed control. Such systems are stabilizable by linear infinite-dimensional state-feedback controllers. For a realistic design, finite-dimensional realizations [1, 6, 22, 30] can be applied. However, finite-dimensional control, which employs, e.g., Galerkin truncation, may lead to local stability results [30]. For linear parabolic systems, mobile collocated sensors and actuators (see [10] and references therein) or adaptive controllers [27, 32] can be used. The above methods are not applicable to the performance analysis.

In many practical applications, measurements of the process outputs are typically available from the sensors at discrete times and not continuously. The frequency at which the measurements are available is dictated by the sampling rate, which is typically constrained by the limitations on the data collection and processing capabilities

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of the sensing devices. In some applications, constraints on the sampling rate may be imposed by the designer to limit the transfer of data over some bandwidth-limited communication channel that connects the sensor and the controller.

Sampled-data H_∞ control of finite-dimensional systems has been studied extensively in recent decades (see, e.g., [7, 3] and the references therein). Three main approaches have been used for sampled-data H_∞ control. The first is based on the lifting technique [2, 36], where the problem is transformed to an equivalent finite-dimensional discrete problem. However, this method may become difficult in the case of uncertain sampling times or uncertain system matrices. The second approach is based on the representation of the system in the form of a hybrid discrete/continuous model [31, 3]. The third is the time-delay approach, where the sampled-data system is presented as a continuous one with input/output delay [18, 33, 13].

Most of the existing references on sampled-data control of distributed parameter systems [8, 28, 29, 34] develop the discrete-time approach treating linear time-invariant systems. Also, a model-reduction-based approach to sampled-data control has been introduced in [20], where a finite-dimensional controller was designed on the basis of a finite-dimensional system that captures the dominant (slow) dynamics of the infinite-dimensional system. The latter approach seems to be not applicable to systems with spatially dependent diffusion coefficients and with uncertain nonlinear terms. The existing sampled-data results are not applicable to the performance analysis of the closed-loop system. In the recent paper [15] sampled-data stabilization of uncertain diffusion equations under the discrete in time and in space measurements was studied. Sufficient conditions in terms of LMIs for the exponential stability with a given decay rate have been derived in [15] via a combination of Lyapunov–Krasovskii functionals and Halanay's inequalities. However, Halanay's inequality is not applicable to the L_2-gain analysis.

In the present paper we study, for the first time, H_∞ sampled-data control of distributed parameter systems. We consider a class of parabolic systems governed by one-dimensional semilinear diffusion-convection equations under the averaged in space measurements. Following [11], we assume that a large number of "point" spatial measurements of the state (e.g., temperature of the rod throughout the reactor) are available so that the averaged measurements are known with sufficient accuracy. Note that a sensor cannot measure exactly in one point: the measuring device relies on some physical phenomenon and, in fact, the sensor measures an average over a certain region occupied by the measuring device. Therefore, we assume that a network of stationary sensing devices provides spatially averaged state measurements over the N sampling spatial intervals. The measurements are supposed to be sampled-data in time, where the sampling intervals in time and in space are bounded.

Our sampled-data static output feedback enters the equation through N shape functions (which are localized in the space) multiplied by the corresponding state measurements. The controller can be implemented by N stationary actuators and by zero-order hold devices. Sufficient conditions for H_∞ control are derived in terms of LMIs in the framework of time-delay approach to sampled-data systems. Some preliminary results will be presented in [14].

1.1. Notation and preliminaries. Throughout the paper $\mathbf{R}^n$ denotes the n-dimensional Euclidean space with the norm $|\cdot|$, $\mathbf{R}^{n\times m}$ is the set of all $n\times m$ real matrices, and the notation $P>0$ with $P\in\mathbf{R}^{n\times n}$ means that P is symmetric and positive definite. The symmetric elements of the symmetric matrix will be denoted by $*$. Functions, continuous (n times continuously differentiable) in all arguments, are referred to as of class C (of class C^n). $L_2(0,l)$ is the Hilbert space of square integrable functions $z(x),\ x\in[0,l]$ with the corresponding norm $\|z\|_{L_2}=\sqrt{\int_0^l z^2(x)dx}$. $L_2(t_0,\infty;L_2(a,b))$ is the Hilbert space of square integrable functions $w:(t_0,\infty)\to L_2(a,b)$ with the corresponding norm

$$\|w\|^2_{L_2(t_0,\infty;L_2(a,b))}=\int_{t_0}^{\infty}\int_0^l w^2(x,t)dxdt<\infty.$$

$\mathcal{H}^1(0,l)$ is the Sobolev space of absolutely continuous scalar functions $z:[0,l]\to R$ with $\frac{dz}{dx}\in L_2(0,l)$. $\mathcal{H}^2(0,l)$ is the Sobolev space of scalar functions $z:[0,l]\to R$ with absolutely continuous $\frac{dz}{dx}$ and with $\frac{d^2z}{dx^2}\in L_2(0,l)$.

For later use, we recall Wirtinger's inequality [23]. Let $g:R\to R$ be 2π-periodic absolutely continuous function with $\frac{dg}{dx}\in L_2(0,2\pi)$ and $\int_0^{2\pi}g(x)dx=0$. Then

$$\int_0^{2\pi}g^2(x)dx\le\int_0^{2\pi}\left[\frac{dg}{dx}(x)\right]^2dx.$$

Assume now that $g\in\mathcal{H}^1(0,\pi)$ is a scalar function with $\int_0^{\pi}g(x)dx=0$. Its symmetric continuation $g(x)\triangleq g(2\pi-x)$ for $x\in[\pi,2\pi]$ results in $g(2\pi)=g(0)$ and thus it can be 2π-periodically continued. Moreover, $\int_0^{2\pi}g^2(x)dx=2\int_0^{\pi}g^2(x)dx$ and $\int_0^{2\pi}[\frac{dg}{dx}(x)]^2dx=2\int_0^{\pi}[\frac{dg}{dx}(x)]^2dx$. Therefore, Wirtinger's inequality leads to

$$\int_0^{\pi}g^2(x)dx\le\int_0^{\pi}\left[\frac{dg}{dx}(x)\right]^2dx.$$

By changing variable we arrive at the following form of Wirtinger's inequality.

LEMMA 1.1. *Let $g\in\mathcal{H}^1(0,l)$ be a scalar function with $\int_0^l g(x)dx=0$. Then*

$$\int_0^l g^2(x)dx\le\frac{l^2}{\pi^2}\int_0^l\left[\frac{dg}{dx}(x)\right]^2dx. \tag{1.1}$$

2. Problem formulation. In the present paper we will start with the sampled-data stabilization of the semilinear convection-diffusion equation

$$z_t(x,t)=\frac{\partial}{\partial x}[a(x)z_x(x,t)]-\beta z_x(x,t)+\phi(z(x,t),x,t)z(x,t)+u(x,t),$$
$$t\ge t_0,\ x\in[0,l], \tag{2.1}$$

where subindexes denote the corresponding partial derivatives. In (2.1) $u(x,t)$ is the control input. The functions a and ϕ are of class C^1 and may be unknown, and the convection coefficient β may be unknown as well. It is assumed that a,ϕ, and β satisfy the inequalities

$$a\ge a_0>0,\ \phi_m\le\phi\le\phi_M,\ 0\le\beta\le\beta_M, \tag{2.2}$$

where $a_0,\phi_m,\ \phi_M$, and β_M are known bounds.

We consider (2.1) under the Dirichlet

$$z(0,t)=z(l,t)=0 \tag{2.3}$$

or under the mixed

$$a(0)z_x(0,t)=g_0z(0,t),\ a(l)z_x(l,t)=-g_lz(l,t) \tag{2.4}$$

boundary conditions, where constants g_0 and g_l satisfy the following bounds:

$$2g_0 > \beta_M, \quad g_l \geq 0. \tag{2.5}$$

It is well known that the open-loop system (2.1) under the above boundary conditions is unstable if ϕ_M is big enough and that a linear infinite-dimensional state feedback $u(x,t) = -Kz(x,t)$ with big enough $K > 0$ exponentially stabilizes the system [9]. In the recent paper [15] the sampled-data controller design under the discrete in time and in space measurements has been considered. However, the results of [15] are not extendable to the H_∞ control under the discrete in time measurements. In the present paper we study, for the first time, sampled-data (in time) H_∞ control under the averaged in space measurements.

Consider (2.1) under the boundary conditions (2.3) or (2.4). Let the points $0 = x_0 < x_1 < \cdots < x_N = l$ divide $[0,l]$ into N sampling intervals. Let $t_0 < t_1 < \cdots < t_k \ldots$ with $\lim_{k\to\infty} t_k = \infty$ be sampling time instants. The sampling intervals in time and in space may be variable but bounded,

$$0 \leq t_{k+1} - t_k \leq h, \; x_{j+1} - x_j \triangleq \Delta_j \leq \Delta. \tag{2.6}$$

We assume that sensors provide the averaged state measurement on the jth spatial interval $[x_j, x_{j+1}]$ at the sampling instant t_k:

$$\begin{aligned} y_{jk} &= \frac{\int_{x_j}^{x_{j+1}} z(\xi, t_k) d\xi}{\Delta_j}, \quad \Delta_j = x_{j+1} - x_j, \\ & j = 0, \ldots, N-1, \; t \in [t_k, t_{k+1}), \; k = 0, 1, 2 \ldots. \end{aligned} \tag{2.7}$$

Our objective is to design for (2.1) an exponentially stabilizing sampled-data controller

$$\begin{aligned} u(x,t) &= -Ky_{jk} = -K\frac{\int_{x_j}^{x_{j+1}} z(\xi, t_k) d\xi}{\Delta_j}, \\ & x \in [x_j, x_{j+1}), \quad j = 0, \ldots, N-1, \quad t \in [t_k, t_{k+1}), \quad k = 0, 1, 2 \ldots, \end{aligned} \tag{2.8}$$

with the gain $K > 0$. The closed-loop system (2.1), (2.8) has the form

$$\begin{aligned} z_t(x,t) &= \frac{\partial}{\partial x}[a(x) z_x(x,t)] - \beta z_x(x,t) \\ & - K\frac{\int_{x_j}^{x_{j+1}} z(\xi, t_k) d\xi}{\Delta_j} + \phi(z(x,t), x, t) z(x,t), \\ & t \in [t_k, t_{k+1}), \; k = 0, 1, 2 \ldots, \; x_j \leq x < x_{j+1}, \; j = 0, \ldots, N-1. \end{aligned} \tag{2.9}$$

We shall use the elementary relation

$$\frac{\int_{x_j}^{x_{j+1}} z(\xi, t_k) d\xi}{\Delta_j} = z(x,t) - f(x,t) - (t - t_k)\rho_j, \quad x_j \leq x < x_{j+1}, \; j = 0, \ldots, N-1,$$

where

$$f(x,t) \triangleq \frac{\int_{x_j}^{x_{j+1}} [z(x,t) - z(\xi,t)] d\xi}{\Delta_j}, \quad \rho_j \triangleq \frac{1}{t - t_k} \frac{\int_{x_j}^{x_{j+1}} \int_{t_k}^{t} z_s(\xi, s) ds d\xi}{\Delta_j}. \tag{2.10}$$

Note that f is piecewise continuous in x and $f_x = z_x$ for $x \neq x_k$. By $\rho_{j|t=t_k}$ we understand the following: $\lim_{t\to t_k^+} \rho_j = \int_{x_j}^{x_{j+1}} z_t(\xi, t_k) d\xi / \Delta_j$.

Then (2.9) can be represented as

$$
\begin{aligned}
z_t(x,t) = {} & \frac{\partial}{\partial x}[a(x)z_x(x,t)] - \beta z_x(x,t) + [\phi(z(x,t),x,t) - K]z(x,t) \\
& + K[f(x,t) + (t-t_k)\rho_j], \\
& x_j \le x < x_{j+1},\ j = 0,\dots,N-1,\ t \in [t_k, t_{k+1}),\ k = 0,1,2\dots.
\end{aligned} \tag{2.11}
$$

By the arguments of [15] it can be shown that there exists a unique strong solution of (2.11), (2.3) (or (2.4)) initialized with $z(\cdot,t_0) \in \mathcal{H}^1(0,l)$, satisfying the boundary conditions (see Appendix A).

