$H\infty$ Control of Distributed and Discrete Delay Systems *via* Discretized Lyapunov Functional

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The discretized Lyapunov functional method is extended to linear systems with both, discrete and distributed delays, and to $H\infty$ control. The coefficients associated with the distributed delay are assumed to be piecewise constant. A new Bounded Real Lemma (BRL) is derived in terms of Linear Matrix Inequalities (LMIs) via descriptor approach. In three numerical examples considered for retarded type systems, the resulting values of $H\infty$-norm converge to the exact ones. The analysis results are applied to state-feedback $H\infty$ control of linear neutral systems with discrete and distributed delays, where the controller may be either instantaneous or may contain discrete or distributed delay terms. A numerical example illustrates the efficiency of the design method and the advantage of using distributed delay term in the feedback for $H\infty$ control of systems with state delay.

Keywords: Time-delay, distributed delay, Lyapunov Krasovskii functional, discretized Lyapunov functional method, $H\infty$ control

1. Introduction

Systems with both, discrete and distributed delays, appear in different applications (see e.g. [2], [3], [12], [19], [22]). Moreover, it is well-known that the optimal linear quadratic regulator for state-delay systems (see e.g. [19]) as well as $H\infty$ state-feedback controller that

results from Riccati equations [6] possess a distributed delay term. Therefore, the distributed delay term in the state feedback may improve the performance of the system with state delay.

Robust control of systems with discrete and distributed delays has been studied *via* simple Lyapunov-Krasovskii Functionals (LKFs) only (see e.g. [14], [20], [26–28]). The necessary condition for the application of simple LKFs is the asymptotic stability of the closed-loop non delayed system. If the latter conditions does not hold, the complete LKF should be applied. Stability and $H\infty$-norm of linear retarded systems with discrete and distributed delays have been analyzed *via* complete LKF

$$V_c(x_t) = x^T(t)P_1 x(t) + 2x^T(t)\int_{-r}^{0} Q(\xi)x(t+\xi)d\xi$$
$$+ \int_{-r}^{0}\int_{-r}^{0} x^T(t+s)R(s,\xi)dsx(t+\xi)d\xi,$$
$$P_1 > 0,\ R(\xi,\eta) = R^T(\eta,\xi) \qquad (1)$$

in [7], where Riccati partial differential equations have been derived.

LMI stability conditions *via* complete LKF and discretization were introduced by K. Gu [9] and appeared to be very efficient, leading in some examples to results close to analytical ones. The discretized LKF method has been developed for stability analysis of either systems with discrete delays [15], [19] or distributed delays [10], [11].

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Parameter-dependent LMIs for stability and $H\infty$ control via complete LKF were derived for linear systems with discrete and distributed delays in [1], [18], [25]. Some technique for reduction of these LMIs to a finite number of parameter-independent LMIs was suggested. However, it was not shown that in some examples the analysis results can approach to analytical ones, while the design procedure was based on the restrictive assumption that $Q(\xi) \equiv P_1$.

Recently a *descriptor discretized* LKF method was introduced in [5], which combined the application of the complete LKF and the discretization procedure of Gu [9] with the descriptor model transformation [4]. In the descriptor approach both $x(t)$ and $\dot{x}(t)$ are the state variables, which allows to avoid some terms in the LKF derivative condition (since $\dot{x}(t)$ is not substituted everywhere by the right hand side of the system). As a result, the descriptor discretized LKF method leads to simpler conditions and can be easily applied to design problems. In [5] the state-feedback stabilization of systems with a single discrete delay was considered.

The objective of this paper is to extend the discretized LKF to $H\infty$ control and to systems with both, single discrete and piecewise constant distributed delays by applying descriptor approach. The method is extended also to nonuniform mesh, which was not relevant in the case of single delay. Numerical examples (one of them is $H\infty$ control of combustion in rocket motor chambers) illustrate the efficiency of the new method and show that the distributed delay term in the feedback improves the $H\infty$ performance.

Notation: Throughout the paper the superscript "T" stands for matrix transposition, $\mathcal{R}^n$ denotes the n dimensional Euclidean space with vector norm $\| \cdot \|$, $\mathcal{R}^{n \times m}$ is the set of all $n \times m$ real matrices, and the notation $P > 0$, for $P \in \mathcal{R}^{n \times n}$ means that P is symmetric and positive definite. The symmetric elements of the symmetric matrix will be denoted by $*$.

