[15] Q.-C. Zhong and L. Mirkin, "Control of integral processes with dead-time. Part 2: Quantitative analysis," *IEE Proc. Control Theory Appl.*, vol. 149, pp. 291–296, Apr. 2002.

[16] E. Nobuyama, S. Shin, and T. Kitamori, "Deadbeat control of continous-time systems: MIMO case," in *Proc. 35th IEEE Conf. Decision Control*, Kobe, Japan, 1996, pp. 2110–2113.

[17] K. Watanabe, E. Nobuyama, and A. Kojima, "Recent advances in control of time delay systems—A tutorial review," in *Proc. 35th IEEE Conf. Decision Control*, Kobe, Japan, 1996, pp. 2083–2089.

[18] K. Zhou and J. C. Doyle, *Essentials of Robust Control*. Upper Saddle River, NJ: Prentice-Hall, 1997.

[19] Z. J. Palmor, "Time-delay compensation—Smith predictor and its modifications," in *The Control Handbook*, S. Levine, Ed. Boca Raton, FL: CRC, 1996, pp. 224–237.

[20] H. Dym, T. Georgiou, and M. C. Smith, "Explicit formulas for optimally robust controllers for delay systems," *IEEE Trans. Automat. Contr.*, vol. 40, pp. 656–669, Apr. 1995.

[21] L. Y. Pao, "Multi-input shaping design for vibration reduction," *Automatica*, vol. 35, no. 1, pp. 81–89, 1999.

[22] Q.-C. Zhong, J. Y. Xie, and Q. Jia, "Time delay filter-based deadbeat control of process with dead time," *Ind. Eng. Chem. Res.*, vol. 39, no. 6, pp. 2024–2028, 2000.

[23] M. Bodson, "Adaptive algorithm for the tuning of two input shaping methods," *Automatica*, vol. 34, pp. 771–776, 1998.

[24] H. X. Li and P. P. J. Van Den Bosch, "A robust disturbance-based control and its application," *Int. J. Control*, vol. 58, no. 3, pp. 537–554, 1993.

[25] O. J. M. Smith, *Feedback Control Systems*. New York: McGraw-Hill, 1958.

[26] Q.-C. Zhong, "Robust stability analysis of simple systems controlled over communication networks," *Automatica*, 2003, submitted for publication.

Robust H_∞ Filtering of Linear Systems With Time-Varying Delay

E. Fridman, U. Shaked, and L. Xie

***Abstract*—A robust delay-dependent H_∞ filtering design is proposed for linear continuous systems with parameter uncertainty and time-varying delay. The resulting filter is of the general linear observer type and it guarantees that the induced L_2-norm of the system, relating the exogenous signals to the estimation error, is less than a prescribed level for all possible parameters that reside in a given polytope. Our design is based on the application of the descriptor model transformation and Park's inequality for the bounding of cross terms and is expected to be the least conservative as compared to existing design methods. A numerical example indeed demonstrates this advantage of the new filtering scheme.**

***Index Terms*—Bounded-real lemma, delay-dependent stability, linear matrix inequalities (LMIs), robust H_∞ filtering, time-delay systems.**

I. Introduction

H_∞ estimation has been attracting much interest in the past decades [1], [2]. One of its main advantages is the fact that it is insensitive to the exact knowledge of the statistics of the noise signals. This estimation procedure ensures that the $\mathcal{L}_2$-induced gain from the noise signals to the estimation error will be less than a prescribed level, where the noise signals are arbitrary energy-bounded signals. Several approaches have been proposed to solve the H_∞ filtering problem ([1], [2], [4]).

The potential of H_∞ estimation lies far beyond its insensitivity to the noise statistics. It had been recognized in [3] that the H_∞ filtering scheme is also less sensitive than its H_2 counterpart to uncertainty in system parameters. The H_∞ filtering approach was adopted for systems with norm-bounded type of parameter uncertainty using a Riccati equation approach in [5] and a convex optimization approach in [6]. Recently, the H_∞ filtering for systems with a polytopic type of parameter uncertainty has been addressed in [7] and [8] where sufficient conditions in terms of linear matrix inequalities (LMIs) are given.

The aforementioned works consider systems without time delays. The existence of the delays is frequently encountered in many dynamic systems [9] and their presence must be taken into account in a realistic design. Moreover, stability and noise attenuation level guaranteed by a H_∞ filter that was designed without considering time delays can collapse if the system actually possesses such delays. In [10] a H_∞ filter design for precisely known systems with a single time delayed measurement was introduced. The H_∞ observer design for precisely known systems with state delays was considered in [11], where a sufficient condition based on an algebraic Riccati equation was derived. Robust H_∞ filtering of uncertain systems with state delays has been considered in [12] and [13] where both delay independent and delay dependent sufficient conditions have been given.

Unfortunately, all the proposed methods for robust filtering of processes with delay are conservative. This conservatism stems from two main sources. The first is the fact that a system with time-delay is, in fact, infinite dimensional. Any attempt to analyze it via finite-dimensional models and criteria must therefore entail an overdesign. The second source of conservatism comes from the uncertainty. The treatment of norm-bounded uncertainties as an additional disturbance [5] or the polytopic uncertainty via a single Lyapunov function [12] leads to conservative results.

While the constraint of a single Lyapunov function has been somehow relaxed by considering parameter-dependent Lyapunov functions [14], the main source of conservatism that is caused by the distributed nature of the delay has not been successfully tackled.

Recently, a new approach to H_∞ filtering has been introduced [15]. This approach applies a Lyapunov–Krasovskii functional [9] and is based on representing the system by a descriptor type model [16] and deriving a bounded-real lemma (BRL) [17] for the corresponding adjoint system. The new BRL was found to be very efficient and it considerably reduced the achievable attenuation level as compared to other results reported in the literature. By assuming a Leunberger-type estimator, the new BRL was applied to the resulting estimation error system and provided a much less conservative filtering estimate in [15]. In spite of the advantage of the new filter design, it still entails a significant amount of conservatism stemming from the overbounding of mixed terms in the proof of the BRL in [15].

