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Robust sampled-data stabilization of linear systems: an input delay approach ☆

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Abstract

A new approach to robust sampled-data control is introduced. The system is modelled as a continuous-time one, where the control input has a piecewise-continuous delay. Sufficient linear matrix inequalities (LMIs) conditions for sampled-data state-feedback stabilization of such systems are derived via descriptor approach to time-delay systems. The only restriction on the sampling is that the distance between the sequel sampling times is not greater than some prechosen $h > 0$ for which the LMIs are feasible. For $h \rightarrow 0$ the conditions coincide with the necessary and sufficient conditions for continuous-time state-feedback stabilization. Our approach is applied to two problems: to sampled-data stabilization of systems with polytopic type uncertainities and to regional stabilization by sampled-data saturated state-feedback.

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1. Introduction

Modelling of continuous-time systems with digital control in the form of continuous-time systems with delayed control input was introduced by Mikheev, Sobolev, and Fridman (1988), Astrom and Wittenmark (1989) and further developed by Fridman (1992). The digital control law may be represented as delayed control as follows:

$$u(t) = u_{\mathrm{d}}(t_k) = u_{\mathrm{d}}(t - (t - t_k)) = u_{\mathrm{d}}(t - \tau(t)),$$

$$t_k \leqslant t < t_{k+1}, \quad \tau(t) = t - t_k, \tag{1}$$

where u_{d} is a discrete-time control signal and the time-varying delay $\tau(t) = t - t_k$ is piecewise-linear with derivative $\dot{\tau}(t) = 1$ for $t \neq t_k$. Moreover, $\tau \leqslant t_{k+1} - t_k$. Based on such a model, for small enough sampling intervals $t_{k+1} - t_k$, asymptotic approximations of the trajectory

(Mikheev et al., 1988) and of the optimal solution to the sampled-data LQ finite horizon problem (Fridman, 1992) were constructed.

Robust stability conditions for systems with time-varying delays were derived via Lyapunov–Krasovskii functionals in the case where the derivative of the delay is less than one (see e.g. Niculescu, de Souza, Dugard, & Dion, 1998). The stability issue in the case of time-varying delay without any restrictions on the derivative of the delay has been treated mainly via Lyapunov–Razumikhin functions, which usually lead to conservative results (see e.g. Hale & Lunel, 1993; Cao, Sun, & Cheng, 1998; Kolmanovskii & Myshkis, 1999). Only recently for the first time this case was treated by Lyapunov–Krasovskii technique (Fridman & Shaked, 2002).

Two main approaches have been used to the sampled-data robust stabilization problem (see e.g. Dullerud & Glover, 1993; Sivashankar & Khargonekar, 1994; Basar & Bernard, 1995; Oishi, 1997). The first one is based on the lifting technique (Bamieh et al., 1991; Yamamoto, 1990) in which the problem is transformed to equivalent finite-dimensional discrete problem. However, this approach does not work in the cases with uncertain sampling times or uncertain system matrices.

The second approach is based on the representation of the system in the form of hybrid discrete/continuous model. Application of this approach to linear systems leads to necessary and sufficient conditions for stability and L_2-gain analysis in the form of differential equations (or inequalities) with jumps (see e.g. Sivashankar & Khargonekar, 1994). The latter approach has been applied recently to sampled-data stabilization of linear uncertain systems for the case of equidistant sampling (Hu, Cao, & Shao, 2002; Hu, Lam, Cao, & Shao, 2003). To overcome difficulties of solving differential inequalities with jumps, a piecewise-linear in time Lyapunov function has been suggested. As a result in Hu et al. (2002) a countable sequence of matrix inequalities has been derived, which has been proposed to solve by iterative method. The feasibility of these matrix inequalities is not guaranteed even for small sampling periods. In Hu et al. (2003) LMIs have been derived (see Corollary 2) which do not depend on the sampling interval and thus are very conservative.