2. BRL *via* Descriptor Discretized LKF Method

Consider a linear system

$$\dot{x}(t) = A_0 x(t) + A_1 x(t - r) + F\dot{x}(t - g)$$
$$+ \int_{-r}^{0} A_d(\theta)x(t + \theta)d\theta + B_1 w(t),$$
$$z(t) = C_0 x(t) + C_1 x(t - r)$$
$$+ \int_{-r}^{0} C_d(\theta)x(t + \theta)d\theta, \qquad (2)$$

where $x(t) \in \mathcal{R}^n$, $w(t) \in \mathcal{R}^q$, $r > 0$ is constant time-delay. A_0, A_1, C_0, C_1, and F are constant matrices. A_d and C_d are piecewise constant matrices. It is assumed that the eigenvalues of F are inside the unit circle.

For a prechosen $\gamma > 0$, we consider the following performance index:

$$J = \int_{0}^{\infty} [z^T(t)z(t) - \gamma^2 w^T(t)w(t)]dt. \qquad (3)$$

We are looking for conditions which guarantee that Eq. (2) is internally stable and has $H\infty$-norm less than γ, i.e. that $J < 0$ for all $0 \neq w(t) \in \mathcal{L}_2$.

We note that distributed delay appears for example in the model of combustion in rocket motor chambers [2], [3]. State-feedback stabilization and $H\infty$ control of combustion will be considered in section 3 and is based on stability and $H\infty$-norm analysis of system (2). Another example is a model of a mechanical rotational cutting process [21], where stability analysis is reduced to the stability of linear comparison system with distributed delay (see Example 2.3).

We apply a complete LKF

$$V(x_t) = V_0(x_t) + \int_{-g}^{0} \dot{x}^T(t + \xi)U\dot{x}(t + \xi)d\xi,$$
$$U > 0, V_0(x_t) = x^T(t)P_1 x(t)$$
$$+ 2x^T(t) \int_{-r}^{0} Q(\xi)x(t + \xi)d\xi$$
$$+ \int_{-r}^{0} \int_{-r}^{0} x^T(t + s)R(s, \xi)ds x(t + \xi)d\xi$$
$$+ \int_{-r}^{0} x^T(t + \xi)S(\xi)x(t + \xi)d\xi, \ P_1 > 0,$$
$$(4)$$

where $Q(\xi) \in R^{n \times n}$, $R(\xi, \eta) = R^T(\eta, \xi) \in R^{n \times n}$, $S(\xi) = S^T(\xi) \in R^{n \times n}$, and Q, R, S are continuous matrix functions. LKF V_0 is of the same form as in [9], [10], and it corresponds to the retarded type system (2) with $F = 0$. The last (nonnegative) term in Eq. (4) is added due to the neutral-type system.

We apply the descriptor complete LKF, which means that V satisfies the following derivative condition along (2):

$$\dot{V}(x_t) + z^T(t)z(t) - \gamma^2 w^T(t)w(t)$$
$$\leq - \epsilon \left(\|x(t)\|^2 + \|\dot{x}(t)\|^2 + \|w(t)\|^2 \right), \qquad (5)$$

where $\epsilon > 0$ is some constant. Inequality (5) guarantees that (2) is internally stable and $J < 0$.

Differentiating LKF (4) along (2) we have

$$\dot{V}(x_t) = 2\dot{x}^T(t)[P_1 x(t) + \int_{-r}^{0} Q(\xi)x(t+\xi)d\xi]$$

$$+2x^T(t)\int_{-r}^{0} Q(\xi)\dot{x}(t+\xi)d\xi$$

$$+2\int_{-r}^{0}\int_{-r}^{0} \dot{x}^T(t+s)R(s,\xi)ds\,x(t+\xi)d\xi$$

$$+2\int_{-r}^{0} \dot{x}^T(t+\xi)S(\xi)x(t+\xi)d\xi$$

$$+\dot{x}^T(t)U\dot{x}(t) - \dot{x}(t-g)U\dot{x}(t-g). \quad (6)$$

Adding to $\dot{V}(x_t)$ the right part of the expression

$$0 = 2[x^T(t)P_2^T + \dot{x}^T(t)P_3^T]$$
$$\times [A_0 x(t) - \dot{x}(t) + A_1 x(t-r) + F\dot{x}(t-g)$$
$$+\int_{-r}^{0} A_d(\theta)x(t+\theta)d\theta + B_1 w(t)], \quad (7)$$

