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Sliding-mode control of uncertain systems in the presence of unmatched disturbances with applications

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Sliding-mode control of uncertain systems in the presence of unmatched disturbances with applications

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This article considers the development of constructive sliding-mode control strategies based on measured output information only for linear, time-delay systems with bounded disturbances that are not necessarily matched. The novel feature of the method is that linear matrix inequalities are derived to compute solutions to both the existence problem and the finite time reachability problem that minimise the ultimate bound of the reduced-order sliding-mode dynamics in the presence of state time-varying delay and unmatched disturbances. The methodology provides guarantees on the level of closed-loop performance that will be achieved by uncertain systems which experience delay. The methodology is also shown to facilitate sliding-mode controller design for systems with polytopic uncertainties, where the uncertainty may appear in all blocks of the system matrices. A time-delay model with polytopic uncertainties from the literature provides a tutorial example of the proposed method. A case study involving the practical application of the design methodology in the area of autonomous vehicle control is also presented.

Keywords: sliding-mode control; output feedback; state delay; LMIs

1. Introduction

The control of time-delay systems is known to be of practical significance. Problems largely fall into two categories. The first category arises because of the need to model systems more accurately given increasing performance expectations. Many processes, such as manufacturing processes, include such after effect phenomena in their dynamics and time delay is also produced via the actuators, sensors and networks involved in the practical implementation of feedback control strategies. The second class of problems arises when time delays are used as a modelling tool to simplify some infinite-dimensional systems. This approach is used for constructing models of distributed systems modelled by partial differential equations where a set of finite-dimensional state variables with appropriate time-delay characteristics can be used to represent heat exchange processes, for example.

The application of sliding-mode control (SMC) to the problem of systems with time delay is a far from trivial problem generically, involving the combination of delay phenomenon with relay actuators which has the potential to induce oscillations around the sliding surface during the sliding-mode. There are a number of papers which have considered the problem. The development of sliding-mode controllers for operation in the presence of single or multiple, constant or time-varying state delays was solved by Gouaisbaut, Dambrine, and Richard (2002). This uses the equivalent control method where the exact knowledge of block matrices must be known and the full-state measurable. The work was further extended to include polytopic uncertainties (Gouaisbaut, Blanco, and Richard 2004), however full-state feedback was required to yield finite reachability design and the upper bound on the states spanning the input space is assumed to be known for solution of the reachability problem. The problem was also considered by Li and DeCarlo (2003) where a class of uncertain time-delay systems with multiple fixed delays in the system states is considered. The method of equivalent control is used and measurement of the state at the current time is assumed. It is also assumed that the delayed state is bounded with a bound dependent on the current state which is restrictive. Even though an adaptive law was used to estimate the bound of the uncertainty in the reaching phase, a switching gain which is again assumed to be large enough rather than calculated explicitly is thought to guarantee the reachability design. The work in Jafarov (2005) considers

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sliding-mode control of an uncertain system in the presence of fixed state delay, but again full-state feedback is assumed.

The assumption of full-state feedback is a limiting one in practice as it may be prohibitively expensive, and indeed, sometimes impossible, to measure all the state variables. One approach to solve this problem is to implement the controller with an observer, where the observer provides state estimates for use by the controller. However, the implementation of the controller–observer is more involved and the theoretical frameworks to ensure stability across the range of practical operation of the plant may be challenging. A more straightforward approach is to use only the subset of state information that is available, i.e. the measured output, within the control design paradigm.

There are typically two facets to the design of a static output feedback sliding-mode control. One is the existence problem, i.e. the design of a switching surface in the output vector space which is usually of lower order than the state vector space. Consider first the switching surface design problem for uncertain systems, where time-delay effects are not involved. Two different methods of designing sliding surfaces using eigenvalue assignment and eigenvector techniques were proposed in Zak and Hui (1993) and El-Khazali and Decarlo (1995). A canonical form via which the static output feedback sliding-mode control design problem is converted to a static output feedback stabilisation problem for a particular subsystem triple was provided in Edwards and Spurgeon (1995). However, the solution to the general static output feedback problem, even for linear time-invariant systems, is still open. Iterative linear matrix inequality (LMI) approaches have been exploited to solve the static output feedback problem using a bilinear matrix inequality formulation, see Cao, Lam, and Sun (1998), Choi (2002) and Huang and Nguang (2006). In Edwards (2004), where the regular form was not used for synthesis of the control law, LMIs were derived for switching function design whilst minimising the cost function associated with the control. Sufficient conditions for static output feedback controller design using LMIs have also been sought. Although only sufficient, the solutions have the advantage of being linear and, hence, easily tractable using standard optimisation techniques, see Crusius and Trofino (1999) and Shaked (2003). The second facet in the design of a sliding-mode output feedback controller is the control, or reachability, synthesis problem whereby a control is determined to ensure the sliding surface is attractive. It is non-trivial to synthesise a control law only using the output vector, even for the situation where time-delay effects are not considered, since the derivative of the sliding surface is always related to the unmeasured states and this derivative appears in the reachability condition. As well as within the existence problem, LMI methods have also been considered within the context of developing a sliding-mode control strategy which solves the reachability problem for a given sliding surface. For example, LMI methods which yield reachability conditions for designing static sliding-mode output feedback controllers are presented in Edwards, Akoachere, and Spurgeon (2001).

In the context of output feedback sliding-mode control for time-delay systems, the existence and reachability problems for systems in the presence of matched uncertainty are considered in Han, Fridman, Spurgeon, and Edwards (2009). The delay is assumed to be time-varying and bounded where the upper bound is known. In line with the development of output feedback controllers in the non-delayed case, LMIs are used to select all the parameters of the closed-loop sliding-mode controller. However, no explicit calculation of the switching gain in the nonlinear part of the control was given, it was only assumed to be large enough to induce the sliding mode. While asymptotic stability in the presence of matched disturbances can be achieved by sliding-mode control, unmatched disturbances usually lead to only bounded stability. For example, in Fernando and Fridman (2006) robustness properties of integral sliding-mode controllers are studied where the Euclidean norm of the unmatched perturbation is minimised by selecting a projection matrix.

The contribution of the proposed work is that all the design parameters, including the switching gain, are derived from LMIs in spite of the presence of state delay and unmatched disturbances. The method is able to deal with polytopic type uncertainties in all blocks of the system matrices. No additional assumption is made on the bound of the uncertain states in the reachability design, as required by other work. Some preliminary results of this article are presented in Han et al. (2009). Central to the work presented is the descriptor approach (Fridman 2001), which is applied to derive LMIs for the solution of the sliding-mode output feedback control problem in the presence of both matched and *unmatched* bounded disturbances and time-varying state delays. It is demonstrated that the state trajectories of the system converge towards a ball with a prespecified exponential convergence rate. In Section 2, the problem formulation is described and an appropriate general framework to accomplish the output feedback sliding-mode control design is formulated in Section 3. A constructive solution to the existence problem is presented in Section 4 and Section 5 shows the formulation of the reachability problem which will ensure that the sliding-mode is reached. A problem from the literature is used to provide a tutorial example of how the paradigm can be used to solve both the existence and reachability problems for practical design. A case study relating to the control of an autonomous vehicle is used to further illustrate the design process in Section 6.

Notation: $\mathcal{R}^n$ denotes the n-dimensional Euclidean space with vector norm $\|\cdot\|$ or the induced matrix norm, $\mathcal{R}^{n\times m}$ is the set of all $n\times m$ real matrices. $P>0$, for $P\in\mathcal{R}^{n\times n}$, denotes that P is symmetric and positive definite whereas $*$ means the symmetric entries of an LMI.

2. Problem formulation

Consider an uncertain dynamical system of the form

$$\begin{aligned} \dot{x}(t) &= Ax(t) + A_d x(t-\tau(t)) + Bu(t) + B_1 w(t),\\ y(t) &= Cx(t),\\ x(t_0-\tau(t)) &= \phi(\tau(t)) \quad \text{for } \tau(t)\in[0,h], \end{aligned} \tag{1}$$

where $x\in\mathcal{R}^n$, $u\in\mathcal{R}^m$, $w\in\mathcal{R}^k$ and $y\in\mathcal{R}^p$ with $m<p<n$, ϕ is absolutely continuous with square integrable $\dot{\phi}$, h is an upper bound on the time-delay function $(0\le\tau(t)\le h\ \forall t\ge 0)$. The time-varying delay may be either slowly varying (i.e. a differentiable delay with $\dot{\tau}(t)\le d<1$) or fast varying (piecewise-continuous delay). Assume that the nominal linear system (A, A_d, B, B_1, C) is known and that the input and output matrices B and C are both of full rank. The disturbance is assumed to be bounded whereby $\|w(t)\|\le\Delta$ with a known upper bound $\Delta>0$. A control strategy will be sought which induces an ideal sliding motion with desirable performance characteristics on the surface

$$\mathcal{S} = \{x\in\mathcal{R}^n : s(t) = FCx(t) = 0\} \tag{2}$$

for some selected matrix $F\in\mathcal{R}^{m\times p}$ so that the motion, when restricted to $\mathcal{S}$, is stable.