In the present paper we suggest a new approach to the robust sampled-data stabilization. We find a solution by solving the problem for a continuous-time system with uncertain but bounded (by the maximum sampling interval) time-varying delay in the control input. The conditions which we obtain are robust with respect to different samplings with the only requirement that the maximum sampling interval is not greater than h. Moreover, the feasibility of the LMIs is guaranteed for small h if the corresponding continuous-time controller stabilizes the system. As a by-product we show that for $h \to 0$ the conditions coincide with the necessary and sufficient conditions for the continuous-time stabilization. Such convergence in H_2 framework and related results were proved by Mikheev et al. (1988), Chen and Francis (1991), Fridman (1992), Osborn and Bernstein (1995), Trentelman and Stoorvogel (1995) and Oishi (1997).

For the first time the new approach allows to develop different robust control methods for the case of sampled-data control. The LMIs are affine in the system matrices and thus for the systems with polytopic type uncertainty the stabilization conditions readily follow. We consider the regional stabilization by sampled-data saturated state-feedback, where we give an estimate on the domain of attraction. For continuous-time stabilization of state-delayed systems by saturated-feedback see e.g. Tarbouriech and Gomes da Silva (2000), Cao et al. (2002).

Notation. Throughout the paper the superscript 'T' stands for matrix transposition, $\mathscr{R}^n$ denotes the n dimensional Euclidean space with vector norm $|\cdot|$, $\mathscr{R}^{n \times m}$ is the set of all $n \times m$ real matrices, and the notation $P > 0$, for $P \in \mathscr{R}^{n \times n}$ means that P is symmetric and positive definite. Given $\bar{u} = [\bar{u}_1, \ldots, \bar{u}_m]^T, 0 < \bar{u}_i, i = 1, \ldots, m$, for any $u = [u_1, \ldots, u_m]^T$ we denote by $sat(u, \bar{u})$ the vector with coordinates $sign(u_i)min(|u_i|, \bar{u}_i)$. By stability of the system we understand the asymptotic stability of it.

2. Sampled-data stabilization of uncertain systems

2.1. Problem formulation

Consider the system

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{2}$$

where $x(t) \in \mathscr{R}^n$ is the state vector, $u(t) \in \mathscr{R}^m$ is the control input. We are looking for a piecewise-constant control law of the form $u(t) = u_d(t_k)$, $t_k \leqslant t < t_{k+1}$, where u_d is a discrete-time control signal and $0 = t_0 < t_1 < \cdots < t_k < \ldots$ are the sampling instants. Our objective is to find a state-feedback controller given by

$$u(t) = Kx(t_k), \qquad t_k \leqslant t < t_{k+1}, \tag{3}$$

which stabilizes the system.

We represent a piecewise-constant control law as a continuous-time control with a time-varying piecewise-continuous (continuous from the right) delay $\tau(t) = t - t_k$ as given in (1). We will thus look for a state-feedback controller of the form: $u(t) = Kx(t - \tau(t))$. Substituting the latter controller into (2), we obtain the following closed-loop system:

$$\dot{x}(t) = Ax(t) + BKx(t - \tau(t)),$$

$$\tau(t) = t - t_k, \; t_k \leqslant t < t_{k+1}. \tag{4}$$

We assume that
A1: $t_{k+1} - t_k \leqslant h \quad \forall k \geqslant 0$.
From A1 it follows that $\tau(t) \leqslant h$ since $\tau(t) \leqslant t_{k+1} - t_k$. We will further consider (4) as the system with uncertain and bounded delay.