We will first study the exponential stability analysis of (2.11) via the Lyapunov–Krasovskii method. Next, sampled-data H_∞ control of the perturbed version of (2.1) under the perturbed measurements (2.7) will be considered. Finally, the results will be extended to the coupled system of convection-diffusion equations studied, e.g., in [30].

Remark 2.1. In [15] sampled-data stabilization of (2.1) with $\beta = 0$ was considered under the discrete in time and in space measurements of the state:

$$
y^d_{jk} = z(\bar{x}_j, t_k),\ \bar{x}_j = \frac{x_{j+1}+x_j}{2}, \quad j = 0,\dots,N-1,\ t \in [t_k,t_{k+1}),\ k = 0,1,2\dots. \tag{2.12}
$$

The control law $u^d(x,t) = -Ky^d_{jk},\ x_j \le x < x_{j+1}, t \in [t_k,t_{k+1})$ leads to the following closed-loop system:

$$
\begin{aligned}
z_t(x,t) = {} & \frac{\partial}{\partial x}[a(x)z_x(x,t)] + \phi(z(x,t),x,t)z(x,t) - Kz(\bar{x}_j,t_k), \\
& t \in [t_k,t_{k+1}),\ k = 0,1,2\dots\ x_j \le x < x_{j+1},\ j = 0,\dots,N-1.
\end{aligned} \tag{2.13}
$$

Similar to (2.11), (2.13) can be represented as

$$
\begin{aligned}
z_t(x,t) = {} & \frac{\partial}{\partial x}[a(x)z_x(x,t)] + \phi(z(x,t),x,t)z(x,t) \\
& - Kz(\bar{x}_j,t) + K[z(\bar{x}_j,t) - z(\bar{x}_j,t_k)],
\end{aligned} \tag{2.14}
$$

but it is not clear how to treat the last term of (2.14) via the Lyapunov method. A different representation,

$$
\begin{aligned}
z_t(x,t) = {} & \frac{\partial}{\partial x}[a(x)z_x(x,t)] + \phi(z(x,t),x,t)z(x,t) - Kz(x,t_k) \\
& + K[z(x,t_k) - z(\bar{x}_j,t_k)],
\end{aligned}
$$

was considered in [15], where the Lyapunov–Krasovskii method was combined with Halanay's inequality. However, Halanay's inequality is not applicable to H_∞ control.

Note that the continuous in time results of [15] under the discrete in space measurements (that do not use Halanay's inequality) can be extended to H_∞ control, similar to Proposition 4.1 below.

3. Sampled-data H_∞ control of diffusion equations.

3.1. Continuous-time exponential stabilization. We will start with the stabilization under the continuous-time measurements

$$
\begin{aligned}
y_j(x,t) = {} & \frac{\int_{x_j}^{x_{j+1}} z(\xi,t)d\xi}{\Delta_j},\ x \in [x_j,x_{j+1}), \\
& \Delta_j = x_{j+1} - x_j \le \Delta,\ j = 0,\dots,N-1,\ t \ge t_0,
\end{aligned}
$$

via a continuous-time controller

$$u(x,t) = -K\frac{\int_{x_j}^{x_{j+1}} z(\xi,t)d\xi}{\Delta_j}, \quad x \in [x_j, x_{j+1}), \quad j = 0, \dots, N-1. \tag{3.1}$$

Consider the closed-loop system

$$z_t(x,t) = \frac{\partial}{\partial x}[a(x)z_x(x,t)] - \beta z_x(x,t) + [\phi(z(x,t),x,t) - K]z(x,t) + Kf(x,t),$$
$$x_j \le x < x_{j+1},\ j = 0,\dots,N-1,\ t \in [t_k, t_{k+1}),\ k = 0,1,2\dots, \tag{3.2}$$

where f is defined in (2.10), under the Dirichlet or the mixed boundary conditions. By using the Lyapunov function

$$V(t) = \int_0^l z^2(x,t)dx, \tag{3.3}$$

we will derive conditions that guarantee $\dot V(t) + 2\delta V(t) \le 0$, where $\delta > 0$ is some scalar, along (3.2), (2.3) (or (2.4)). The latter inequality yields $V(t) \le e^{-2\delta(t-t_0)}V(t_0)$ or

$$\int_0^l z^2(x,t)dx \le e^{-2\delta(t-t_0)} \int_0^l z^2(x,t_0)dx \tag{3.4}$$

for the strong solutions of (3.2), (2.3) (or (2.4)) initialized with $z(\cdot,t_0) \in \mathcal{H}^1(0,l)$, satisfying the boundary conditions. If (3.4) holds, we will say that (3.2) under (2.3) (or (2.4)) is exponentially stable with the decay rate δ.

Differentiating V along (3.2) we find

$$\begin{aligned}\dot V(t) = 2\int_0^l z(x,t)z_t(x,t)dx &\le 2\int_0^l z(x,t)\Big[\tfrac{\partial}{\partial x}[a(x)z_x(x,t)] - \beta z_x(x,t) \\ &+ [\phi_M - K]z(x,t)\Big]dx + 2\sum_{j=0}^{N-1}\int_{x_j}^{x_{j+1}} Kz(x,t)f(x,t)dx.\end{aligned}$$

Integration by parts and substitution of the boundary conditions lead to

$$\begin{aligned}&2\int_0^l z(x,t)\{\tfrac{\partial}{\partial x}[a(x)z_x(x,t)] - \beta z_x(x,t)\}dx \\ &= 2az(x,t)z_x(x,t)\Big|_0^l - \beta z^2(x,t)\Big|_0^l - 2\int_0^l a(x)z_x^2(x,t)dx \\ &\le -[2g_l + \beta]z^2(l,t) - [2g_0 - \beta]z^2(0,t) - 2a_0\int_0^l z_x^2(x,t)dx\end{aligned} \tag{3.5}$$

under (2.4) and to

$$2\int_0^l z(x,t)\{\tfrac{\partial}{\partial x}[a(x)z_x(x,t)] - \beta z_x(x,t)\}dx \le -2a_0\int_0^l z_x^2(x,t)dx \tag{3.6}$$

under the Dirichlet boundary conditions. Therefore,

$$\begin{aligned}&\dot V(t) + 2\delta V(t) \\ &\le -2a_0\int_0^l z_x^2(x,t)dx + 2\sum_{j=0}^{N-1}\int_{x_j}^{x_{j+1}} Kz(x,t)f(x,t)dx \\ &\quad + 2\int_0^l (\delta + \phi_M - K)z^2(x,t)dx \\ &= \sum_{j=0}^{N-1}\int_{x_j}^{x_{j+1}} \Big[-2a_0z_x^2(x,t) + 2Kz(x,t)f(x,t) + 2[\delta + \phi_M - K]z^2(x,t)\Big]dx.\end{aligned} \tag{3.7}$$

Note that the function $f(x,t)$ has the zero average $\int_{x_j}^{x_{j+1}} f(x,t)dx = 0$ and $f_x = z_x$. Then, application of Wirtinger's inequality (1.1) yields

$$-2a_0 \int_{x_j}^{x_{j+1}} z_x^2(x,t)dx \le -\frac{2a_0\pi^2}{\Delta^2} \int_{x_j}^{x_{j+1}} f^2(x,t)dx. \tag{3.8}$$

Denote $\eta^T = [z(x,t) \quad f(x,t)]$. We find from (3.7), (3.8) that

$$\dot{V}(t) + 2\delta V(t) \le \sum_{j=0}^{N-1} \int_{x_j}^{x_{j+1}} \eta^T \Phi \eta dx \le 0 \tag{3.9}$$

if

$$\Phi \triangleq \begin{bmatrix} 2(\delta + \phi_M - K) & K \\ * & -\frac{2a_0\pi^2}{\Delta^2} \end{bmatrix} \le 0. \tag{3.10}$$

Note that inequality (3.10) is feasible for small enough $\delta > 0$ and $\Delta > 0$ iff $K > \phi_M$. Moreover, if given $\Delta > 0$ the strong version of (3.10), i.e., $\Phi < 0$ is feasible, then for small enough $\delta > 0$ we have $\Phi < 0$, i.e., (3.2) is exponentially stable. We have proved the following.

PROPOSITION 3.1. *Given $\Delta > 0$, let there exist $\delta > 0$ and $K > \phi_M$ such that the linear scalar inequalities* (3.10) *are feasible. Then the closed-loop system* (3.2) *under the Dirichlet or under the mixed boundary conditions is exponentially stable with the decay rate δ* (*in the sense of* (3.4))*. Moreover, if the strong inequality* (3.10) *is feasible for $\delta = 0$, then* (3.2) *is exponentially stable with a small enough decay rate.*

Remark 3.1. As in [15], under the Dirichlet boundary condition the system can be stabilized by a smaller gain $K > \phi_M - \frac{a_0\pi^2}{l^2}$. This can be seen from the following arguments. Applying Young's inequality to the cross terms in the right-hand side of (3.7), we have for any $R > 0$

$$2K \sum_{j=0}^{N-1} \int_{x_j}^{x_{j+1}} z(x,t)f(x,t)dx \le K \left[\frac{\Delta R}{\pi} \int_0^l z^2(x,t)dx \right.$$
$$\left. + \frac{\pi}{\Delta} R^{-1} \sum_{j=0}^{N-1} \int_{\bar{x}_j}^{x_{j+1}} f(x,t)^2 dx \right].$$

Then (3.7) together with Wirtinger's inequality (1.1) leads to

$$\dot{V}(t) + 2\delta V(t) \le \left(R^{-1}K\frac{\Delta}{\pi} - 2a_0 \right) \int_0^l z_x^2(x,t)dx$$
$$+ \left(RK\frac{\Delta}{\pi} + 2\delta + 2(\phi_M - K) \right) \int_0^l z^2(x,t)dx.$$

By taking advantage of the Dirichlet boundary condition (2.3), under which another Wirtinger's inequality holds,

$$\int_0^l z^2(x,t)dx \le \frac{l^2}{\pi^2} \int_0^l z_x^2(x,t)dx,$$

we conclude finally that $\dot{V}(t) + 2\delta V(t) \le 0$ if

$$R^{-1}K\frac{\Delta}{\pi} - 2a_0 \le 0, \quad RK\frac{\Delta}{\pi} + 2(\delta + \phi_M - K) + \frac{\pi^2}{l^2}\left(R^{-1}K\frac{\Delta}{\pi} - 2a_0 \right) \le 0.$$

The latter inequalities (which coincide with the conditions of Proposition 3.1 in [15]) are not LMIs due to R. Moreover, an additional Wirtinger's inequality cannot be applied under the mixed conditions (2.4). (Note that in [15] the mixed conditions were different with $z(l,t)=0$.) Therefore, in the present paper we consider $K>\phi_M$, where simple LMIs will be derived for both boundary conditions.