where P_2 and P_3 are $n \times n$ matrices, which is equivalent to descriptor model transformation of [4], we integrate by parts in Eq. (6). We find

$$\dot{V}(x_t) = \zeta^T \Xi \zeta + 2\dot{x}^T(t)\int_{-r}^{0}[Q(\xi)$$

$$+P_3^T A_d(\xi)]x(t+\xi)d\xi$$

$$-\int_{-r}^{0}\int_{-r}^{0} x^T(t+\xi)(\frac{\partial}{\partial\xi}R(\xi,\theta)$$

$$+\frac{\partial}{\partial\theta}R(\xi,\theta))x(t+\theta)d\theta d\xi$$

$$+2x^T(t)\int_{-r}^{0}[P_2^T A_d(\xi) - \dot{Q}(\xi)$$

$$+R(0,\xi)]x(t+\xi)d\xi$$

$$-2x^T(t-r)\int_{-r}^{0} R(-r,\theta)x(t+\theta)d\theta$$

$$-\int_{-r}^{0} x^T(t+\xi)\dot{S}(\xi)x(t+\xi)d\xi$$

$$+\dot{x}^T(t)U\dot{x}(t) - \dot{x}^T(t-g)U\dot{x}(t-g)$$

$$+2[x^T(t)P_2^T + \dot{x}^T(t)P_3^T][F\dot{x}(t-g)$$

$$+B_1 w(t)], \quad (8)$$

where

We apply the discretization of Gu [10]. Divide the delay interval $[-r, 0]$ into N segments $[\theta_p, \theta_{p-1}]$, $p = 1, ..., N$ of length $h_p = \theta_{p-1} - \theta_p$ in such a way that

$$A_d(\theta) = A_{dp}, \; p = 1, ..., N, \; \theta \in [\theta_p, \theta_{p-1}], \quad (10)$$

where A_{dp} are constant matrices. This divides the square $[-r, 0] \times [-r, 0]$ into $N \times N$ small squares $[\theta_p, \theta_{p-1}] \times [\theta_q, \theta_{q-1}]$. Each small square is further divided into two triangles.

The continuous matrix functions $Q(\xi)$ and $S(\xi)$ are chosen to be linear within each segment and the continuous matrix function $R(\xi, \theta)$ is chosen to be linear within each triangular:

$$Q(\theta_p + \alpha h_p) = (1-\alpha)Q_p + \alpha Q_{p-1}, \; S(\theta_p + \alpha h_p)$$
$$= (1-\alpha)S_p + \alpha S_{p-1}, \; \alpha \in [0,1],$$

$$R(\theta_p + \alpha h_p, \theta_q + \beta h_q)$$

$$= \begin{cases} (1-\alpha)R_{pq} + \beta R_{p-1,q-1} \\ +(\alpha-\beta)R_{p-1,q}, \; \alpha \geq \beta, \\ (1-\beta)R_{pq} + \alpha R_{p-1,q-1} \\ +(\beta-\alpha)R_{p,q-1}, \; \alpha < \beta. \end{cases} \quad (11)$$

Thus the LKF is completely determined by $P_1, Q_p, S_p, R_{pq}, p, q = 0, 1, ..., N$.

The LKF condition $V(x_t) \geq V_0(x_t) \geq \epsilon_0 \|x(t)\|^2$ is satisfied for some $\epsilon_0 > 0$ (see 10) if $S_p > 0$, $p = 0, 1, ..., N$ and

$$\begin{bmatrix} P_1 & \tilde{Q} \\ * & \tilde{R}+\tilde{S} \end{bmatrix} > 0, \quad (12)$$

where

$$\tilde{Q} = [Q_0 Q_1 ... Q_N], \tilde{S} = diag\left\{\frac{1}{\tilde{h}_0}S_0, \frac{1}{\tilde{h}_1}S_1, ..., \frac{1}{\tilde{h}_N}S_N\right\},$$

$$\tilde{h}_0 = h_1, \tilde{h}_p = \max\{h_p, h_{p+1}\}, p = 1, ..., N-1, \tilde{h}_N = h_N,$$

$$\tilde{R} = \begin{bmatrix} R_{00} & R_{01} & ... & R_{0N} \\ R_{10} & R_{11} & ... & R_{1N} \\ ... & ... & ... & ... \\ R_{N0} & R_{N1} & ... & R_{NN} \end{bmatrix}. \quad (13)$$