2.2. Stability of the closed-loop system

Similarly to Fridman and Shaked (2002), where the continuous delay was considered, we obtain for the case of piecewise-continuous delay the following result:

Lemma 2.1. *Given a gain matrix K, system* (4) *is stable for all the samplings satisfying* A1, *if there exist $n \times n$ matrices $0 < P_1, P_2, P_3, Z_1, Z_2, Z_3$ and $R > 0$ that satisfy the following LMIs:*

$$\Psi_1 < 0, \quad and \quad \begin{bmatrix} R & [0 \; K^T B^T]P \\ * & Z \end{bmatrix} \geqslant 0, \tag{5a,b}$$

where

$$P = \begin{bmatrix} P_1 & 0 \\ P_2 & P_3 \end{bmatrix}, \quad Z = \begin{bmatrix} Z_1 & Z_2 \\ * & Z_3 \end{bmatrix},$$

$$\Psi_1 = \Psi_0 + hZ + \begin{bmatrix} 0 & 0 \\ 0 & hR \end{bmatrix},$$

$$\Psi_0 = P^T \begin{bmatrix} 0 & I \\ A + BK & -I \end{bmatrix} + \begin{bmatrix} 0 & I \\ A + BK & -I \end{bmatrix}^T P.$$

Proof. Proof is based on the following descriptor representation of (4):

$$\dot{x}(t) = y(t), \quad 0 = -y(t) + Ax(t) + BKx(t - \tau(t)), \tag{6a,b}$$

or equivalently

$$\dot{x}(t) = y(t),$$

$$0 = \begin{cases} -y(t) + Ax(t) + BKx(t - \tau(t)), & \text{if } t \in [0, h), \\ -y(t) + (A + BK)x(t) \\ \quad -BK \displaystyle\int_{t-\tau(t)}^{t} y(s)\,\mathrm{d}s, & \text{if } t \geqslant h, \end{cases} \tag{7a,b}$$

which is valid in the case of piecewise-continuous delay $\tau(t)$ for $t \geqslant 0$. Given a matrix K and initial condition $x(t) = \phi(t)$ $(t \in [-h, 0])$, where ϕ is a continuous function, $x(t)$ satisfies (4) for $t \geqslant 0$ iff it satisfies (6) (or equivalently (7)). Note that the descriptor system (6) has no impulsive solutions since in (6) $y(t)$ is multiplied by the nonsingular matrix I (Fridman, 2002).

We apply the Lyapunov–Krasovskii functional of the form

$$V(t) = V_1 + V_2, \quad V_1 = \bar{x}^\mathrm{T}(t)EP\bar{x}(t),$$

$$V_2 = \int_{-h}^{0}\int_{t+\theta}^{t} y^\mathrm{T}(s)Ry(s)\,\mathrm{d}s\,\mathrm{d}\theta, \tag{8}$$

where

$$\bar{x}(t) = col\{x(t), y(t)\}, \quad E = \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix},$$

$$P = \begin{bmatrix} P_1 & 0 \\ P_2 & P_3 \end{bmatrix}, \quad P_1 = P_1^\mathrm{T} > 0, \tag{9a–d}$$

which satisfies the following inequalities:

$$a|x(t)|^2 \leqslant V(t) \leqslant b \sup_{s \in [-h, 0]} |\bar{x}(t+s)|^2,$$

$$a > 0, \quad b > 0. \tag{10}$$

Differentiating $V(t)$ along the trajectories of (7) for $t \geqslant h$ we find (see Fridman & Shaked, 2002) that

$$\dot{V}(t) < \bar{x}^\mathrm{T}(t)\Psi_1\bar{x}(t) < -c|x(t)|^2, \quad c > 0, \ t \geqslant h, \tag{11}$$

provided that (5a,b) hold. Integrating (11) we have

$$V(t) - V(h) \leqslant -c \int_{h}^{t} |x(s)|^2\,\mathrm{d}s \tag{12}$$

and, hence, (10) yields

$$|x(t)|^2 \leqslant V(t)/a \leqslant V(h)/a \leqslant b/a \sup_{s \in [-h, 0]} |\bar{x}(h+s)|^2.$$