A new over-bounding technique has recently been proposed that produces tighter bounds [18]. In [19], this technique was applied to reduce the over-design entailed in the approach of [15]. It cannot be used however when uncertainty is encountered because it is based on using a Leunberger type filter to cancel the effect of the system states on the dynamics of the estimation error.

In this note, we solve the robust H_∞ filtering problem for systems with time-varying multiple delays and polytopic type uncertainties. A general full order filter is sought that guarantees the required estimation accuracy over the entire uncertainty polytope. We obtain sufficient

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conditions for the robust filtering for both the delay dependent and the delay independent cases. It turns out that the condition for the latter case is a special case of that for the former. An example, taken from [13], is solved. The advantages of our results are clearly demonstrated there.

In comparison with the results in [15] and [17], this note deals with filters of general structure that allows the consideration of parameter uncertainty. It applies the new bounding technique of [18] which considerably reduces the overdesign entailed in robust estimation and it deals with time varying delays.

Notation: Throughout the note the superscript T stands for matrix transposition, $\mathcal{R}^n$ denotes the n dimensional Euclidean space, $\mathcal{R}^{n\times m}$ is the set of all $n \times m$ real matrices and the notation $P > 0$, for $P \in \mathcal{R}^{n\times n}$ means that P is symmetric and positive definite. The space of functions in $\mathcal{R}^q$ that are square integrable over $[0\ \infty)$ is denoted by $\mathcal{L}_2^q[0,\ \infty)$.

II. Problem Formulation

Consider the following system:

$$\dot{x}(t) = A_0 x(t) + \sum_{i=1}^{2} A_i x(t-\tau_i) + Bw(t) \tag{1a}$$

$$z(t) = Lx(t) + Dw(t) \tag{1b}$$

where $x(t) \in \mathcal{R}^n$ is the system state vector with $x(t) = 0$ for any $t \le 0$, $w(t) \in \mathcal{L}_2^q[0,\ \infty)$ is the exogenous disturbance signal and $z(t) \in \mathcal{R}^p$ is the signal to be estimated.

The measurement is given by

$$y(t) = \mathrm{col}\{C_0 x(t), C_1 x(t-\tau_1), C_2 x(t-\tau_2)\} + D_{21} w(t) \tag{2}$$

where $y(t) \in \mathcal{R}^r$. The time varying delays $\tau_i(t) > 0$ satisfy $\tau_i(t) \le h_i$ over $[0\ \infty)$ where h_i, $i = 1, 2$ are known. The matrices A_i, C_i, $i = 0, 1, 2$, B, D, D_{21}, and L are constant matrices of appropriate dimensions ($C_i \in \mathcal{R}^{r_i \times n}$, $r_0 + r_1 + r_2 = r$). The exact values of these matrices are not precisely known. Denoting

$$\Omega = [\, A_0 \quad A_1 \quad A_2 \quad C_0 \quad C_1 \quad C_2 \quad B \quad D \quad D_{21} \quad L \,]$$

we assume that $\Omega \in \mathcal{C}o\{\Omega_j, j = 1, \ldots N\}$, namely

$$\Omega = \sum_{j=1}^{N} f_j \Omega_j \quad \text{for some} \quad 0 \le f_j \le 1,\ \sum_{j=1}^{N} f_j = 1 \tag{3}$$

where the N vertices of the polytope are described by

$$\Omega_j = [\, A_0^{(j)}\ A_1^{(j)}\ A_2^{(j)}\ C_0^{(j)}\ C_1^{(j)}\ C_2^{(j)}\ B^{(j)}\ D^{(j)}\ D_{21}^{(j)}\ L^{(j)} \,].$$

Remark 1 : In (2), we allowed for the general case of delayed measurements. It will be shown below that a delay in the measurement, if exists, will result in a nonlinear optimization problem. It will also be clarified below how to cope with delayed measurements, indirectly, in a way that allows a solution via LMIs. For simplicity, only two delays are considered in this note. The results obtained can, however, be easily generalized to any finite number of delays.

We seek a filter of the form

$$\begin{aligned} \dot{\hat{x}}(t) &= A_f \hat{x}(t) + B_f y(t) \quad \hat{x}(0) = 0 \\ \hat{z}(t) &= C_f \hat{x}(t) + [\, D_f \quad 0 \quad 0 \,] y(t) \end{aligned} \tag{4}$$

where $D_f \in \mathcal{R}^{p\times r_0}$, which ensures, for a prescribed value of γ, that the performance index

$$J(w) = \int_0^\infty \left(\tilde{z}^T \tilde{z} - \gamma^2 w^T w \right) dt \tag{5}$$

where $\tilde{z}(t) \triangleq z(t) - \hat{z}(t)$, is negative $\forall 0 \ne w(t) \in \mathcal{L}_2^q[0,\ \infty)$ and for all the possible plant parameters in the above uncertainty polytope.

We will now treat two cases of time-varying delays.

Case 1: $\tau_i(t)$, $i = 1, 2$ are differentiable functions, satisfying, for all $t \ge 0$ and for given scalars $0 \le d_i$, $\dot{\tau}_i(t) \le d_i$, $i = 1, 2$.

Case 2: $\tau_i(t)$, $i = 1, 2$ are continuous for all $t \ge 0$ (satisfying the above bound of h_i). In this case, very fast changes in the time delay are allowed.