$$\Xi = \begin{bmatrix} P^T\begin{bmatrix} 0 & I \\ A_0 & -I \end{bmatrix} + \begin{bmatrix} 0 & A_0^T \\ I & -I \end{bmatrix}P + \begin{bmatrix} Q(0)+Q^T(0)+S(0) & 0 \\ 0 & 0 \end{bmatrix} & P^T\begin{bmatrix} 0 \\ A_1 \end{bmatrix} - \begin{bmatrix} Q(-r) \\ 0 \end{bmatrix} \\ * & -S(-r) \end{bmatrix},$$

$$\zeta^T = \begin{bmatrix} x^T(t) & \dot{x}^T(t) & x^T(t-r) \end{bmatrix}, P = \begin{bmatrix} P_1 & 0 \\ P_2 & P_3 \end{bmatrix}. \quad (9a-c)$$

Eqs (8), (9), and (11) imply (cf. [10])

$$
\dot{V}(x_t) = \zeta^T \Xi \zeta - \int_0^1 \phi^T(\alpha) S_d \phi(\alpha) d\alpha + \dot{x}^T(t) U \dot{x}(t) - \dot{x}(t-g) U \dot{x}(t-g)
$$

$$
+ 2[x^T(t) P_2^T + \dot{x}^T(t) P_3^T][F\dot{x}(t-g) + B_1 w(t)] - \left[\int_0^1 \phi(\alpha) d\alpha\right]^T R_{ds}\left[\int_0^1 \phi(\alpha) d\alpha\right] \tag{14}
$$

$$
- \int_0^1 \left[\int_0^\alpha (\phi^T(\alpha)\ \phi^T(\beta)) \begin{pmatrix} 0 & R_{da} \\ R_{da}^T & 0 \end{pmatrix} \begin{pmatrix} \phi(\alpha) \\ \phi(\beta) \end{pmatrix} d\alpha\right] d\beta + 2\zeta^T \int_0^1 [D^s + (1-2\alpha)D^a]\phi(\alpha) d\alpha,
$$

where ζ is given by (9b) and

$$
\phi^T(\alpha) = [x^T(t + \theta_1 + \alpha h_1)\ x^T(t + \theta_2 + \alpha h_2)\ ...\ x^T(t + \theta_N + \alpha h_N)],
$$

$$
\Xi = \begin{bmatrix} P^T \begin{bmatrix} 0 & I \\ A_0 & -I \end{bmatrix} + \begin{bmatrix} 0 & A_0^T \\ I & -I \end{bmatrix} P + \begin{bmatrix} Q_0 + Q_0^T + S_0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} P_2^T A_1 \\ P_3^T A_1 \end{bmatrix} - \begin{bmatrix} Q_N \\ 0 \end{bmatrix} \\ * & -S_N \end{bmatrix},
$$

$$
S_d = diag\{S_0 - S_1, S_1 - S_2, ..., S_{N-1} - S_N\}
$$

$$
R_{ds} = \begin{bmatrix} R_{ds11} & R_{ds12} & ... & R_{ds1N} \\ R_{ds21} & R_{ds22} & ... & R_{ds2N} \\ ... & ... & ... & ... \\ R_{dsN1} & R_{dsN2} & ... & R_{dsNN} \end{bmatrix}, \quad R_{da} = \begin{bmatrix} R_{da11} & R_{da12} & ... & R_{da1N} \\ R_{da21} & R_{da22} & ... & R_{da2N} \\ ... & ... & ... & ... \\ R_{daN1} & R_{daN2} & ... & R_{daNN} \end{bmatrix}, \tag{15a−k}
$$

$$
R_{dspq} = 1/2[h_p + h_q](R_{p-1,q-1} - R_{pq}) + 1/2[h_p - h_q](R_{p,q-1} - R_{p-1,q}),
$$

$$
R_{dapq} = 1/2[h_p - h_q](R_{p-1,q-1} - R_{p-1,q} - R_{p,q-1} + R_{p,q}), \quad p, q = 1, 2, ..., N,
$$

$$
D^s = [D_1^s\ D_2^s\ ...\ D_N^s], \quad D^a = [D_1^a\ D_2^a\ ...\ D_N^a],
$$

$$
D_p^s = \begin{bmatrix} h_p P_2^T A_{dp} + \frac{h_p}{2}(R_{0,p-1} + R_{0p}) - (Q_{p-1} - Q_p) \\ h_p P_3^T A_{dp} + \frac{h_p}{2}(Q_{p-1} + Q_p) \\ -\frac{h_p}{2}(R_{N,p-1} + R_{Np}) \end{bmatrix}, \quad D_p^a = \begin{bmatrix} -\frac{h_p}{2}(R_{0,p-1} - R_{0p}) \\ -\frac{h_p}{2}(Q_{p-1} - Q_p) \\ \frac{h_p}{2}(R_{N,p-1} - R_{Np}) \end{bmatrix}
$$

