Chapter 10

STEADY MODES IN RELAY SYSTEMS WITH DELAY

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10.1 Introduction

The present lecture is devoted to relay control systems with a relatively big time delay in the control element.

The relay control systems are widely used thanks to the following main reasons:

- the relay control law is one of the simplest control algorithms;
- relay controllers are robust;
- there are control systems in which only sign of variables is observable ([8], [22]);
- sliding motions on a discontinuity surface, a special kind of motions in discontinuous systems, are quite useful for design of an efficient control.

On the other hand, time delay in control systems is usually present and must be taken into account. In practice, time delay is caused by the following:

- *Measuring devices have time delay.* An example of such systems is the controllers of exhausted gas in the fuel injector automotive control systems (see for example [8], [22]).
- *Actuators have a time delay.* An example of such a system is the controller for stabilization of the fingers of an underwater manipulator [4].

We distinguish between the two classes of the relay control system with delay:

- *Systems with time delay in the state.*
- *Systems with time delay in the input.*

The usual approach to the systems with delay in the state consists of two steps ([5],[6]):
(i) definition of the sliding equation;
(ii) application of the sliding mode technique.

We shall concentrate on systems with time delay in the input and describe in detail what kind of stabilization can be achieved, though the standard sliding mode technique does not apply here.

The following simple example shows that the time delay in the relay control law does not allow to realize an ideal sliding mode and underlines the meaning of the general results presented in the sequel.

The simplest example of steady modes

The equation

$$\dot{x}(t) = -\text{sign } [x(t-1)] \tag{SE}$$

has a 4-periodic solution

$$g_0(t) = \begin{cases} t, & \text{for } -1 \leq t \leq 1, \\ 2-t, & \text{for } 1 \leq t \leq 3. \end{cases}$$

$$g_0(t+4k) = g_0(t) \ , \quad k \in \mathbb{Z} \ .$$

Since

$$\dot{g}_0(t) = -\text{sign}[g_0(t-1-4n)] \ ,$$

one we can substitute t for $(4n+1)t$ and obtain

$$\frac{1}{4n+1}[g_0((4n+1)t)]' = -\text{sign } [\frac{1}{4n+1} g_0((4n+1)t)] \ ,$$

hence a $4/(4n+1)$-periodic solution to (SE)

$$g_n(t) = \frac{1}{4n+1} g_0((4n+1)t) \ , \quad t \in \mathbb{R} \ ,$$

for each integer $n \geq 1$. This means that there exists a countable set of periodic solution, so-called *steady modes* (briefly, SM).

We shall show later that any solution $x(t) \not\equiv 0$, of (SE) is equivalent to $g_n(t+\alpha)$ for some $n \geq 0$, $\alpha \in \mathbb{R}$; moreover, a solution $g_n(t)$ is stable for $n = 0$, and unstable for $n \geq 1$. These crucial features persist in more general situations.

Statement of the problem

Consider the equation

$$\dot{x}(t) = -\text{sign}\,[x(t-1)] + F(x(t),t), \quad t \geq 0 \tag{10.1}$$

$$|F(x,t)| \leq p < 1, \quad F \in C^1(\mathbb{R}^2)\ , \tag{10.2}$$

$$x(t) = \varphi(t),\ t \in [-1;0],\ \varphi \in C[-1,0]. \tag{10.3}$$

Under condition (10.2), for any $\varphi \in C[-1;0]$, there exists a unique continuous solution $x_\varphi(t)$, $t \in [-1;\infty)$, of the problem (10.1), (10.3) [21]. We will consider further only such solutions.

The time delay does not allow to realize an ideal sliding mode, but implies the oscillations, whose stability is determined by one discrete parameter – *oscillation frequency*, which is the number of zeroes on the time interval with length of delay preceding some zero of $x_\varphi(t)$. The basic property of the frequency, its monotone decrease, has been observed in other situations (see [23], [25]). A specific topic for discontinuous delay equations, infinite frequency oscillations, have been studied in [27],[1],[26],[9]. Some problem of qualitative behavior of solutions of relay equation with delay was considered in [20]. The relay control algorithms for systems with delay have been suggested in [8], [22],[2]. We show also that any motion of system (10.1) turns into a steady mode, a motion with a constant frequency, as it happens in the case of usual sliding modes. At the same time this means that there are no asymptotically decreasing solutions.

All these observations are used in our approach to the following main questions on relay controllers with delay:

1. What steady modes are stable?

2. How could relay controllers with delay be used for stabilization of unstable systems?

3. How could relay controllers with delay be used for stabilization of oscillations in a small neighborhood of constraints for stable systems, in which perturbations accumulate and take the system rather far from constraints?

Organization of the material

Section 10.2.1 contains one of the main results, Theorem 4, which says that any solution of the equation (10.1) can be basically characterized by one discrete parameter, an average oscillation frequency, which is the number of zeroes on the time interval, preceding some zero of the solution, of length equal to the delay. A similar result was obtained for the smooth system in [23]. It is shown that each solution of the equation (10.1) is equivalent to steady mode, which is a *solution with a constant frequency.* That means we have *finite time* of input in steady mode. Moreover, in the autonomous case, there exists a countable set of periodic SM generating all other SM by translations in t. Another important result consists in a description of classes of stable and unstable SM (section 10.2.2). A multidimensional singularly perturbed relay system with time delay is studied in section 10.3, where we prove the existence of slow stable periodic solutions, which is a generalization of a similar result for system (SE).

The algorithms of stabilization are presented in section 10.4. After this section we discuss possible generalization and open problems. The proofs are presented in section 10.6.

10.2 Steady Modes and Stability

10.2.1 Steady Modes

The main object of this section is a special characteristic of a solution, its *oscillation frequency*. Our main result (Theorem 1) states that, for any solution, its frequency becomes constant after a period of time. Two solutions are called *equivalent* if they coincide after some time moment. So, each solution is equivalent to some steady mode, a solution with a constant frequency.

Here we formulate and discuss the statements. The proofs are presented in appendix.

Let Z_φ denote a set of zeros of $x_\varphi(t)$. Put $Z^+_\varphi = Z_\varphi \cap [0; +\infty)$.

Lemma 1 *For any $\varphi \in C[-1;0]$ the set Z_φ is non-empty and unbounded.*

Hence we can define the frequency function $\nu_\varphi : Z_\varphi^+ \longrightarrow \mathbb{N} \cup \{0\} \cup \{\infty\}$ by

$$\nu_\varphi(t) = \text{card } (Z_\varphi \cap (t-1;t)) \ , \quad t \in Z_\varphi^+ \ .$$

Theorem 1 *For any $\varphi \in C[-1;0]$ the function ν_φ is non-increasing, and hence there exists a limit*

$$N_\varphi \stackrel{\text{def}}{=} \lim_{\substack{t\to\infty \\ t\in Z_\varphi^+}} \nu_\varphi(t).$$

Lemma 2 *If $N_\varphi < \infty$ then N_φ is even, and $C[-1;0]$ is divided into sets*

$$\mathcal{U}_\infty = \{\varphi \in C[-1;0] \ : \ N_\varphi = \infty\},$$

$$\mathcal{U}_n = \{\varphi \in C[-1;0] \ : \ N_\varphi = 2n\}, \quad n \geq 0 \ .$$

Introduce the following subset of $C[-1;0]$:

$$\mathcal{F} = \{\varphi \in C[-1;0] \ : \ \varphi^{-1}(0) \text{ is finite}\}$$

It follows immediately from Theorem 1 that

$$\mathcal{F} \subset \bigcup_{0\leq n<\infty} \mathcal{U}_n$$

Definition 3 *A solution $x_\varphi(t)$ with $\nu_\varphi \equiv \text{const}$ is called steady mode (SM).*

The set of SM is represented naturally as the union of disjoint sets $\mathcal{S}_n = \{x_\varphi(t) \ : \ \nu_\varphi \equiv 2n\}$, $n \geq 0$, $\mathcal{S}_\infty = \{x_\varphi(t) | \nu_\varphi \equiv \infty\}$.

Theorem 2 *For any integer $n \geq 0$ and real $T \geq 0$ there exists $g(t) \in \mathcal{S}_n$ such that*

$$g(T) = 0 \ , \quad \dot{g}(T) > 0 \ . \tag{10.4}$$

If $n = 0$ then such SM is unique.

In the autonomous case, we give a more precise description of the SM set:

Theorem 3 *In the autonomous case for any $n \geq 0$ the SM are unique in following sense: there are periodic steady modes $g_0, g_1 \ldots g_n, \ldots$ such that*

$$\mathcal{S}_n = \{g_n(t+\alpha) \,:\, \alpha \in \mathbb{R}, \quad n \geq 0\},$$

and their periods satisfy inequalities

$$\tau_0 > 2, \quad n^{-1} > \tau_n > (n+1)^{-1}, \quad n \geq 1 \,. \tag{10.5}$$

Remark 4 *In fact, in the autonomous case $\mathcal{S}_\infty = \emptyset$ if $F(0) \neq 0$, and $\mathcal{S}_\infty = \{0\}$ if $F(0) = 0$. This was recently proved by Akian, Bliman [1] and Nussbaum, Shustin [26]. For the non-autonomous case see [9, 27].*

As a consequence of the above statements we obtain

Theorem 4 *Any solution $x_\varphi(t)$ of the (10.1), (10.3) is equivalent to a suitable SM.*

10.2.2 Stability

Here we study the stability of solutions of our equation with respect to the standard metric in the space $C[-1;0]$ of initial functions. First we show that the zero steady frequency is stable, then from this we derive the non-asymptotic stability of zero-frequency SM in the autonomous case and give a condition of the closeness to the autonomous case, where the same type of stability is present. Finally, we establish that SM with positive frequency are unstable.