III. H_∞ Filtering for Case 1

A. Delay-Dependent Filtering

Defining $e = x - \hat{x}$ and $\xi = \mathrm{col}\{x, e\}$ we derive from (1), (4), and (2) the following augmented model, where:

$$\dot{\xi}(t) = \bar{A}_0 \xi(t) + \sum_1^2 \bar{A}_i \xi(t-\tau_i) + \bar{B} w(t) \tag{6a}$$

$$\tilde{z}(t) = \bar{L}\xi(t) + \bar{D} w(t) \tag{6b}$$

where

$$\bar{A}_0 = \begin{bmatrix} A_0 & 0 \\ A_0 - A_f - B_f \begin{bmatrix} C_0 \\ 0 \\ 0 \end{bmatrix} & A_f \end{bmatrix} \tag{7a}$$

$$\bar{A}_1 = \begin{bmatrix} A_1 & 0 \\ A_1 - B_f \begin{bmatrix} 0 \\ C_1 \\ 0 \end{bmatrix} & 0 \end{bmatrix} \tag{7b}$$

$$\bar{A}_2 = \begin{bmatrix} A_2 & 0 \\ A_2 - B_f \begin{bmatrix} 0 \\ 0 \\ C_2 \end{bmatrix} & 0 \end{bmatrix} \tag{7c}$$

$$\bar{B} = \begin{bmatrix} B \\ B - B_f D_{21} \end{bmatrix} \tag{7d}$$

$$\bar{L} = [\, L - C_f - D_f C_0 \quad C_f \,] \tag{7e}$$

$$\bar{D} = D - [\, D_f \quad 0 \quad 0 \,] D_{21}. \tag{7f}$$

The filtering problem thus becomes one of finding the filter parameters such that the induced L_2-norm of the system (6) will be less than the prescribed γ for all the points in the uncertainty polytope and for all the delays that correspond to case 1. To ensure the induced norm we have to use the appropriate BRL. Unfortunately, the standard BRL for systems with time delays (see for example [17]) is not suitable for solving the filtering problem. Extending the arguments of [15] it can be shown that the $\mathcal{L}_2$-induced norms of the system described by (6) and the following system are equal:

$$\dot{\zeta}(\bar{t}) = \bar{A}_0^T \zeta(\bar{t}) + \sum_{i=1}^{2} \bar{A}_i^T \zeta(\bar{t}-\tau_i) + \bar{L}^T \tilde{z}(\bar{t}) \tag{8a}$$

$$\zeta(\bar{t}) = 0\ \forall \bar{t} \le 0,\ \tilde{w}(\bar{t}) = \bar{B}^T \zeta(\bar{t}) + \bar{D}^T \tilde{z}(\bar{t}) \tag{8b}$$

where $\zeta(\bar{t}) \in \mathcal{R}^n$, $\tilde{z}(\bar{t}) \in \mathcal{R}^p$, and $\tilde{w}(\bar{t}) \in \mathcal{R}^q$. Note that the latter system represents the forward adjoint of (6) (as defined in [20]).

An equivalent descriptor form representation of (8a) is given by [16]

$$\begin{aligned} \dot{\zeta}(t) &= \eta(t) \\ 0 &= -\eta(t) + \sum_{i=0}^{2} \bar{A}_i^T \zeta(t) \\ &\quad - \sum_{i=1}^{2} \bar{A}_i^T \int_{t-\tau_i}^{t} \eta(s) ds + \bar{L}^T \tilde{z}(t) \end{aligned} \tag{9}$$

In association with (9), we adopt the following Lyapunov–Krasovskii functional:

$$V(t) = \left[\zeta^T(t)\,\eta^T(t)\right] EP \begin{bmatrix}\zeta(t)\\ \eta(t)\end{bmatrix} + \sum_{i=1}^{2}\int_{t-\tau_i}^{t}\zeta^T(\bar t)S_i^{-1}\zeta(\bar t)d\bar t + \sum_{i=1}^{2}\int_{-h_i}^{0}\int_{t+\theta}^{t}\eta^T(s)\bar A_i R_i^{-1}\bar A_i^T\eta(s)dsd\theta \qquad (10)$$

where

$$E = \begin{bmatrix} I_n & 0\\ 0 & 0\end{bmatrix} \qquad (11a)$$

$$P = \begin{bmatrix} P_1 & 0\\ P_2 & P_3\end{bmatrix} \quad P_1 > 0, \qquad S_i > 0,\ R_i > 0,\ i = 1, 2. \qquad (11b)$$

The first term of (10) corresponds to the descriptor system ([17]) while the second and the third terms—to the delay-dependent conditions with respect to the distributed delays (with respect to τ_i). The third term is included in order to compensate the term that emerges when Park's inequality [18] is used to obtain the required BRL.

Using this, along the lines of [20], an expression for $\dot V(t)$ is first derived and a term in the resulting expression which mixes ζ and η is then bounded using inequality of [18]. The following BRL is thus obtained.

Lemma 1: Consider the system of (6). For a prescribed $\gamma > 0$ and for given $A_f,\ B_f,\ C_f$ and D_f, the cost function (5) achieves $J(w) < 0$ for all nonzero $w \in \mathcal{L}_2^q[0,\ \infty)$ and for all the parameters that belong to the uncertainty polytope, if there exist

$$P = \begin{bmatrix} P_1 & 0\\ P_2 & P_3\end{bmatrix} \quad P_1 = P_1^T > 0, P_1 \in \mathcal{R}^{2n\times 2n}$$

$$W_i = \begin{bmatrix} -P_1 & 0\\ W_{i1} & W_{i2}\end{bmatrix},\ W_{i2} \in \mathcal{R}^{2n\times 2n},\ i = 1,\ 2 \qquad (12)$$

and $2n \times 2n$-matrices S_i and R_i, $i = 1, 2$ that satisfy the inequality over the uncertainty polytope Ω of (3), as shown in (13) at the bottom of the page, where

$$\Psi \triangleq P^T \begin{bmatrix} 0 & I_{2n}\\ \sum_{i=1}^{2}\bar A_i^T & -I_{2n}\end{bmatrix} + \begin{bmatrix} 0 & \sum_{i=1}^{2}\bar A_i\\ I_{2n} & -I_{2n}\end{bmatrix} P + \begin{bmatrix} \sum_{i=1}^{2} S_i^{-1} & 0\\ 0 & \sum_{i=1}^{2} h_i\bar A_i R_i^{-1}\bar A_i^T\end{bmatrix} + \sum_{i=1}^{2} W_i^T \begin{bmatrix} 0 & 0\\ \bar A_i^T & 0\end{bmatrix} + \sum_{i=1}^{2}\begin{bmatrix} 0 & \bar A_i\\ 0 & 0\end{bmatrix} W_i \quad \Phi_i^T = [0 \quad I_{2n}]\,[W_i + P].$$