Applying arguments of [10], [12] to (14) we verify that for any matrices $\tilde{U} > 0$ and $W > 0$ the following holds:

$$
\dot{V}(x_t) = - \int_0^1 \left[\zeta^T[D^s + (1-2\alpha)D^a]\ \phi^T(\alpha)\right] \begin{bmatrix} \tilde{U} & -I_N \\ -I_N & S_d - W \end{bmatrix} \begin{bmatrix} [D^s + (1-2\alpha)D^a]^T \zeta \\ \phi(\alpha) \end{bmatrix} d\alpha
$$

$$
+ \zeta^T(\Xi + D^s \tilde{U} D^{sT} + \frac{1}{3} D^a \tilde{U} D^{aT})\zeta - \int_0^1 \phi^T(\alpha) S_d \phi(\alpha) d\alpha - \left[\int_0^1 \phi(\alpha) d\alpha\right]^T R_{ds}\left[\int_0^1 \phi(\alpha) d\alpha\right]
$$

$$
- \int_0^1 \left[\int_0^\alpha (\phi^T(\alpha)\ \phi^T(\beta)) \begin{pmatrix} W & R_{da} \\ R_{da}^T & W \end{pmatrix} \begin{pmatrix} \phi(\alpha) \\ \phi(\beta) \end{pmatrix} d\alpha\right] d\beta
$$

$$
+ \dot{x}^T(t) U \dot{x}(t) - \dot{x}(t-g) U \dot{x}(t-g) + 2[x^T(t) P_2^T + \dot{x}^T(t) P_3^T][F\dot{x}(t-g) + B_1 w(t)]. \tag{16}
$$

We assume that

$$\begin{bmatrix} \tilde{U} & -I_N \\ -I_N & S_d - W \end{bmatrix} > 0. \qquad (17)$$

Then applying to the first integral in the right side of (16) Jensen's inequality (see e.g. [12]), we find that

$$\dot{V}(x_t) \le -\int_0^1 [\zeta^T[D^s + (1-2\alpha)D^a] \quad \phi^T(\alpha)] d\alpha \begin{bmatrix} \tilde{U} & -I_N \\ -I_N & S_d - W \end{bmatrix} \int_0^1 \begin{bmatrix} [D^s + (1-2\alpha)D^a]^T\zeta \\ \phi(\alpha) \end{bmatrix} d\alpha$$

$$+ \zeta^T(\bar{\Xi} + D^s\tilde{U}D^{sT} + \frac{1}{3}D^a\tilde{U}D^{aT})\zeta - \int_0^1 \phi^T(\alpha)S_d\phi(\alpha)d\alpha - \left[\int_0^1 \phi(\alpha)d\alpha\right]^T R_{ds}\left[\int_0^1 \phi(\alpha)d\alpha\right]$$

$$- \int_0^1 \int_0^\alpha (\phi^T(\alpha) \quad \phi^T(\beta)) \begin{pmatrix} W & R_{da} \\ R_{da}^T & W \end{pmatrix} \begin{pmatrix} \phi(\alpha) \\ \phi(\beta) \end{pmatrix} d\alpha d\beta + \dot{x}^T(t)U\dot{x}(t) - \dot{x}(t-g)U\dot{x}(t-g)$$

$$+ 2[x^T(t)P_2^T + \dot{x}^T(t)P_3^T][F\dot{x}(t-g) + B_1 w(t)] \qquad (18)$$

$$= [\zeta^T \quad \int_0^1 \phi^T(\alpha)d\alpha] \begin{bmatrix} \bar{\Xi} + \frac{1}{3}D^a\tilde{U}D^{aT} & D^s \\ D^{sT} & -R_{ds} - S_d + W \end{bmatrix} \begin{bmatrix} \zeta \\ \int_0^1 \phi(\alpha)d\alpha \end{bmatrix}$$