Theorem 5 *The set $\mathcal{U}_0$ has nonempty interior. Moreover, Int $\mathcal{U}_0$ contains the non-empty set*

$$\tilde{\mathcal{U}_0} = \mathcal{U}_0 \cap \{\varphi \in C[-1;0] \;:\; \text{mes}(\varphi^{-1}(0)) = 0\}.$$

In particular, we get that the function $N(\varphi) = N_\varphi = 0$ is stable if $\mathrm{mes}(\varphi^{-1}(0)) = 0$.

Corollary 6 *In the autonomous case all the solutions $x_\varphi(t), \varphi \in \tilde{\mathcal{U}}_0$, are non-asymptotically stable.*

Theorem 7 *If*

$$\int_0^\infty \max_x \left|\frac{\partial F(x,t)}{\partial t}\right| dt < \infty \tag{10.6}$$

then all solutions $x_\varphi(t), \varphi \in \tilde{\mathcal{U}}_0$, are non-asymptotically stable.

We should underline that there are unstable solutions $x_\varphi(t)$ with $\varphi \in \mathcal{U}_0$. For example, let $\psi \in \mathcal{U}_n,\ n \geq 1$, then $\varphi(t) = \max\{0; \psi(t)\} \in \mathcal{U}_0$, but $\varphi_\tau(t) = \varphi(t) + \tau\psi(t) \in \mathcal{U}_n$, for any $\tau > 0$.

Theorem 8 *If*

$$\sup\left|\frac{\partial F}{\partial x}\right| = M_x < 2(1-p)^2(1+p)^{-3} \tag{10.7}$$

or

$$\sup\left|\frac{\partial F}{\partial t}\right| = M_t < 2(1-p)^2(1+p)^{-2} \tag{10.8}$$

then all solutions $x_\varphi(t), \quad \varphi \in \bigcup_{1\leq n\leq\infty} \mathcal{U}_n$, are unstable.

Note that conditions of Theorems 7 and 8 are fulfilled in the autonomous case.

10.3 Singular perturbation in relay systems with time delay

10.3.1 Existence of stable zero frequency periodic steady modes for a singularly perturbed multidimensional system

Here we study a multidimensional generalization of system (SE). Consider the system

$$\mu\frac{dz}{dt} = f(z,s,x,u), \quad \frac{ds}{dt} = g(z,s,x,u), \quad \frac{dx}{dt} = h(z,s,x,u), \tag{10.9}$$

where $z \in \mathbb{R}^m, s \in \mathbb{R}, x \in \mathbb{R}^n, u(s) = \text{sign}[s(t-1)]; f,g,h \in \mathcal{C}^2(\bar{Z}), Z \subset \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^n \times [-1,1]$; μ is a small parameter.

Ignoring additional dynamics, accepting $\mu = 0$ and expressing z_0 from the equation

$$g(z_0, s, x, u(s)) = 0 \ ,$$

we obtain from the formula $z_0 = \varphi(s,x,u)$ that

$$\frac{ds}{dt} = g(\varphi(s,x,u),s,x,u) = G(s,x,u) \ ,$$

$$\frac{dx}{dt} = h(\varphi(s,x,u),s,x,u) = H(s,x,u) \ , \tag{10.10}$$

which satisfy the sufficient conditions for the existence of a zero frequency steady mode.

Suppose that

C1: the function $z_0 = \varphi(s,x,u)$, for all $(s,x,u) \in \bar{S}; S \subset \mathbb{R}\times\mathbb{R}^n\times[-1,1]$, is a uniformly asymptotically stable isolated equilibrium point of system $dz/d\tau = f(z,s,x,u)$; moreover, the matrix $\frac{\partial f(z,s,x,u)}{\partial z}$ is stable at all $(s,x,u) \in \bar{S}$, and the inequality $Re\, Spec\, \frac{\partial f(z,s,x,u)}{\partial z} < -\alpha < 0$ holds.

Under condition C1 we design the point mapping of surface $s = 0$ into itself determined by system (10.10).

Namely, consider the solution to (10.10) for $u = 1$:

$$\frac{d\bar{s}_0^+}{dt} = G(\bar{s}_0^+, \bar{x}_0^+, 1), \quad \frac{d\bar{x}_0^+}{dt} = H(\bar{s}_0^+, \bar{x}_0^+, 1) \tag{10.10+}$$

with the initial conditions

$$\bar{s}_0^+(0) = 0,\ \bar{s}_0^+(t) < 0,\ t \in [-1, 0);\ \bar{x}_0^+(0) = \xi,\ \xi \in V \subset S^+ = \{\xi : G(0, \xi, 1) > 0\}.$$

Suppose that, for $t = 1$, the relay control u changes its value from $+1$ to -1 so that the behavior of a solution to (10.10) is described by the system

$$\frac{d\bar{s}_0^-}{dt} = G(\bar{s}_0^-, \bar{x}_0^-, -1), \quad \frac{d\bar{x}_0^-}{dt} = H(\bar{s}_0^-, \bar{x}_0^-, -1), \tag{10.10−}$$

$$\bar{s}_0^-(1) = \bar{s}_0^+(1),\ \bar{x}_0^-(1) = \bar{x}_0^+(1)\ .$$

Suppose that, for all $\xi \in V$, there exists the smallest root of equation $\bar{s}_0^-(\theta(\xi)) = 0$ such that $G(0, \bar{x}_0^-(\theta(\xi)), -1) < 0$ and consequently for $t = \theta(\xi) + 1$ the control low u changes its value from -1 to $+1$. Then the behavior of solution of (10.10) the behavior of the system for $t > \theta(\xi) + 1$ is described by system (10.10+) with initial condition

$$\bar{s}_0^+(\theta(\xi) + 1) = \bar{s}_0^-(\theta(\xi) + 1), \quad \bar{x}_0^+(\theta(\xi) + 1) = \bar{x}_0^-(\theta(\xi) + 1)\ .$$

Suppose also that, for all $\xi \in V$, there exists $T(\xi)$, the smallest root of the equation $\bar{s}_0^+(T(\xi)) = 0$, such that $T(\xi) > \theta(\xi)+1$ and $G(0, \bar{x}_0^+(T(\xi)), 1) > 0$. Then the point mapping $\Psi(\xi) : \xi \to \bar{x}_0^+(T(\xi))$ is the point mapping of the domain V on the surface $s = 0$ produced by system (10.10).

Introduce now the following assumptions

C2: system (10.10) has an isolated zero frequency steady mode $(s_0(t), x_0(t))$, which has exactly two intersection points with the surface $s = 0$ such that

$$s_0(0) = 0,\ \frac{ds_0}{dt}(0) > 0, \quad s_0(\theta_0) = 0,\ \frac{ds_0}{dt}(\theta_0) < 0\ ;$$

C3: the point mapping $\Psi(x)$ of the surface $s = 0$ into itself, which made by system (10.10) , has the stable isolated equilibrium point x_0 corresponding $(s_0(t), x_0(t))$, moreover

$$\|\frac{\partial \Psi(x_0)}{\partial x}\| < q < 1 \ ;$$

C4: the points $\varphi(s_0(1), x_0(1), -1)$ and $\varphi(s_0(\theta_0+1), x_0(\theta_0+1), 1)$ are situated in the attractive domains of stable equilibrium points $\varphi(s_0(1), x_0(1), 1)$ and $\varphi(s_0(\theta_0 + 1), x_0(\theta_0 + 1), -1)$, respectively.

Theorem 9 *Under conditions C1-C4 system (10.9) has an orbitally asymptotically stable isolated periodic solution close to* $(s_0(t), x_0(t))$ *with a period* $T(\mu)$ *which tends to* T *as* $\mu \to 0$, *and the boundary layers close to* $t = 1$, $t = \theta_0 + 1$.

Remark 5 *An algorithm for the asymptotic representation of a zero frequency periodic steady mode [34], based on the boundary layer method is suggested in [17].*

10.3.2 Existence of stable zero frequency steady modes in systems of arbitrary order

Consider the system

$$\mu\frac{dz_1}{dt} = -z_1 + u, \quad u(s) = -\text{sign}[s(t-1)] \ ,$$
$$\mu\frac{dz_2}{dt} = z_1 - z_2, ..., \mu\frac{dz_k}{dt} = z_{k-1} - z_k; \qquad (10.11)$$
$$\frac{ds}{dt} = x, \quad \frac{dx}{dt} = -x + z_k,$$

where $z_1, ..., z_k, s, x \in R, \mu$ is the small parameter. It is obvious that system (10.11) is system with relative degree $(k + 2)$ with respect to output variable s. Let's show that for system (10.11) the conditions of Theorem 9 hold and consequently system (10.11) has an orbitally asymptotically stable zero frequency steady mode at least for the small μ.