3.2. Sampled-data exponential stabilization. Consider the Lyapunov functional

$$
\begin{aligned}
V(t) = {} & p_1\int_0^l z^2(x,t)dx + \int_0^l [a(x)p_3 z_x^2(x,t) \\
& + r(t_{k+1}-t)\int_{t_k}^t e^{2\delta(s-t)} z_s^2(x,s)ds]dx + qz^2(0,t) + q_l z^2(l,t), \\
& t\in[t_k,t_{k+1}),\quad p_3>0,\ p_1>0,\ r>0,
\end{aligned}
\tag{3.11}
$$

where $q=q_l=0$ corresponds to the Dirichlet and $q=p_3g_0, q_l=p_3g_l$ corresponds to the mixed boundary conditions. It is continuous in time since

$$
\begin{aligned}
V(t_k) = {} & p_1\int_0^l z^2(x,t_k)dx + \int_0^l a(x)p_3 z_x^2(x,t_k)dx \\
& + qz^2(0,t_k) + q_l z^2(l,t_k) = V(t_k^-).
\end{aligned}
\tag{3.12}
$$

PROPOSITION 3.2. *Consider* (2.11) *under the Dirichlet or the mixed boundary conditions with the initial functions* $z(\cdot,t_0)\in\mathcal{H}^1(0,l)$ *satisfying the corresponding boundary conditions. Given positive scalars* δ, Δ, $K>\phi_M$, *and* h, *let there exist scalars* $p_1>0$, p_2, p_3, r, *and* λ *satisfying the following five LMIs:*

$$
p_2\left[1-\frac{\beta_M}{2g_0}\right]-\delta p_3\ge 0, \tag{3.13}
$$

$$
{\Phi^i}_{|\phi=\phi_m}\le 0,\ {\Phi^i}_{|\phi=\phi_M}\le 0,\ i=0,1, \tag{3.14}
$$

where

$$
\Phi^0 \triangleq \begin{bmatrix} \Phi_{11} & \Phi_{12} & p_2K & 0 \\ * & hr-2p_3 & p_3K & p_3\beta_M \\ * & * & -\lambda\frac{\pi^2}{\Delta^2} & 0 \\ * & * & * & -2a_0(p_2-\delta p_3)+\lambda \end{bmatrix}, \tag{3.15}
$$

$$
\Phi^1 \triangleq \begin{bmatrix} \Phi_{11} & \Phi_{12} & p_2K & hp_2K & 0 \\ * & -2p_3 & p_3K & hp_3K & p_3\beta_M \\ * & * & -\lambda\frac{\pi^2}{\Delta^2} & 0 & 0 \\ * & * & * & -hre^{-2\delta h} & 0 \\ * & * & * & * & -2a_0(p_2-\delta p_3)+\lambda \end{bmatrix},
$$

$$
\Phi_{11}=2\delta p_1+2p_2(\phi-K),\ \Phi_{12}=p_1-p_2+p_3(\phi-K).
$$

Then strong solutions of (2.11) *satisfy the inequality*

$$
\begin{aligned}
& p_1\int_0^l z^2(x,t)dx + p_3\int_0^l a(x)z_x^2(x,t)dx \le e^{-2\delta(t-t_0)}\left[p_1\int_0^l z^2(x,t_0)dx\right. \\
& \left. +\ p_3\int_0^l a(x)z_x^2(x,t_0)dx + qz^2(0,t_0)+q_l z^2(l,t_0)\right],\ t\ge t_0,
\end{aligned}
$$

where $q = q_l = 0$ corresponds to the Dirichlet and $q = p_3 g_0, q_l = p_3 g_l$ correspond to the mixed boundary conditions. Moreover, if the strong inequalities (3.14) *are feasible for $\delta = 0$, then* (2.11) *is exponentially stable with a small enough decay rate.*

Proof. We prove the result under the mixed boundary conditions (2.4). Under the Dirichlet boundary conditions the proof is similar. Note that the feasibility of LMIs (3.14) implies that $p_2 > 0, p_3 > 0, r > 0$, and $\lambda > 0$. We consider first $z^{(0)} \in \mathcal{D}(A)$, where

$$\mathcal{D}(A) = \{\omega \in \mathcal{H}^2(0, l) : \omega(0) = \omega(l) = 0\}.$$

Differentiating V in (3.11), where $t \in [t_k, t_{k+1})$, we find

$$\begin{aligned} \dot{V}(t) + 2\delta V(t) &= 2p_1 \int_0^l z(x,t) z_t(x,t) dx \\ &+ 2p_3 \int_0^l a(x) z_x(x,t) z_{xt}(x,t) dx + r \int_0^l (t_{k+1} - t) z_t^2(x,t) dx \\ &- r \int_0^l \int_{t_k}^t e^{2\delta(s-t)} z_s^2(x,s) ds dx + 2p_3 g_0 z(0,t) z_t(0,t) + 2p_3 g_l z(l,t) z_t(l,t) \\ &+ 2\delta[p_3 g_0 z^2(0,t) + p_3 g_l z^2(l,t)] + 2\delta \int_0^l [p_1 z^2(x,t) + p_3 a(x) z_x^2(x,t)] dx. \end{aligned} \tag{3.16}$$

Applying Jensen's inequality [21] twice (first to the internal integral and later to the external one), we have

$$\begin{aligned} & -r \int_0^l \int_{t_k}^t e^{2\delta(s-t)} z_s^2(x,s) ds dx \\ &\le -r \frac{1}{t - t_k} e^{-2\delta h} \int_0^l \left[\int_{t_k}^t z_s(x,s) ds \right]^2 dx \\ &= -\frac{r}{t - t_k} e^{-2\delta h} \sum_{j=0}^{N-1} \int_{x_j}^{x_{j+1}} \left[\int_{t_k}^t z_s(x,s) ds \right]^2 dx \\ &\le -\frac{r}{t - t_k} e^{-2\delta h} \sum_{j=0}^{N-1} \frac{1}{\Delta_j} \left[\int_{x_j}^{x_{j+1}} \left[\int_{t_k}^t z_s(x,s) ds \right] dx \right]^2 \\ &= -r(t - t_k) e^{-2\delta h} \Delta_j \sum_{j=0}^{N-1} \rho_j^2 = -r(t - t_k) e^{-2\delta h} \sum_{j=0}^{N-1} \int_{x_j}^{x_{j+1}} \rho_j^2 dx. \end{aligned} \tag{3.17}$$

We apply further the descriptor method [12] to (2.11), where the left-hand side of

$$\begin{aligned} & 2\int_0^l [p_2 z(x,t) + p_3 z_t(x,t)] \left[-z_t(x,t) + \frac{\partial}{\partial x}[a(x) z_x(x,t)] \right. \\ & \left. - \beta z_x(x,t) + \phi(z(x,t), x, t) - K \right] z(x,t) dx \\ & + 2K \sum_{j=0}^{N-1} \int_{x_j}^{x_{j+1}} [p_2 z(x,t) + p_3 z_t(x,t)][f(x,t) + (t - t_k)\rho_j] dx = 0 \end{aligned} \tag{3.18}$$

with some free scalar p_2 is added to $\dot{V}(t) + 2\delta V(t)$. Integrating by parts we have

(3.19)
$$2p_3\int_0^l z_t(x,t)\frac{\partial}{\partial x}[a(x)z_x(x,t)]dx$$
$$= -2p_3g_lz_t(l,t)z(l,t) - 2p_3g_0z_t(0,t)z(0,t) - 2p_3\int_0^l a(x)z_{tx}(x,t)z_x(x,t)dx.$$

From Remark A.1 (see Appendix A) the equality
$$\int_0^l a(x)z_{tx}(x,t)z_x(x,t)dx = \int_0^l a(x)z_{xt}(x,t)z_x(x,t)dx$$
holds almost for all t. Then (3.16)–(3.19) and (3.5) imply
$$\begin{aligned}
&\dot V(t) + 2\delta V(t) \le 2p_1\int_0^l z(x,t)z_t(x,t)dx - 2a_0(p_2-\delta p_3)\int_0^l {z_x}^2(x,t)dx\\
&+ 2\int_0^l [p_2z(x,t)+p_3z_t(x,t)][-z_t(x,t)+[\phi(z(x,t),x,t)-K]z(x,t)]dx\\
&+ 2K\sum_{j=0}^{N-1}\int_{x_j}^{x_{j+1}}[p_2z(x,t)+p_3z_t(x,t)][f(x,t)+(t-t_k)\rho_j]dx\\
&+ r\int_0^l (t_{k+1}-t)z_t^2(x,t)dx - re^{-2\delta h}(t-t_k)\sum_{j=0}^{N-1}\int_{x_j}^{x_{j+1}}\rho_j^2dx\\
&+ 2\int_0^l [\delta p_1z^2(x,t)+p_3\beta z_t(x,t)z_x(x,t)]dx - [2g_l(p_2-\delta p_3)+p_2\beta]z^2(l,t) - \psi,
\end{aligned}$$
where $\psi = [2g_0(p_2-\delta p_3) - p_2\beta]z^2(0,t) \ge 0$ due to (3.13). Set
$$\eta_s = col\{z(x,t), z_t(x,t), f(x,t), \rho_j,\ z_x(x,t)\}.$$
From the latter inequality and from (3.8) we have

(3.20)
$$\dot V(t) + 2\delta V(t) \le \dot V(t) + 2\delta V(t) + \sum_{j=0}^{N-1}\lambda\int_{x_j}^{x_{j+1}}[z_x^2(x,t) - \frac{\pi^2}{\Delta^2}f^2(x,t)]dx \le \sum_{j=0}^{N-1}\int_{x_j}^{x_{j+1}}\eta_s^T\bar\Phi^s\eta_sdx \le 0$$

if $\bar\Phi^s \le 0$, where
(3.21)
$$\bar\Phi^s \triangleq \begin{bmatrix} \Phi_{11} & \Phi_{12} & p_2K & (t-t_k)p_2K & 0\\ * & (t_{k+1}-t)r - 2p_3 & p_3K & (t-t_k)p_3K & p_3\beta_M\\ * & * & -\lambda\frac{\pi^2}{\Delta^2} & 0 & 0\\ * & * & * & -(t-t_k)re^{-2\delta h} & 0\\ * & * & * & * & -2a_0(p_2-\delta p_3)+\lambda \end{bmatrix}.$$

We will prove next that four LMIs (3.14) yield $\bar\Phi^s \le 0$. Matrices Φ^0 and Φ^1 given by (3.15) are affine in ϕ. Therefore, $\Phi^j \le 0$ for all $\phi \in [\phi_m, \phi_M]$ if LMIs (3.14) are satisfied. For $t - t_k \to 0$ and $t - t_k \to h$ the matrix inequality $\bar\Phi_s \le 0$ leads to $\Phi^0 \le 0$ and $\Phi^1 \le 0$ with notation given in (3.15). Denote by $\eta_0 = col\{z(x,t), z_t(x,t), f(x,t), z_x(x,t)\}$. Then $\Phi^0 \le 0$ and $\Phi^1 \le 0$ imply for $t \in [t_k, t_{k+1})$
$$\frac{t_{k+1}-t}{t_{k+1}-t_k}\eta_0^T\Phi^0\eta_0 + \frac{t-t_k}{t_{k+1}-t_k}\eta_s^T\Phi^1\eta_s = \eta_s^T\Phi_h\eta_s \le 0\ \forall\eta_s \ne 0,$$

where

$$\Phi_h \triangleq \begin{bmatrix} \Phi_{11} & \Phi_{12} & p_2 K & h\frac{t-t_k}{t_{k+1}-t_k} p_2 K & 0 \\ * & h\frac{t_{k+1}-t}{t_{k+1}-t_k} r - 2p_3 & p_3 K & h\frac{t-t_k}{t_{k+1}-t_k} p_3 K & p_3\beta_M \\ * & * & -\lambda\frac{\pi^2}{\Delta^2} & 0 & 0 \\ * & * & * & -h\frac{t-t_k}{t_{k+1}-t_k} r e^{-2\delta h} & 0 \\ * & * & * & * & -2a_0(p_2-\delta p_3)+\lambda \end{bmatrix} \leq 0.$$

Since $\frac{h}{t_{k+1}-t_k} \geq 1$, the feasibility of $\Phi_h \leq 0$ (by Schur complements) implies $\bar{\Phi}^s \leq 0$. Therefore, inequalities $\bar{\Phi}^s \leq 0$ yield $V(t) \leq e^{-2\delta(t-t_0)}V(t_0)$ for $z^{(0)} \in \mathcal{D}(\mathcal{A})$. Since $\mathcal{D}(\mathcal{A})$ is dense in $\mathcal{H}^1(0,l)$ the same estimate remains true (by continuous extension) for any initial conditions $z^{(0)} \in \mathcal{H}^1(0,l)$ satisfying the boundary conditions, which completes the proof. □

3.3. Sampled-data H_∞ control. Consider the perturbed version of (2.1)

$$z_t(x,t) = \frac{\partial}{\partial x}[a z_x(x,t)] - \beta z_x(x,t) + \phi(z(x,t),x,t)z(x,t) + u(x,t) + w(x,t),\ t \geq t_0,\ 0 \leq x \leq l, \tag{3.22}$$

where $w(x,t) \in L_2(0,\infty; L_2(0,l))$ is an external disturbance. As previously, we assume that the points $0 = x_0 < x_1 < \cdots < x_N = l$ divide $[0,l]$ into N sampled in space intervals with $\Delta_j = x_{j+1} - x_j \leq \Delta$. Sensors provide the weighted average in a spatial variable state, which is sampled in time instants $t_0 < t_1 < \cdots < t_k$ with $t_{k+1} - t_k \leq h$. The measurements are also supposed to be perturbed

$$y_{jk} = \frac{\int_{x_j}^{x_{j+1}} z(\xi,t_k)d\xi}{\Delta_j} + w_{jk}, \quad \Delta_j = x_{j+1} - x_j \leq \Delta,\quad j = 0,\dots,N-1,\ t \in [t_k, t_{k+1}),\ k = 0,1,2\dots, \tag{3.23}$$

where $w_{jk} \in l_2$, i.e., $\sum_{k=0}^{\infty}\sum_{j=0}^{N-1} w_{jk}^2 < \infty$.