3. A general framework for design

The first problem considered is how to choose F in (2) so that the associated sliding motion is stable. A control law will then be determined to guarantee the existence of a sliding motion. A convenient system representation closely allied to the usual regular form used for sliding-mode control design is employed. It can be shown that if $\text{rank}(CB)=m$ and system triple (A,B,C) are minimum phase, there exists a coordinate system $x_r=T_r x$, $x_r=[x_1\ \ x_2]^T$, in which the system (A, A_d, B, B_1, C) has the transformed structure

$$A_r = \begin{bmatrix} A_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix},\quad A_{d_r} = \begin{bmatrix} A_{d11} & A_{d12}\\ A_{d21} & A_{d22}\end{bmatrix},$$
$$B_r = \begin{bmatrix} 0\\ B_2\end{bmatrix},\quad B_{1_r} = \begin{bmatrix} B_{11}\\ B_{12}\end{bmatrix},\quad C_r = \begin{bmatrix} 0 & T\end{bmatrix}, \tag{3}$$

where $B_2\in\mathcal{R}^{m\times m}$ is nonsingular and $T\in\mathcal{R}^{p\times p}$ is orthogonal (Edwards and Spurgeon 1995). Furthermore, $A_{11}, A_{d11}\in\mathcal{R}^{(n-m)\times(n-m)}$ and the remaining sub-blocks are partitioned accordingly. Let

$$\begin{bmatrix} F_1 & F_2\end{bmatrix} = FT, \tag{4}$$

where $F_1\in\mathcal{R}^{m\times(p-m)}$ and $F_2\in\mathcal{R}^{m\times m}$. As a result

$$FC_r = \begin{bmatrix} F_1 C_1 & F_2\end{bmatrix}, \tag{5}$$

where

$$C_1 = \begin{bmatrix} 0_{(p-m)\times(n-p)} & I_{(p-m)}\end{bmatrix}. \tag{6}$$

It is straightforward to see that $FC_rB_r = F_2B_2$ and the square matrix F_2 is non-singular. By assumption, the system contains both matched and unmatched uncertainties and therefore the sliding motion is independent of the matched uncertainty but dependent on the unmatched uncertainty. In terms of the coordinate framework defined above, the reduced-order sliding-mode dynamics are governed by the following reduced-order system

$$\begin{aligned} \dot{x}_1(t) &= (A_{11} - A_{12}KC_1)x_1(t) + (A_{d11} - A_{d12}KC_1)\\ &\quad\times x_1(t-\tau(t)) + B_{11}w(t). \end{aligned} \tag{7}$$

The response of this system must therefore be ultimately bounded, where $K = F_2^{-1}F_1$, and the problem of hyperplane design is equivalent to a static output feedback problem for the system $(A_{11}, A_{d11}, A_{12}, A_{d12}, C_1)$, where $(A_{11}+A_{d11}, A_{12}+A_{d12})$ is assumed controllable and $(A_{11}+A_{d11}, C_1)$ observable. Note that the presence of the unmatched uncertainty means that, in general, asymptotic stability cannot be attained by the system (7). This is formalised in terms of the existence problem, which must be solved to determine the switching surface, in the following section.

4. Existence problem

It will be shown that the system (3) is exponentially attracted to a bounded region in $\mathcal{R}^n$ if the reduced-order system (7) is exponentially attracted to a bounded domain in $\mathcal{R}^{n-m}$. Consider the Lyapunov–Krasovskii functional below for the exponential stability analysis of (7)

$$V(t) = x_1^T(t)Px_1(t) + \int_{t-h}^{t} e^{\alpha(s-t)} x_1^T(s)Ex_1(s)\mathrm{d}s + \int_{t-\tau(t)}^{t} e^{\alpha(s-t)} x_1^T(s)Sx_1(s)\mathrm{d}s + h\int_{-h}^{0}\int_{t+\theta}^{t} e^{\alpha(s-t)}\dot{x}_1^T(s)R\dot{x}_1(s)\mathrm{d}s\,\mathrm{d}\theta \quad (8)$$

with $(n-m)\times(n-m)$-matrices $P>0$ and $E\geq 0$, $S\geq 0$, $R\geq 0$. To prove exponential stability of the system (7) using (8), we will use the following lemma.

Lemma 4.1 (Fridman and Dambrine 2009)**:** *Let* V: $[0,\infty)\to R^+$ *be an absolutely continuous function. If there exist* $\alpha>0$ *and* $b>0$ *such that the derivative of* V *satisfies almost everywhere the inequality*

$$\frac{\mathrm{d}}{\mathrm{d}t}V(t) + \alpha V(t) - b\|w(t)\|^2 \leq 0$$

then it follows that for all $\|w(t)\|\leq\Delta$

$$V(t) \leq e^{-\alpha(t-t_0)}V(t_0) + \frac{b}{\alpha}\Delta^2, \quad t\geq t_0.$$

Differentiating $V(t)$ from (8) yields

$$\begin{aligned} M \leq\ & 2x_1^T(t)P\dot{x}_1(t) + h^2\dot{x}_1^T(t)R\dot{x}_1(t) \\ & - he^{-\alpha h}\int_{t-h}^{t}\dot{x}_1^T(s)R\dot{x}_1(s)\mathrm{d}s + x_1^T(t)(E+S)x_1(t) \\ & - x_1^T(t-h)Ex_1(t-h)e^{-\alpha h} + \alpha x_1^T(t)Px_1(t) \\ & - (1-\dot{\tau}(t))x_1^T(t-\tau(t))Sx_1(t-\tau(t))e^{-\alpha\tau(t)} \\ & - bw^T(t)w(t). \end{aligned} \quad (9)$$

Further using the identity

$$-h\int_{t-h}^{t}\dot{x}_1^T(s)R\dot{x}_1(s)\mathrm{d}s = -h\int_{t-h}^{t-\tau(t)}\dot{x}_1^T(s)R\dot{x}_1(s)\mathrm{d}s - h\int_{t-\tau(t)}^{t}\dot{x}_1^T(s)R\dot{x}_1(s)\mathrm{d}s \quad (10)$$

and applying Jensen's inequality

$$\int_{t-\tau(t)}^{t}\dot{x}_1^T(s)R\dot{x}_1(s)\mathrm{d}s \geq \frac{1}{h}\int_{t-\tau(t)}^{t}\dot{x}_1^T(s)\mathrm{d}sR\int_{t-\tau(t)}^{t}\dot{x}_1(s)\mathrm{d}s \quad (11)$$

and

$$\int_{t-h}^{t-\tau(t)}\dot{x}_1^T(s)R\dot{x}_1(s)\mathrm{d}s \geq \frac{1}{h}\int_{t-h}^{t-\tau(t)}\dot{x}_1^T(s)\mathrm{d}sR\int_{t-h}^{t-\tau(t)}\dot{x}_1(s)\mathrm{d}s \quad (12)$$

then Equation (9) becomes

$$\begin{aligned} M \leq\ & 2x_1^T(t)P\dot{x}_1^T(t) + \alpha x_1^T(t)Px_1(t) + h^2\dot{x}_1^T(t)R\dot{x}_1(t) \\ & - \Big[\big(x_1(t)-x_1(t-\tau(t))\big)^T R\big(x_1(t)-x_1(t-\tau(t))\big) \\ & - \big(x_1(t-\tau(t))-x_1(t-h)\big)^T \\ & \times R\big(x_1(t-\tau(t))-x_1(t-h)\big)\Big]e^{-\alpha h} \\ & + x_1^T(t)(E+S)x_1(t) - x_1^T(t-h)Ex_1(t-h)e^{-\alpha h} \\ & - (1-d)x_1^T(t-\tau(t))Sx_1(t-\tau(t))e^{-\alpha h} \\ & - bw^T(t)w(t). \end{aligned} \quad (13)$$

Using the descriptor method as in Fridman (2001) and the free-weighting matrices technique from He, Wang, Lin, and Wu (2007)

$$0 \equiv 2(x_1^T(t)P_2^T + \dot{x}_1^T(t)P_3^T)[-\dot{x}_1(t) + (A_{11}-A_{12}KC_1)x_1(t) + (A_{d11}-A_{d12}KC_1)x_1(t-\tau(t)) + B_{11}w(t)], \quad (14)$$

where matrix parameters $P_2, P_3=\varepsilon P_2\in\mathcal{R}^{n-m}$ are added to the right-hand side of (13). Setting $\eta(t)=\mathrm{col}\{x_1(t),\dot{x}_1(t),x_1(t-h),x_1(t-\tau(t)),w(t)\}$, then $M\leq\eta^T(t)\Theta\eta(t)\leq 0$ if the matrix $\Theta<0$. Multiplying matrix Θ from the right and the left by $\mathrm{diag}\{P_2^{-1}, P_2^{-1}, P_2^{-1}, P_2^{-1}, I\}$ and its transpose, respectively, and denoting

$$Q_2 = P_2^{-1}, \quad \widehat{P} = Q_2^TPQ_2, \quad \widehat{R} = Q_2^TRQ_2, \quad \widehat{E} = Q_2^TEQ_2, \quad \widehat{S} = Q_2^TSQ_2$$