For $\mu = 0$ system (10.11) has the form

$$\bar{z}_1 = \bar{z}_2 = ... = \bar{z}_k = u,$$

$$\frac{d\bar{s}_0}{dt} = \bar{x}_0, \ \frac{d\bar{x}_0}{dt} = -\bar{x}_0 + u. \tag{10.12}$$

Then for the solution of (10.12) with initial conditions

$$\bar{x}_0^+(0) = \xi, \quad \bar{s}_0^+(0) = 0,$$

$$\text{sign}\,[\bar{s}_0^+(t-1)] = -1, \quad u = 1 \quad \text{for} \quad t \in [-1, 0]$$

we have

$$\bar{x}_0{}^+(t, \xi) = e^{-t}(\xi - 1) + 1; \quad \bar{s}_0^+(t, \xi) = (1 - e^{-t})(\xi - 1) + t;$$

and consequently

$$\bar{x}_0^+(1, \xi) = e^{-1}(\xi - 1) + 1; \quad \bar{s}_0^+(1, \xi) = (1 - e^{-1})(\xi - 1) + 1.$$

For $t > 1, u = -1$ and until switching of sign(u)

$$\bar{x}_0^-(t, \xi) = e^{-(t-1)}(\bar{x}_0^+(1, \xi) + 1) - 1;$$

$$\bar{s}_0^-(t, \xi) =$$

$$= (1 - e^{-(t-1)})(\bar{x}_0^+(1, \xi) + 1) - (t - 1) + (1 - e^{-1})(\xi - 1) + 1.$$

In this case the switching moment $\theta(\xi)$ is defined by equation $s_0^-(\theta(\xi), \xi) = 0$. Taking into account the symmetry of system (10.12) with respect to the point $s = x = 0$, we can conclude that the semi-period of the desired periodic solution θ_0 and the fixed point ξ_0 of the point mapping $\Psi(\xi)$ are described by equation

$$\bar{s}_0^-(\theta_0, \xi_0) = 0, \quad \bar{x}_0^-(\theta_0, \xi_0) = -\xi_0\ ,$$

hence

$$\xi_0 = 1 - 2\frac{e^{-\theta_0+1}}{1 + e^{-\theta_0}}; \quad \theta_0 = 4 - 4\frac{e^{-\theta_0+1}}{1 + e^{-\theta_0}}.$$

This system has the solution $\theta_0 \approx 3,75, \xi_0 \approx 0,87$. Here ξ_0 is the fixed point of point mapping $\Psi(\xi)$, corresponding the $2\theta_0-$ periodic solution of (10.12) determined by the equations

$$(\bar{s}_0(t), \bar{x}_0(t)) = \begin{cases} (\bar{s}\,^+_0(t,\xi_0), \bar{x}\,^+_0(t,\xi_0)), & \text{for} \quad -\theta_0 + 1 \le t \le 1, \\ (\bar{s}\,^-_0(t,\xi_0), \bar{x}\,^-_0(t,\xi_0)), & \text{for} \quad 1 \le t \le \theta_0 + 1. \end{cases}$$

Moreover,

$$\frac{d\Psi}{d\xi}(\xi_0) = \Big(\frac{dx^-(\theta(\xi),\xi)}{d\xi}(\theta_0,\xi_0)\Big)^2 = \Big(e^{-\theta_0} - 2\frac{e^{-\theta_0+1}(1-e^{-\theta_0})}{e^{-\theta_0}+1-2e^{-\theta_0+1}}\Big)^2 \approx 0,0144.$$

Then the conditions of Theorem 9 hold for system (10.11), therefore system (10.11) has an orbitally asymptotically stable periodic zero frequency steady mode at least for the small μ. This means that for any k there exists at list one orbitally asymptotically stable zero frequency periodic steady mode of $(k+2)$-th order.

10.4 Design of delay controllers of relay type

10.4.1 Stabilization of the simplest unstable system

Consider the stabilization problem for the simplest unstable system

$$\dot{x} = kx, \quad (x \in \mathbb{R},\, k > 0) \tag{US}$$

by means of a delay relay control law of the form $u = -\text{sign}\,[x(t-\gamma)]$, where γ is time delay. In this case the equation for control system has the form

$$\dot{x}(t) = -\text{sign}\,[x(t-\gamma)] + kx, \tag{CS}$$

Let us compute the constant $A > 0$ for which the system (CS) with initial function

$$\varphi(t) = A, \quad t \in [-\gamma, 0] \tag{IF}$$

has stable periodic solution for $t > 0$.

Before the switching moment we have

$$x(t) = \frac{1}{k} + (A - \frac{1}{k})\, e^{kt}.$$

The function $x(t)$ could change its sign if and only if the condition

$$A - \frac{1}{k} < 0$$

holds. In this case we can rewrite equation for $\tau -$ which is the root of equation $x(\tau) = 0$ in form $e^{k\tau} = \frac{1}{1-kA}$. From periodicity of $x(t)$ we have the equation for the switching moment of the control law in the form $x(\tau + \gamma) = -A$. Then

$$\frac{1}{k} + (A - \frac{1}{k})e^{k\tau}e^{k\gamma} = -A,$$

and consequently $A = (e^{k\gamma} - 1)/k$. This means that sufficient condition for existence of the periodic solution has he form

$$k\gamma < \log 2. \tag{SC}$$

This implies that for any positive feedback coefficient k we can choose the time delay γ for which there exist zero frequency stable periodic steady mode of (CS). Moreover the equation (CS) has a countable set of steady modes in the interior of the strip $|x| < (e^{k\gamma} - 1)/k$. System (CS) has unstable solutions $x = \pm 1/k$, and unbounded solutions in the regions $|x| > 1/k$.

This means that Cauchy problem (CS), (10.3) has bounded solution if for any $t \in [0, \gamma]$, $k|x_\varphi(t)| < 1$. This means that if $\varphi(0) > 0$, then

$$k|x_\varphi(t)| = |-1 + (k\varphi(0) + 1)e^{kt}| < 1.$$

This implies

Theorem 10 *If condition (CS) holds and* $|\varphi(0)| < \frac{2-e^{k\gamma}}{ke^{k\gamma}}$*, than the solution* $x_\varphi(t)$ *of (CS), (10.3) is bounded.*

10.4.2 Stable systems with bounded perturbation and relay controllers with delay

Consider the simplest stable system with bounded perturbations

$$\dot{x} = -kx + F(t, x), \quad (x \in \mathbb{R},\, k > 0). \tag{PS}$$

Here $|F(t, x)| \leq \varepsilon$ is bounded perturbation. Suppose that we have the possibility to use the relay control with delay γ in form $u(s) = -\lambda \cdot \text{sign}[x(t-\gamma)]$, $\lambda > \varepsilon$. The behavior of the control system is described by equation

$$\dot{x} = -kx + F(t, x) - \lambda \cdot \text{sign}[x(t-\gamma)]. \tag{CPS}$$

Than for the amplitude of the we have the following estimation

$$|x(\gamma)| \leq \int_0^{\gamma} e^{-k(\gamma-\tau)}(|\lambda| + |F(\tau, x(\tau)|)d\tau \leq \frac{\lambda+\varepsilon}{k}(1-e^{-k\gamma}) \leq \gamma(\lambda+\varepsilon).$$

It allows us to conclude that the motions in stable systems are in the $O(\varepsilon)$ neighborhood from constraints. If we are using the relay control with delay, the amplitude of oscillation is $O((\lambda+\varepsilon)\gamma)$. It is important in the case of sufficiently small $\lambda, \varepsilon, \gamma$.

10.4.3 Statement of the adaptive control problem

Consider the system

$$\dot{x}(t) = F(x, t) + u(t). \tag{10.13}$$

$$u(t) = \alpha(t) \cdot \text{sign}\,[x(t-1)].$$

A real controller operates with an unavoidable time delay. Here we develop the direct adaptive delay control of relay type $u(t) = -\alpha \cdot \text{sign}\,[x(t-1)]$ with a step function α depending on the only information on the time interval $(-1, t-1)$ provides exponentially decreasing oscillations even in the presence of disturbances. Here we restrict ourselves to those systems satisfying (cf. [20])

$$F(0, t) \equiv 0\ .$$

and everywhere below in section 10.3 we suppose this equality.