$$- \int_0^1 \int_0^\alpha (\phi^T(\alpha)\phi^T(\beta)) \begin{pmatrix} W & R_{da} \\ R_{da}^T & W \end{pmatrix} \begin{pmatrix} \phi(\alpha) \\ \phi(\beta) \end{pmatrix} d\alpha d\beta + \dot{x}^T(t)U\dot{x}(t) - \dot{x}(t-g)U\dot{x}(t-g)$$

$$+ 2[x^T(t)P_2^T + \dot{x}^T(t)P_3^T][F\dot{x}(t-g) + B_1 w(t)].$$

Eliminating $\tilde{U}$ from the latter inequality we conclude that if

$$\begin{bmatrix} W & R_{da} \\ * & W \end{bmatrix} > 0, \qquad (19)$$

then

$$\dot{V}(x_t) < \psi^T \Phi \psi + 2[x^T(t)P_2^T + \dot{x}^T(t)P_3^T]B_1 w(t),$$

$$\psi^T = [\zeta^T \int_0^1 \phi^T(\alpha)d\alpha \int_0^1 \phi^T(\alpha)d\alpha$$

$$\dot{x}^T(t-g) \quad \dot{x}^T(t-g)], \qquad (20)$$

where

$$\Phi \triangleq \begin{bmatrix} \bar{\Xi} & D^s & D^a & \begin{bmatrix} P_2^T F \\ P_3^T F \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ U \\ 0 \end{bmatrix} \\ * & -R_{ds} - S_d + W & 0 & 0 & 0 \\ * & * & -3(S_d - W) & 0 & 0 \\ * & * & * & -U & 0 \\ * & * & * & * & -U \end{bmatrix}. \qquad (21)$$

Assuming that

$$C_d(\theta) = C_{dp}, \ p = 1, ..., N, \ \theta \in [\theta_p, \theta_{p-1}], \qquad (22)$$

where C_{dp} are constant matrices, we find (after application of Schur complements to $z^T z$) that $\dot{V} + z^T z - \gamma^2 w^T w < 0$ if

$$\begin{bmatrix} | & P_2^T B_1 & C_0^T \\ | & P_3^T B_1 & 0 \\ | & 0 & C_1^T \\ \Phi\ | & 0 & h_1 C_{d1}^T \\ | & ... & ... \\ | & 0 & h_N C_{dN}^T \\ | & 0 & 0 \\ * & -\gamma^2 I_q & 0 \\ * & * & -I_n \end{bmatrix} < 0. \qquad (23)$$

Moreover, $\Phi < 0$ implies that $S_0 1 > ...S_N > 0$. Hence, (13) guarantees $V(x_t) \ge \epsilon\|x(t)\|^2$, $\epsilon > 0$. We thus proved the following BRL:

Lemma 2.1: *The system (2) is internally stable and has H∞-norm less than γ if there exist $n \times n$-matrices $P_1 > 0$, P_2, P_3, U, $S_p = S_p^T$, Q_p, $R_{pq} = R_{qp}^T$, $p = 0, 1, ..., N$, $q = 0, 1, ..., N$ and $nN \times nN$-matrix W such*

that LMIs (12), (19) and (23) are satisfied with notations defined in (13), (15) and (21).

Notice that in the case of uniform mesh, one can choose $W = 0$ and (19) can be omitted.

Remark 2.1. *In the case of system matrices from the uncertain time-invariant polytope*

$$\Omega = \sum_{j=1}^{M} f_j \Omega_j \qquad 0 \le f_j \le 1, \ \sum_{j=1}^{M} f_j = 1,$$
$$\Omega_j = \left[A_i^{(j)} \quad A_{dp}^{(j)} \quad F^{(j)} \quad B_1^{(j)} \quad C_i^{(j)} \quad C_{dp}^{(j)} \right], \tag{24}$$

where $i = 0, 1$, $p = 1, ..., N$ by the descriptor discretized method one have to solve the LMIs (12), (23), (19) simultaneously for all the M vertices Ω_j, applying the same matrices P_2 and P_3 and solving for the M vertices.