Our objective is to find a linear static output feedback

$$u(x,t) = -Ky_{jk} = -K\frac{\int_{x_j}^{x_{j+1}} z(\xi,t_k)d\xi}{\Delta_j} - Kw_{jk},\quad x \in [x_j, x_{j+1}),\quad j = 0,\dots,N-1,\quad t \in [t_k, t_{k+1}),\quad k = 0,1,2\dots, \tag{3.24}$$

with a constant gain K that internally stabilizes the parabolic system, i.e., exponentially stabilizes the disturbance-free system (where $w = 0,\ w_{jk} = 0$). While internally stabilizing the parabolic process, the influence of the admissible external disturbance $w(x,t) \in L_2(0,\infty; L_2(0,l))$ on the controlled output

$$\zeta(x,t) = [c(x,t,z(x,t))z(x,t), \qquad d(t,z(x,t))u(t)]^T, \tag{3.25}$$

is to be attenuated through (3.24), where d and c are continuous and uniformly bounded functions

$$|c(x,t,z)| \leq c_1,\ |d(t,z)| \leq d_1 \tag{3.26}$$

for all $(x,t,z) \in [0,l] \times R^2$ and where $c_1 \geq 0$ and $d_1 \geq 0$ are some constants.

The disturbances $w(x,t) \in L_2(t_0,\infty; L_2(0,l))$, $w_{jk} \in l_2$ are said to be admissible if the closed-loop system (3.22), (3.24) possesses a unique strong solution being initialized with the zero data $z(x,t_0)=0$ and if this solution is globally continuable to the right. We note that if w is C^1 in x,t and is uniformly bounded, then by arguments of the Appendix A, the strong solutions of the closed-loop system (3.22), (3.24) under the boundary conditions (2.3) (or (2.4)) exist and they are continuable for $t \geq t_0$.

Denote

$$w_0(x,t) = w_{jk}, \quad x \in [x_j, x_{j+1}), \quad j = 0,\dots,N-1, \\ t \in [t_k, t_{k+1}), \quad k = 0,1,2\dots. \tag{3.27}$$

The following H_∞ control problem is thus under study. Given $\gamma > 0$, it is required to find a linear static output feedback (3.24) that internally stabilizes (3.22) and leads to a negative performance index

$$J(T) = \int_{t_0}^{T}\int_0^l [|\zeta(x,t)|^2 - \gamma^2[w^2(x,t) + w_0^2(x,t)]dxdt < 0 \quad \forall T > t_0 \tag{3.28}$$

for all admissible external disturbances $w(x,t) \in L_2(t_0,\infty; L_2(0,l))$, $w_{jk} \in l_2$ with $\int_0^l w^2(x,t)dx + \sum_{j=0}^{N-1} w_{jk}^2 > 0$ and for the zero initial condition $z(x,t_0) \equiv 0$. Whenever the closed-loop system (3.22), (3.24) satisfies the above inequality, it is said to have L_2-gain less than γ. Thus, corresponding feedback leads to L_2-gain less than γ.

Remark 3.2. The above H_∞ performance extends the indexes of [33] for sampled-data H_∞ control of finite-dimensional systems to the diffusion equation. It takes into account the updating rates of the measurement and is related to the energy of the measurement noise.

Since

$$u(x,t) = -Kz(x,t) + Kf(x,t) + K(t-t_k)\rho_j - Kw_{jk},$$

the closed-loop system has the form

$$\begin{aligned} z_t(x,t) &= \frac{\partial}{\partial x}[a(x)z_x(x,t)] - \beta z_x(x,t) + [\phi(z(x,t),x,t) - K]z(x,t) \\ &\quad + K[f(x,t) + (t-t_k)\rho_j] + w(x,t) - Kw_{jk}, \\ \zeta^T(x,t) &= [cz(x,t),\ dK[-z(x,t) + f(x,t) + (t-t_k)\rho_j - w_{jk}]], \\ &\quad x_j \leq x < x_{j+1},\ j = 0,\dots,N-1, \quad t \in [t_k,t_{k+1}),\ k = 0,1,2\dots, \end{aligned} \tag{3.29}$$

where f and ρ_j are given by (2.10). In order to solve the problem we carry out conditions that guarantee

$$W(t) \triangleq \frac{d}{dt}V + \sum_{j=0}^{N-1}\int_{x_j}^{x_{j+1}} [\zeta^T(x,t)\zeta(x,t) \\ - \gamma^2[w^2(x,t) + w_{jk}^2]dx < 0,\ t \in [t_k,t_{k+1}) \tag{3.30}$$

if $\int_0^l w^2(x,t)dx + \sum_{j=0}^{N-1} w_{jk}^2 > 0$, where V is given by (3.11) and the temporal derivative is computed along the trajectories of (3.29). Then integrating (3.30) in t from t_0 to T and taking into account that $V \geq 0$ and $V(t_0) = 0$ would yield (3.28).

Similar to (3.20) (see Appendix B for details), we find that

$$W(t) \le W(t) + \lambda \int_{x_j}^{x_{j+1}} \left[z_x^2(x,t) - \frac{\pi^2}{\Delta^2} f^2(x,t) \right] dx \le \sum_{j=0}^{N-1} \int_{x_j}^{x_{j+1}} \eta_\gamma^T \Psi_\gamma \eta_\gamma dx + \int_0^l \zeta^T(x,t)\zeta(x,t)dx, \tag{3.31}$$

where $\eta_\gamma = [z(x,t) \; z_t(x,t) \; f(x,t) \;\; \rho_j \; z_x(x,t) \; w(x,t) \; w_{jk}]^T$ and

$$\Psi_\gamma \triangleq \begin{bmatrix} & & p_2 & -p_2K \\ \bar{\Phi}^s_{|\delta=0} & & p_3 & -p_3K \\ & & 0 & 0 \\ & & 0 & 0 \\ & & 0 & 0 \\ & * & -\gamma^2 & 0 \\ & * & * & -\gamma^2 \end{bmatrix}$$

with $\bar{\Phi}^s$ given by (3.21). It is worth noticing that

$$\int_0^l \zeta^T(x,t)\zeta(x,t)dx \le \int_0^l c_1^2 z^2(x,t)dx + d_1^2 K^2 \sum_{j=0}^{N-1} \int_{x_j}^{x_{j+1}} [-z(x,t) + f(x,t) + (t-t_k)\rho_j - w_{jk}]^2 dx, \quad t \in [t_k, t_{k+1}). \tag{3.32}$$

Applying further Shur complements we find that $W(t) < 0$ if

$$\Psi^{s\gamma} \triangleq \begin{bmatrix} & p_2 & -p_2K & -d_1K & c_1 \\ \bar{\Phi}^s_{|\delta=0} & p_3 & -p_3K & 0 & 0 \\ & 0 & 0 & d_1K & 0 \\ & 0 & 0 & (t-t_k)d_1K & 0 \\ & 0 & 0 & 0 & 0 \\ * & -\gamma^2 & 0 & 0 & 0 \\ * & * & -\gamma^2 & -d_1K & 0 \\ * & * & * & -1 & 0 \\ * & * & * & * & -1 \end{bmatrix} < 0.$$

We have proved the following.

THEOREM 3.3. *Given positive scalars γ, Δ, $K > \phi_M$, and h, let there exist scalars $p_1 > 0, p_2, p_3,$ and $r > 0$, satisfying four LMIs*

$$\Psi^{s\gamma}{}_{|\phi=\phi_m, t\to t_k} < 0, \;\; \Psi^{s\gamma}{}_{|\phi=\phi_M, t\to t_k} < 0,$$
$$\Psi^{s\gamma}{}_{|\phi=\phi_m, t\to t_{k+1}} < 0, \;\; \Psi^{s\gamma}{}_{|\phi=\phi_M, t\to t_{k+1}} < 0.$$

Then the sampled-data static output feedback (3.24) *internally exponentially stabilizes the boundary-value problem* (3.22), (2.3) (*or* (2.4)) *and leads to L_2-gain less than γ.*

Remark 3.3. For the nonzero $z(\cdot, t_0) \in \mathcal{H}^1(0,l)$ satisfying the boundary conditions, LMIs of Theorem 3.3 guarantee the feasibility of the inequality

$$J(T) = \int_{t_0}^T \int_0^l \left[|\zeta(x,t)|^2 - \gamma^2 [w^2(x,t) + w_0^2(x,t)] \right] dxdt < V(t_0) = \int_0^l [p_1 z^2(x,t_0) + p_3 a(x) z_x^2(x,t_0)]dx \quad \forall T > t_0$$

TABLE 3.1
Sampled-data stabilization.

$\Delta \setminus h$	0	0.1	0.2	0.3	0.4	0.42
[15]	2.09	1	0.65	0.3	-	-
Prop 2	2.294	2.155	1.767	1.197	0.426	0.112

for all admissible external disturbances $w(x,t) \in L_2(t_0,\infty;L_2(0,l))$, $w_{jk} \in l_2$ with $\int_0^l w^2(x,t)dx + \sum_{j=0}^{N-1} w_{jk}^2 > 0$. This follows from the integration of (3.30) in t from t_0 to T.

Example 3.1. Consider the controlled diffusion equation (2.1) with $a(x) \geq a_0 = 1$, $\beta = 0$, and $\phi \leq 1.8$ (as in [15]) under the Dirichlet (2.3) or under the mixed (2.4) (with $g_0 = g_l = 0$) boundary conditions. The continuous in time controller (3.1) and the sampled-data one (2.8) are chosen with $K = 3 > 1.8$.

We study first stability of the closed-loop system under the continuous in time controller (3.1) by applying Proposition 3.1. We find that the closed-loop system remains exponentially stable till $\Delta \leq 2.29$. Therefore, the continuous in time controller exponentially stabilizes the system if the spatial domain is divided into two subdomains with $x_{j+1} - x_j \leq 2.29$. Moreover, if we choose $x_1 = \frac{\pi}{2}$ in the middle of $[0,\pi]$, then the above controller exponentially stabilizes the system with the decay rate $\delta = 0.6375$.

We consider further the sampled-data controller (2.8). Assume additionally that ϕ is lower (and not only upper) bounded $0 \leq \phi \leq 1.8$ and apply Proposition 3.2 to the closed-loop system. Table 3.1 shows the resulting maximum values of Δ, which preserve the exponential stability of the system, as the function of h. Comparing our results with the ones in [15], where the discrete in time and in space measurements were considered, we see that Proposition 3.2 (the sampled-data controller under the averaged measurements) guarantees the stability under greater values of Δ and h. The latter means that the system can be stabilized by a smaller number of actuators and under a bigger sampling rate (which is preferable, e.g., for the network-based control in the presence of communication constraints). However, this can be achieved on the account of a bigger number of sensors that provide the averaged in space measurements (instead of the discrete in space measurements in [15]).

We proceed with numerical simulations of the solutions to the closed-loop system under the Dirichlet boundary conditions, where we choose $a \equiv 1, z(x,0) = \sin^2 x$ and either $\phi(z) = 1.8\cos^2 z$ or $\phi \equiv 1.8$. We use a finite difference method. Simulations of solutions under the sampled-data controller with $x_{j+1} - x_j = \pi/2,\ j = 0,1$, where the space domain is divided into two subdomains, show that the closed-loop system is exponentially stable. This confirms the behavior predicted by Proposition 3.1. Moreover, for $x_{j+1} - x_j = \pi/2,\ j = 0,1$, the sampled-data controller preserves the stability for $t_{k+1} - t_k \leq 0.82$. (See Figure 3.1, where $t_{k+1} - t_k = 0.237, \phi(z) = 1.8\cos^2 z$.) The latter illustrates the conservatism of Proposition 3.2, where for $t_{k+1} - t_k = 0.237$ the corresponding value of the maximum Δ is $\frac{\pi}{2}$.