it follows $\Theta<0 \Leftrightarrow \widehat{\Theta}<0$ where

$$\widehat{\Theta} = \begin{bmatrix} \widehat{\theta}_{11} & \widehat{\theta}_{12} & 0 & \widehat{\theta}_{14} & \widehat{\theta}_{15} \\ * & \widehat{\theta}_{22} & 0 & \widehat{\theta}_{24} & \widehat{\theta}_{25} \\ * & * & \widehat{\theta}_{33} & \widehat{\theta}_{34} & 0 \\ * & * & * & \widehat{\theta}_{44} & 0 \\ * & * & * & * & \widehat{\theta}_{55} \end{bmatrix} < 0, \quad (15)$$

and

$$\begin{aligned} \widehat{\theta}_{11} &= (A_{11}-A_{12}KC_1)Q_2 + \alpha\widehat{P} + Q_2^T(A_{11}-A_{12}KC_1)^T + \widehat{E} + \widehat{S} - \widehat{R}e^{-\alpha h}, \\ \widehat{\theta}_{12} &= \widehat{P} - Q_2 + \epsilon Q_2^T(A_{11}-A_{12}KC_1)^T, \\ \widehat{\theta}_{14} &= (A_{d11}-A_{d12}KC_1)Q_2 + \widehat{R}e^{-\alpha h}, \\ \widehat{\theta}_{15} &= B_{11}, \quad \widehat{\theta}_{22} = -\varepsilon Q_2 - \varepsilon Q_2^T + h^2\widehat{R}, \\ \widehat{\theta}_{24} &= \varepsilon(A_{d11}-A_{d12}KC_1)Q_2, \quad \widehat{\theta}_{25} = \varepsilon B_{11}, \\ \widehat{\theta}_{33} &= -(\widehat{E}+\widehat{R})e^{-\alpha h}, \quad \widehat{\theta}_{34} = \widehat{R}e^{-\alpha h}, \\ \widehat{\theta}_{44} &= -2e^{-\alpha h}\widehat{R} - (1-d)\widehat{S}e^{-\alpha h}, \quad \widehat{\theta}_{55} = -bI. \end{aligned} \quad (16)$$

Select the LMI variable Q_2 in the following form

$$Q_2 = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{22}\mathcal{M} & \delta Q_{22} \end{bmatrix}, \quad (17)$$

where Q_{22} is a $(p-m)\times(p-m)$ matrix, $\mathcal{M}$ is a $(p-m)\times(n-p)$ tuning matrix and δ is a tuning parameter to be selected by the designer. It follows that

$$KC_1Q_2 = \begin{bmatrix} KQ_{22}\mathcal{M} & \delta KQ_{22} \end{bmatrix}.$$

Defining $Y = KQ_{22}$ it follows that

$$KC_1Q_2 = \begin{bmatrix} Y\mathcal{M} & \delta Y \end{bmatrix}. \quad (18)$$

To construct K, substitute (18) into (16) to yield

$$\begin{aligned}
\widehat{\theta}_{11} &= A_{11}Q_2 - A_{12}[Y \quad \delta Y] + Q_2^TA_{11}^T + \alpha\widehat{P}, \\
&\quad - [Y\mathcal{M} \quad \delta Y]^TA_{12}^T + \widehat{E} + \widehat{S} - \widehat{R}e^{-\alpha h}, \\
\widehat{\theta}_{12} &= \widehat{P} - Q_2 + \varepsilon Q_2^TA_{11}^T - \varepsilon[Y\mathcal{M} \quad \delta Y]^TA_{12}^T, \\
\widehat{\theta}_{14} &= A_{d11}Q_2 - A_{d12}[Y\mathcal{M} \quad \delta Y] + \widehat{R}e^{-\alpha h}, \\
\widehat{\theta}_{15} &= B_{11}, \quad \widehat{\theta}_{22} = -\varepsilon Q_2 - \varepsilon Q_2^T + h^2\widehat{R}, \\
\widehat{\theta}_{24} &= \varepsilon A_{d11}Q_2 - \varepsilon A_{d12}[Y\mathcal{M} \quad \delta Y], \quad \widehat{\theta}_{25} = \varepsilon B_{11}, \\
\widehat{\theta}_{33} &= -(\widehat{E} + \widehat{R})e^{-\alpha h}, \quad \widehat{\theta}_{34} = \widehat{R}e^{-\alpha h}, \\
\widehat{\theta}_{44} &= -2e^{-\alpha h}\widehat{R} - (1-d)\widehat{S}e^{-\alpha h}, \quad \widehat{\theta}_{55} = -bI.
\end{aligned} \quad (19)$$

The following proposition can now be stated:

Proposition 4.2: *Given scalars $h > 0$, $d < 1$, $\alpha > 0$, ε, δ, b and a matrix $\mathcal{M} \in \mathcal{R}^{(p-m)\times(n-p)}$, if there exist $(n-m)\times(n-m)$ matrices $\widehat{P} > 0$, $\widehat{E} \geq 0$, $\widehat{S} \geq 0$, $\widehat{R} \geq 0$ and matrices $Q_{22} \in \mathcal{R}^{(p-m)\times(p-m)}$, $Q_{11} \in \mathcal{R}^{(n-p)\times(n-p)}$, $Q_{12} \in \mathcal{R}^{(n-p)\times(p-m)}$, $Y \in \mathcal{R}^{m\times(p-m)}$ such that the LMI* (15) *with matrix entries* (19) *holds, then the reduced-order system* (7)*, where $K = YQ_{22}^{-1}$, is exponentially attracted to the ellipsoid*

$$x_1^T(t)Px_1(t) \leq \frac{b}{\alpha}\Delta^2, \quad (20)$$

where $P = Q_2^{-T}\widehat{P}Q_2^{-1}$, for all differentiable delays $0 \leq \tau(t) \leq h$, $\dot{\tau}(t) \leq d < 1$. Moreover, the reduced order dynamics (7) *is exponentially stable for all piecewise-continuous delays $0 \leq \tau(t) \leq h$, if the LMI* (15) *is feasible with $\widehat{S} = 0$.*

Remark 1: Since the LMI (15) is affine in the system matrices A, A_d and B_1, the results are applicable to the case where these matrices are uncertain. Denote $\Omega = [A \ A_d \ B_1]$ and assume that $\Omega \in \mathcal{C}o\{\Omega_j, \ j = 1, \ldots, N\}$, namely $\Omega = \sum_{j=1}^N f_j(t)\Omega_j$ for some $0 \leq f_j(t) \leq 1$, $\sum_{j=1}^N f_j(t) = 1$, where the N vertices of the polytope are described by $\Omega_j = \begin{bmatrix} A^{(j)} & A_d^{(j)} & B_1^{(j)} \end{bmatrix}$. One has to solve the LMIs simultaneously for all the N vertices, applying the same decision matrices for all vertices.

Example 4.3: Consider the following simple system which is in regular form, with polytopic uncertainties and unknown (bounded) perturbations $\beta(t)$ and $f(t)$

$$\begin{aligned}
\dot{x}(t) &= \begin{bmatrix} -3 & 2 & 1 \\ 2 & 1 + \sin(x_3(t)) & 1 \\ 1 & 1 & x_2^2(t) + 1 \end{bmatrix} x(t) \\
&\quad + \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 1 & 0.2 \\ -0.2 & -0.5 & 1 \end{bmatrix} x(t-\tau) \\
&\quad + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \bar{u}(t) + \begin{bmatrix} \beta(t)x_1(t) + 0.5w(t) \\ -0.5\beta(t)x_1(t-\tau) - 0.5w(t) \\ 0.2\beta(t)x_2(t) + w(t) \end{bmatrix}, \\
y(t) &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} x(t)
\end{aligned} \quad (21)$$

with $0 \leq \beta(t) \leq 2$ and disturbance $w(t) \in [-1, 1]$. The delay is assumed to be time-varying. In order to present (21) in the form of (1) with uncertain matrices, define the control variable $\bar{u}(t)$ as follows:

$$\bar{u}(t) = u(t) + (x_2^2(t) + 1)x_3(t), \quad (22)$$

where $u(t)$ is the sliding-mode control variable of the form (26). This change is possible because $x_2(t)$ and $x_3(t)$ are measured. Considering next $\sin(x_3(t))$ as uncertainty $\gamma(t) = \sin(x_3(t)) \in [-1, 1]$, the above system is represented as a polytopic system with four vertices defined by $\gamma = \pm 1$, $\beta = 0$ and $\beta = 2$

$$\dot{x}(t) = \sum_{j=1}^4 f_j(t)\left[A^{(j)}x(t) + A_d^{(j)}x(t-\tau)\right] + B\bar{u}(t) + B_1w(t), \quad (23)$$

where

$$\begin{aligned}
A^{(1)} &= \begin{bmatrix} -3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \quad A^{(2)} = \begin{bmatrix} -3 & 2 & 1 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \\
A^{(3)} &= \begin{bmatrix} -1 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1.4 & 0 \end{bmatrix}, \quad A^{(4)} = \begin{bmatrix} -1 & 2 & 1 \\ 2 & 0 & 1 \\ 1 & 1.4 & 0 \end{bmatrix}, \\
A_d^{(1)} &= A_d^{(2)} = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 1 & 0.2 \\ -0.2 & -0.5 & 1 \end{bmatrix}, \\
A_d^{(3)} &= A_d^{(4)} = \begin{bmatrix} 0.5 & 0 & 0 \\ -1 & 1 & 0.2 \\ -0.2 & -0.5 & 1 \end{bmatrix}, \\
B &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0.5 \\ -0.5 \\ 1 \end{bmatrix}.
\end{aligned} \quad (24)$$

Note that state-feedback sliding-mode control of the above system without unmatched disturbances and polytopic uncertainties $0.2\beta(t)x_2(t)$ in $\dot{x}_3(t)$ was considered in Gouaisbaut et al. (2004). The work employs LMI methods for the solution of the existence problem, and is suitable for uncertain systems where the polytopic uncertainties appear only in the subsystem (7). The control law is derived based on the assumption that those states varying in the span of the control input are bounded so that a large enough switching gain can induce the sliding motion in finite time. An appropriate switching gain must usually be determined by trial and error.