Since $\sup_{s \in [-h, 0]} |x(h + s)| \leqslant c_1 \sup_{s \in [-h, 0]} |\phi(s)|, c_1 > 0$ (cf. Hale & Lunel, 1993, p. 168) and thus $\dot{x}$, defined by the right-hand side of (4), satisfies $\sup_{s \in [-h, 0]} |\dot{x}(h + s)| \leqslant c_2 \sup_{s \in [-h, 0]} |\phi(s)|, \ c_2 > 0$, we obtain that

$$|x(t)|^2 \leqslant V(h)/a \leqslant c_3 \sup_{s \in [-h, 0]} |\phi(s)|, \quad c_3 > 0.$$

Hence, (4) is stable (i.e. $x(t)$ is bounded and small for small ϕ).

To prove asymptotic stability we note that $x(t)$ is uniformly continuous on $[0, \infty)$ (since $\dot{x}(t)$ defined by the right-hand side of (4) is uniformly bounded). Moreover, (12) yields that $|x(t)|^2$ is integrable on $[h, \infty)$. Then, by Barbalat's lemma, $x(t) \to 0$ for $t \to \infty$. □

Applying now the continuous state-feedback $u(t) = Kx(t)$ to (2) we obtain the system

$$\dot{x}(t) = (A + BK)x(t). \tag{13}$$

It is clear that the stability of (13) is equivalent to the stability of its equivalent descriptor form

$$\dot{x}(t) = y(t), \quad 0 = -y(t) + (A + BK)x(t), \tag{14}$$

which coincides with (6) for $h = 0$. It is well-known (Takaba, Morihira, & Katayama, 1995) that the stability of the latter system is equivalent to the condition $\Psi_0 < 0$. If there exists P of the form (9c,d) which satisfies $\Psi_0 < 0$, then for small enough $h > 0$ LMIs of Lemma 2.1 are feasible (take e.g. $Z = I_{2n}$ and $R = [0 \ K^\mathrm{T}B^\mathrm{T}]PP^\mathrm{T}[0 \ K^\mathrm{T}B^\mathrm{T}]^\mathrm{T}$). We, therefore, obtain the following result:

Corollary 2.2. *If the continuous-time state-feedback* $u(t) = Kx(t)$ *stabilizes the linear system* (2), *then the sampled-data state-feedback* (3) *with the same gain K stabilizes* (2) *for all small enough h.*

Remark 1. In the case where the matrices of the system are not exactly known, we denote $\Omega = [A \ B]$ and assume that $\Omega \in \mathscr{C}o\{\Omega_j, j = 1, \ldots, N\}$, namely, $\Omega = \sum_{j=1}^{N} f_j\Omega_j$ for some $0 \leqslant f_j \leqslant 1, \sum_{j=1}^{N} f_j = 1$, where the N vertices of the polytope are described by $\Omega_j = [A^{(j)} \ B^{(j)}]$. In order to guarantee the stability of (2) over the entire polytope one can use the result of Lemma 2.1 by applying the same matrices P_2 and P_3 and solving (5a,b) for the N vertices only.

2.3. Sampled-data stabilization

LMIs of Lemma 2.1 are bilinear in P and K. In order to obtain LMIs we use P^{-1}. It is obvious from the requirement of $0 < P_1$, and the fact that in (5) $-(P_3 + P_3^\mathrm{T})$ must be negative definite, that P is nonsingular. Define

$$P^{-1} = Q = \begin{bmatrix} Q_1 & 0 \\ Q_2 & Q_3 \end{bmatrix} \quad \text{and} \quad \Delta = diag\{Q, I\}. \tag{15a,b}$$

Applying Schur formula to the term hR in (5a), we multiply (5a,b) by Δ^T and Δ, on the left and on the right, respectively. Denoting $\bar{R} = R^{-1}$ and $\bar{Z} = Q^\mathrm{T}ZQ$ we obtain, similarly to Fridman and Shaked (2002), the following

Lemma 2.3. *The control law of* (3) *stabilizes* (2) *for all the samplings with the maximum sampling interval not greater than h and for all the system parameters that reside in the uncertainty polytope Ω, if there exist:*