In order to obtain a convex optimization problem, we restrict ourselves to the case of

$$[W_{i1} \quad W_{i2}] = \varepsilon_i\,[0 \quad I_{2n}]\,P \qquad (14)$$

where $\varepsilon_i = \mathrm{diag}\{\bar\varepsilon_i,\ \tilde\varepsilon_i\}$ is a diagonal matrix, $\bar\varepsilon_i, \tilde\varepsilon_i \in \mathcal{R}^{n\times n}$, $i = 1, 2$. For $\varepsilon_i = -I_{2n}$ (13) yields the delay-independent condition since in this case $\Phi_1 = \Phi_2 = 0$. For $\varepsilon_i = 0$ we have $-W_i^T[0\ \bar A_i]^T = 0$ and therefore (13) implies the delay-dependent condition which does not depend on d_i.

Observe from (13) that $-(P_3 + P_3^T)$ must be negative definite as $\Psi < 0$, which in conjunction with the requirement of $0 < P_1$, implies that P is nonsingular. Defining

$$P^{-1} = Q = \begin{bmatrix} Q_1 & 0\\ Q_2 & Q_3\end{bmatrix} \quad \Delta = \mathrm{diag}\left\{Q, I_{8n+p+q}\right\} \qquad (15a\text{-}b)$$

we multiply (13) by Δ^T and Δ, on the left and on the right, respectively. Applying Schur formula to the quadratic term in Q we obtain the following.

Lemma 2: Consider the system of (6). For a prescribed $\gamma > 0$ and for given $A_f,\ B_f,\ C_f$ and D_f, the cost function (5) achieves $J(w) < 0$ for all nonzero $w \in \mathcal{L}_2^q[0,\ \infty)$ and for all the parameters that belong to the uncertainty polytope Ω, if for some diagonal matrices ε_1 and $\varepsilon_2 \in \mathcal{R}^{2n\times 2n}$, there exist $Q_1 > 0,\ S_1,\ S_2,\ Q_2,\ Q_3,\ R_1,\ R_2 \in \mathcal{R}^{2n\times 2n}$ that satisfy the inequality over the uncertainty polytope Ω of (3), shown in (16) at the bottom of the next page, where

$$\Xi = Q_3 - Q_2^T + Q_1\left(\sum_{i=0}^{2}\bar A_i + \sum_{i=1}^{2}\bar A_i\varepsilon_i\right).$$

The inequality of the last lemma is affine in Q_1, Q_{j+1}, S_j, and R_j, $j = 1, 2$. Thus, for given γ, ε_1 and ε_2 the estimation performance can be verified, for a given filter of the form of (4), over the whole uncertainty polytope, if the matrices $Q_1 > 0$, Q_{j+1}, S_j and R_j, $j = 1, 2$ simultaneously satisfy the N inequalities of the type of (16) where each corresponds to a vertex, Ω_j, of the uncertainty polytope Ω and where the parameters of Ω_j are substituted to construct in (16) the corresponding elements of (7).

We do not know, however, what are the optimal choice of the filter parameters, and we will thus have to solve the inequality of (16) for $Q_1 > 0$, Q_{j+1}, S_j, R_j, $j = 1, 2$, $A_f,\ B_f,\ C_f$, and D_f, simultaneously for the N vertices of Ω.

In order to linearize the resulting optimization problem we look for Q_1 that has the following block diagonal structure:

$$Q_1 = \mathrm{diag}\left\{Q_{11}, Q_{12}\right\}. \qquad (17)$$

This restriction is required to entangle the bilinear terms that appear in (16). It introduces an additional conservatism to the solution proposed, but one should bear in mind that in the standard H_∞ filtering problem (without uncertainty), if one uses the Luenberger observer and solves

$$\begin{bmatrix} \Psi & P^T\begin{bmatrix}0\\ \bar L^T\end{bmatrix} & h_1\Phi_1 R_1 & h_2\Phi_2 R_2 & -W_1^T\begin{bmatrix}0\\ \bar A_1^T\end{bmatrix}S_1 & -W_2^T\begin{bmatrix}0\\ \bar A_2^T\end{bmatrix}S_2 & \begin{bmatrix}\bar B\\ 0\end{bmatrix}\\ * & -\gamma^2 I_q & 0 & 0 & 0 & 0 & \bar D\\ * & * & -h_1R_1 & 0 & 0 & 0 & 0\\ * & * & * & -h_2R_2 & 0 & 0 & 0\\ * & * & * & * & -(1-d_1)S_1 & 0 & 0\\ * & * & * & * & * & -(1-d_2)S_2 & 0\\ * & * & * & * & * & * & -I_p \end{bmatrix} < 0 \qquad (13)$$

the estimation problem for the augmented system for $\mathrm{col}\{x, e\}$, the solution of corresponding Riccati equation indeed possesses the structure of (17).

Applying (17) it is readily seen that if $\bar{A}_1$ or $\bar{A}_2$ possesses in (16) a term with B_f the resulting inequality will still be nonlinear. We, therefore, assume that in (2) $r = r_0$, namely, that the output does not include any delayed measurements. We shall see later how to cope with the more general case of delayed measurements. For this case of instantaneous measurement, by considering (17) and (7), the following result follows from (16).