Consider next the perturbed diffusion equation (3.22) with $a_0 = 1,\ \beta = 0, 0 \leq \phi \leq 1.8$ under the sampled-data controller (2.8) with $K = 3$ and the performance index (3.28), where we choose $d_1 = 0.1$, $c_1 = 1$. For $h = 0.1$ and $\Delta_{max} = \frac{\pi}{2}$ Theorem 3.3 guarantees L_2-gain $\gamma = 6.178$.

Simulations of the solutions of the closed-loop system starting from the origin with $a \equiv 1, K = 3$, $\Delta = \frac{\pi}{2}$, $w(x,t) = (1+x)e^{-0.01t}$, $w_{jk} = cos(t_k)e^{-0.01t_k}$, $\phi(z) = 1.8cos^2 z$, $d \equiv 0.1$, $c \equiv 1$, $t_{k+1} - t_k = 0.1$ under the Dirichlet boundary conditions confirm that the resulting $J(T)$ is negative. Moreover, for the chosen disturbances, $J(T)$ remains

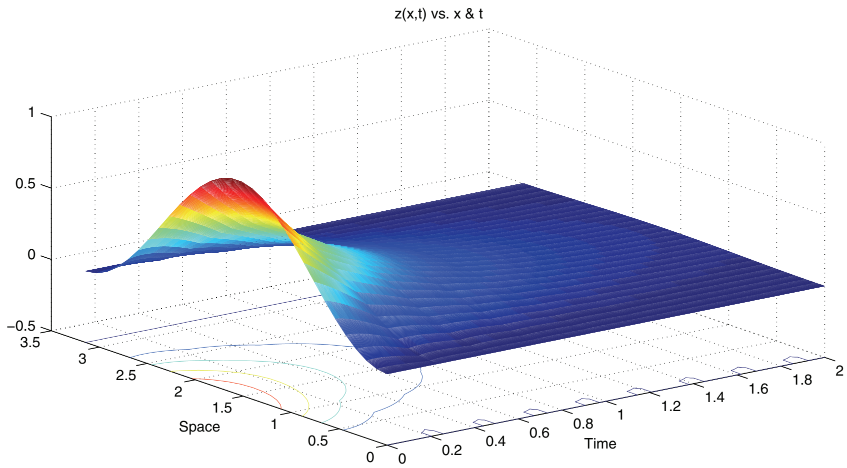

FIG. 3.1. *Solution under the Dirichlet b.c. with* $\Delta = \pi/2, h = 0.237, \phi(z) = 1.8\cos^2 z,$ *and* $\beta = 0.$

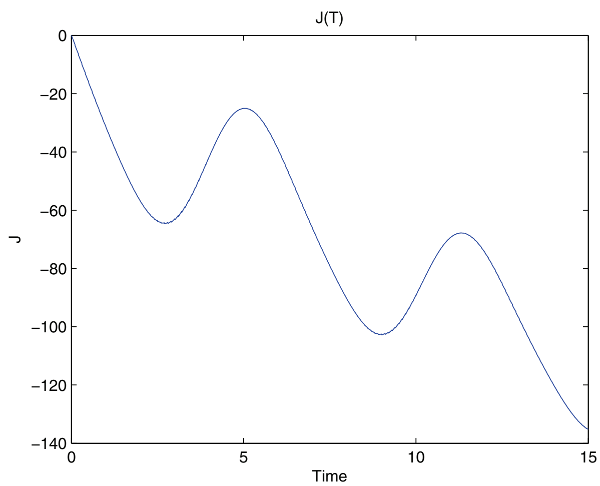

FIG. 3.2. $J(T)$: $\gamma = 1.1, \Delta = \pi/2, h = 0.1, \phi = 1.8\cos^2 z, \beta = 0,$ $w = (1+x)e^{-0.01t}, w_{jk} = cos(t_k)e^{-0.01t_k}$.

negative till $\gamma = 1.1$ (see Figure 3.2), which may mirror the conservatism of the presented method.

4. H_∞ sampled-data control of the coupled system of convection-diffusion equations. In this section we extend the results to the second-order transport-reaction system. Consider the transport-reaction system

$$\begin{aligned}(4.1)\qquad z_t(x,t) &= \frac{\partial}{\partial x}[a_1(x)z_x(x,t)] - \beta_1 z_x(x,t) + b_2 v(x,t)\\ &\quad + \phi(t,x,z(x,t),v(x,t))\cdot z(x,t) + u(x,t) + w_1(x,t),\\ v_t(x,t) &= \frac{\partial}{\partial x}[a_2(x)v_x(x,t)] - \beta_2 v_x(x,t) + b_3 z(x,t) + b_4 v(x,t) + w_2(x,t),\\ &\quad t \geq t_0, x \in [0,l], l > 0,\end{aligned}$$

where $w_i(x,t) \in L_2(0,\infty;L_2(0,l))\,(i=1,2)$ are external disturbances and w_i is assumed to be C^1 in x,t. Functions a_1, a_2, and ϕ are assumed to be of class C^1 and may be unknown, b_3, b_4 are given constants, whereas convection coefficients β_1 and β_2 are constant and may be unknown, and u is control input. It is assumed that

$$a_1(x) \geq a_{1m} > 0,\ a_2(x) \geq 0,\ b_4 < 0,$$
$$\beta_i \in [0, \beta_{iM}],\ i=1,2,\ \phi_m \leq \phi \leq \phi_M,$$

where a_{1m} , β_{iM}, ϕ_m, and ϕ_M are given bounds.

The system (4.1) is subject to the boundary conditions

$$\begin{aligned} &a_1(0)z_x(0,t) = g_{z0}z(0,t), \quad a_1(l)z_x(l,t) = -g_{zl}z(l,t),\\ &a_2(0)v_x(0,t) = g_{v0}v(0,t), \quad v_x(l,t) = -g_{vl}v(l,t). \end{aligned} \tag{4.2}$$

We assume that

$$2g_{z0} \geq \beta_{1M}, \quad 2g_{v0} \geq \beta_{2M},\ g_{zl} \geq 0,\ g_{vl} \geq 0. \tag{4.3}$$

The unperturbed version of the model (4.1), (4.2) with constant diffusion coefficients a_1 and a_2 was studied in [30]. This model accounts for an activator z, which undergoes reaction (expressed as $b_2 v + \phi z$), advection, and diffusion, and for a fast inhibitor v, which may be advected by the flow. Here β_i $(i = 1,2)$ are convective velocities.

Following [30] we consider the measurements of z only. Note that under the additional assumption $a_2 \geq a_{2m} > 0$ our results can be easily extended to the case of measurements in both states, z and v. As previously, we assume that the points $0 = x_0 < x_1 < \cdots < x_N = l$ divide $[0,l]$ into N sampled in space intervals with $\Delta_j = x_{j+1} - x_j \leq \Delta$. The sensors provide the weighted average in spatial variable state z, which is sampled in time instants $t_0 < t_1 < \cdots < t_k$ with $t_{k+1} - t_k \leq h$, given by (3.23), where $w_{jk} \in l_2$ is the measurement disturbance.

While internally stabilizing the parabolic process, the influence of the admissible external disturbance $w(x,t) \in L_2(0,\infty;L_2(0,l))$ on the controlled output

$$\zeta(x,t) = [c^z z(x,t), c^v v(x,t), du(x,t)]^T \tag{4.4}$$

is to be attenuated through the static output feedback (3.24), where $d = d(x,t,z,v)$, $c^z = c^z(x,t,z,v)$, and $c^v = c^v(x,t,z,v)$ are continuous and uniformly bounded functions

$$|c^z| \leq c_1,\ |c^v| \leq c_2,\ |d| \leq d_1 \tag{4.5}$$

for all $(x,t,z,v) \in [0,l] \times R^3$. Here $c_i \geq 0,\ i=1,2$, and $d_1 \geq 0$ are given constants.

The following H_∞ control problem is thus under study. Given $\gamma > 0$, it is required to find a linear static output feedback of (3.24) that exponentially stabilizes the unperturbed process (4.1), (4.2) and leads the perturbed one to the negative performance index $J(T)$ given by (3.28) for all $T > t_0$, where $w^2(x,t) = \sum_{i=1}^2 w_i^2(x,t)$, for all admissible external disturbances $w_i(x,t) \in L_2(t_0,\infty;L_2(0,l))$ and $w_{jk} \in l_2$ with $\int_0^l w^2(x,t)dx + \sum_{j=0}^{N-1} w_{jk}^2 > 0$ and for the zero initial condition $z(x,t_0) = v(x,t_0) \equiv 0$. We note that if u is stabilizing and w_i is C^1 in x,t and uniformly bounded (i.e., $|w_i(x,t)| \leq b_1$ for all $(x,t) \in [0,l] \times R$ and some $b_1 > 0$), then by arguments of Appendix A, the strong solutions of (4.1), (4.2) exist and they are continuable for $t \geq t_0$.

4.1. Continuous-time H_∞ control. Consider first the continuous-time counterpart of the H_∞ control problem, where the continuous-time measurements are taken as

$$y_j(x,t) = \frac{\int_{x_j}^{x_{j+1}} z(\xi,t)d\xi}{\Delta_j} + w_j(t), \quad x \in [x_j, x_{j+1}), \tag{4.6}$$
$$\Delta_j = x_{j+1} - x_j \le \Delta, \quad j = 0, \dots, N-1, \ t \ge t_0.$$

Here $w_j \in L_2(t_0, \infty)$ is the measurement disturbance. Denote

$$w_0(x,t) = w_j(t), \quad \Delta_j = x_{j+1} - x_j \le \Delta.$$

The controlled output is given by (4.4). The following H_∞ control problem is under study. Given $\gamma > 0$, it is required to find a linear static output feedback of

$$u(x,t) = -Ky_j(x,t) = -K\left[\frac{\int_{x_j}^{x_{j+1}} z(\xi,t)d\xi}{\Delta_j} + w_j(t)\right], \tag{4.7}$$
$$x \in [x_j, x_{j+1}), \quad j = 0, \dots, N-1, \ t \ge t_0,$$

that exponentially stabilizes the unperturbed process (4.1), (4.2) and leads the perturbed one to the negative performance index

$$J_c(T) = \int_{t_0}^{T}\int_0^l \left[\zeta^T(x,t)\zeta(x,t) - \gamma^2\left[\sum_{i=0}^{2} w_i^2(x,t)\right]\right] dxdt < 0 \quad \forall T \ge t_0 \tag{4.8}$$

for all admissible $w_i \in L_2(t_0, \infty; L_2(0,l))$, $w_j \in L_2(0,\infty)$ with $\int_0^l [w_1^2 + w_2^2]dx + \sum_{j=0}^{N-1} w_j^2 > 0$ and for the zero initial condition $z(x,t_0) = v(x,t_0) \equiv 0$. We note that if w_1 and w_2 are C^1 in x,t and are uniformly bounded, whereas w_j, $j = 0, \dots, N-1$, are Lipschitz-continuous in t, then the unique strong solution initialized with the zero data exists and is continuable for all $t \ge t_0$.