$Q_1 > 0$, $Q_2^{(j)}$, $Q_3^{(j)}$, $\bar{R}$, $\bar{Z}_1^{(j)}$, $\bar{Z}_2^{(j)}$, $\bar{Z}_3^{(j)} \in \mathcal{R}^{n \times n}$, $\bar{Y} \in \mathcal{R}^{q \times n}$ *that satisfy the following nonlinear matrix inequalities*:

$$\begin{bmatrix} Q_2^{(j)}+Q_2^{(j)\mathrm{T}}+h\bar{Z}_1^{(j)} & \hat{\Xi}^{(j)} & hQ_2^{(j)\mathrm{T}} \\ * & -Q_3^{(j)}-Q_3^{(j)\mathrm{T}}+h\bar{Z}_3^{(j)} & hQ_3^{(j)\mathrm{T}} \\ * & * & -h\bar{R} \end{bmatrix} < 0,$$

$$\begin{bmatrix} Q_1\bar{R}^{-1}Q_1 & 0 & \bar{Y}^{\mathrm{T}}B^{(j)\mathrm{T}} \\ * & \bar{Z}_1^{(j)} & \bar{Z}_2^{(j)} \\ * & * & \bar{Z}_3^{(j)} \end{bmatrix} \geqslant 0, \tag{16a,b}$$

where

$$\hat{\Xi}^{(j)} = Q_3^{(j)} - Q_2^{(j)\mathrm{T}} + Q_1A^{(j)\mathrm{T}} + h\bar{Z}_2^{(j)} + \bar{Y}^{\mathrm{T}}B^{(j)\mathrm{T}},$$

$$j = 1, 2, \ldots, N. \tag{17}$$

The state-feedback gain is then given by $K = \bar{Y}Q_1^{-1}$.

For solving (16) there exist *two methods*. The *first* uses the assumption

$$\bar{R} = \varepsilon Q_1, \quad \varepsilon > 0, \tag{18}$$

and thus leads to $2N$ LMIs with tuning parameter ε:

$$\begin{bmatrix} Q_2^{(j)}+Q_2^{(j)\mathrm{T}}+h\bar{Z}_1^{(j)} & \hat{\Xi}^{(j)} & hQ_2^{(j)\mathrm{T}} \\ * & -Q_3^{(j)}-Q_3^{(j)\mathrm{T}}+h\bar{Z}_3^{(j)} & hQ_3^{(j)\mathrm{T}} \\ * & * & -\varepsilon hQ_1 \end{bmatrix} < 0,$$

$$\begin{bmatrix} \varepsilon Q_1 & 0 & \varepsilon\bar{Y}^{\mathrm{T}}B^{(j)\mathrm{T}} \\ * & \bar{Z}_1^{(j)} & \bar{Z}_2^{(j)} \\ * & * & \bar{Z}_3^{(j)} \end{bmatrix} \geqslant 0, \tag{19a,b}$$

where $\hat{\Xi}^{(j)}$ and j are given by (17).

Similarly to Corollary 2.2 we can show that if system (2) is quadratically stabilizable by a continuous-time state-feedback $u(t) = Kx(t)$, then for all small enough h the latter LMIs are feasible and the sampled-data state-feedback with the same gain stabilizes the system.

The *second* method for solving (16) is based on the iterative algorithm developed recently by Gao and Wang (2003). This method is preferable in the cases of comparatively large h, since it leads to less conservative results. However it may take more computer time due to iterative process. In the sequel we shall adopt the first method for solving (16).