Theorem 1: Consider the system of (1) and (2) with $r = r_0$. For a prescribed $\gamma > 0$ the cost function (5) achieves $J(w) < 0$ for all nonzero $w \in \mathcal{L}_2^q[0, \infty)$ and for all the parameters that belong to the uncertainty polytope Ω of (3), if for some diagonal matrices $\bar{\varepsilon}_1, \tilde{\varepsilon}_1, \bar{\varepsilon}_2$ and $\tilde{\varepsilon}_2 \in \mathcal{R}^{n\times n}$, there exist $S_1, S_2, Q_2, Q_3, R_1, R_2 \in \mathcal{R}^{2n\times 2n}$, $0 < Q_{11}, 0 < Q_{12}$ and $Z_a \in \mathcal{R}^{n\times n}$, $Z_b \in \mathcal{R}^{n\times r}$, $C_f \in \mathcal{R}^{p\times n}$ and $D_f \in \mathcal{R}^{p\times r}$ that satisfy the set of LMIs for $j = 1, 2, \ldots, N$ as shown in (18) at the bottom of the next page, where $\phi_i = \mathrm{diag}\{\bar{\varepsilon}_i + I_n, \tilde{\varepsilon}_i + I_n\}$, $i = 1, 2$.

If a solution to this set of LMIs exists then the filter that guarantees the estimation error level of γ is given by (4) with

$$A_f = Q_{12}^{-1} Z_a,\ B_f = Q_{12}^{-1} Z_b,\ C_f\ and\ D_f.$$

The previous theorem explicitly requires measurements without any delay. For the case where delay is encountered in the measurements ($r > r_0$ in (2)) an additional component can be placed in series with the delayed components of y. The state-space model of this component is given by

$$\dot{\eta}(t) = -\rho I_{r-r_0}\eta(t) + [\,0 \quad \rho I_{r-r_0}\,]\, y(t) \tag{19}$$

for $1 \ll \rho$. Denoting the augmented state vector by $\bar{\xi}(t) = \mathrm{col}\{x(t), \eta(t)\}$, the augmented system is then described by

$$\dot{\bar{\xi}}(t) = \sum_{i=0}^{2} \tilde{A}_i \bar{\xi}(t - h_i) + \tilde{B} w \tag{20}$$

where

$$\tilde{A}_0 = \begin{bmatrix} A_0 & 0 \\ 0 & -\rho I_{r-r_0} \end{bmatrix} \quad \tilde{A}_1 = \begin{bmatrix} A_1 & 0 \\ \begin{bmatrix} 0 \\ \rho C_1 \\ 0 \end{bmatrix} & 0 \end{bmatrix}$$

$$\tilde{A}_2 = \begin{bmatrix} A_2 & 0 \\ \begin{bmatrix} 0 \\ \rho C_2 \end{bmatrix} & 0 \end{bmatrix} \quad \tilde{B} = \begin{bmatrix} B \\ \rho\,[\,0 \quad I_{r-r_0}\,]\, D_{21} \end{bmatrix}.$$

The results of Theorem 1 can readily be applied to the resulting system of (20). If a solution is found to the corresponding set of LMIs, for large enough ρ, it will provide a good approximation to the required filter (see the arguments in [15]).

B. Delay-Independent Rate-Dependent Filtering

A filtering solution that is valid for all positive values of $h_i, i = 1, 2$ is readily obtained from the theory of the last subsection. Choosing in (14) $\varepsilon_i = -I_{2n}$ and $R_i = h_i^{-1}\rho I_{2n}$, $0 < \rho \to \infty$, $i = 1, 2$, the delay-independent version of Lemma 2 becomes the following.

Lemma 3: Consider the system of (6). For a prescribed $\gamma > 0$ and for given rates of delays d_1 and d_2 and filter parameters $A_f,\ B_f,\ C_f$ and D_f, the cost function (5) achieves $J(w) < 0$, independently of the delay lengths, for all nonzero $w \in \mathcal{L}_2^q[0,\ \infty)$ and for all the parameters that belong to the uncertainty polytope Ω, if there exist $Q_1 > 0, S_1, S_2, Q_2, Q_3, R_1, R_2 \in \mathcal{R}^{2n\times 2n}$ that satisfy the inequality over the uncertainty polytope Ω of (3), as shown in (21) at the bottom of the next page. Based on this lemma, the corresponding equivalent to Theorem 1 is as follows.

$$\left[\begin{array}{cccccccccccc}
Q_2 + Q_2^T & \Xi & 0 & 0 & 0 & 0 & 0 & Q_1 & Q_1 & Q_1\bar{B} & h_1 Q_2^T \bar{A}_1 & h_2 Q_2^T \bar{A}_2 \\
* & -Q_3 - Q_3^T & \bar{L}^T & h_1(\varepsilon_1 + I_{2n}) R_1 & h_2(\varepsilon_2 + I_{2n}) R_2 & \varepsilon_1 \bar{A}_1^T S_1 & \varepsilon_2 \bar{A}_2^T S_2 & 0 & 0 & 0 & h_1 Q_3^T \bar{A}_1 & h_2 Q_3^T \bar{A}_2 \\
* & * & -\gamma^2 I_q & 0 & 0 & 0 & 0 & 0 & 0 & \bar{D} & 0 & 0 \\
* & * & * & -h_1 R_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & -h_2 R_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & -(1-d_1) S_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & -(1-d_2) S_2 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & -S_1 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & -S_2 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & -I_q & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & -h_1 R_1 & 0 \\
* & * & * & * & * & * & * & * & * & * & * & -h_2 R_2
\end{array}\right] < 0 \tag{16}$$