Note that here we lose the restriction (10.2), and solutions may be unbounded and inextensible to the infinite interval. On the other hand, there are SM with sufficiently big frequency and small amplitude. It turns out that the existence of stable SM with *zero frequency* implies the existence of a wide class of bounded solutions. Namely,

Lemma 6 *Let*

$$F(0,t) \equiv 0 \tag{10.14}$$

$$\frac{\partial F}{\partial x}(x,t) \le k < \ln 2, t \in \mathbb{R}, \ |x| < \alpha/k. \tag{10.15}$$

Then all the solutions of equation

$$\dot{x}(t) = F(x,t) + \alpha \cdot \text{sign}\,[x(t-1)]$$

with initial condition (10.3), where

$$|x(0)| = |\varphi(0)| < \alpha(2\exp(-k) - 1)/k, \tag{10.16}$$

are extensible to the interval $(-1;\infty)$ *and satisfy inequalities*

$$|x_\varphi(t)| \le \frac{\alpha}{k}(e^k - 1), \quad |\dot{x}_\varphi(t)| \le \alpha e^k. \tag{10.17}$$

These solutions hold all properties mentioned in sections 10.2.

In a non-ideal situation we'll use the following simple estimate:

Proposition 7 *Under the conditions of Lemma 6, let* T *be a zero of some solution* $x(t)$ *of (10.1), and* $|T^* - T| < \delta$. *Then*

$$|x(T^*)| < \alpha(e^{k\delta} - 1)/k \ .$$

10.4.4 The case of definite systems

Assume $F(x,t)$ holds (10.15), and we know a complete information on $F(x,t)$ and have the observer, which indicates zeros of $x(t)$ and signs of $x(t)$ with the delay 1. We design the desired control by means of the following algorithm.

Let (10.16) hold with some constant $\alpha = \alpha_0$. Put $\alpha(t) = \alpha_0$, $t \geq 0$, and consider the equation

$$\dot{x}(t) = -\alpha_0 \cdot \operatorname{sign}[x(t-1)] + F(x(t),t), \quad t \geq 0.$$

We fix a time moment $t_1 + 1$, when the observer indicates the first zero t_1 of $x(t)$ greater than 1. Using the distribution of zeros and signs of $x(t)$ on the segment $[0; t_1]$, we extrapolate $x(t)$ on the interval $t > t_1$ and compute the first zero t_2 of $x(t)$ greater than $t_1 + 1$. Now in the ideal situation we can put

$$\alpha(t) = \alpha_1, \quad t \geq t_2$$

where α_1 is an anyhow small positive constant, and, according to (10.17), we obtain a solution $x(t)$ anyhow close to zero.

Assume we compute t_2 with error δ. Let δ satisfy the condition

$$\rho \stackrel{\text{def}}{=} \frac{e^{k\delta} - 1}{2e^{-k} - 1} < 1 \Longleftrightarrow \delta < \frac{\ln 2}{k} - 1. \tag{10.18}$$

From Proposition 7 it follows immediately the property (10.16) at t_2 with the constant $\alpha = \alpha_0 \rho$. Now we put $\alpha(t) = \alpha_0 \rho$, $t \geq t_2$ and repeat our algorithm from the beginning. After m steps we get from (10.17)

$$|x(t)| \leq \frac{e^k - 1}{k} \alpha_0 \rho^m = \frac{k^{m-1}(e^k - 1)}{(2e^{-k} - 1)^m} \delta^m + O(\delta^{m+1}) \tag{10.19}$$

The left side of (10.19) tend to zero for $m \to \infty$.

10.4.5 The case of indefinite systems

Having the error δ_0 of the observer and the property (10.15) as only an information on $F(x,t)$, we really have to solve a single problem in using the previous algorithm: to construct a zeros sequence on an interval $(t;\infty)$ having a zeros sequence on $(-1;t-1)$.

In the autonomous case Theorem 8 provides (with the probability 1) turning any bounded solution of the equation

$$\dot{x}(t) = -\alpha \cdot \mathrm{sign}[x(t-1)] + F(x(t))$$

into some zero frequency SM. Assume that by the time moment $t_{2n}+1$ our observer indicated consequent zeros $t_0, t_1, \ldots, t_{2n}$ such that $t_i + 1 < t_{i+1}$, $i = 0, \ldots, 2n-1$. According to periodicity of SM (see Theorem 4), the following zero equals $t_{2n+1} = t_{2n-1} + (t_{2n} - t_0)/n > t_{2n} + 1$ with error $\delta = \delta_0(1 + 2/n)$. If δ satisfies (10.18) then, repeating such steps, we stabilize the zero solution as above.

10.5 Generalizations and open problems

10.5.1 The case when $|F(x)| > 1$ for some x

In [29] it was shown that the results of section 10.2 for system (10.1) hold for the case when for some x the function $F(x)$ has values out of $[-1,1]$, but satisfies the following conditions.
$(i)\, x^+_{-1} \leq x^+_1$ or $\int_0^{x_1^+} \frac{dx}{1+F(x)} > 1$,
and
$(ii)\, x^-_1 \geq x^-_{-1}$ or $\int_{x^-_{-1}}^0 \frac{dx}{1-F(x)} > 1$,
where
$x^+_1 = inf\{x > 0 \,:\, F(x) = 1\}, \quad x^+_{-1} = inf\{x > 0 \,:\, F(x) = -1\},$
$x^-_1 = inf\{x < 0 \,:\, F(x) = 1\}, \quad x^-_{-1} = inf\{x < 0 \,:\, F(x) = -1\}.$

Systems with different delays are more complicated [3]. This is a very interesting subject for study.

10.5.2 Systems and steady modes of the second order.

Relay control systems with delay of second order are considered in the form

$$\frac{d^2x}{dt^2} = -\frac{dx}{dt} + F(x) - \operatorname{sign}[x(t-1)], \qquad \text{(SONRCSD)}$$

$$x(t) = \varphi(t), t \in [-1, 0], \varepsilon = \text{const} > 0, \dot{x}(0) = x_0$$

$$|F(x,t)| < p < 1,\ F \in \mathcal{C}^1(\mathbb{R}^2).$$

was considered in [18]. It is shown that if the frequency ν of solution of (SONRCSD) is even, then ν does not increase. If the frequency ν is odd it could increase by 1. This allows us to introduce the notion of frequency for the second order relay control systems with delay in the form $\psi = [(\nu+1)/2]$ (here $[\cdot]$ is the entire part), which is a non-increasing function. It is shown that for each solution of (SONRCSD) there exists the limit value of frequency $N = \lim_{t\to\infty} \psi$. It is proved that in the case when F is autonomous for any integer $\psi > 0$ there exists a periodic steady mode.

Second order linear relay control system with delay

$$\varepsilon\frac{d^2x}{dt^2} = -\frac{dx}{dt} + kx - \operatorname{sign}[x(t-1)] \qquad \text{(SOLRCSD)}$$

was considered in [15], [16], [18], [19], [33]. For such system there have been found conditions providing that

- the frequency ψ is non-increasing;
- there exists a countable set of periodic steady modes for any integer nonnegative value of ψ;
- the zero frequency periodic steady modes are orbitally asymptotically stable.

The natural sufficient conditions for orbital asymptotic stability of zero frequency steady modes for (SOLDRCS) was found in [15],[16],[18], [19],[33]. For second order relay control systems with delay the problem of instability of steady modes with nonzero frequency is still open.

10.5.3 Stability and instability of steady modes for multidimensional case.

In [14] was considered the multidimensional relay control systems with delay in form

$$\begin{cases} \dot{s}(t) = -\text{sign}[s(t-1)] + F(s(t), x(t)), \\ \dot{x}(t) = As(t) + Bx(t), \end{cases} \tag{MDRCSD}$$

$$s \in \mathbb{R},\ x \in \mathbb{R}^n,\ |F(s,x)| < p < 1.$$

It is shown that if B is a stable matrix, then, for any even value of frequency, there exists a periodic steady mode. In [14] the problem of stability of zero frequency steady modes of (MDRDCS) is reduced to the problem of contractibility of point mapping of the surface $s = 0$ into itself made by the original system. In fact, it is practically impossible to check this property of (MDRCSD) and the problem of stability is open. As in the previous case the problem of instability is open too.

Conclusions

1. The notions of frequency and steady modes are introduced. The existence of steady modes for any even frequency are established.
2. The steady modes possess properties similar to that of sliding modes:

- (i) the set of switches of any steady mode is unbounded, thus, a steady mode is not equivalent to any solution of a continuous part of the given equation;

- (ii) for any solution there exists a finite time preceding its input into a steady mode;

- (iii) the shift operator is not invertible;

- (iv) the properties (i)-(iii) are invariant with respect to bounded perturbations which satisfy the conditions (10.2).

3. Stability criteria for steady modes with zero frequency are established.
4. It is proved that all steady modes with the positive frequency are unstable under some mild conditions.
5. The existence of a slow stable periodic solution of the multidimensional singularly perturbed relay system with time delay, which corresponds to the stable zero frequency steady mode, is proved.
6. A direct adaptive control of relay type with time delay that extinguishes parasite auto-oscillations in this model is designed.

10.6 Appendix. Proofs

Lemma 1 is obvious.