Example 1. We consider (2) with the following matrices:

$$A = \begin{bmatrix} 1 & 0.5 \\ g_1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1+g_2 \\ -1 \end{bmatrix},$$

where $|g_1| \leqslant 0.1$, $|g_2| \leqslant 0.3$. It is verified by using (19) that for all uncertainties the system is stabilizable by a sampled-data state-feedback with the maximum sampling

interval $h \leqslant 0.35$. Thus, for $h = 0.35$ the resulting $K = [-2.6884 - 0.6649]$ (with $\varepsilon = 0.7$). Simulation results for the closed-loop system with the latter gain and the uniform samplings show that for the sampling period not greater than 0.35 and $g_1 = 0.1$ *sint*, $g_2 = 0.3$ *cost* the system is stable.

3. Stabilization by saturated sampled-data controller

3.1. Problem formulation

Consider the system (2) with the sampled-data control law (3) which is subject to the following amplitude constraints: $|u_i(t)| \leqslant \bar{u}_i$, $0 < \bar{u}_i$, $i = 1, \ldots, m$. We represent the state-feedback in the delayed form $u(t) = sat(Kx(t - \tau(t)), \bar{u})$, $\tau(t) = t - t_k$, $t_k \leqslant t < t_{k+1}$. Applying the latter control, law the closed-loop system obtained is

$$\dot{x}(t) = Ax(t) + B\,sat(Kx(t - \tau(t)), \bar{u}),$$

$$\tau(t) = t - t_k, \ t_k \leqslant t < t_{k+1}. \tag{20}$$

Though the closed-loop system (20) has a delay, in the case of sampled-data control the initial condition is defined in the point $t = 0$ and not in the segment $[-h, 0]$. Denote by $x(t, x(0))$ the state trajectory of (20) with the initial condition $x(0) \in R^n$. Then the domain of attraction of the origin of the closed-loop system (20) is the set $\mathscr{A} = \{x(0) \in R^n : \lim_{t \to \infty} x(t, x(0)) = 0\}$. We seek conditions for the existence of a gain matrix K which leads to a stable closed loop. Having met these conditions, a simple procedure for finding the gain K should be presented. Moreover, we obtain an estimate $\mathscr{X}_\beta \subset \mathscr{A}$ on the domain of attraction, where

$$\mathscr{X}_\beta = \{x(0) \in R^n : x^{\mathrm{T}}(0)P_1x(0) \leqslant \beta^{-1}\}, \tag{21}$$

and where $\beta > 0$ is a scalar and $P_1 > 0$ is an $n \times n$ matrix.

Reducing the original problem to the problem with input delay, we solve it by modifying derivations of Fridman, Pila, and Shaked (2003), where the case of state delay was considered.

3.2. A linear system representation with polytopic type uncertainty

For the estimation of $\mathscr{A}$ we can restrict ourself to the following initial functions $\phi(s)$, $s \in [-h, 0]$:

$$\phi(0) = x(0), \quad \phi(s) = 0, \qquad s \in [-h, 0), \tag{22}$$

because the initial condition for (20) is defined in the point $t = 0$. Denoting the ith row by k_i, we define the polyhedron

$$\mathscr{L}(K, \bar{u}) = \{x \in \mathscr{R}^n : |k_ix| \leqslant \bar{u}_i, \ i = 1, \ldots, m\}.$$

If the control and the disturbance are such that $x \in \mathscr{L}(K, \bar{u})$ then system (20) admits the linear representation. Following Cao et al. (2002), we denote the set of all diagonal matrices in $\mathscr{R}^{m \times m}$ with diagonal elements that are either 1 or 0 by Υ, then there are 2^m elements D_i in Υ, and for every $i = 1, \ldots, 2^m$ $D_i^- \triangleq I_m - D_i$ is also an element in Υ.

Lemma 3.1 (Cao et al., 2002). *Given K and H in $\mathscr{R}^{m\times n}$. Then*

$$sat(Kx(t),\bar{u})\in\mathscr{C}o\{D_iKx+D_i^-Hx,\ i=1,\ldots,2^m\}$$

for all $x\in\mathscr{R}^n$ that satisfy $|h_ix|\leqslant\bar{u}_i,\ i=1,\ldots,2^m$.

The following is obtained from Lemma 3.1.