Theorem 2: Consider the system of (1) and (2) with $r = r_0$. For a prescribed $\gamma > 0$ and given delay rates d_1 and d_2, the cost function (5) achieves $J(w) < 0$ for all nonzero $w \in \mathcal{L}_2^q[0, \infty)$, for all the parameters that belong to the uncertainty polytope Ω of (3) and for all delay lengths, if there exist $S_1, S_2, Q_2, Q_3 \in \mathcal{R}^{2n \times 2n}$, $0 < Q_{11}$, $0 < Q_{12}$ and $Z_a \in \mathcal{R}^{n \times n}$, $Z_b \in \mathcal{R}^{n \times r}$, $C_f \in \mathcal{R}^{p \times n}$ and $D_f \in \mathcal{R}^{p \times r}$ that satisfy the set of LMIs for $j = 1, 2 \ldots, N$, as shown in (22) on the next page. If a solution to this set of LMIs exists then a filter that

$$\left[\begin{array}{cccccccccccc}
Q_2 + Q_2^T & Q_3 - Q_2^T + \begin{bmatrix} Q_{11}\left(\Sigma_{i=0}^2 A_i^{(j)} + \Sigma_{i=1}^2 A_i^{(j)} \bar{\varepsilon}_i\right) & 0 \\ Q_{12}\left(\Sigma_{i=0}^2 A_i^{(j)} + \Sigma_{i=1}^2 A_i^{(j)} \bar{\varepsilon}_i\right) - Z_a - Z_b C_0 & Z_a \end{bmatrix} & 0 & 0 & 0 & 0 & 0 & \mathrm{diag}\{Q_{11}, Q_{12}\} & \mathrm{diag}\{Q_{11}, Q_{12}\} & \begin{bmatrix} Q_{11} B^{(j)} \\ Q_{12} B^{(j)} - Z_b D_{21}^{(j)} \end{bmatrix} & h_1 Q_2^T \begin{bmatrix} I_n \\ I_n \end{bmatrix} A_1^{(j)} [I_n \;\; 0] & h_2 Q_2^T \begin{bmatrix} I_n \\ I_n \end{bmatrix} A_2^{(j)} [I_n \;\; 0] \\
* & -Q_3 - Q_3^T & \begin{bmatrix} L^{(j)T} - C_f^T - C_0^{(j)T} D_f^T \\ C_f^T \end{bmatrix} & h_1 \phi_1 R_1 & h_2 \phi_2 R_2 & \begin{bmatrix} \bar{\varepsilon}_1 A_1^{(j)T} \\ 0 \end{bmatrix} [I \;\; I] S_1 & \begin{bmatrix} \bar{\varepsilon}_2 A_2^{(j)T} \\ 0 \end{bmatrix} [I \;\; I] S_2 & 0 & 0 & 0 & h_1 Q_3^T \begin{bmatrix} I_n \\ I_n \end{bmatrix} A_1^{(j)} [I_n \;\; 0] & h_2 Q_3^T \begin{bmatrix} I_n \\ I_n \end{bmatrix} A_2^{(j)} [I_n \;\; 0] \\
* & * & -\gamma^2 I_p & 0 & 0 & 0 & 0 & 0 & 0 & D^{(j)} - D_f D_{21}^{(j)} & 0 & 0 \\
* & * & * & -h_1 R_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & -h_2 R_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & -(1 - d_1) S_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & -(1 - d_2) S_2 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & -S_1 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & -S_2 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & -I_q & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & -h_1 R_1 & 0 \\
* & * & * & * & * & * & * & * & * & * & * & -h_2 R_2
\end{array}\right] \tag{18}$$

$$\begin{bmatrix}
Q_2 + Q_2^T & Q_1 \bar{A}_0 & 0 & 0 & 0 & Q_1 & Q_1 & Q_1 \bar{B} \\
* & -Q_3 - Q_3^T & \bar{L}^T & \bar{A}_1^T S_1 & \bar{A}_2^T S_2 & 0 & 0 & 0 \\
* & * & -\gamma^2 I_q & 0 & 0 & 0 & 0 & \bar{D} \\
* & * & * & -(1 - d_1) S_1 & 0 & 0 & 0 & 0 \\
* & * & * & * & -(1 - d_2) S_2 & 0 & 0 & 0 \\
* & * & * & * & * & -S_1 & 0 & 0 \\
* & * & * & * & * & * & -S_2 & 0 \\
* & * & * & * & * & * & * & -I_q
\end{bmatrix} < 0. \tag{21}$$

guarantees the estimation error level of γ, independently of the lengths of the time delays, is given by (4) with

$$A_f = Q_{12}^{-1} Z_a,\ B_f = Q_{12}^{-1} Z_b,\ C_f\ and\ D_f.$$

IV. H_∞ FILTERING FOR CASE 2

Case 2 is characterized by time delays that may vary very fast. Derivation of sufficient conditions for this case follows lines similar to those used in deriving Theorem 1, where the second term in the Lyapunov–Krasovski functional of (10) is omitted and where $\varepsilon_i = 0$, $i = 1, 2$. We first define the set of LMIs, shown in (23) at the bottom of the page.

Theorem 3: Consider the system of (1) and (2) with $r = r_0$ where the time-varying delays are continuous for all $0 \leq t$ and satisfy $0 < \tau_i(t) \leq h_i$. For a prescribed $\gamma > 0$, the cost function (5) achieves $J(w) < 0$ for all nonzero $w \in \mathcal{L}_2^q[0, \infty)$ and for all the parameters that belong to the uncertainty polytope Ω of (3), if there exist $Q_2, Q_3, R_1, R_2 \in \mathcal{R}^{2n\times 2n}$, $0 < Q_{11}$, $0 < Q_{12}$ and $Z_a \in \mathcal{R}^{n\times n}$, $Z_b \in \mathcal{R}^{n\times r}$, $C_f \in \mathcal{R}^{p\times n}$ and $D_f \in \mathcal{R}^{p\times r}$ that satisfy the set of LMIs in (23) for $j = 1, 2, \ldots, N$. Further, if a solution to this set of LMIs exists then a filter that guarantees the estimation error level of γ is given by (4) with