The advantage of the proposed method is that it facilitates analysis of polytopic uncertainties appearing in the input channel and the switching gain is derived from LMIs, ensuring finite time reachability onto the sliding surface with a prescribed decay rate.

The initial function is taken as $x(t)=[1,1-1]^T$ for $t\in[-\tau,0]$. To construct K for the reduced-order system (7) according to Proposition 4.2, the parameter settings in the LMI (15) with entries (19) are selected with the delay upper bound $h=1\,$s and the rate of change of the time-varying delay $\dot{\tau}\le d=0.1$. For $\delta=2$, $\varepsilon=0.3$, $M=2$ and choosing $\alpha=0.1$, $b=0.005$, then it is obtained that the LMI variables

$$\widehat{P}=\begin{bmatrix} 949.4 & 39.5 \\ * & 925.9 \end{bmatrix}, \quad Y=1237.4,$$
$$Q_{22}=175, \quad K=7.07.$$

Once a stable sliding-mode dynamics has been designed, the next step is to find a controller which ensures the closed-loop system reaches the prescribed sliding surface in finite time. This will now be considered in general terms.

5. Reachability problem

It can be shown (Edwards et al. 2001) that the following system transformation and control structure exist such that $z(t)=T_1x_r(t)$, where $T_1=\left[\begin{smallmatrix} I_{n-m} & 0 \\ KC_1 & I_m \end{smallmatrix}\right]$ so that the system $(\bar{A},\bar{A}_d,\bar{B},F\bar{C})$ has the property

$$\bar{A}=\begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix}, \quad \bar{A}_d=\begin{bmatrix} \bar{A}_{d11} & \bar{A}_{d12} \\ \bar{A}_{d21} & \bar{A}_{d22} \end{bmatrix},$$
$$\bar{B}=\begin{bmatrix} 0 \\ I_m \end{bmatrix}, \quad \bar{B}_1=\begin{bmatrix} \bar{B}_{11} \\ \bar{B}_{12} \end{bmatrix}, \quad F\bar{C}=\begin{bmatrix} 0 & I_m \end{bmatrix}, \quad (25)$$

where $z_1(t)=x_1(t)$, $z_2(t)=s(t)$, $\bar{A}_{11}=A_{11}-A_{12}KC_1$ and $\bar{A}_{d11}=A_{d11}-A_{d12}KC_1$ exhibit the reduced-order sliding-mode dynamics. Also, $\bar{C}=[0 \quad \bar{T}]$, where $\bar{T}\in\mathcal{R}^{p\times p}$ is nonsingular. The control law is considered as

$$u(t)=-Gy(t)-v_y(t), \quad (26)$$

where

$$G=\begin{bmatrix} G_1 & G_2 \end{bmatrix}\bar{T}^{-1} \quad (27)$$

$$v_y(t)=\begin{cases} \rho\dfrac{Fy(t)}{\|Fy(t)\|}, & \text{if } Fy(t)\neq 0, \\ 0, & \text{otherwise}, \end{cases} \quad (28)$$

where $G_1\in\mathcal{R}^{m\times(p-m)}$, $G_2\in\mathcal{R}^{m\times m}$, $F=[K \ I_m]T^{-1}$. The uncertain system (1) becomes

$$\dot{z}(t)=\bar{A}z(t)+\bar{A}_dz(t-\tau(t))+\bar{B}u(t)+\bar{B}_1w(t). \quad (29)$$

Closing the loop in the system (29) with the control law (26) yields

$$\dot{z}(t)=A_0z(t)+\bar{A}_dz(t-\tau(t))-\bar{B}v_y(t)+\bar{B}_1w(t), \quad (30)$$

where $A_0=\bar{A}-\bar{B}G\bar{C}$. Let $\bar{P}$ be a symmetric positive definite matrix partitioned conformably with (25) so that $\bar{P}=\left[\begin{smallmatrix} \bar{P}_1 & 0 \\ 0 & \bar{P}_2 \end{smallmatrix}\right]$. It follows that $\bar{P}\bar{B}=(F\bar{C})^TP_2$ and from (25) $Fy(t)=z_2(t)$. It can be shown that

$$\begin{aligned}\psi&=\bar{P}A_0+A_0^T\bar{P}\\ &=\begin{bmatrix} \bar{P}_1\bar{A}_{11}+\bar{A}_{11}^T\bar{P}_1 & \bar{P}_1\bar{A}_{12}+(\bar{A}_{21}-G_1C_1)^T\bar{P}_2 \\ * & \left\{\begin{matrix}\bar{P}_2\bar{A}_{22}+\bar{A}_{22}^T\bar{P}_2 \\ -\bar{P}_2G_2-(\bar{P}_2G_2)^T\end{matrix}\right\} \end{bmatrix}\\ &=\begin{bmatrix} \bar{P}_1\bar{A}_{11}+\bar{A}_{11}^T\bar{P}_1 & \bar{P}_1\bar{A}_{12}+\bar{A}_{21}^T\bar{P}_2-(L_1C_1)^T \\ * & \bar{P}_2\bar{A}_{22}+\bar{A}_{22}^T\bar{P}_2-L_2-(L_2)^T \end{bmatrix},\end{aligned} \quad (31)$$

where $L_1=\bar{P}_2G_1$ and $L_2=\bar{P}_2G_2$. A stability condition for the full order closed-loop system can be derived using the following Lyapunov–Krasovskii functional

$$\begin{aligned}V(t)&=z^T(t)\bar{P}z(t)+\int_{t-h}^{t}e^{\bar{\alpha}(s-t)}z^T(s)\bar{E}z(s)\mathrm{d}s\\ &\quad+\int_{t-\tau(t)}^{t}e^{\bar{\alpha}(s-t)}z^T(s)\bar{S}z(s)\mathrm{d}s\\ &\quad+h\int_{-h}^{0}\int_{t+\theta}^{t}e^{\bar{\alpha}(s-t)}\dot{z}^T(s)\bar{R}\dot{z}(s)\mathrm{d}s\,\mathrm{d}\theta,\end{aligned} \quad (32)$$

where $\bar{E}\ge 0$, $\bar{S}\ge 0$ and $\bar{R}=\left[\begin{smallmatrix} \bar{R}_1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$ where $\bar{R}_1\ge 0$ (as it is desired to determine a stability condition for the time-delay system which is delay independent of $z_2(t)$). Then

$$\begin{aligned}\bar{M}&=\dot{V}+\bar{\alpha}V-\bar{b}w^T(t)w(t)\\ &\le 2z^T(t)\bar{P}\dot{z}^T(t)+\bar{\alpha}z^T(t)\bar{P}z(t)+h^2\dot{z}^T(t)\bar{R}\dot{z}(t)\\ &\quad-\left[(z(t)-z(t-\tau(t)))^T\bar{R}(z(t)-z(t-\tau(t)))\right.\end{aligned}$$

$$+ (z(t-\tau(t)) - z(t-h))^T \times \bar{R}(z(t-\tau(t)) - z(t-h))\Big]e^{-\bar{\alpha}h} + z^T(t)(\bar{E}+\bar{S})z(t) - z^T(t-h)\bar{E}z(t-h)e^{-\bar{\alpha}h} - (1-d)z^T(t-\tau(t))\bar{S}z(t-\tau(t))e^{-\bar{\alpha}\tau(t)} - \bar{b}w^T(t)w(t). \quad (33)$$

Substitute the right-hand side of Equation (30) into (33). Setting $\varsigma(t) = \mathrm{col}\{z(t), z(t-h), z(t-\tau(t)), w(t)\}$, then

$$\dot{V}(t) \le \varsigma(t)^T\Phi_h\varsigma(t) + h^2\dot{z}^T(t)\bar{R}\dot{z}(t) + 2z^T\bar{P}\bar{B}(\bar{B}_{12}w(t) - v_y(t)) < 0 \quad (34)$$

is satisfied if $\varsigma^T(t)\Phi_h\varsigma(t) + h^2\dot{z}^T(t)\bar{R}\dot{z}(t) < 0$ and $2z^T\bar{P}\bar{B}(\bar{B}_{12}w(t) - v_y(t)) < 0$, where

$$\Phi_h = \begin{bmatrix} \phi_{11} & 0 & \bar{P}\bar{A}_d + \bar{R}e^{-\bar{\alpha}h} & \begin{bmatrix} \bar{P}_1\bar{B}_{11} \\ 0 \end{bmatrix} \\ * & \phi_{22} & \bar{R}e^{-\bar{\alpha}h} & 0 \\ * & * & -2e^{-\bar{\alpha}h}\bar{R} - (1-d)\bar{S}e^{-\bar{\alpha}h} & 0 \\ * & * & * & -\bar{b}I \end{bmatrix} \quad (35)$$

with

$$\phi_{11} = \psi + \bar{\alpha}\bar{P} + \bar{S} + \bar{E} - \bar{R}e^{-\bar{\alpha}h}; \quad \phi_{22} = -(\bar{E}+\bar{R})e^{-\bar{\alpha}h}.$$