Example 2.1. *[7] Consider the system*

$$\dot{x}(t) - f\dot{x}(t-g) = -0.7x(t) - 0.3x(t-r)$$
$$- \int_{-r}^{0} x(t+\theta)d\theta + 0.5w(t), \quad z(t) = x(t),$$

where $f = 0$. For the values of r given in Table 1 it has been verified in [7] that the system is internally stable and has H_∞-norm γ^* given in Table 1, where γ^* is found as the peak value of the frequency response of the transfer function

$$T_{zw}(s) = 0.5s[s^2 + fs^2 \cdot exp(-gs) + s \cdot 0.7s$$
$$+ 0.3s \cdot exp(-rs) + 1 - exp(-rs)]^{-1}.$$

By applying Lemma 2.1 for N = 1 N = 1, 2, 3 we find the same (for N = 1, 2, 3) values we find the achievable value of γ_1 given in Table 1, which are close to γ^*.

Considering next $f \ne 0$, we find the corresponding values of γ^* (for g = r) and γ_1 (for all $g \ge 0$). As it is

expected, for greater values of f the influence of the term $f\ddot{x}(t-g)$ on the performance of the system becomes greater and, thus, the gap increases between the actual value of γ^* for g = r and g-independent value of achievable γ_1.

Example 2.2: *Consider (2), where*

$$A_0 = \begin{bmatrix} -1.5 & 0 \\ 0.5 & -1 \end{bmatrix}, A_1 = 0, \ C_0 = [1\ 1],$$

$$C_1 = 0, \ C_d = 0, \ B_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

$$A_d(\theta) = \begin{bmatrix} 2 & 2.5 \\ 0 & -0.5 \end{bmatrix}, \theta \in [0, \frac{r}{2}],$$

$$A_d(\theta) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \ \theta \in [\frac{r}{2}, r].$$

Stability of the latter system was studied in [10] via discretized Lyapunov functional and asymptotic stability was guaranteed for all $r \le r_{max}$, where for N = 2,4 the following values of r_{max} were found: $r_{max}^{N=2} = 1.97$ and $r_{max}^{N=4} = 1.99$. The analytical value is $r_{max} = 2$. In this example, application of Lemma 2.1 (with uniform mesh) leads to a slightly slower speed of convergence: $r_{max}^{N=2} = 1.84$, $r_{max}^{N=4} = 1.97$ and $r_{max}^{N=6} = 1.99$.

Applying next Lemma 2.3 with N = 2 and N = 4 we calculate for different values of r the minimum achievable values of γ_N given in Table 2. For $r \le 1.5$ the resulting values of γ are close to the exact values γ^* (see Table 2), obtained in the frequency domain.

Example 2.3: *The following model of a mechanical rotation cutting process has been considered in [21] (see also the references therein):*

$$\ddot{x}(t) + 2\xi\omega_n\dot{x}(t) + \omega_n^2 x(t)$$
$$= \frac{k}{m}(x(t-\tau(t)) - x(t)), \ k > 0, \tag{25}$$

where $m = 100, \omega_n = 632.45, \xi = 0.039585$. The periodic delay has a form $\tau(t) = \tau_0 + \delta f_1(\Omega t), \delta > 0, \Omega > 0$, where $f_1(t)$ is a sawtooth function (cf. [21]). For $\tau(t) \equiv$

Table 1. Example 2.1.

	r	0.1	0.2	0.3	0.4	0.5	0.6
$f = 0$	γ^*	0.4545	0.4167	0.3846	0.3571	0.3333	0.3125
$f = 0$	γ_1	0.4548	0.4167	0.3847	0.3572	0.3334	0.3129
$f = 0.2$	γ^*	0.4227	0.4154	0.3837	0.3565	0.3329	0.3123
$f = 0.2$	γ_1	0.4647	0.4279	0.3964	0.3723	0.3581	0.3613
$f = 0.5$	γ^*	0.4253	0.4151	0.3835	0.3563	0.3328	0.3121
$f = 0.5$	γ_1	0.5320	0.4993	0.4813	0.4864	0.5216	0.5843

Table 2. Example 2.2.

r	0.1	0.5	1	1.5	1.7
γ^*	2.10	2.65	3.95	7.79	12.62
γ_2	2.11	2.67	4.01	8.03	17.22
γ_4	2.11	2.67	4.00	8.00	13.34

0 *the system is not stable. It was shown in [21] that (25) is asymptotically stable for high enough frequency Ω, if the following comparison system with distributed delay (and $w=0$) is asymptotically stable:*

$$\ddot{x}(t) + 2\xi\omega_n\dot{x}(t) + \omega_n^2 x(t) + \omega_n^2 w(t)$$
$$= \frac{k}{m}\left[\int_{t-\tau_0-\delta}^{t-\tau_0+\delta}\frac{x(\theta)}{2\delta}d\theta - x(t)\right] \qquad (26)$$

Choosing as in [21] $\delta = 0.05\tau_0$ and applying Lemma 2.1 with $N=21$, we find that (26) is asymptotically stable for all $0 < k \le 10^6$, which is close to results of [21].