$$A_f = Q_{12}^{-1} Z_a,\ B_f = Q_{12}^{-1} Z_b,\ C_f\ and\ D_f.$$

$$\begin{bmatrix}
Q_2+Q_2^T & Q_3-Q_2^T+\begin{bmatrix} Q_{11}A_0^{(j)} & 0 \\ Q_{12}A_0^{(j)}-Z_a-Z_bC_0 & Z_a\end{bmatrix} & 0 & 0 & 0 & \mathrm{diag}\{Q_{11},Q_{12}\} & \mathrm{diag}\{Q_{11},Q_{12}\} & \begin{bmatrix} Q_{11}B^{(j)} \\ Q_{12}B^{(j)}-Z_bD_{12}^{(j)}\end{bmatrix} \\
* & -Q_3-Q_3^T & \begin{bmatrix} L^{(j)T}-C_f^T-C_0^{(j)T}D_f^T \\ C_f^T\end{bmatrix} & \begin{bmatrix} A_1^{(j)T} \\ 0\end{bmatrix}[I \quad I]\,S_1 & \begin{bmatrix} A_2^{(j)T} \\ 0\end{bmatrix}[I \quad I]\,S_2 & 0 & 0 & 0 \\
* & * & -\gamma^2 I_p & 0 & 0 & 0 & 0 & D^{(j)}-D_fD_{21}^{(j)} \\
* & * & * & -(1-d_1)\,S_1 & 0 & 0 & 0 & 0 \\
* & * & * & * & -(1-d_2)\,S_2 & 0 & 0 & 0 \\
* & * & * & * & * & -S_1 & 0 & 0 \\
* & * & * & * & * & * & -S_2 & 0 \\
* & * & * & * & * & * & * & -I_q
\end{bmatrix} < 0. \tag{22}$$

$$\begin{bmatrix}
Q_2+Q_2^T & Q_3-Q_2^T+\begin{bmatrix} Q_{11}\left(\Sigma_{i=0}^2 A_i^{(j)}\right) & 0 \\ Q_{12}\left(\Sigma_{i=0}^2 A_i^{(j)}\right)-Z_a-Z_bC_0 & Z_a\end{bmatrix} & 0 & 0 & 0 & \begin{bmatrix} Q_{11}B^{(j)} \\ Q_{12}B^{(j)}-Z_bD_{12}^{(j)}\end{bmatrix} & h_1Q_2^T\begin{bmatrix} I_n \\ I_n\end{bmatrix}A_1^{(j)}[I_n \quad 0] & h_2Q_2^T\begin{bmatrix} I_n \\ I_n\end{bmatrix}A_2^{(j)}[I_n \quad 0] \\
* & -Q_3-Q_3^T & \begin{bmatrix} L^{(j)T}-C_f^T-C_0^{(j)T}D_f^T \\ C_f^T\end{bmatrix} & h_1R_1 & h_2R_2 & 0 & h_1Q_3^T\begin{bmatrix} I_n \\ I_n\end{bmatrix}A_1^{(j)}[I_n \quad 0] & h_2Q_3^T\begin{bmatrix} I_n \\ I_n\end{bmatrix}A_2^{(j)}[I_n \quad 0] \\
* & * & -\gamma^2 I_p & 0 & 0 & D^{(j)}-D_fD_{21}^{(j)} & 0 & 0 \\
* & * & * & -h_1R_1 & 0 & 0 & 0 & 0 \\
* & * & * & * & -h_2R_2 & 0 & 0 & 0 \\
* & * & * & * & * & -I_q & 0 & 0 \\
* & * & * & * & * & * & -h_1R_1 & 0 \\
* & * & * & * & * & * & * & -h_2R_2
\end{bmatrix} < 0. \tag{23}$$

TABLE I
VALUES OF THE MINIMUM γ, AS A FUNCTION OF THE BOUND $d_1 = d_2 = d$

d	0	.3	.5	.6	.7	.778
γ_{min}	.77	.9716	1.2895	1.6545	2.5125	4.4899

V. EXAMPLE

Consider the system given in (1) where [13]

$$A_0 = \begin{bmatrix} 0 & 2 \\ -3 & -4+\rho \end{bmatrix}, \; A_1 = \begin{bmatrix} -.1 & 0 \\ .2 & -.2+\phi \end{bmatrix}$$
$$A_2 = \begin{bmatrix} 0 & .1 \\ -.2 & -.3+\phi \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
$$C_0 = [1 \quad 0], \; C_1 = 0, \; C_2 = 0, \; D = 0, \; D_{21} = 1$$
$$L = [1 \quad 2].$$

The uncertain parameters satisfy $|\rho| \leq 2$ and $|\phi| \leq 0.1$.

Applying the delay-independent criterion of Theorem 2, assuming that the delays are constants, i.e., $d_1 = d_2 = 0$, we obtain that (21) indeed has a solution and that the minimum achievable bound on the estimation error is $\gamma = 0.7869$ which is essentially the same as $\gamma = 0.7843$ obtained in [13]. The corresponding matrices of the filter are

$$A_f = \begin{bmatrix} -0.0753 & 2.429 \\ -2.4764 & -0.7912 \end{bmatrix}$$
$$B_f = \begin{bmatrix} 0.0528 \\ 0.593 \end{bmatrix}$$
$$C_f = [0.0528 \quad 0.5930]$$
$$D_f = 0.7778.$$

Allowing for delay to change in time we obtain the values of the minimum γ, as a function of the bound $d_1 = d_2 = d$, that are described in Table I. For $d \geq 0.78$, the delay-independent criterion becomes infeasible.

The delay-dependent criterion of Theorem 1 is applied for $h_1 = 0.4$ and $h_2 = 0.5$ (for $d_1 = d_2 = 0$). A minimum value of $\gamma = 0.6784$ is obtained (for the simple choice of $\varepsilon_1 = \varepsilon_2 = -0.52I_4$), compared to the minimum achievable value of $\gamma = 0.9725$ that was obtained in [13]. The result that we have obtained is not only 70% of the other result, but it is also significantly smaller than the one obtained by the delay-independent method. The corresponding matrices of the H_∞ filter are

$$A_f = \begin{bmatrix} -0.7674 & 2.4320 \\ -3.2390 & -1.2999 \end{bmatrix}$$
$$B_f = \begin{bmatrix} 0.2067 \\ 0.5157 \end{bmatrix}$$
$$C_f = [0.3853 \quad 0.9931]$$
$$D_f = 0.5722.$$

Considering next the maximum value of $h = h_1 = h_2$ for which the delay-dependent criterion is still feasible, the value of $h_{\max} = 0.92$ was reported in [13]. With our method we can achieve any value of h by simply letting ε_1 and ε_2 tend to $-I_4$. For, say, $h = 1.35$ it is enough to choose $\varepsilon_1 = \varepsilon_2 = -.6I_4$ in order to obtain a feasible solution.