Setting $\xi(t) = \mathrm{col}\{z(t), z(t-h), z(t-\tau(t)), w(t), v_y(t)\}$ and $\bar{I} = \begin{bmatrix} I_{(n-m)} & 0 \end{bmatrix}^T$, it is obtained that

$$\begin{aligned} h^2\dot{z}^T(t)\bar{R}\dot{z}(t) &= \Big[z^T(t)A_0^T + z^T(t-\tau(t))\bar{A}_d^T - v_y^T(t)\bar{B}^T + w^T(t)\bar{B}_1^T\Big] \\ &\quad \times h^2\bar{R}\big[A_0z(t) + \bar{A}_dz(t-\tau(t)) - \bar{B}v_y(t) + \bar{B}_1w(t)\big] \\ &= \xi^T(t)\begin{bmatrix} A_0^T \\ 0 \\ \bar{A}_d^T \\ \bar{B}_1^T \\ \bar{B}^T \end{bmatrix}\bar{I}h^2\bar{R}_1\bar{I}^T\begin{bmatrix} A_0^T \\ 0 \\ \bar{A}_d^T \\ \bar{B}_1^T \\ \bar{B}^T \end{bmatrix}^T\xi(t). \end{aligned} \quad (36)$$

Using the Schur complement, $\xi^T(t)\Phi_h\xi(t) + h^2\dot{z}^T(t)\bar{R}\dot{z}(t) < 0$ holds if

$$\begin{bmatrix} \Phi_h & \begin{matrix} hA_0^T\begin{bmatrix} I_{(n-m)} \\ 0 \end{bmatrix}\bar{R}_1 \\ 0 \\ h\bar{A}_{\mathrm{d}}^T\begin{bmatrix} I_{(n-m)} \\ 0 \end{bmatrix}\bar{R}_1 \\ h\bar{B}_1^T\begin{bmatrix} I_{(n-m)} \\ 0 \end{bmatrix}\bar{R}_1 \end{matrix} \\ * \; * \; * \; * & -\bar{R}_1 \end{bmatrix} < 0 \quad (37)$$

for some $\bar{\alpha} > 0$, $\bar{b} > 0$ and $0 \le \tau(t) \le h$, i.e. to ensure the exponential attractiveness of (30) to the ellipsoid $z^T(t)\bar{P}z(t) \le \frac{\bar{b}}{\bar{\alpha}}\Delta^2$. Given the control structure in (27), then

$$\begin{aligned} &2z^T(t)\bar{P}\bar{B}(\bar{B}_{12}w(t) - v_y(t)) \\ &= 2z_2^T(t)\bar{P}_2(\bar{B}_{12}w(t) - v_y(t)) \\ &\le -2\rho\bar{P}_2\|z_2(t)\| + 2\bar{P}_2\|\bar{B}_{12}\|\|z_2(t)\|\Delta \\ &< 0. \end{aligned}$$

The latter inequality implies exponential attractivity of the ellipsoid $z^T(t)\bar{P}z(t) \le \frac{\bar{b}}{\bar{\alpha}}\Delta^2$, thus for $t \to \infty$, $z^T(t-\tau(t))\bar{P}z(t-\tau(t)) \le \frac{\bar{b}}{\bar{\alpha}}\Delta^2$ holds. The following proposition can now be stated:

Proposition 5.1: *Given scalars $h > 0$, $d < 1$, $\bar{\alpha} > 0$, $\bar{b} > 0$, assume there exist $n \times n$ matrices $\bar{P} = \mathrm{diag}\{\bar{P}_1, \bar{P}_2\} > 0$ with $P_2 \in \mathcal{R}^{m\times m}$, $\bar{E} \ge 0$, $\bar{S} \ge 0$, a $(n-m) \times (n-m)$-matrix $\bar{R}_1 \ge 0$, $L_1 \in \mathcal{R}^{m\times(p-m)}$, $L_2 \in \mathcal{R}^{m\times m}$ such that LMI (37) is feasible. Then for $\rho > \|\bar{B}_{12}\|\Delta$ the closed-loop system (30), where $G_1 = \bar{P}_2^{-1}L_1$, $G_2 = \bar{P}_2^{-1}L_2$, is exponentially attracted to the ellipsoid $z^T(t)\bar{P}z(t) \le \frac{\bar{b}}{\bar{\alpha}}\Delta^2$ for all $\tau(t) \in [0, h]$. Consequently it also holds that $z^T(t-\tau(t))\bar{P}z(t-\tau(t)) \le \frac{\bar{b}}{\bar{\alpha}}\Delta^2$ for $t \to \infty$.*

Denote

$$A_0^L = [0 \; I_m]A_0, \quad \bar{A}_d^L = [0 \; I_m]\bar{A}_d, \quad \beta = \frac{\bar{b}}{\bar{\alpha}}\Delta^2. \quad (38)$$

Given $\delta_1 > 0$, $\delta_2 > 0$, conditions will now be derived that guarantee the solutions of (25) satisfy the bound

$$\|A_0^Lz(t)\| < \delta_1, \quad \|A_d^Lz(t-\tau(t))\| < \delta_2 \quad (39)$$

for $t \to \infty$. The following inequalities

$$\begin{aligned} &z^T(t)(A_0^L)^T(A_0^L)z(t) \\ &\quad \le \delta_1^2\frac{z^T(t)\bar{P}z(t)}{\beta} \\ &z^T(t-\tau(t))(A_d^L)^T(A_d^L)z(t-\tau(t)) \\ &\quad \le \delta_2^2\frac{z^T(t-\tau(t))\bar{P}z(t-\tau(t))}{\beta} \end{aligned} \quad (40)$$

guarantee (39). Hence equivalently

$$(A_0^L)^T(A_0^L) \le \frac{\delta_1^2\bar{P}}{\beta}, \qquad (A_d^L)^T(A_d^L) \le \frac{\delta_2^2\bar{P}}{\beta} \quad (41)$$

or by Schur complements

$$\begin{bmatrix} -\frac{\delta_1^2\bar{P}}{\beta} & (A_0^L)^T \\ * & -I \end{bmatrix} < 0, \quad \begin{bmatrix} -\frac{\delta_2^2\bar{P}}{\beta} & (A_d^L)^T \\ * & -I \end{bmatrix} < 0. \quad (42)$$

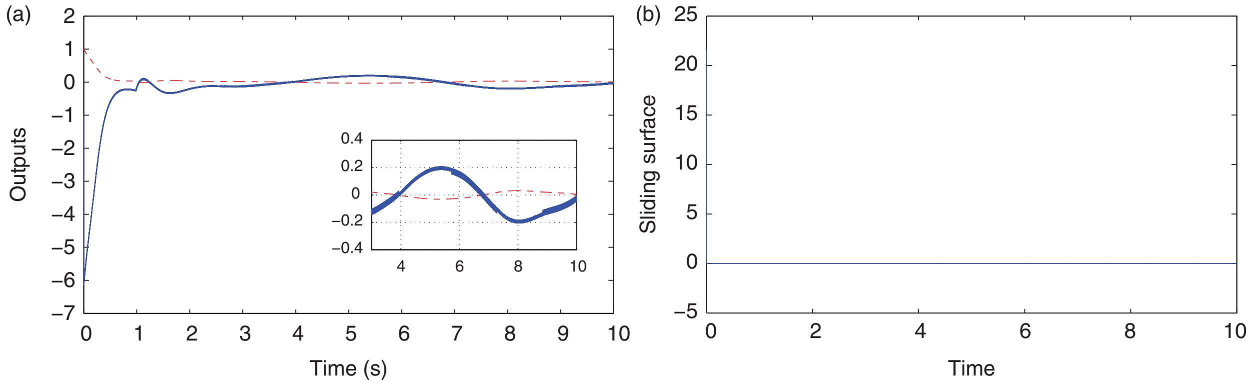

Figure 1. Closed-loop response with delay $h = 1$ s: (a) outputs and (b) sliding surface.