Lemma 3.2. *Given $\beta>0$, assume that there exists H in $\mathscr{R}^{m\times n}$ such that $|h_ix|\leqslant\bar{u}_i$ for all $x(t)\in\mathscr{X}_\beta$. Then for $x(t)\in\mathscr{X}_\beta$ the system* (20) *admits the following representation.*

$$\dot{x}(t)=Ax(t)+\sum_{j=1}^{2^m}\lambda_j(t)A_jx(t-\tau(t)), \tag{23}$$

where

$$A_j=B(D_jK+D_j^-H)\ j=1,\ldots,2^m,$$

$$\sum_{j=1}^{2^m}\lambda_j(t)=1,0\leqslant\lambda_j(t),\quad\forall 0<t. \tag{24}$$

We denote

$$\Omega_\alpha=\sum_{j=1}^{2^m}\lambda_j\Omega_j\quad\text{for all}\quad 0\leqslant\lambda_j\leqslant 1,\ \sum_{j=1}^{2^m}\lambda_j=1, \tag{25}$$

where the vertices of the polytope are described by $\Omega_j=[A_j],\ j=1,\ldots,2^m$. The problem becomes one of finding $\mathscr{X}_\beta$ and a corresponding H such that $|h_ix|\leqslant\bar{u}_i,\ i=1,\ldots,2^m$ for all $x\in\mathscr{X}_\beta$ and that the state of the system

$$\dot{x}(t)=Ax(t)+A_jx(t-\tau(t)),$$

$$\tau(t)=t-t_k,\ t_k\leqslant t<t_{k+1}, \tag{26}$$

remains in $\mathscr{X}_\beta$.

3.3. Regional stabilization

By using the first method for solving the stabilization matrix inequalities (with tuning parameter ε), we obtain the following result:

Theorem 3.3. *Consider system* (2) *with the constrained sampled-data control law* (3). *The closed-loop system* (20) *is stable with $\mathscr{X}_\beta$ inside the domain of attraction for all the samplings with the maximum sampling interval not greater than h, if there exist $0<Q_1,Q_2^{(j)},Q_3^{(j)},\bar{Z}_1^{(j)},\bar{Z}_2^{(j)},\bar{Z}_3^{(j)}\in\mathscr{R}^{n\times n}$, $\bar{Y},G\in\mathscr{R}^{m\times n}$, $\varepsilon>0$ and $\beta>0$ that satisfy LMI* (19a) *for $j=1,\ldots,2^m$, where*

$$\Sigma_j=Q_3^{(j)}-Q_2^{T(j)}+Q_1A^T+(\bar{Y}^TD_j+G^TD_j^-)B^T$$

$$+h\bar{Z}_2^{(j)},\quad B^{(j)}=B \tag{27}$$

and

$$\begin{bmatrix}\varepsilon Q_1 & 0 & \varepsilon(\bar{Y}^TD_j+G^TD_j^-)B^T\\ * & \bar{Z}_1^{(j)} & \bar{Z}_2^{(j)}\\ * & * & \bar{Z}_3^{(j)}\end{bmatrix}\geqslant 0,\ j=1,\ldots,2^m,$$

$$\begin{bmatrix}\beta & g_i\\ * & \bar{u}_i^2Q_1\end{bmatrix}\geqslant 0,\ i=1,\ldots,m. \tag{28a,b}$$

The feedback gain matrix which stabilizes the system is given by $K=\bar{Y}Q_1^{-1}$.

Proof. For V given by (8) conditions are sought to ensure that $\dot{V}<0$ for any $x(t)\in\mathscr{X}_\beta$. As in Fridman et al. (2003), the inequalities (28b) guarantee that $|h_ix|\leqslant\bar{u}_i,\forall x\in\mathscr{X}_\beta,\ i=1,\ldots,m$, where $g_i\triangleq h_iQ_1,\ i=1,\ldots,m$ and $Q_1\triangleq P_1^{-1}$, and the polytopic system representation of (26) is thus valid. Moreover, (19a,b) guarantee that $\dot{V}<0$.