It should be noted that we have specialized the free scaling parameters ε_1 and ε_2 to be scaled identity matrices. Further improvement on filtering performance can be made via optimization over these free matrix parameters.

VI. CONCLUSION

Delay-independent and delay-dependent sufficient conditions are presented which guarantee that the L_2-induced norm of the estimation error process that results from the application of a general full-order filter will be less than a prescribed value over the entire range of uncertainty. The results obtained are less conservative than corresponding results in the literature due to the efficient BRL that was derived for time delay systems based on an equivalent descriptor representation of the system and due to the Park's efficient overbounding method. The example has clearly indicated the less conservatism of our design. The approach of the present note can be applied to various problems with time-delay, including control and estimation of stochastic systems and general H_∞ output-feedback control.

REFERENCES

[1] K. M. Nagpal and P. P. Khargonekar, "Filtering and smoothing in an $\boldsymbol{H}_\infty$ setting," *IEEE Trans. Automat. Contr.*, vol. 36, pp. 152–166, Feb. 1991.

[2] I. Yaesh and U. Shaked, "Game theory approach to optimal linear estimation in the minimum $\boldsymbol{H}_\infty$ norm sense," in *Proc. 28th IEEE Conf. Decision Control*, Tampa, FL, Dec. 1989, pp. 421–425.

[3] U. Shaked and Y. Theodor, "$\boldsymbol{H}_\infty$ -optimal estimation: A tutorial," in *Proc. 31st IEEE Conf. Decision Control*, Tucson, TX, Dec. 1992, pp. 2278–2286.

[4] R. N. Banavar and J. L. Speyer, "A linear-quadratic game approach to estimation and smoothing," in *Proc. 1991 Amer. Control Conf.*, Boston, MA, June 1991, pp. 2818–2822.

[5] M. Fu, C. E. de Souza, and L. Xie, "$\boldsymbol{H}_\infty$ estimation for uncertain systems," *Int. J. Robust Nonlinear Control*, vol. 2, pp. 87–105, 1992.

[6] H. Li and M. Fu, "A linear matrix inequality approach to robuat $\boldsymbol{H}_\infty$ filtering," *IEEE Trans. Signal Processing*, vol. 45, pp. 2338–2350, Sept. 1997.

[7] J. C. Geromel and M. C. de Oliviera, "$\boldsymbol{H}_2$ and $\boldsymbol{H}_\infty$ robust filtering for convex bounded uncertain systems," in *Proc. 37th IEEE Conf. Decision Control*, Tampa, FL, Dec. 1998, pp. 146–161.

[8] R. M. Palhares and Pl. L. D. Peres, "Robust $\boldsymbol{H}_\infty$ filtering design with pole placement constraint via LMIs," *J. Optim. Theory Appl.*, vol. 102, pp. 239–261, 1999.

[9] M. Malek-Zavarei and M. Jamshidi, *Time-Delay Systems: Analysis, Optimization and Applications*. Amsterdam, The Netherlands: North-Holland, 1987.

[10] A. Pila, U. Shaked, and C. de Souza, "$\boldsymbol{H}_\infty$ filtering for continuous-time linear systems with delay," *IEEE Trans. Automat. Contr.*, vol. 44, pp. 1412–1417, July 1999.

[11] A. Fattou, O. Sename, and J. Dion, "$\boldsymbol{H}_\infty$ observer design for time-delay systems," in *Proc. 37th IEEE Conf. Decision Control*, Tampa, FL, Dec. 1998, pp. 4545–4546.

[12] C. E. de Souza, R. M. Palhares, and P. L. D. Peres, "Robust $\boldsymbol{H}_\infty$ filtering for uncertain linear continuous-time systems with multiple time-varying state delays: An LMI approach," presented at the 38th Conf. Decision Control, Phoenix, AZ, Dec. 1999.

[13] R. Palhares, P. Peres, and C. de Souza, "Robust $\boldsymbol{H}_\infty$ filtering for linear continuous-time uncertain systems with multiple delays: An LMI approach," presented at the 3rd IFAC Conf. Robust Control Design, Prague, Czech Republic, July 2000.

[14] C. E. de Souza, A. Trofino, and J. de Oliveira, "Robust $\boldsymbol{H}_\infty$ control of uncertain linear systems via parameter-dependent Lyapunov functions," in *Proc. 39th IEEE Conf. Decision Control*, Sydney, Australia, Dec. 2000, pp. 3194–3199.

[15] E. Fridman and U. Shaked, "A new $\boldsymbol{H}_\infty$ filter design for linear time-delay systems," *IEEE Trans. Signal Processing*, vol. 49, pp. 2839–2843, Nov. 2001.

[16] E. Fridman, "New Lyapunov-Krasovskii functionals for stability of linear retarded and neutral type systems," *Syst. Control Lett.*, vol. 43, no. 4, pp. 309–319, 2001.

[17] E. Fridman and U. Shaked, "New bounded real lemma representations for time-delay systems and their applications," *IEEE Trans. Automat. Contr.*, vol. 46, pp. 1973–1979, Dec. 2001.

[18] P. Park, "A delay-dependent stability criterion for systems with uncertain time-invariant delays," *IEEE Trans. Automat. Contr.*, vol. 44, pp. 876–877, Apr. 1999.

[19] E. Fridman and U. Shaked, "Delay-dependent $\boldsymbol{H}_\infty$ filtering of linear neutral systems," *IEEE Trans. Signal Processing* .

[20] ——, "Delay-dependent stability and $\boldsymbol{H}_\infty$ control: Constant and time-varying delays," *Int. J. Control* .