Proof of Theorem 1. If $t_1 < t_2,\ t_1, t_2 \in Z_\varphi^+$, then, according to Rolle's theorem and (10.1), (10.2), there exists $\xi \in (t_1 - 1; t_2 - 1) \cap Z_\varphi$. Therefore

$$\text{card}\,(Z_\varphi \cap (t_1 - 1; t_2 - 1)) \geq \text{card}\left(Z_\varphi^+ \cap (t_1; t_2)\right) + 1\ ,$$

hence

$$\nu_\varphi(t_1) = \text{card}\,(Z_\varphi \cap (t_1 - 1; t_1)) \geq \text{card}\,(Z_\varphi \cap (t_2 - 1; t_2)) = \nu_\varphi(t_2).$$

Proof of Lemma 2. Let $\nu_\varphi(t) = N_\varphi < \infty$, when $t \geq T$. Then $x_\varphi(t)$ changes its sign at every point $t \in Z_\varphi \cap [T; +\infty)$. Indeed, if $t_1 < t_2$ are neighboring points from $Z_\varphi \cap [T+1; \infty)$ then, according to above assumption, there is a unique $z \in (t_1 - 1; t_2 - 1) \cap Z_\varphi$, and hence $x_\varphi(t)$ changes its sign at z. Let us suppose, for example, that $z \in Z_\varphi^+$ and $x_\varphi(z)$ change its sign at some point from plus to minus. Hence $\dot{x}(z)$ is negative. This means that $x_\varphi(z-1)$ is positive. It is possible only in case when number of switches is even.

Proof of Theorem 2. In the case $N = 0$ the desired statement is obvious. Fix even $N > 0$. Put

$$\Sigma = \{(a_0, \ldots, a_N) \in R^{N+1}\ :\ a_0 \geq 0, \ldots a_N \geq 0,\ , a_0 + \cdots + a_N = 1\}\ .$$

Let $Z_\varphi \cap [T;+\infty)$ be locally finite, and

$$T = t_1 < t_2 < t_3 < \dots$$

be all zeros of $x_\varphi(t)$ in $[T;+\infty)$. Let us define the operators of 'step forward" and "step back". Assume that $\nu_\varphi(t_k) = \nu_\varphi(t_{k+1}) = N$. Define the following vectors of sign changes: $\overline{a} = (a_0, \dots, a_N), \overline{b} = (b_0, \dots, b_N) \in \Sigma$, where

$$a_0 = t_k - t_{k-1}, \quad a_1 = t_{k-1} - t_{k-2}, \dots, a_{N-1} = t_{k-N+1} - t_{k-N},$$

$$a_N = t_{k-N} - (t_k - 1)$$

$$b_0 = t_{k+1} - t_k, b_1 = t_k - t_{k-1}, \dots, b_{N-1} = t_{k-N+2} - t_{k-N+1},$$

$$b_N = t_{k-N+1} - (t_{k+1} - 1).$$

Hence we obtain a correspondence

$$\Gamma : (\overline{a}, \alpha, \varepsilon) \to (\overline{b}, \beta, -\varepsilon) \ ,$$

where $\alpha = t_k,\ \beta = t_{k+1}, \varepsilon = \text{sign}\ \dot{x}_\varphi(t_k)$.

Proposition 8 *For a fixed ε, the correspondence inverse to Γ, is a smooth map*

$$M_\varepsilon : \Sigma \times R \to \Sigma \times R.$$

Proof. Denote by $x_\varepsilon(t_0, x_0, a)$, $\varepsilon = \pm 1$, the solution of the Cauchy problem

$$\frac{dx}{da} = \varepsilon + F(x, t_0 + a), \quad x(0) = x_0.$$

Define functions $T = \lambda_\varepsilon(t, a), \varepsilon = \pm 1$, by equations

$$x_{-\varepsilon}(t + a, x_\varepsilon(t, 0, a),\ b) = 0, \quad T = t + a + b\ . \tag{10.20}$$

It is easy to see that for a fixed t_0, the function $\lambda_\pm(t_0, a)$ increases strictly, and $\lambda_\pm(t_0, a) > a$ if $a > 0$. Therefore, for a fixed t_0, we can define positive functions of $b > 0$:

- $\rho_\varepsilon(t_0, b)$ inverse to $b = \lambda_\varepsilon(t_0, \rho_\varepsilon)$;
- $\sigma_\varepsilon(t_0, b) = b - \rho_\varepsilon(t_0, b)$.

Hence $(\overline{a}, \alpha) = M_\varepsilon(\overline{b}, \beta)$ can be defined as

$$a_0 = b_1, \quad a_1 = b_2, \ldots, a_{N-2} = b_{N-1} \ ,$$

$$a_{N-1} = b_N + \sigma_\varepsilon(\beta - b_0,\ b_0),\ a_N = \rho_\varepsilon(\beta - b_0, b_0),$$

$$\alpha = \beta - b_0 \tag{10.21}$$

Thereby Proposition 8 defines the operator of step back with a constant frequency independently from initial assumption $\nu_\varphi(t_k) = \nu_\varphi(t_{k+1}) = N$.

So, given a triple $(\overline{a}, \alpha, \varepsilon)$, we can construct a solution of (10.1) for $t \geq \alpha$, and using maps $M_\pm$ we can extend this solution on the interval $(-\infty, \alpha)$ with a constant frequency function. Now let us introduce the decreasing sequence of closed connected sets

$$\Pi_0 = \Sigma \times R, \quad \Pi_{n+1} = (M_- M_+)(\Pi_n), n \geq 0.$$

The set $\Pi = \Pi_0 \cap \Pi_1 \cap \Pi_2 \cap \cdots$ is an invariant set of operator of step back. The statement of Theorem 2 is equivalent to $\Pi \cap (\Sigma \times \{\alpha\}) \neq \emptyset$ for any $\alpha \in R$. It is obvious that, for any $k > 0$,

$$\Pi_k \cap (\Sigma \times \{\beta\}) \neq \emptyset,$$

for β both anyhow big and anyhow small, because the time decrease in one step is absolutely bounded. Then (10.21) is fulfilled for any $k \geq 0, \beta \in R$, because $\Pi_k,\ k \geq 0$, are connected. Thus, $\Pi \cap (\Sigma \times \{\alpha\}) \neq \emptyset$, because $\Pi_k \cap (\Sigma \times \{\alpha\}) \neq \emptyset$, $k \geq 0$, are non-empty compacts.

Proof of Theorem 3. We shall prove that, for any $n \geq 1$ and a fixed $T \in R$, there is a unique $g_{n,T} \in \mathcal{S}_n$ with property (10.4). Since M_ε, defined by (10.21), doesn't depend on β we get a map $M_\varepsilon : \Sigma \to \Sigma$ such that

$$\overline{a} = M_\varepsilon(\overline{b});\ \overline{a},\ \overline{b} \in \Sigma,$$

$$a_0 = b_1,\ a_1 = b_2, \dots, a_{N-2} = b_{N-1},$$

$$a_{N-1} = b_N + \sigma_\varepsilon(b_0), a_N = \rho_\varepsilon(b_0) \tag{10.22}$$

where $N = 2n$ and according to the definition of $\rho_\varepsilon, \sigma_\varepsilon$ (see Proposition 8) and (10.2)

$$\frac{1-p}{2} \le \rho'_\varepsilon(b) \le \frac{1+p}{2}, \quad \frac{1-p}{2} \le \sigma'_\varepsilon(b) \le \frac{1+p}{2}. \tag{10.23}$$

We have to show that the intersection of a decreasing sequence of compacts

$$(M_- \circ M_+)^k(\Sigma), \quad k \ge 0,$$

is one point.

Proposition 9 *For the metric*

$$\|\overline{a} - \overline{b}\| = \sum_{i=0}^{N} |a_i - b_i|$$

the operator

$$M = (M_- \circ M_+)^{N^2-1} : \Sigma \longrightarrow \Sigma$$

is a contraction with a coefficient $1-\gamma$, *where*

$$\gamma = \frac{1}{N}\left(\frac{1-p}{2}\right)^{N^2-1}.$$

Proof. If $\overline{a}, \overline{b} \in \Sigma$ then the vector $\overline{a} - \overline{b}$ has at least one pair of coordinates with different signs. Let

$$a_j - b_j = \max_i\{a_i - b_i\} > 0\ , \quad a_k - b_k = \min_i\{a_i - b_i\} < 0.$$

It is easy to see that

$$a_j - b_j \ge \frac{\|\overline{a} - \overline{b}\|}{2N}\ , \quad b_k - a_k \ge \frac{\|\overline{a} - \overline{b}\|}{2N}.$$