Theorem 5.2: *Given scalars $\bar{\alpha} > 0$, $\bar{b} > 0$, let there exist $n \times n$ matrices $\bar{P} = \mathrm{diag}\{\bar{P}_1, \bar{P}_2\} > 0$, $\bar{E} \geq 0$, $\bar{S} \geq 0$, a $(n-m) \times (n-m)$-matrix $\bar{R}_1 \geq 0$, $L_1 \in \mathcal{R}^{m \times (p-m)}$, $L_2 \in \mathcal{R}^{m \times m}$ such that the LMI* (37) *is feasible for $0 \leq \tau(t) \leq h$, $\dot{\tau}(t) \leq d < 1$. Let δ_1 and δ_2 satisfy* (42) *with the notation given in* (38). *Then for*

$$\rho > \|B_{12}\|\Delta + \delta_1 + \delta_2 \tag{43}$$

an ideal sliding motion takes place on the surface $\mathcal{S}$. The closed-loop system (26), (29) *is ultimately bounded by*

$$\limsup_{t\to\infty} z^T(t)\bar{P}z(t) \leq \frac{\bar{b}}{\bar{\alpha}}\Delta^2.$$

Proof: Substituting the control law it follows from (29) that

$$\dot{s}(t) = F\bar{C}A_0 z(t) + F\bar{C}\bar{A}_d z((t - \tau(t)) + (\bar{B}_{12}w(t) - v_y(t)).$$

Let $V_c : \mathcal{R}^m \to \mathcal{R}$ be defined by $V_c(s) = s^T(t)\bar{P}_2 s(t)$. It follows that

$$\bar{P}_2 F\bar{C}A_0 = \bar{P}_2 A_0^L, \quad \bar{P}_2 F\bar{C}A_d = \bar{P}_2 A_d^L.$$

Starting from initial condition $z(t_0)$, it can be verified that there exists $t_1 > 0$ such that for all $t \geq t_1$,

$$\begin{aligned}\dot{V}_c(s) &= 2s^T(t)\bar{P}_2 A_0^L z(t) + 2s^T(t)\bar{P}_2 A_d^L z(t - \tau(t)) \\ &\quad + 2s^T(t)\bar{P}_2(\bar{B}_{12}w(t) - v_y(t)) \\ &\leq 2\|s(t)\|\|\bar{P}_2\|(\|A_0^L z(t)\| + \|A_d^L z(t - \tau(t))\|) \\ &\quad - 2(\delta_1 + \delta_2)\|s(t)\|\|\bar{P}_2\| \\ &< -2\eta\|s(t)\|,\end{aligned} \tag{44}$$

where $\eta = \delta_1 + \delta_2 - \|A_0^L z(t)\| - \|A_d^L z(t - \tau(t))\|$. A sliding motion will thus be attained in finite time. □

Remark 2: Since LMIs (15), (37) and (42) are affine in the system matrices A, A_d and B_1, the results are applicable to the case where these matrices are uncertain with polytopic type uncertainties (see Remark 1). One has to solve the LMIs simultaneously for all the N vertices, applying the same decision matrices for all vertices. In contrast to the existing methods in the literature (Gouaisbaut et al. 2004; Seuret, Edwards, Spurgeon, and Fridman 2009), polytopic type uncertainties can be incorporated in all the blocks of A, A_d, B_1 and not only in A_{11}, A_{d11} because the switching gain ρ (and not only the sliding surface) is found using LMIs.

Example 5.3: Following on from Example 4.3, where a sliding surface prescribing stable dynamics has been designed for the uncertain system (21), then the control law in (27) will have the sliding function matrix $F = [7.07, 1]$. A control gain G must be designed which will bring the closed-loop system into a bounded region centred at the sliding surface. Setting $\bar{\alpha} = 0.3$, $\bar{b} = 5$ in Proposition 5.1, it is obtained that

$$\bar{P} = \begin{bmatrix} 22.4 & -12.8 & 0 \\ * & 29 & 0 \\ * & * & 0.68 \end{bmatrix}, \quad L_1 = -6.08, \quad L_2 = 85.4,$$

which gives

$$G = \begin{bmatrix} -9, & 126.6 \end{bmatrix}\bar{T}^{-1}, \quad \text{where } \bar{T} = \begin{bmatrix} 1 & 0 \\ -7.07 & 1 \end{bmatrix}.$$

Once the state of the closed-loop system has entered the sliding patch $z^T(t)\bar{P}z(t) \leq \frac{\bar{b}}{\bar{\alpha}}\Delta^2$, the switching gain $\rho = 753$ derived from LMI (42) will ensure the sliding surface is reached in finite time. Figure 1 shows that the sliding surface is reached in finite time and the outputs of the system are stable with ultimate bound $\|y(t)\| \leq 0.2$.

Sliding-mode control has the advantage over linear control for its absolute rejection of the

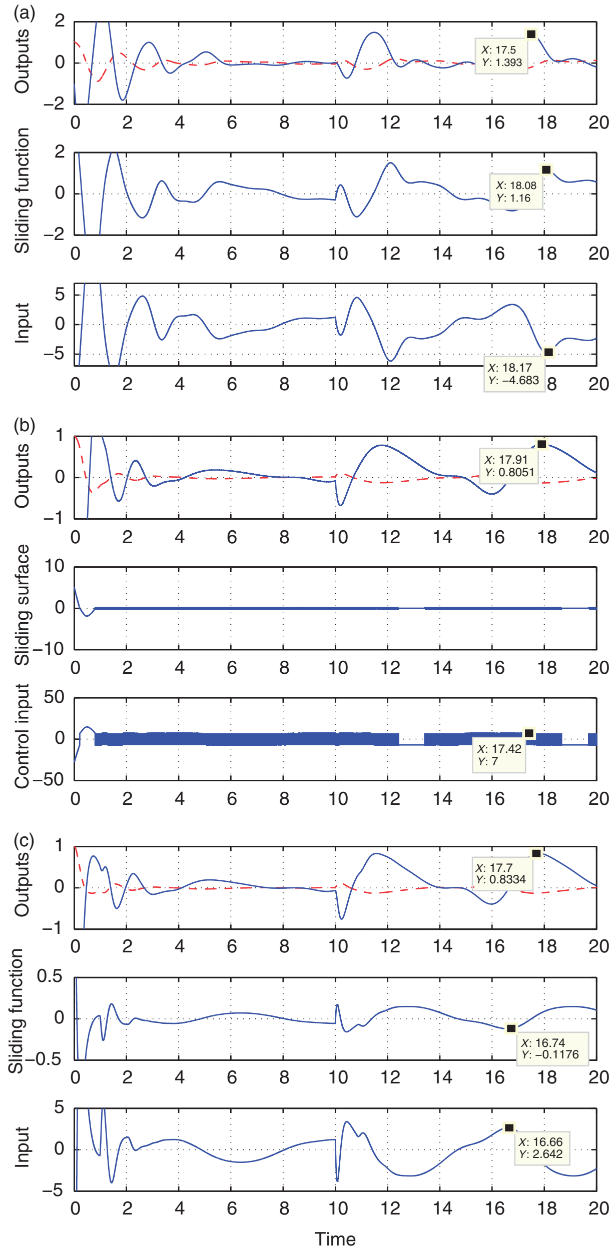

Figure 2. Closed-loop response with linear control and SMC: (a) linear control with G, (b) SMC with linear part G and (c) linear control with $\hat{G}$.

matched uncertainties. To verify the statement, suppose there is a change of the matched disturbance at time 10 s of magnitude from $1 \rightarrow 50$, then the sliding surface remains unaffected so that $B_1 = [0\ 0\ 50]^T$. While keeping the same control parameters obtained so far comparisons between using sliding-mode control and only the linear control G for the new closed-loop design with only matched disturbances are made in Figure 3. As can be seen, the sliding-mode control is more robust than the linear control to matched disturbances. The difference between using the linear part of the control G alone and SMC with switching for a system with unmatched disturbances can be demonstrated below. For the same uncertain system, suppose the unmatched disturbances are changed from $B_1 = [0.5, -0.5, 1]^T$ to $B_1 = [2, -2, 1]^T$ after the initial 10 s. Using the linear control $G = [25, 4]$ alone in the feedback, the responses for the outputs $y(t)$, sliding function $s(t)$ and control input $u(t)$ are plotted in Figure 2(a). As can be seen the outputs $\|y(t)\| \leq 1.4$, sliding function is bounded as $\|s(t)\| \leq 1.16$ and control input $\|u(t)\| \leq 4.7$. If SMC is used with the same linear gain and a switching gain $\rho = 7$, the system responses are plotted in Figure 2(b), where $\|y(t)\| \leq 0.81$, $s(t) = 0$, $\|u(t)\| \leq 7$. Therefore for a system with unmatched disturbances, SMC can give an ideal sliding surface response rather than the bounded sliding function given by its linear control part. As a result, a smaller bound of the outputs can be obtained. For linear control to yield the similar bound on the outputs as by SMC, the linear gain needs to be increased from $G = [25, 4]$ to $\hat{G} = [134, 21]$ as seen in Figure 2(c), where $\|y(t)\| \leq 0.83$. To conclude, for a linear control to give similar output performance in the presence of unmatched disturbances, the linear gain needs to be larger, but not substantially larger than the linear part of the SMC. In another words, a linear control design for a system with unmatched disturbances can give a similar output bound as SMC if the linear control is large enough.