The closed-loop system (4.1), (4.7) is given by

$$\begin{aligned}
z_t(x,t) &= \frac{\partial}{\partial x}[a_1(x)z_x(x,t)] - \beta_1 z_x(x,t) + [\phi_1 - K]z(x,t) \\
&\quad + Kf(x,t) + b_2 v(x,t) + w_1(x,t) - Kw_0(x,t), \\
v_t(x,t) &= \frac{\partial}{\partial x}[a_2(x)v_x(x,t)] - \beta_2 v_x(x,t) + b_3 z(x,t) + b_4 v(x,t) + w_2(x,t), \\
\zeta^T(x,t) &= \Big[c^z z(x,t),\ c^v v(x,t), dK[-z(x,t) + f(x,t) - w_0(x,t)]\Big], \\
&\quad x_j \le x < x_{j+1},\ j = 0, \dots, N-1,
\end{aligned} \tag{4.9}$$

where f is defined by (2.10). Consider

$$V(t) = \int_0^l [p_z z^2(x,t) + p_v v^2(x,t)]dx,\ p_z > 0,\ p_v > 0.$$

We have along (4.9)

$$\begin{aligned}
\frac{d}{dt}\int_0^l v^2(x,t)dx = 2\int_0^l \bigg[&\frac{\partial}{\partial x}[a_2(x)v_x(x,t)] - \beta_2 v_x(x,t) \\
&+ b_3 z(x,t) + b_4 v(x,t) + w_2(x,t)\bigg] v(x,t)dx.
\end{aligned} \tag{4.10}$$

Integrating by parts and taking into account the boundary conditions (4.2) and inequality (4.3) we obtain

$$\begin{aligned} &2\int_0^l \frac{\partial}{\partial x}[a_2(x)v_x(x,t)]v(x,t)dx - 2\beta_2\int_0^l v_x(x,t)v(x,t)dx \\ &\quad = 2a_2(x)v_x(x,t)v(x,t)|_0^l - 2\int_0^l a_2(x)v_x^2(x,t)dx \\ &\quad - \beta_2 v^2(x,t)|_0^l \le -(2g_{v0}-\beta_2)v^2(0,t)\le 0. \end{aligned} \tag{4.11}$$

Similar to (3.7) we find along (4.9)

$$\begin{aligned} \frac{d}{dt}\int_0^l z^2(x,t)dx \le \sum_{j=0}^{N-1}\int_{x_j}^{x_{j+1}} &[-2a_0 z_x^2(x,t) + 2Kz(x,t)f(x,t) \\ &+ 2z(x,t)[b_2 v(x,t) + w_1(x,t) - Kw_0(x,t)] + 2[\phi_M - K]z^2(x,t)]dx. \end{aligned} \tag{4.12}$$

Here we have

$$\begin{aligned} \int_0^l \zeta^T(x,t)\zeta(x,t)dx \le \int_0^l &[c_1^2 z^2(x,t) + c_2^2 v^2(x,t)]dx \\ &+ d_1^2K^2\sum_{j=0}^{N-1}\int_{x_j}^{x_{j+1}}[-z(x,t)+f(x,t)-w_0(x,t)]^2dx. \end{aligned} \tag{4.13}$$

From (3.8) and (4.10)–(4.13), by applying the Schur complements to the last terms of (4.13) we conclude that

$$\dot V(t) + \sum_{j=0}^{N-1}\int_{x_j}^{x_{j+1}}\left[|\zeta(x,t)|^2 - \gamma^2\left[\sum_{i=0}^{2} w_i^2(x,t)\right]\right]dx < 0$$

along (4.9) if $\sum_{i=0}^2\int_0^l w_i^2(x,t)dx > 0$ and

$$\begin{bmatrix} \Phi_{11} & \Phi_{12} & p_zK & p_z & -p_zK & 0 & -d_1K \\ * & 2p_vb_4 + c_2^2 & 0 & 0 & 0 & p_v & 0 \\ * & * & -\frac{2p_za_{1m}\pi^2}{\Delta^2} & 0 & 0 & 0 & d_1K \\ * & * & * & -\gamma^2 & 0 & 0 & 0 \\ * & * & * & * & -\gamma^2 & 0 & -d_1K \\ * & * & * & * & * & -\gamma^2 & 0 \\ * & * & * & * & * & * & -1 \end{bmatrix} < 0, \tag{4.14}$$
$$\Phi_{11} = 2p_z(\phi_M - K) + c_1^2, \quad \Phi_{12} = p_zb_2 + p_vb_3.$$

Proposition 4.1. *Given positive scalars γ, Δ, and $K > \phi_M$, let there exist scalars $p_z > 0, p_v > 0$, satisfying LMI* (4.14)*. Then the continuous-time static output feedback* (4.7) *internally stabilizes the boundary-value problem* (4.1), (4.2) (*subject to* (4.3)) *and leads to an L_2-gain less than γ.*

4.2. Sampled-data H_∞ control. The closed-loop system under the sampled-data controller can be represented as follows:

$$
\begin{aligned}
z_t(x,t) &= \frac{\partial}{\partial x}[a_1(x)z_x(x,t)] - \beta_1 z_x(x,t) + [\phi_1 - K]z(x,t) \\
&\quad + K[f(x,t) + (t-t_k)\rho_j] + b_2 v(x,t) + w_1(x,t) - Kw_{jk}, \\
v_t(x,t) &= \frac{\partial}{\partial x}[a_2(x)v_x(x,t)] - \beta_2 v_x(x,t) + b_3 z(x,t) + b_4 v(x,t) + w_2(x,t), \\
\zeta^T(x,t) &= \left[c^z z(x,t),\ c^v v(x,t), dK[-z(x,t) + f(x,t) + (t-t_k)\rho_j - w_{jk}]\right], \\
&\quad x_j \le x < x_{j+1},\ j = 0,\dots,N-1,\ t \in [t_k, t_{k+1}),\ k = 0,1,2\dots,
\end{aligned}
\tag{4.15}
$$

where f and ρ_j are given by (2.10).

Consider

$$
\begin{aligned}
V(t) &= p_z \int_0^l z^2(x,t)dx + p_v \int_0^l v^2(x,t)dx + \int_0^l a_1(x)p_3 z_x^2(x,t)dx \\
&\quad + r(t_{k+1} - t)\int_0^l \int_{t_k}^t z_s^2(x,s)dsdx + p_3 g_{z0} z^2(0,t) + p_3 g_{zl} z^2(l,t), \\
&\quad t \in [t_k, t_{k+1}), \quad p_3 > 0,\ p_1 > 0,\ r > 0.
\end{aligned}
\tag{4.16}
$$

In order to solve the problem we derive conditions that guarantee

$$
\begin{aligned}
W(t) \triangleq \frac{d}{dt}V + \sum_{j=0}^{N-1} \int_{x_j}^{x_{j+1}} &\left[\zeta^T(x,t)\zeta(x,t)\right. \\
&\left. - \gamma^2[w_1^2(x,t) + w_2^2(x,t) + w_{jk}^2]\right] dx < 0, \quad t \in [t_k, t_{k+1})
\end{aligned}
$$

along (4.15). Similarly to Theorem 3.3 and Proposition 4.1 (by using the descriptor method in z and applying the Schur complements as shown in Appendix C) we arrive at

$$
\begin{aligned}
W(t) &\le W(t) + \lambda \int_{x_j}^{x_{j+1}} \left[z_x^2(x,t) - \frac{\pi^2}{\Delta^2} f^2(x,t)\right] dx \\
&\le \sum_{j=1}^{N} \int_{x_j}^{x_{j+1}} \eta_j^T \Psi \eta_j dx < 0,\ t \in [t_k, t_{k+1})
\end{aligned}
\tag{4.17}
$$

for $\int_0^l w^2(x,t)dx + \sum_{j=0}^{N-1} w_{jk}^2 > 0$ if

$$
\Psi \triangleq \begin{bmatrix}
\Psi^{TR} & \begin{matrix} p_2 & -p_2K & 0 & -d_1K \\ p_3 & -p_3K & 0 & 0 \\ 0 & 0 & 0 & d_1K \\ 0 & 0 & 0 & (t-t_k)d_1K \\ 0 & 0 & p_v & 0 \\ 0 & 0 & 0 & 0 \end{matrix} \\
\begin{matrix} * \\ * \\ * \\ * \end{matrix} & \begin{matrix} -\gamma^2 & 0 & 0 & 0 \\ * & -\gamma^2 & 0 & -d_1K \\ * & * & -\gamma^2 & 0 \\ * & * & * & -1 \end{matrix}
\end{bmatrix} < 0,
\tag{4.18}
$$

where

(4.19)
$$\Psi^{TR} \triangleq \begin{bmatrix} \Psi_{11} & \Psi_{12} & p_2K & (t-t_k)p_2K & p_2b_2+p_vb_3 & 0 \\ * & \Psi_{22} & p_3K & (t-t_k)p_3K & p_3b_2 & p_3\beta_{1M} \\ * & * & -\lambda\frac{\pi^2}{\Delta^2} & 0 & 0 & \\ & * & * & -(t-t_k)r & 0 & 0 \\ * & * & * & * & 2p_vb_4+c_2^2 & 0 \\ * & * & * & * & * & -2a_{1m}p_2+\lambda \end{bmatrix},$$
$$\Psi_{11}=2p_2(\phi_1-K)+c_1^2,\ \Psi_{12}=p_z-p_2+p_3(\phi_1-K),\ \Psi_{22}=(t_{k+1}-t)r-2p_3,$$
$$\eta_j^T=\begin{bmatrix} z & z_t & f & \rho_j & v & z_x & w_1 & w_{jk} & w_2 & 1\end{bmatrix},\ x\in[x_j,x_{j+1}).$$

THEOREM 4.2. *Given positive scalars γ, Δ, $K>\phi_M$, and h, let there exist scalars $p_z>0, p_v>0, p_2, p_3,$ and $r>0$, satisfying four LMIs,*

$$\Psi_{|\phi_1=\phi_m,t\to t_k}<0,\ \Psi_{|\phi_1=\phi_M,t\to t_k}<0,$$
$$\Psi_{|\phi_1=\phi_m,t\to t_{k+1}}<0,\ \Psi_{|\phi_1=\phi_M,t\to t_{k+1}}<0.$$

Then the sampled-data static output feedback (3.24) *internally exponentially stabilizes the boundary-value problem* (4.1), (4.2) (*subject to* (4.3)) *and leads to an L_2-gain less than γ.*

Remark 4.1. For $\beta_{1M}=0$ and $h\to 0$, the conditions of Theorem 4.2 and of Proposition 4.1 are equivalent, whereas for $\beta_{1M}>0$ the conditions of Theorem 4.2 are more conservative. Moreover, the conditions of Proposition 4.1 do not depend on ϕ_m. Thus, for $\phi_m=-\infty$ the sample data in time controller leads to the local results, whereas the continuous one leads to the global results.

Remark 4.2. In the case of constant and known diffusion and convection coefficients $a_1>a_2\geq 0$, $2\beta_2\geq\beta_1>0$, by changing the variables in (4.1), (4.2) (see, e.g., [32])

$$\bar z(x,t)=z(x,t)e^{-\frac{\beta_1}{2a_1}x},\quad \bar v(x,t)=v(x,t)e^{-\frac{\beta_1}{2a_1}x},$$

we arrive at the system

$$\bar z_t(x,t)=a_1\bar z_{xx}(x,t)+\phi_1\bar z(x,t)+b_2\bar v(x,t)+e^{-\frac{\beta_1}{2a_1}x}u(x,t)+e^{-\frac{\beta_1}{2a_1}x}w_1(x,t),$$
$$\bar v_t(x,t)=a_2\bar v_{xx}(x,t)-\bar\beta_2\bar v_x(x,t)+b_3\bar z(x,t)+\bar b_4\bar v(x,t)+e^{-\frac{\beta_1}{2a_1}x}w_2(x,t),$$
$$t\geq t_0, x\in[0,l], l>0,\ a_1>0, a_2\geq 0$$

(with $\bar\beta_1=0$) under the boundary conditions

$$a_1\bar z_x(0,t)=\bar g_{z0}\bar z(0,t),\quad \bar z_x(l,t)=-\bar g_{zl}\bar z(l,t),$$
$$a_2\bar v_x(0,t)=\bar g_{v0}\bar v(0,t),\quad \bar v_x(l,t)=-\bar g_{vl}\bar v(l,t),$$

where due to (4.3)

$$\phi_m-\frac{\beta_1^2}{4a_1}\leq\phi_1=\phi-\frac{\beta_1^2}{4a_1}\leq\phi_M-\frac{\beta_1^2}{4a_1},$$
$$\bar\beta_2=\beta_2-\frac{a_2\beta_1}{a_1}\geq 0,\quad \bar b_4=b_4+\frac{a_2\beta_1^2}{4a_1^2}-\frac{\beta_1\beta_2}{2a_1}<0,$$

$$\bar{g}_{z0} = g_{z0} - \frac{\beta_1}{2} \geq 0, \quad \bar{g}_{zl} = g_{zl} + \frac{\beta_1}{2a_1} \geq 0,$$
$$\bar{g}_{v0} = g_{v0} - \frac{a_2\beta_1}{2a_1} \geq \frac{\bar{\beta}_2}{2}, \quad \bar{g}_{vl} = g_{vl} + \frac{\beta_1}{2a_1} \geq 0.$$