According to (10.22), $\overline{c} = M_\varepsilon(\overline{a}) - M_\varepsilon(\overline{b})$ can be defined by

$$c_0 = \rho'_\varepsilon(\theta) \cdot (a_0 - b_0), c_1 = a_1 - b_1, \ldots, c_{N-1} = a_{N-1} - b_{N-1},$$

$$c_N = a_N - b_N + \sigma'_\varepsilon(\theta) \cdot (a_0 - b_0).$$

Thus, the transformation $\overline{a} - \overline{b} \mapsto \overline{c}$ can be described as a multiplication by a matrix $\{\alpha_{ij}\}$ (depending on $\overline{a}, \overline{b}$), where according to (10.23)

$$\alpha_{ij} \geq 0, \quad 0 \leq i, j \leq N \quad \sum_{i=0}^{N} \alpha_{ij} = 1, \quad j = 0, \ldots, N \tag{10.24}$$

with

$$\min\{\alpha_{ij} \ : \ \alpha_{ij} > 0\} \geq \frac{1-p}{2}. \tag{10.25}$$

A product of matrices of type (10.24) is of the same type. Also it is not difficult to see that the product of $N+1$ matrices of type (10.24) does not contain zeros on the principal diagonal and on the next upper diagonal. Hence the product of the $N^2 - 1$ matrices of type (10.24) contains the first string with, by (10.25),

$$\min_{k=0,\ldots,N} \{m_{0k}\} \geq \left(\frac{1-p}{2}\right)^{N^2-1} = N\gamma.$$

This implies immediately that

$$|M(\overline{a}) - M(\overline{b})\| = \sum_{i=0}^{N} \Big| \sum_{q=0}^{N} m_{iq}(a_q - b_q) \Big| \leq$$

$$\leq \left(\sum_{q=0}^{N} m_{0q}|a_q - b_q| - 2N\gamma \cdot \frac{\|\overline{a} - \overline{b}\|}{2N} \right) + \sum_{i=1}^{N} \sum_{q=0}^{N} m_{iq}\|a_q - b_q\| \leq$$

$$< \|\overline{a} - \overline{b}\| - \gamma \cdot \|\overline{a} - \overline{b}\| = (1-\gamma)\|\overline{a} - \overline{b}\|.$$

This uniqueness and the autonomy imply the equality $g_{n,T}(t) = g_{n,0}(t - T)$, $t, T \in R$, as well as the periodicity of $g_{n,0}$. Inequalities (10.5) follow from that the frequency of $g_{n,0}$ is equal to $2n$.

Proof of Theorem 4. It is easy to deduce from the proof of Proposition 8 that every solution $g(t),\ t \geq T$, of (10.1) with a constant finite frequency can be extended on $[-1;\infty)$ with the same frequency. That finishes the proof according to Lemmas 1, 2 and Theorem 2.

Proof of Theorem 5 and Corollary 6. The set $\tilde{\mathcal{U}}_0$ is non-empty, because it contains $\mathcal{S}_0 \neq \emptyset$. Now let $\varphi \in \mathcal{U}_0$, and $\mathrm{mes}(\varphi^{-1}(0)) = 0$. Then $x_\varphi(t) = g_{0,T}(t),\ t \geq T$, for a relevant $T \in R$. That means

$$x_\varphi(T) = 0, \quad \dot{x}_\varphi(t) > 0, \quad t \in \left(T; T + \frac{2}{1+p}\right).$$

If $\psi \in C[-1;0]$ is close to φ, then $\psi^{-1}(0)$ is contained in a sufficiently small neighborhood of $\varphi^{-1}(0)$, and

$$\mathrm{mes}(\{\varphi > 0\} \circ \{\psi > 0\}), \quad \mathrm{mes}(\{\varphi < 0\} \circ \{\psi < 0\})$$

are small enough, where $A \circ B$ denotes $(A\backslash B) \cup (B \backslash A)$. Hence $Z_\psi \cap [0; T+2]$ is contained in a sufficiently small neighborhood of $Z_\varphi \cap [0; T+2]$. Therefore

$$x_\psi(t) > 0, \quad t \in \left(T + \delta; T + \frac{2}{1+p} - \delta\right), \quad 2\delta < \frac{2}{1+p} - 1$$

that implies $\psi \in \mathcal{U}_0$. The statement of Corollary 6 follows from this immediately.

Proof of Theorem 7. Let $\varphi \in \tilde{\mathcal{U}}_0$, and $x_\varphi(t) = g_{0\alpha}(t),\ t \geq T$. We have just showed that if ψ is sufficiently close to φ then $x_\psi(t) = g_{0\beta}(t),\ t \geq T$, where $|\beta - \alpha|$ is small enough. Let

$$\alpha = t_1 < t_2 < \cdots, \quad \beta = t'_1 < t'_2 < \cdots$$

be all zeros of the functions $g_{0\alpha}, g_{0\beta}$ respectively in the interval $[T;\infty)$. It is enough to prove that

$$C_1 \cdot |\beta - \alpha| < |t_k - t'_k| < C_2 \cdot |\beta - \alpha|, \quad C_1, C_2 = \mathrm{const}, \quad k = 1, 2, \ldots$$

According to the definition of the functions $\lambda_\pm(t_0, a)$

$$t_{k+1} = \lambda_\pm(t_k, 1) \ , \quad t'_{k+1} = \lambda_\pm(t'_k, 1),$$

hence

$$t'_{k+1} - t_{k+1} = \frac{\partial\lambda_\pm(\theta_k, 1)}{\partial t} \cdot (t'_k - t_k) \ , \quad |\theta_k - t_k| < |t'_k - t_k| \ , \quad k \geq 1,$$

$$t'_n - t_n = \prod_{k=1}^{n-1} \frac{\partial\lambda_\pm(\theta_k, 1)}{\partial t} \cdot (\beta - \alpha).$$

The desired statement follows from

Proposition 10 *Under condition (10.6), the product*

$$\prod_{k=1}^{\infty} \frac{\partial\lambda_\pm}{\partial t}(\theta_k, 1)$$

converges uniformly when

$$\theta_{k+1} \geq \theta_k + 1 \ , \quad k = 1, 2, 3, \ldots \tag{10.26}$$

Proof. We will show that the series

$$\sum_{k=1}^{\infty} \left(\frac{\partial\lambda_\pm}{\partial t}(\theta_k, 1) - 1 \right)$$

converges uniformly. Put

$$\mu(t) = \max_x \left| \frac{\partial F}{\partial t}(x, t) \right| \ , \quad t \geq 0.$$

It follows from (10.20) and well-known formulae for the derivatives of solutions with respect to initial data , that

$$\frac{\partial\lambda_\varepsilon}{\partial t}(t, a) = 1 - (-\varepsilon + F(0, \tau))^{-1} \cdot \exp \int_{t+a}^{\tau} \frac{\partial F}{\partial x}(x_{-\varepsilon}, t) dt$$

$$\times\left(\int_{t+a}^{\tau}\frac{\partial F}{\partial t}(x_{-\varepsilon},t)dt+\int_{t}^{t+a}\frac{\partial F}{\partial t}(x_{\varepsilon},t)dt\cdot\exp\int_{t}^{t+a}\frac{\partial F}{\partial x}(x_{\varepsilon},t)dt\right),$$

where $\tau=\lambda_{\varepsilon}(t,a)$, hence

$$\Big|\frac{\partial\lambda_{\varepsilon}}{\partial t}(\theta,1)-1\Big|\leq\frac{1}{1-p}\exp\int_{\theta+1}^{\tau}\frac{\partial F}{\partial x}(x_{-\varepsilon},t)dt$$

$$\times\left(\int_{\theta+1}^{\tau}\mu(t)dt+\int_{\theta}^{\theta+1}\mu(t)dt\cdot\exp\int_{\theta}^{\theta+1}\frac{\partial F}{\partial x}(x_{\varepsilon},t)dt\right). \tag{10.27}$$

According to (10.26), one may admit

$$\theta\gg 0\ ,\quad \int_{\theta}^{\infty}\mu(t)dt\leq 1.$$

Then

$$\int_{\theta+1}^{\tau}\frac{\partial F}{\partial x}(x_{-\varepsilon},t)dt=\int_{\theta+1}^{\tau}\frac{dF}{dt}\cdot(-\varepsilon+F(x_{-\varepsilon},t))^{-1}dt$$

$$-\int_{\theta+1}^{\tau}\frac{\partial F}{\partial t}\cdot(-\varepsilon+F(x_{-\varepsilon},t))^{-1}dt\leq\log\frac{1+p}{1-p}+\frac{1}{1-p}\cdot\int_{\theta+1}^{\tau}\mu(t)dt\leq\log\frac{1+p}{1-p}+\frac{1}{1-p}\ ,$$

$$\int_{\theta}^{\theta+1}\frac{\partial F}{\partial x}(x_{\varepsilon},t)dt\leq\log\frac{1+p}{1-p}+\frac{1}{1-p}.$$

Put $q=\exp(2p+1/(1-p))$, $N=[(1+p)/(1-p)]+1$. Then (10.27) implies

$$\Big|\frac{\partial\lambda_{\varepsilon}}{\partial t}(\theta,1)-1\Big|\leq\frac{q^2}{1-p}\int_{\theta}^{\tau}\mu(t)dt,$$

$$\sum_{\theta_i>\theta}\Big|\frac{\partial\lambda_{\pm}}{\partial t}(\theta_i,1)-1\Big|\leq\frac{q^2N}{1-p}\int_{\theta}^{\infty}\mu(t)dt\overset{\theta\to\infty}{\longrightarrow}0\ ,$$

because $\tau\leq\theta+(1+p)/(1-p)$ according to (10.2), that completes the proof of Theorem 7.