6. Autonomous vehicle control

Autonomous vehicles are expected to operate effectively in time-varying and uncertain conditions. Here a case study described in Yao, Spurgeon, and Edwards (2006) is considered, where the elevation angle of the gun barrel of a vehicle in space should be maintained while the hull of the vehicle is subject to external disturbances resulting from the motion of the vehicle across rough terrain. To meet the high control specifications, sliding-mode control has been considered within the application domain for its robustness against friction and disturbances and its ease of implementation for motor drive control. The existence problem must determine a sliding surface that minimises the ultimate bound of the reduced-order dynamics in the presence of time-varying state delay and unmatched disturbances relating to frictional effects. A fully nonlinear simulation model of the system is available for controller analysis and testing (Yao et al. 2006). The model is physically based and is known to

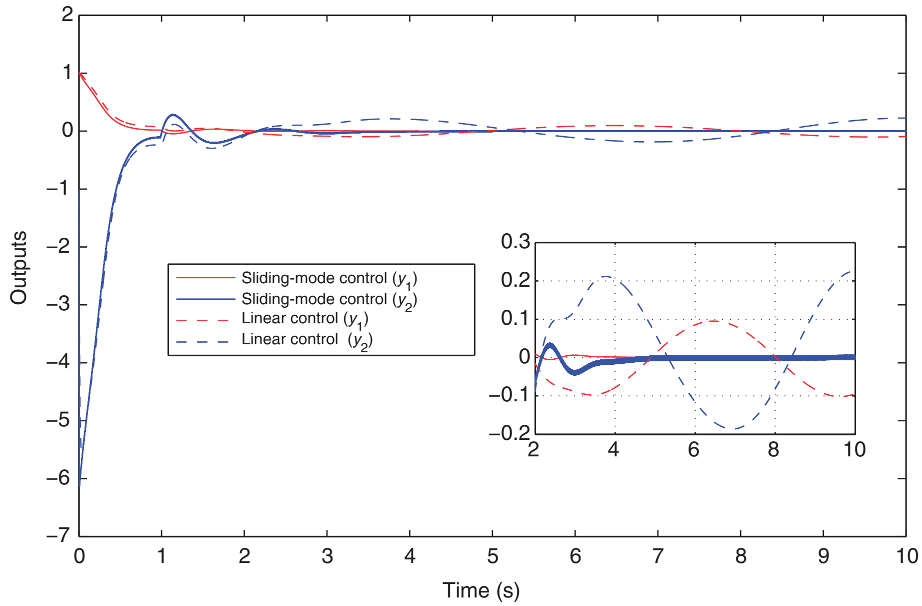

Figure 3. Comparison between sliding mode control and linear control in the presence of matched disturbance and delay. Available in colour online.

represent with high fidelity the dynamics and behaviour of the real system:

$$\dot{x}(t) = \underbrace{\begin{bmatrix} -\frac{D_m}{J_m N^2} & -\frac{K_m}{J_m N} & \frac{D_m}{J_m N} & 0 & 0 \\ \frac{1}{N} & 0 & -1 & 0 & 0 \\ \frac{D_m}{J_1 N} & \frac{K_m}{J_1} & \frac{-D_m - D_{12}}{J_1} & \frac{-K_{12}}{J_1} & \frac{D_{12}}{J_1} \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & \frac{D_{12}}{J_2} & \frac{K_{12}}{J_2} & \frac{-D_{12}}{J_2} \end{bmatrix}}_{A} x(t) + \underbrace{\begin{bmatrix} \frac{K_t}{J_m} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{B} u(t) + \underbrace{\begin{bmatrix} -\frac{D_m(N-1)}{J_m N^2} & -\frac{1}{J_m} & 0 & \frac{1}{J_m N^2} & 0 \\ \frac{N-1}{N} & 0 & 0 & 0 & 0 \\ \frac{D_m(N-1)}{J_1 N} & 0 & -\frac{1}{J_1} & 0 & \frac{1}{J_1} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{B_{11}} w(t), \tag{45}$$

where the state vector $x(t) = [\dot{\theta}_m(t), \theta_{mb}(t), \dot{\theta}_b(t), \theta_{bl}(t), \dot{\theta}_l(t)]^T$, and $w(t) = [\dot{\theta}_p(t), \omega_{1m}(t), \omega_{1l}(t), F_{mb} \times \mathrm{sign}(\dot{\theta}_m(t) - \dot{\theta}_p(t)), F_{mb}\,\mathrm{sign}(\dot{\theta}_b(t) - \dot{\theta}_p(t))]^T$.The friction related signals are

$$F_{mb}(t) = K_\theta \left| \frac{D_m}{N}\dot{\theta}_m(t) + \frac{D_m(N-1)}{N}\dot{\theta}_p(t) + K_m\theta_{mb}(t) \right|,$$
$$\omega_{1m}(t) = f_{ms} \cdot \mathrm{sign}(\tau_{am}(t) - J_m\ddot{\theta}_p(t)),$$
$$\omega_{1l}(t) = f_{ls} \cdot \mathrm{sign}(\tau_{a1}(t) - J_1\ddot{\theta}_p(t)),$$

where the second and third states of the disturbance from $w(t)$ are a function of the friction level fd, where fd can take values 0, 1, 2, 3 (Yao et al. 2006). Note

θ_{mb}	$= \frac{1}{N}\theta_m - \theta_b + (1 - \frac{1}{N})\theta_p$;
$\dot{\theta}_b$	breech velocity;
θ_m	motor position;
$\dot{\theta}_l$	muzzle velocity;
$\dot{\theta}_p$	pitch rate disturbance;
J_m	motor inertia;
N	gearbox of ratio;
J_1	elevation inertia on load one;
K_m and D_m	stiffness and damping between the motor and the load;
K_{12} and D_{12}	stiffness and damping between the load one and the load two;
τ_{am} and τ_{al}	applied torque to the motor and the load;
ω_{1m} and ω_{1l}	motor friction and the load friction;

f_{ms} and f_{ls}	motor coulomb friction and the load coulomb friction;
u	control input defined as the voltage input to the power amplifier;
$\dot{\theta}_p$	disturbance input defined as pitch rate disturbance;
ω_{1m} and $\text{sign}(\dot{\theta}_m - \dot{\theta}_p)$	motor friction disturbances;
ω_{1l} and $\text{sign}(\dot{\theta}_b - \dot{\theta}_p)$	load friction disturbances

The parameter values used in Yao et al. (2006) define

$$A = \begin{bmatrix} -338.14 & -2.55\times 10^7 & 50942 & 0 & 0 \\ 0.0066 & 0 & -1 & 0 & 0 \\ 0.66 & 50000 & -110.1 & -15000 & 10 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 7.69 & 11538 & -7.69 \end{bmatrix},$$

$$B = \begin{bmatrix} 4523.1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$$B_{11} = \begin{bmatrix} -50604 & -769 & 0 & 5.1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 99 & 0 & -0.01 & 0 & 0.01 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \tag{46}$$

and the disturbance is known to be

$$\|\dot{\theta}_p\| \le 0.07, \quad \|\omega_{1m}\| \le 0.5\times fd, \quad \|\omega_{1l}\| \le 10\times fd,$$
$$\|F_{mb}\,\text{sign}(\dot{\theta}_m - \dot{\theta}_p)\| \le 4, \quad \|F_{mb}\,\text{sign}(\dot{\theta}_b - \dot{\theta}_p)\| \le 4. \tag{47}$$

The vehicle dynamics is augmented with an extra state related to the breech position, where the desired breech position is zero. Denote $C_a = [0\ 0\ 1\ 0\ 0]$, the state space representation of the augmented system is given by

$$A_c = \begin{bmatrix} 0 & -C_a \\ 0 & A \end{bmatrix}, \quad B_c = \begin{bmatrix} 0 \\ B \end{bmatrix}, \quad C_c = \begin{bmatrix} 1 & 0 \\ 0 & C \end{bmatrix}. \tag{48}$$

6.1 *Sliding-mode control design*

A design which does not incorporate knowledge of delay effects is first performed to yield a benchmark level of performance. Firstly it is necessary to construct K for the reduced-order system (7) according to Proposition 4.2. The parameter settings in LMI (15) with entries (19) are selected as $A_d = 0$, $h = 0$ s. If

$$\delta = 11.5, \quad \varepsilon = 0.0000002, \quad M = \begin{bmatrix} 0.15 & 0.015 \\ 0.21 & 0.06 \\ 0.009 & 0.0003 \end{bmatrix}$$

and choosing $\alpha = 7.2$, $b = 0.0005$, then it is obtained that

$$K = \begin{bmatrix} 0.9, & 28, & 327 \end{bmatrix}. \tag{49}$$

The poles of the corresponding reduced order system are

$$\begin{bmatrix} -4.15 \pm j106, & -389.6 \pm j160.7, & -2823.7 \end{bmatrix}. \tag{50}$$

The control law in (27) will have the sliding function matrix

$$F = \begin{bmatrix} -327 & 0.0002 & 28 & 0.9 \end{bmatrix}. \tag{51}$$

A control G is designed which will bring the closed-loop system into a bounded region centred about the sliding surface. Setting $\bar{\alpha} = 0.8$, $\bar{b} = 3.88\times 10^{-6}$ in Proposition 5.1, it is obtained that

$$G = \begin{bmatrix} -1.02\times 10^7, & 7.6, & 910610, & 27834 \end{bmatrix}. \tag{52}$$

The closed-loop poles of $A_c - B_c G C_c$ are

$$\begin{bmatrix} -31257 & -2810.4 & -390.8 \pm j160.9 & -4.2 \pm j106 \end{bmatrix}.$$

The switching gain $\rho = 1561$, which is derived from LMI (42), will ensure the sliding surface is reached in finite time. Figure 4 shows the position error and rate error for $fd = 1, 2, 3$ using the proposed controller. In the original case study (Yao et al. 2006), an observer was used to estimate the effect of the disturbance and the equivalent control method was used to synthesise the control law, which was augmented with an additional PI control. With this strategy the position and rate errors were $\|e_p(t)\| \le 0.2\times 10^{-3}$ rad and $\|e_r(t)\| \le 0.01$ rad/s respectively. Setting a fixed sampling frequency of 10 kHz and choosing *ode*3 solver in simulink, $\|e_p(t)\| \le 0.8\times 10^{-5}$ rad, $\|e_r(t)\| \le 0.4\times 10^{-3}$ rad/s was achieved, as seen in Figure 4, for $fd = 1, 2, 3$ with the proposed control scheme. The output feedback sliding-mode control approach presented in this section has thus improved the tracking accuracy over previous results in Yao et al. (2006). The ultimate bound of the outputs is a function of the unmatched disturbance, but it can be seen that the effect of the friction disturbance on the control performance after changing $fd = 1, \rightarrow 3$ is very small.