From $\dot{V}<0$ it follows that $V(t)<V(0)$ and therefore for the initial conditions of the form (22)

$$x^T(t)P_1x(t)\leqslant V(t)<V(0)=x^T(0)P_1x(0)\leqslant\beta^{-1}. \tag{29}$$

Then for all initial values $x(0)\in\mathscr{X}_\beta$, the trajectories of $x(t)$ remain within $\mathscr{X}_\beta$, and the polytopic system representation (26) is valid. Hence $x(t)$ is a trajectory of the linear system (26) and $\dot{V}<0$ along the trajectories of the latter system which implies that $\lim_{t\to\infty}x(t)=0$. $\square$

Example 2. We consider (2) with the following matrices (Cao et al., 2002), where $h=0$):

$$A=\begin{bmatrix}1.1 & -0.6\\ 0.5 & -1\end{bmatrix},\quad B_1=\begin{bmatrix}1\\ 1\end{bmatrix}$$

and where $\bar{u}=5$. Applying Theorem 3.3 a stabilizing gain was obtained for all samplings with the maximum sampling interval $h\leqslant 0.75$. In order to 'enlarge' the volume of the ellipse we minimized the value of β (to improve the result we also added the inequality $Q_1>\alpha I$ and chose such $\alpha>0$ that enlarged the resulting ellipse). The ellipse volume increases when h decreases. For, say, $h=0.75$ we obtain $K=[-1.6964\ 0.5231]$ (with $\varepsilon=0.325,\ \beta=0.1261,P_1=\begin{bmatrix}0.9132 & -0.2816\\ -0.2816 & 0.0868\end{bmatrix},\alpha=1$) and we show (see Fig. 1) that a trajectory starting on the periphery of the ellipse (for the case of the uniform sampling with the sampling period $t_{k+1}-t_k=0.75$) never leaves this ellipse and converges to the origin, while a trajectory starting not far from the ellipse remains outside the ellipse.

4. Conclusions

A new approach to robust sampled-data stabilization of linear continuous-time systems is introduced. This approach is based on the continuous-time model with time-varying
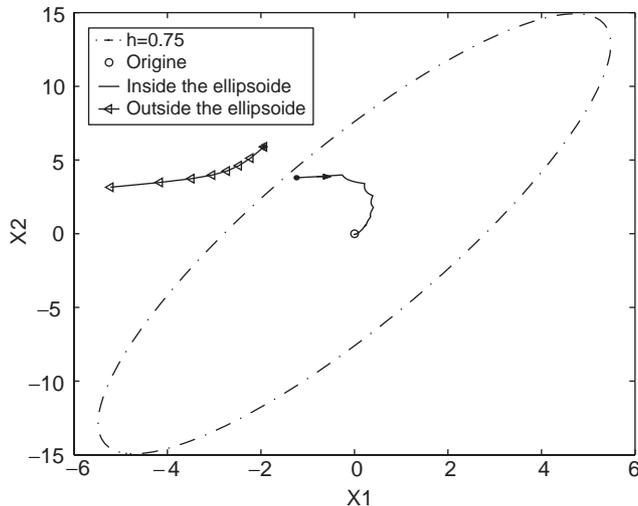
Fig. 1. Trajectories and estimate of the domain of attraction for $h = 0.75$.

input delay. Under assumption that the maximum sampling interval is not greater than $h > 0$, the h-dependent sufficient LMIs conditions are derived for stabilization of systems with polytopic type uncertainty and for regional stabilization of systems with sampled-data saturated state-feedback.

The derived conditions are conservative since they guarantee stabilization for all sampling intervals not greater than h and due to assumption (18). However, these conditions are simple. The problem of reducing their conservatism is currently under study. The input delay approach may be applied to a wide spectrum of robust sampled-data control problems.

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