In this case the measurements can be taken as weighted averages

$$y_{jk} = \frac{\int_{x_j}^{x_{j+1}} e^{-\frac{\beta_1}{2a_1}\xi} z(\xi, t_k) d\xi}{\Delta_j} + w_{jk}, \quad \Delta_j = x_{j+1} - x_j \leq \Delta,$$
$$j = 0, \dots, N-1, \ t \in [t_k, t_{k+1}), \ k = 0, 1, 2 \dots.$$

The control law can be modified as follows:

$$u(x,t) = -Ke^{\frac{\beta_1}{2a_1}x} y_{jk} = -e^{\frac{\beta_1}{2a_1}x} K \left[\frac{\int_{x_j}^{x_{j+1}} \bar{z}(\xi, t_k) d\xi}{\Delta_j} + w_{jk} \right],$$
$$x \in [x_j, x_{j+1}), \quad j = 0, \dots, N-1, \quad t \in [t_k, t_{k+1}), \quad k = 0, 1, 2 \dots. \tag{4.20}$$

Consider the controlled output of the form

$$\bar{\zeta}(x,t) = e^{-\frac{\beta_1}{2a_1}x} \zeta(x,t) = [c^z \bar{z}(x,t), c^v \bar{v}(x,t), e^{-\frac{\beta_1}{2a_1}x} du]^T \tag{4.21}$$

and $\bar{J}(T)$ of (3.28), where ζ is changed by $\bar{\zeta}$. The closed-loop system (4.1), (4.20) has the form of (4.15), where $z, v, \beta_1, \beta_2, b_4$, and ϕ should be replaced by $\bar{z}, \bar{v}, 0, \bar{\beta}_2, \bar{b}_4$, and ϕ_1, respectively, and where u and w_i are multiplied by $e^{-\frac{\beta_1}{2a_1}x}$. Note that $\phi_m - \frac{\beta^2}{4a} \leq \phi_1 \leq \phi_M - \frac{\beta^2}{4a}$. The conditions of Theorem 4.2 can be applied to the closed-loop system (4.1), (4.20). This change of variables improves the decay rate and the value of γ (for given Δ and h), but in the different norm. Thus, the exponential bounds are given on $\|\bar{z}\|_{L_2}, \|\bar{v}\|_{L_2}$ with

$$\|z\|_{L_2} \leq e^{\frac{\beta_1}{2a_1}l} \|\bar{z}\|_{L_2}, \ \|v\|_{L_2} \leq e^{\frac{\beta_1}{2a_1}l} \|\bar{v}\|_{L_2},$$

which leads to the big constant $e^{\frac{\beta_1 l}{2a_1}}$ for big l.

Remark 4.3. The developed technique can be extended to observer-based control. For example, consider (4.1), (4.2) subject to (4.3) under the measurements of z given by (3.23), where a_i, ϕ, β_i are known with $a_i \geq a_{im} > 0, \ i = 1, 2$. The performance index is given by (3.28) with ζ of (4.4). It is possible to construct an estimate $\hat{v}$ of v:

$$\hat{v}_t(x,t) = \frac{\partial}{\partial x}[a_2(x)\hat{v}_x(x,t)] - \beta_2 \hat{v}_x(x,t) + b_3 y_{jk} + b_4 \hat{v}(x,t), \tag{4.22}$$
$$a_2(0)\hat{v}_x(0,t) = g_{v0}\hat{v}(0,t), \quad \hat{v}_x(l,t) = -g_{vl}\hat{v}(l,t),$$
$$t \geq t_0, x \in [x_j, x_{j+1}), \ j = 0, \dots, N-1.$$

Then the estimation error $e = v - \hat{v}$ satisfies the following boundary-value problem:

$$e_t(x,t) = \frac{\partial}{\partial x}[a_2(x)e_x(x,t)] - \beta_2 e_x(x,t) + b_3[f(x,t) + (t - t_k)\rho_j - w_{jk}] + b_4 e(x,t), \tag{4.23}$$
$$a_2(0)e_x(0,t) = g_{v0}e(0,t), \quad e_x(l,t) = -g_{vl}e(l,t),$$
$$t \geq t_0, x \in [x_j, x_{j+1}), \ j = 0, \dots, N-1.$$

The sampled-data control of the form

$$u(x,t) = -Ky_{jk} - K_v \frac{\int_{x_j}^{x_{j+1}} \hat{v}(\xi, t_k) d\xi}{\Delta_j},$$
$$t \in [t_k, t_{k+1}),\ k = 0, 1, 2, \dots,\ x \in [x_j, x_{j+1}),\ j = 0, \dots, N-1,$$

where K and K_v are constants, can be designed by extending the developed technique to the closed-loop system given by (4.22), (4.23), and

$$\begin{aligned} z_t(x,t) &= \frac{\partial}{\partial x}[a_1(x) z_x(x,t)] - \beta_1 z_x(x,t) + [\phi_1 - K] z(x,t) + b_2[\hat{v}(x,t) + e(x,t)] \\ &\quad + K[f(x,t) + (t - t_k)\rho_j - w_{jk}] + w_1(x,t) - K_v \frac{\int_{x_j}^{x_{j+1}} \hat{v}(\xi, t_k) d\xi}{\Delta_j}, \\ \zeta^T(x,t) &= \Bigg[c^z z(x,t),\ c^v v(x,t), dK[-z(x,t) + f(x,t) + (t - t_k)\rho_j - w_{jk}], \\ &\quad - dK_v \frac{\int_{x_j}^{x_{j+1}} \hat{v}(\xi, t_k) d\xi}{\Delta_j} \Bigg], \\ &\quad x_j \le x < x_{j+1},\ j = 0, \dots, N-1,\ t \in [t_k, t_{k+1}),\ k = 0, 1, 2 \dots . \end{aligned}$$

Example 4.1. Consider the system [30]

$$\begin{aligned} z_t(x,t) &= z_{xx}(x,t) - \beta z_x(x,t) + v(x,t) \\ &\quad + \phi(z(x,t)) z(x,t) + u(x,t) + w_1(x,t), \\ v_t(x,t) &= -100\beta v_x(x,t) - 45 z(x,t) - 20 v(x,t) \\ &\quad + w_2(x,t), \quad t \ge 0,\ x \in [0, 10],\ \beta \ge 0\ a_1 \ge 1, \end{aligned} \tag{4.24}$$

where we changed the time $\bar{t}$ of [30] to the slow one $t = 0.01\bar{t}$, under the boundary conditions

$$\begin{aligned} z_x(0,t) &= \beta z(0,t), \quad & z_x(l,t) &= 0, \\ v(0,t) &= 0, & v_x(l,t) &= 0. \end{aligned} \tag{4.25}$$

Note that (4.25) satisfy the assumptions (4.3). In [30] ϕ was chosen to be $\phi = 1 - z^2 \le 1$ and it was shown that the linearized system is unstable for $\beta > 0.9$ with maximum unstable eigenvalues for $\beta \in (0.9, 1.4)$.

Consider (4.24), where $\phi \le 1$, $\beta \ge 0$, under the continuous-time controller (4.7) with $K = 1.5$ and under the continuous-time controller which corresponds to the change of variables (as in Remark 4.2). The continuous-time controller without change of variables needs $N = 4$ actuators, whereas the one with the change of variables needs $N = 3$ actuators for the global exponential stabilization of the nonperturbed system for all $\beta \ge 0$. (Note that in [30] the continuous-time controller leads to local stability.)

Consider next (4.24), where $\phi \in [0, 1]$, $\beta \in (0, 1.4]$, $K = 1.5$, $N = 5$, under the sampled-data controller (3.24). For the bounded ϕ Theorem 4.2 guarantees global results. Applying Theorem 4.2 to the resulting closed-loop system, we find that the latter system is internally exponentially stable for $t_{k+1} - t_k \le h = 0.163$. Choosing next $h = 0.05$, $|d| \le 0.1$, $|c^z| \le 1, |c^v| \le 1$, we find by Theorem 4.2 that the closed-loop system achieves the L_2-gain $\gamma = 8.56$. (For $h \to 0$ we arrive to a smaller value $\gamma = 6.66$.)

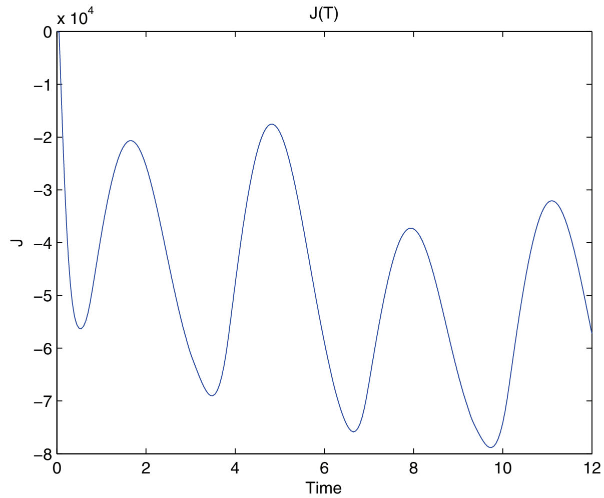

FIG. 4.1. $J(T)$: $\gamma = 1.56, \Delta = 2, h = 0.05, \phi = \cos^2 z, \beta = 0.5,\ w_i = (i+x)e^{-0.01t}\ (i = 1,2), w_{jk} = 100\cos(t_k)e^{-0.01t_k}$.

Simulations of the solutions of the closed-loop system (4.24), (3.24) starting from the origin with $\beta = 0.5$, $K = 1.5$, $N = 5$, $w_1(x,t) = (1+x)e^{-0.01t}$, $w_2(x,t) = (2+x)e^{-0.01t}$, $w_{jk} = 100\cos(t_k)e^{-0.01t_k}$, $\phi(z) = \cos^2 z$, $d \equiv 0.1$, $c^z = c^v \equiv 1$, $t_{k+1} - t_k = 0.05$ confirm that the resulting $J(T)$ is negative for $T \geq t_0 = 0$. Moreover, for the chosen disturbances, $J(T)$ remains negative till $\gamma = 1.56$ (see Figure 4.1), which may mirror the conservatism of the presented method.

5. Conclusions. We have introduced the sampled-data distributed H_∞ control of semilinear convection-diffusion equations. A network of stationary sensing devices provides spatially averaged state measurements over the sampling spatial intervals. These measurements are sampled-data in time. It is supposed that the sampling intervals are bounded. Sufficient conditions for the H_∞ stabilization are derived in terms of LMIs. By solving these LMIs, upper bounds are found on the sampling intervals that preserve the internal stability and lead to a prescribed L_2-gain. Numerical examples illustrate the efficiency of the method. This is the first result that achieves a desired performance for distributed parameter systems in the presence of significant uncertainties/disturbances and nonlinearities under the discrete-time measurements, which is of theoretical and practical importance. Extension of the method to robust sampled-data control and observation of various classes of parabolic systems may be a topic for the future research.