Speed control of a motor in the presence of uncertainties such as friction normally exhibit delays due to the fact that the mechanical response of the

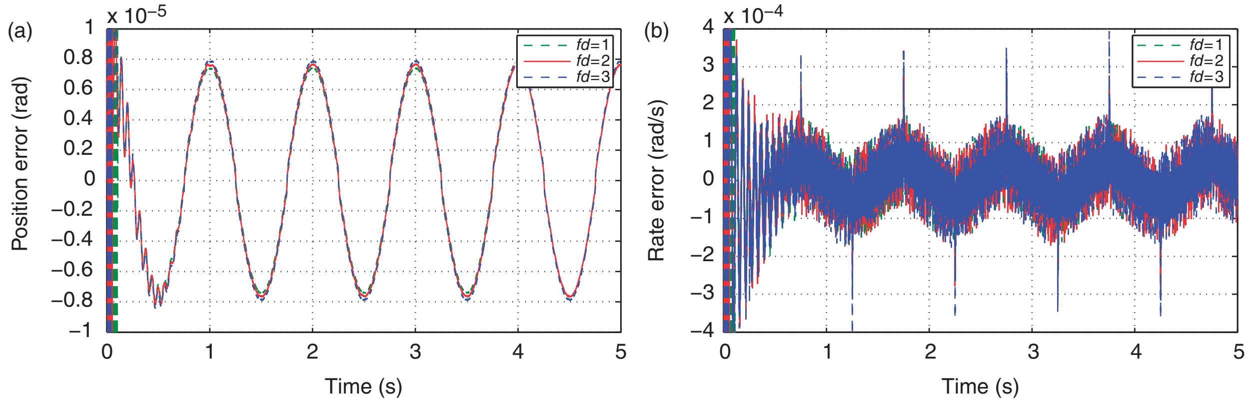

Figure 4. Closed-loop response without delay: (a) position error and (b) rate error. Available in colour online.

motor is slower than the electrical command. The size of the delay will depend upon the physical parameters of the actuator and can vary from milliseconds to several seconds, depending on the application. To take account of such delay effects in the actual system for the control design purpose, it is desirable to introduce a system model incorporating delay into the system model used for design. This will provide a means to analyse the potential delay effect on the stability of the closed-loop system at the design stage. Assume the delay matrix

$$A_d = \begin{bmatrix} -10 & 20 & 0 & 0 & 0 \\ 0.007 & 0 & 0.1 & 0 & 0 \\ 0 & 20 & -2 & 1 & 0 \\ 0 & 0 & 0.1 & 0 & 0.1 \\ 0 & 0 & 2 & 1 & 0 \end{bmatrix}$$

$$\text{and the augmented matrix} \quad A_{d_c} = \begin{bmatrix} 0 & 0 \\ 0 & A_d \end{bmatrix}. \tag{53}$$

The open-loop tests on system (46) with the delay matrix A_d in (53) shows that the vehicle system with state delay $h \le 3$ ms yields a breech position error of $\|e_p(t)\| \le 0.01$ rad/s as expected from the known system response. The augmented linear system with delay has dynamics close to those of the original plant.

Designing a controller without considering explicitly possible delay effects within the control design process can lead to deterioration of the system performance and sometimes even instability. Suppose there is a constant delay $h = 3$ ms in the system where the values of F and G are taken as in (51) and (52) respectively, with the delay distribution matrix in (53). In this case the position error will increase from $\|e_p(t)\| \le 0.8 \times 10^{-5}$ to $\|e_p(t)\| \le 1.4 \times 10^{-3}$ rad. The closed-loop system becomes unstable for $h \ge 4$ ms when delay effects are not incorporated in the design process.

A controller will now be designed based on a model incorporating delay effects. Firstly to construct K for the reduced-order system (7) according to Proposition 4.2, the parameter settings in LMI (15) with entries (19) are selected with the delay upper bound $h = 10$ ms and the rate of change of the time-varying delay $\dot{\tau} \le d = 0$. If for

$$\delta = 3, \quad \varepsilon = 0.0013, \quad M = \begin{bmatrix} 0.0018 & 0.002 \\ 0.22 & 0.22 \\ 7.54 & 1.1 \end{bmatrix}$$

and choosing $\alpha = 0.9$, $b = 0.0048$, then it is obtained that

$$K = \begin{bmatrix} 0.01, & 16, & 16.8 \end{bmatrix}. \tag{54}$$

The poles of the reduced-order system are

$$\begin{bmatrix} -5.23 \pm j92.9, & -71.3 \pm j254.8, & -477.8 \end{bmatrix}. \tag{55}$$

Thus the control law in (27) will have the sliding function matrix

$$F = \begin{bmatrix} -16.8, & 0.0002, & 16, & 0.01 \end{bmatrix}.$$

The control G is designed to bring the closed-loop system into a bounded region centred about the sliding surface. Setting $\bar{\alpha} = 0.8$, $\bar{b} = 3.88 \times 10^{-6}$ in Proposition 5.1, it is obtained that

$$G = \begin{bmatrix} -2.02 \times 10^6, & 26.7, & 1.94, & 0.002 \end{bmatrix}.$$

The closed-loop poles of $A_c - B_c G C_c$ are

$$\begin{bmatrix} -120700, & -481.1, & -75.2 \pm j253.2, & -5.43 \pm j92.9 \end{bmatrix}.$$

The switching gain $\rho = 68{,}632$, which is derived from LMI (42), will ensure the sliding surface is reached in finite time. The initial function was chosen as $x(t_0 - \tau) = 0$ for $\tau \in [0\ h]$ in the simulation. The closed-loop performance is shown in Figure 5 for $fd = 1, 2, 3$. The position and rate errors are kept within the bound

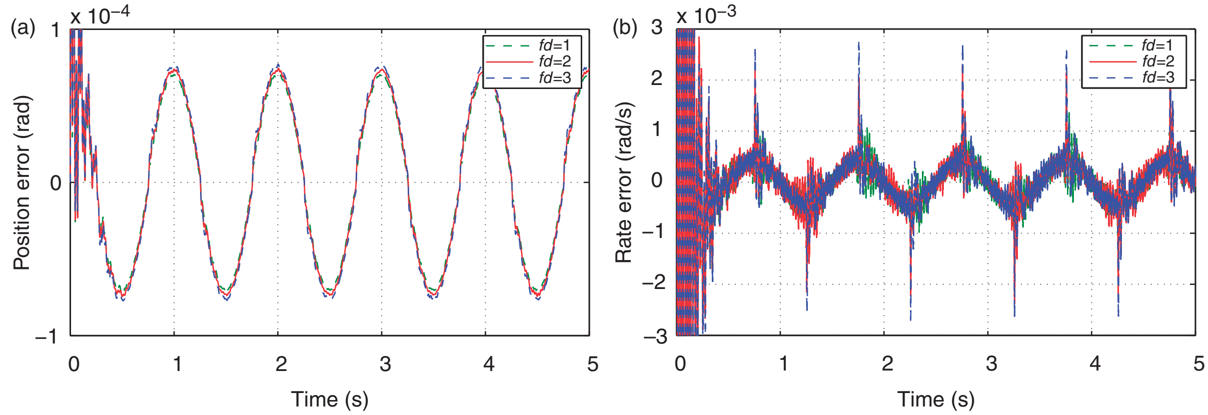

Figure 5. Closed-loop response with constant delay $h=10$ ms present in the plant where the control has been designed based on a time-delay system model: (a) position error and (b) rate error. Available in colour online.

$\|e_p(t)\| \leq 0.8 \times 10^{-4}$ rad, $\|e_r(t)\| \leq 0.1 \times 10^{-4}$ rad/s in the presence of delay. Despite the effect of the friction disturbance on the nonlinear model which is not fully rejected, the controller is seen to be robust to the disturbance even in the presence of delay. This has demonstrated the efficiency of the proposed control scheme on a system of practical interest.

7. Conclusion

The development of output feedback-based sliding-mode control schemes for systems in the presence of state delay and both matched and unmatched disturbances has been presented. A descriptor Lyapunov functional approach has been used for switching function design. The methodology has been implemented using LMIs and can give desirable sliding-mode dynamics. The advantage of the method is that for the first time and despite only output feedback being available, not only the switching function is derived from LMIs but also the switching gain required to solve the reachability problem is determined using LMIs. The method allows polytopic uncertainties to be included in all blocks of A, A_d, B_1 and not only in A_{11}, A_{d11} as with other methods. This is novel even for systems without delay. As well as an example incorporating polytopic uncertainties, the methodology has also been applied to a nonlinear autonomous vehicle control problem. Nonlinear simulations show that the gun barrel is maintained at the desired position, despite variation in the vehicle motion caused by friction.

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