Proof of Theorem 8. We shall use the two following propositions.

Proposition 11 *If*

$$a\leq\frac{1+p}{2} \tag{10.28}$$

and one of (10.7), (10.8) is fulfilled, then

$$\frac{\partial\lambda_{\varepsilon}}{\partial a}(t,a)\geq q>1\ ,\quad \varepsilon=\pm 1. \tag{10.29}$$

Proof. It is not difficult to derive that

$$\frac{\partial\lambda_\varepsilon}{\partial a}(t,a) = 1 + (1-\varepsilon F(0,T))^{-1}\exp\int_{t+a}^{T}\frac{\partial F}{\partial x}(x_{-\varepsilon},t)dt$$

$$\times\left(1+\varepsilon F(x_\varepsilon(t,0,t+a),t+a)+\varepsilon\int_{t+a}^{T}\frac{\partial F}{\partial t}(x_{-\varepsilon},t)dt\right)\ ,$$

where $T=\lambda_\varepsilon(t,a)$. Therefore (10.8) implies

$$1+\varepsilon F(x_\varepsilon(t,0,t+a),t+a)+\varepsilon\int_{t+a}^{T}\frac{\partial F}{\partial t}(x_{-\varepsilon},t)dt$$

$$> 1-p-(T-t-a)M_t \geq 1-p-aM_t\frac{1+p}{1-p} > 0\ ,$$

$$\int_{t+a}^{T}\frac{\partial F}{\partial x}(x_{-\varepsilon},t)dt = \int_{t+a}^{T}\left(\frac{dF}{dt}-\frac{\partial F}{\partial t}\right)\cdot(\dot{x}_{-\varepsilon})^{-1}dt$$

$$= \int_{t+a}^{T}\frac{dF}{dt}\cdot(-\varepsilon+F(x_{-\varepsilon},t))^{-1}dt - \int_{t+a}^{T}\frac{\partial F}{\partial t}\cdot(-\varepsilon+F(x_{-\varepsilon},t))^{-1}dt$$

$$\geq -\log\frac{1+p}{1-p} - M_t\frac{T-t-a}{1-p} \geq -\log\frac{1+p}{1-p} - M_t a\frac{1+p}{(1-p)^2}$$

$$\geq -\log\frac{1+p}{1-p} - M_t\frac{(1+p)^2}{2(1-p)^2}\ ,$$

that implies (10.29). Analogously (10.7) implies

$$1+\varepsilon F(x_\varepsilon(t,0,t+a),t+a)+\varepsilon\int_{t+a}^{T}\frac{\partial F}{\partial t}(x_{-\varepsilon},t)dt$$

$$= 1+\varepsilon F(x_\varepsilon(t,0,t+a),t+a)+\varepsilon\int_{t+a}^{T}\frac{dF}{dt}dt - \varepsilon\int_{t+a}^{T}\frac{\partial F}{\partial x}\cdot\dot{x}_{-\varepsilon}dt$$

$$\geq 1+\varepsilon\cdot F(0,T) - M_x(1+p)(T-t-a) \geq 1-p-M_x a\frac{(1+p)^2}{1-p} > 0,$$

$$\int_{t_0}^{t_0+a}\frac{\partial F}{\partial x}(x_\varepsilon,t)dt \geq -M_x a \geq -M_x(1+p)/2,$$

that implies (10.29).

Proposition 12 *Under conditions of Theorem 8 the measure of the set* Π *from the proof of Theorem 2 is zero.*

Proof. First we show that any $\overline{a} = (a_0, \dots, a_N) = M_\varepsilon(\overline{b}), \overline{b} \in \Sigma$, satisfies $a_N \leq (1+p)/2$. Indeed, we have $a_N \leq a_{N-1}(1+p)/(1-p)$, that implies the above inequality.

Now from (10.21) the Jacobian $|M'_\varepsilon|$ of the map M_ε is equal to

$$\frac{\partial \rho_\varepsilon}{\partial b}(t,b)\Big|_{t=\alpha, b=b_0} = \left(\frac{\partial \lambda_\varepsilon}{\partial a}(t,a)\Big|_{t=\alpha, a=a_N} \right)^{-1} \leq \frac{1}{q} < 1$$

according to Proposition 11. Then

$$|(M_- \circ M_+)' \ :\leq q^{-2} < 1. \tag{10.30}$$

Fix $A \in R$ and $T > A$. Then

$$\Pi \cap (\Sigma \times (-\infty; A]) \subset \bigcup_{k \geq n} (M_- \circ M_+)^k (\Sigma \times [T; T+1]) \ ,$$

where n might be chosen big enough, because $T > A$ is arbitrary. Thus, we obtain from (10.30)

$$\text{mes}(\Pi \cap (\Sigma \times (-\infty; A])) \leq q^{-2(n-1)} \frac{\text{mes}(\Sigma)}{q^2 - 1} \overset{n\to\infty}{\longrightarrow} 0,$$

that completes the proof.

Now we can finish the proof of Theorem 8. Now fix $\varphi \in \mathcal{U}_n$ and a neighborhood V of φ in $C[-1;0]$. The set $\mathcal{F}$ is dense in $C[-1;0]$, evidently. Put

$$m = \min\{k \ : \ \mathcal{F} \cap \mathcal{U}_k \cap V \neq= \emptyset\}.$$

Assume $m \geq 1$, and $\psi \in \mathcal{F} \cap \mathcal{U}_m \cap V$. Then there is $\xi \in \mathcal{S}_m$ such that $x_\psi(t) = \xi(t), \ t \geq T, \ \xi(T) = 0$. Let $2k$ be a number of sign changes of ψ in $[-1;0]$, and $\overline{a} \in \Sigma_k \subset R^{2k+1}$ be a vector of sign changes of ψ, constructed as in the proof of Theorem 2, as well as $\overline{b} \in \Sigma_m \subset R^{2m+1}$ be a vector of sign changes of ξ in $(T-1;T)$. Suppose $\overline{c} \in \Sigma_t, \overline{d} \in \Sigma_s$ are vectors of sign changes of $x_\psi(t)$ in intervals $(t_n - 1; t_n)$ and $(t_{n+1} - 1; t_{n+1})$ respectively. If $r = s$ then, according to the proof of Theorem 2, the equation (10.1) generates a diffeomorphism of neighborhoods of $(\overline{c}, t_n)$, $(\overline{d}, t_{n+1})$ in $\Sigma_r \times R$. If $r < s$ then it is possible to deduce, following arguments from the proof of Theorem 2,

$$c_0 = d_1, \ldots, c_{2s-1} = d_{2s}, \; c_{2r} = \lambda \left(d_0, c_{2s}, \ldots, c_{2r-2}, t_{n+1}\right),$$

$$c_{2r-1} = 1 - c_0 - \ldots - c_{2r-2} - c_{2r}, \; t_n = t_{n+1} - d_0,$$

where Λ is some smooth function. Hence an inverse image of $\left(\overline{d}, t_{n+1}\right)$ in a neighborhood of $(\overline{c}, t_n)$ in $\Sigma_r \times R$ has the codimension $2s+1$. That implies the measure of an inverse image of $\Pi \cap (\Sigma_m \times R)$ in $\Sigma_k \times R$ is zero. Therefore, after a suitable small variation of $(\overline{a}, 0)$ in $\Sigma_k \times R$ an image of $(\overline{a}, 0)$ in $\Sigma_m \times R$ leaves Π, i.e. a limit frequency of the changed solution is less than $2m$, what contradicts to definition of m, and hence to our assumption $m > 0$.

Thus, we get that $\mathcal{U}_0 \cap \mathcal{F}$ is dense in $\mathcal{F}$, and also in $C[-1;0]$, because $\mathcal{F}$ is dense in $C[-1;0]$. According to Theorem 5, it means that $\mathcal{U}_\infty \cup \bigcup_{k \geq 1} \mathcal{U}_k$ is dense nowhere in $C[-1;0]$.

Proof of Lemma 6. From (10.15) we deduce that

$$\frac{F(x,t)}{x} \leq k, x \neq 0 \tag{10.31}$$

In particular, that means: if $x(t)$ is a solution of (10.1) then, for $x(T) \geq 0, x(t) \leq \omega(t), t \geq T$, where $\omega(t) = ((\alpha + kx(T))e^{k(t-T)} - \alpha)/k$ is the solution of Cauchy problem

$$\dot{\omega}(t) = \alpha + k\omega(t), \quad \omega(T) = x(T),$$

and, for $x(T) \leq 0, x(t) \geq \omega(t), t \geq T$, where $\omega(t) = ((-\alpha + kx(T))e^{k(t-T)} + \alpha)/k$ is the solution of Cauchy problem

$$\dot{\omega}(t) = -\alpha + k\omega(t), \quad t \geq T$$

Those inequalities and (10.31) imply that $|F(x,t),t)| < \alpha$ when $t \in [0,1]$ and $x(0) = \varphi(0)$ satisfies (10.16), and that $x(t)$ satisfies (10.17) when $t \in [T, T+1], x(T) = 0$, and secondly, $x(t)$ does not leave the strip $|x| \le \alpha(e^k - 1)/k$ for $t \le T$.