Appendix A. Well-posedness of (2.9). Consider the closed-loop system (2.9) initialized with $z(x,t_0) = z^{(0)}(x)$ under the Dirichlet boundary conditions (2.3). We will use the step method for solution of time-delay systems [26]. For $t \in [t_0, t_1)$ the system has a form

$$
\begin{aligned}
z_t(x,t) &= \frac{\partial}{\partial x}[a(x)z_x(x,t)] - \beta z_x(x,t) \\
&\quad + \phi(z(x,t),x,t)z(x,t) - K\frac{\int_{x_j}^{x_{j+1}} z^{(0)}(\xi)d\xi}{\Delta_j}, \\
&x_j \leq x < x_{j+1}, z(0,t) = z(l,t) = 0.
\end{aligned}
\tag{A.1}
$$

Introduce the Hilbert space $H = L_2(0, l)$ with the norm $\|\cdot\|_{L_2}$ and with the scalar product $\langle\cdot,\cdot\rangle$. The boundary-value problem (A.1), (2.3) can be rewritten as a differential equation

$$\dot{\omega}(t) = A\omega(t) + F(t, \omega(t)), \quad t \geq t_0, \tag{A.2}$$

in H where the operator $A = \frac{\partial[a(x)\frac{\partial}{\partial x}]}{\partial x}$ has the dense domain

$$\mathcal{D}(A) = \{\omega \in \mathcal{H}^2(0, l) : \omega(0) = \omega(l) = 0\}$$

and the nonlinear term $F : R \times \mathcal{H}^1(0, l) \to L_2(0, l)$ is defined on functions $\omega(\cdot, t)$ according to

$$F(t, \omega(\cdot, t)) = \phi(\omega(x, t), x, t)\omega(x, t) - K\omega(x, t) \\ - \beta\omega_x(x, t) + K\frac{\int_{x_j}^{x_{j+1}} z^{(0)}(\xi)d\xi}{\Delta_j}, \; x_j \leq x < x_{j+1}.$$

It is well known that A generates a strongly continuous exponentially stable semigroup T, which satisfies the inequality $\|T(t)\| \leq \kappa e^{-\delta t}$, $(t \geq 0)$, with some constant $\kappa \geq 1$ and decay rate $\delta > 0$ (see, e.g., [9] for details). The domain $H_1 = \mathcal{D}(A) = A^{-1}H$ forms another Hilbert space with the graph inner product $\langle x, y\rangle_1 = \langle Ax, Ay\rangle$, $x, y \in H_1$. The domain $\mathcal{D}(A)$ is dense in H and the inequality $\|A\omega\|_{L_2} \geq \mu\|\omega\|_{L_2}$ holds for all $\omega \in \mathcal{D}(A)$ and some constant $\mu > 0$. Operator $-A$ is positive, so that its square root $(-A)^{\frac{1}{2}}$ with

$$H_{\frac{1}{2}} = \mathcal{D}((-A)^{\frac{1}{2}}) = \{\omega \in \mathcal{H}^1(0, l) : \omega(0) = \omega(l) = 0\}$$

is well defined. Moreover, $H_{\frac{1}{2}}$ is a Hilbert space with the scalar product

$$\langle u, v\rangle_{\frac{1}{2}} = \langle(-A)^{\frac{1}{2}}u, (-A)^{\frac{1}{2}}v\rangle.$$

Denote by $H_{-\frac{1}{2}}$ the dual of $H_{\frac{1}{2}}$ with respect to the pivot space H. Then A has an extension to a bounded operator $A : H_{\frac{1}{2}} \to H_{-\frac{1}{2}}$. We have $H_1 \subset H_{\frac{1}{2}} \subset H$ with continuous embedding and the inequality

$$\|(-A)^{\frac{1}{2}}\omega\|_{L_2} \geq \mu\|\omega\|_{L_2} \quad \forall\omega \in H_{\frac{1}{2}}$$

holds. All relevant material on fractional operator degrees can be found, e.g., in [35].

A function $\omega : [t_0, T) \to H_{\frac{1}{2}}$ is called a strong solution of (A.2) if

$$\omega(t) - \omega(t_0) = \int_{t_0}^{t} [A\omega(s) + F(s, \omega(s))]ds \tag{A.3}$$

holds for all $t \in [t_0, T)$. Here, the integral is computed in $H_{-\frac{1}{2}}$. Differentiating (A.3) we obtain (A.2). Since the function ϕ of class C^1, the Lipschitz condition

$$\|F(t_1, \omega_1) - F(t_2, \omega_2)\|_{L_2} \leq C[|t_1 - t_2| + \|(-A)^{\frac{1}{2}}(\omega_1 - \omega_2)\|_{L_2}] \tag{A.4}$$

with some constant $C > 0$ holds locally in $(t_i, \omega_i) \in R \times H_{\frac{1}{2}}$, $i = 1, 2$. Thus, Theorem 3.3.3 of [24] is applicable to (A.2), and by applying this theorem, a unique strong solution $\omega(t) \in H_{\frac{1}{2}}$ of (A.2), initialized with $z^{(0)} \in H_{\frac{1}{2}}$, exists locally and $\omega(t) \in \mathcal{D}(A)$ for $t > t_0$. Since ϕ is bounded, there exists $C_1 > 0$ such that

$$\|F(t,\omega)\|_{L_2} \le C_1[1 + \|(-A)^{\frac{1}{2}}\omega\|_{L_2}] \quad \forall \omega \in H_{\frac{1}{2}}.$$

Hence, the strong solution initialized with $z^{(0)} \in H_{\frac{1}{2}}$ exists for all $t \in [t_0, t_1)$ [24]. Moreover, if $z^{(0)} \in \mathcal{D}(A)$, then $\omega : [t_0, t_1] \to H_{\frac{1}{2}}$ is of class C^1. By next considering $t \in [t_k, t_{k+1}),\ k = 1, 2, \dots$, we conclude that (2.9) has a unique strong solution for all $t \ge t_0$. Under the mixed boundary conditions the well-posedness is verified by similar arguments.

Remark A.1. From the above, if $z^{(0)} \in \mathcal{D}(A)$, then $\omega : [t_0, t_1] \to H_{\frac{1}{2}}$ is of class C^1, i.e.,

$$\frac{d}{dt}\int_0^l a(x) z_x^2(x,t)dx = 2\int_0^l a(x) z_{tx}(x,t) z_x(x,t)dx$$

is continuous in t, implying that $z_{tx} \in L_2(0, l; t_0, t_0 + T)$ for all $T > 0$. Consider now z_{xt} as a distribution defined by

$$\int_{t_0}^{t_0+T}\int_0^l z_{xt}(x,t)\varphi(x,t)dxdt \triangleq \int_{t_0}^{t_0+T}\int_0^l z(x,t)\varphi_{tx}(x,t)\,dxdt$$

for all test functions $\varphi : R^2 \to R$ of class C^∞ with the support in $[0, l; t_0, t_0 + T]$. Since φ is smooth, we have $\varphi_{tx} = \varphi_{xt}$ and therefore

$$\begin{aligned}\int_{t_0}^{t_0+T}\int_0^l z_{xt}(x,t)\varphi(x,t)dxdt &= \int_{t_0}^{t_0+T}\int_0^l z(x,t)\varphi_{xt}(x,t)\,dxdt \\ &= \int_{t_0}^{t_0+T}\int_0^l z_{tx}(x,t)\varphi(x,t)dxdt.\end{aligned}$$

This proves that z_{xt} is a measurable function coinciding with z_{tx} for almost all x, t. Thus, almost for all t

$$\int_0^l a(x) z_{tx}(x,t) z_x(x,t)dx = \int_0^l a(x) z_{xt}(x,t) z_x(x,t)dx.$$

Appendix B. Verification of (3.31). By the descriptor method applied to (3.29), we add to $\dot{V}$ the left-hand side of the following equation:

$$\begin{aligned}&2\int_0^l [p_2 z(x,t) + p_3 z_t(x,t)]\Big[-z_t(x,t) + \frac{\partial}{\partial x}[a(x) z_x(x,t)] \\ &\quad - \beta z_x(x,t) + \phi(z(x,t),x,t) - K\Big] z(x,t)dx \\ &\quad + 2K\sum_{j=0}^{N-1}\int_{x_j}^{x_{j+1}}[p_2 z(x,t) + p_3 z_t(x,t)][f(x,t) + (t - t_k)\rho_j]dx \\ &\quad + 2\sum_{j=0}^{N-1}\int_{x_j}^{x_{j+1}}[p_2 z(x,t) + p_3 z_t(x,t)][-K w_{jk} + w(x,t)]dx = 0.\end{aligned}$$

Then from (3.16)–(3.19) and (3.5) we obtain

$$\begin{aligned}\dot V(t) \le\ & 2p_1\int_0^l z(x,t)z_t(x,t)dx - 2a_0p_2\int_0^l {z_x}^2(x,t)dx + 2\int_0^l p_3\beta z_t(x,t)z_x(x,t)dx\\ &+ 2\int_0^l [p_2z(x,t)+p_3z_t(x,t)][-z_t(x,t)+[\phi(z(x,t),x,t)-K]z(x,t)]dx\\ &+ 2K\sum_{j=0}^{N-1}\int_{x_j}^{x_{j+1}}[p_2z(x,t)+p_3z_t(x,t)][f(x,t)+(t-t_k)\rho_j]dx\\ &+ r\int_0^l (t_{k+1}-t)z_t^2(x,t)dx - re^{-2\delta h}(t-t_k)\sum_{j=0}^{N-1}\int_{x_j}^{x_{j+1}}\rho_j^2dx\\ &+ 2\sum_{j=0}^{N-1}\int_{x_j}^{x_{j+1}}[p_2z(x,t)+p_3z_t(x,t)][-Kw_{jk}+w(x,t)]dx,\end{aligned}$$

which leads to (3.31).

Appendix C. Verification of (4.18). Consider the closed-loop system (4.15) and the Lyapunov functional (4.16). By the descriptor method applied to the first equation of (4.15), we add to $\dot V$ the left-hand side of the following equation:

$$\begin{aligned}&2\int_0^l [p_2z(x,t)+p_3z_t(x,t)]\left[-z_t(x,t)+\frac{\partial}{\partial x}[a_1(x)z_x(x,t)]\right.\\ &\left.-\beta_1z_x(x,t)+[\phi_1-K]z(x,t)\right]dx\\ &+ 2K\sum_{j=0}^{N-1}\int_{x_j}^{x_{j+1}}[p_2z(x,t)+p_3z_t(x,t)][f(x,t)+(t-t_k)\rho_j]dx\\ &+ 2\sum_{j=0}^{N-1}\int_{x_j}^{x_{j+1}}[p_2z(x,t)+p_3z_t(x,t)][b_2v(x,t)-Kw_{jk}+w_1(x,t)]dx = 0.\end{aligned}$$

Then from (3.5), (3.16)–(3.19), and (4.10), (4.11) we obtain

$$\begin{aligned}\dot V(t) \le\ & 2p_z\int_0^l z(x,t)z_t(x,t)dx - 2a_0p_2\int_0^l {z_x}^2(x,t)dx + 2\int_0^l p_3\beta_1 z_t(x,t)z_x(x,t)dx\\ &+ 2\int_0^l [p_2z(x,t)+p_3z_t(x,t)][-z_t(x,t)+(\phi_1-K)z(x,t)]dx\\ &+ 2K\sum_{j=0}^{N-1}\int_{x_j}^{x_{j+1}}[p_2z(x,t)+p_3z_t(x,t)][f(x,t)+(t-t_k)\rho_j]dx\\ &+ r\int_0^l (t_{k+1}-t)z_t^2(x,t)dx - re^{-2\delta h}(t-t_k)\sum_{j=0}^{N-1}\int_{x_j}^{x_{j+1}}\rho_j^2dx\\ &+ 2\sum_{j=0}^{N-1}\int_{x_j}^{x_{j+1}}[p_2z(x,t)+p_3z_t(x,t)][b_2v(x,t)-Kw_{jk}+w_1(x,t)]dx\\ &+ 2\sum_{j=0}^{N-1}\int_{x_j}^{x_{j+1}}[b_3z(x,t)+b_4v(x,t)+w_2(x,t)]v(x,t)dx.\end{aligned}$$

Finally taking into account

$$\begin{aligned}&\int_0^l \zeta^T(x,t)\zeta(x,t)dx\\&\quad\le \int_0^l [c_1^2 z^2(x,t)+c_2^2 v^2(x,t)]dx\\&\qquad + d_1^2 K^2 \sum_{j=0}^{N-1}\int_{x_j}^{x_{j+1}}[-z(x,t)+f(x,t)+(t-t_k)\rho_j - w_{jk}(x,t)]^2 dx,\\&\quad t\in[t_k,t_{k+1}),\ k=0,1,2\dots,\end{aligned}$$

and applying Schur complements we arrive at (4.17).

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