Proof of Theorem 9. Let us study the point mapping $\Phi(z, x, \mu)$ of the switching surface $s = 0$ into itself induced by the full order system (10.9). First we show that under the conditions of the Theorem 9 there exists a neighborhood of the point $(\varphi(0, x_0, 1), 0, x_0)$ in the z, x space on the surface $s = 0$ mapped into itself.

It follows from the continuous dependence of the solutions to differential equations on the parameters and initial conditions that there exists $\bar{U}(\alpha)$, the closed ball with the center at the point x_0 and radius α on the surface $s = 0$ in the x space such that for some q' for all $x' \in U(\alpha)$

- the point $\varphi(s_0^-(1), x_0^-(1), -1)$ is situated in the interior of the attractive domain of the equilibrium point $\varphi(s_0^-(1), x_0^-(1), 1)$, where $(\bar{s}_0^-(t), \bar{x}_0^-(t))$ is the solution of system (10.10) for $u = -1$ with the initial conditions $\bar{s}_0^-(0) = 0, \quad \bar{x}_0^-(0) = x^0, \bar{s}_0^-(t) < 0, t \in [-1, 0)$;

- there exists the smallest root θ^0 of the equation $\bar{s}_0^+(\theta^0) = 0$ such that $d\bar{s}_0^+(\theta^0)/dt < 0$; here $(\bar{s}_0^+(t), \bar{x}_0^+(t))$ is the solution of system (10.10) for $u = 1$ with the initial conditions $\bar{s}_0^+(1) = \bar{s}_0^-(1), \quad \bar{x}_0^+(1) = \bar{x}_0^-(1)$;

- the point $\varphi(s_0^+(\theta_0 + 1), x_0^+(\theta_0 + 1), 1)$ is situated in the interior of the attractive domain of the equilibrium point $\varphi(s_0^+(\theta_0+1), x_0^+(\theta_0+1), -1)$;

- there exists the smallest root $T(x^0)$ of the equation $\bar{s}_0^-(T(x^0)) = 0$ such that $T(x^0) > \theta + 1, d\bar{s}_0^-(T(x^0))/dt > 0$; here $(s_0^-(t), x_0^-(t))$ is the solution of system (10.10) with $u = -1$ and initial conditions $(s_0^-(\theta_0 + 1), x_0^-(\theta_0 + 1)) = (s_0^+(\theta_0 + 1), x_0^+(\theta_0 + 1))$;

- $\|\partial\Psi(x^0)/\partial x\| < q' < 1$.

Consider the set $A = co(\varphi(0, \bar{U}(\alpha), -1)) \times \bar{U}(\alpha)$ and arbitrary $(z^0, x^0) \in A$. Then according to Tichonov's theorem [34] and the implicit function theorem, there exists $\mu(z^0, x^0)$ such that, for all $\mu \in [0, \mu(z^0, x^0)]$,

- there exists the unique solution $(z^-(t,\mu), s^-(t,\mu), x^-(t,\mu))$ of system (10.9) for $u = -1$ on $[0, 1]$ with the initial conditions

  $$z^-(0,\mu) = z^0, \quad s^-(0,\mu) = 0, \quad x^-(0,\mu) = x^0, \quad s^-(t,\mu) < 0,\ t \in [-1, 0);$$

- the point $z^-(t,\mu)$ is situated in the interior of the attractive domain of the equilibrium point $\varphi(s_0^-(1), x_0^-(1), 1)$
- there exists the smallest root $\theta(\mu, z^0, x^0)$ of the relations

  $$s^+(\theta(\mu, z^0, x^0), \mu) = 0, \qquad \frac{ds^+(\theta(\mu, z^0, x^0), \mu)}{dt} < 0,$$

  where $(z^+(t,\mu), s^+(t,\mu), x^+(t,\mu))$ is the solution of system (10.9) for $u = 1$ with the initial conditions

  $$z^+(1,\mu) = z^-(1,\mu), \quad s^+(1,\mu) = s^-(1,\mu), \quad x^+(1,\mu) = x^-(1,\mu);$$

- the point $z^+(\theta(\mu, z^0, x^0) + 1, \mu)$ is situated in the interior of the attractive domain of the equilibrium point $\varphi(\bar{s}_0^+(\theta_0 + 1), \bar{x}_0^+(\theta_0 + 1), -1)$;
- there exists the smallest root $T(\mu, z^0, x^0)$ of the relations

  $$s^-(T(\mu, z^0, x^0), \mu) = 0, \qquad \frac{d\bar{s}^-(T(\mu, z^0, x^0))}{dt} > 0, \quad T(\mu, z^0, x^0) > \theta(\mu, z^0, x^0) + 1,$$

  where $(z^-(t,\mu), s^-(t,\mu), x^-(t,\mu))$ is the solution of system (10.9) with $u = -1$ and the initial conditions

  $$(z^-(\theta(\mu, z^0, x^0) + 1, \mu), s^-(\theta(\mu, z^0, x^0) + 1, \mu), x^-(\theta(\mu, z^0, x^0) + 1), \mu) =$$

  $$= (z^+(\theta(\mu, z^0, x^0) + 1), \mu), s^+((\theta(\mu, z^0, x^0) + 1, \mu), x^+((\theta(\mu, z^0, x^0) + 1, \mu);$$

- at last,

$$(z^-(T(\mu, z^0, x^0), \mu), x^-(T(\mu, z^0, x^0), \mu)) \in \{\varphi(0, \bar{U}((1+q')\alpha/2), -1), \bar{U}((1+q')\alpha/2)\} \subset$$

This means that the image of the set A by the point mapping

$$\Phi(z^0, x^0, \mu) = (\Phi_1(z^0, x^0, \mu), \Phi_2(z^0, x^0, \mu)) =$$

$$= (z^-(T(\mu, z^0, x^0), \mu), x^-(T(\mu, z^0, x^0), \mu))$$

induced by system (10.9) for all $\mu \in [0, \mu(z^0, x^0)]$ is a subset of the interior of A. Moreover, for all $\mu \in [0, \mu(z^0, x^0)]$,

$$\Phi(z^0, x^0, 0) = \lim_{\mu \to 0} \Phi(z^0, x^0, \mu) = (\varphi(0, x^-(T(x^0)), -1), x^-(T(x^0))), \quad (10.32)$$

and

$$\Phi(\varphi(0, x_0, -1), x_0, 0) = (\varphi(0, x_0, -1), x_0).$$

This means that the point mapping Φ is continuous on $A \times [0, \mu']$, $\mu' > 0$, and at all $\mu \in [0, \mu']$ has a fixed point which corresponds to a periodic solution of system (10.9) close to $(s_0(t), x_0(t))$. Let us show that this periodic solution is stable and unique. The derivative of the point mapping Φ is a smooth function of the derivatives of the functions $\theta(\mu, x^0, z^0)$, $T(\mu, x^0, z^0)$, $z^-(1, \mu)$, $x^-(1, \mu)$, $z^-(T(\mu, x^0, z^0), \mu)$, $x^-(T(\mu, x^0, z^0), \mu)$, $z^+(\theta(\mu, x^0, z^0) + 1, \mu)$, $x^+(\theta(\mu, x^0, z^0) + 1, \mu)$, hence the derivatives of Φ exist and they are continuous.

Let us consider the new variable $\eta = z^0 - \varphi(0, x^-(T(x^0)), -1)$ and the auxiliary point mapping

$$\Xi(\eta, x^0, \mu) = (\Xi_1(\eta, x^0, \mu), \Xi_2(\eta, x^0, \mu))$$

$$= (\Phi_1(\eta + \varphi(0, x^-(T(x^0)), -1), x^0, \mu) - \varphi(0, x^-(T(x^0)), -1),$$

$$\Phi_2(\eta + \varphi(0, x^-(T(x^0)), -1), x^0, \mu)).$$

The point $(0, x^0)$ is a fixed point of Ξ for $\mu = 0$. For a sufficiently small $\mu > 0$ the point mapping takes the set

$$B(\beta, \alpha, \mu") = \{(\eta, x, \mu) : \|\eta\| \leq \beta, x \in \bar{U}(\alpha), \mu \in [0, \mu"]\}$$

into itself.

It follows from (10.32) that the value of $\Xi(\eta, x^0, 0)$ does not depend on η. Then

$$\frac{\partial \Xi}{\partial(\eta, x)} = \begin{pmatrix} O(\mu) & O(\mu) \\ O(\mu) & \partial\Psi/\partial x(x_0) + O(\mu) \end{pmatrix}.$$

This means that for some $q_1 < 1$

$$\sup_{B(\alpha,\beta,\mu'')} \|\frac{\partial \Xi}{\partial(\eta, x)}\| < q_1 < 1.$$

Consequently the point mapping Ξ is a contaction and Φ has a unique fixed point, thus, the desired periodic solution of system (10.9) is unique and orbitally asymptotically stable.

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