It was shown in [21] that "distributing" of the pointwise delay over the interval enlarges the stability region: the stability region of the comparison distributed delay system (26) is larger than the one of the original system (25) with the discrete constant delay $\tau \equiv \tau_0$. Consider (25), where $z(t) = x(t)$, $\tau_0 = 0.06$, $\delta = 0.25\tau_0$, $\xi = 10$, $k = 10^4$ and $\omega_n = 632.45$. Applying Lemma 2.1 for $N=7$ we find the minimum achievable $\gamma = 1.00020$. This is close to the analytical value $\gamma^* = 0.9997$ obtained in the frequency domain. When taking the same setup with the discrete delay $\tau(t) = \tau_0$ and $N=7$ we find a slightly larger value of $\gamma = 1.00023$, whereas the analytical one is $\gamma^* = 1$. In this example "distributing" of the point-wise delay over the interval slightly improves the H∞-norm of the system.

3. H∞ Control

Given the following system:

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-r) + \int_{-r}^{0} A_d(\theta)x(t+\theta)d\theta$$
$$+ F\dot{x}(t-g) + B_1 w(t) + B_2 u(t),$$
$$z(t) = C_0 x(t) + C_1 x(t-r)$$
$$+ \int_{-r}^{0} C_d(\theta)x(t+\theta)d\theta + Du(t),$$

$$(27)$$

where $x(t) \in \mathcal{R}^n$ is the system state vector, $u(t) \in \mathcal{R}^m$ is the control input, $z(t) \in \mathcal{R}^k$ is the controlled output,

$A_i, B_{i+1}, C_i, i = 0, 1$ are constant matrices, A_d and C_d are given by (10) and (22) for some discretization segments $[\theta_p, \theta_{p-1}]$, $p = 1, ..., N$. We are looking for a stabilizing state feedback

$$u(t) = K_0 x(t) + K_1 x(t-r) + \int_{-r}^{0} K_d(\xi)x(t+\xi)d\xi,$$

$$(28)$$

where K_d is piecewise constant:

$$K_d(\theta) = K_{dp}, \ p = 1, ..., N, \ \theta \in [\theta_p, \theta_{p-1}]. \quad (29)$$

The closed-loop system (27), (28) has the form

$$\dot{x}(t) = (A_0 + BK_0)x(t) + (A_1 + BK_1)x(t-r)$$
$$+ B_2\int_{-r}^{0} K_d(\xi)x(t+\xi)d\xi + F\dot{x}(t-g),$$
$$z(t) = (C_0 + DK_0)x(t) + (C_1 + DK_1)x(t-r)$$
$$+ \int_{-r}^{0}(C_d(\xi) + DK_d(\xi))x(t+\xi)d\xi.$$

$$(30)$$

Following [24] we choose $P_3 = \delta P_2, \delta \in R$, where δ is a tuning scalar parameter. Note that P_2 is nonsingular due to the fact that the only matrix which can be negative definite in the second block on the diagonal of (21) is $-\delta(P_2 + P_2^T)$. Defining:

$$\bar{P} = P_2^{-1}, \quad [\bar{P}_1 \ \ \bar{Q}_p \ \ \bar{S}_p \ \ \bar{R}_{pq} \ \ \bar{U}]$$
$$= \bar{P}^T[P_1\bar{P} \ \ Q_p\bar{P} \ \ S_p\bar{P} \ \ R_{pq}\bar{P} \ \ U\bar{P}],$$
$$Y_i = K_i\bar{P}, \ i = 0, 1, \ Y_{dp} = K_{dp}\bar{P},$$
$$p = 1, ..., N, q = 1, ..., N\bar{W}_{kj} = \bar{P}^T W\bar{P},$$
$$j = 1, ..., N, \ k = j, ..., N.$$

$$(31)$$

we multiply (12) and (21), (19) by $diag\{\bar{P}, ..., \bar{P}\}$ and its transpose, from the right and the left, we obtain:

Theorem 3.1: *Given $\gamma > 0$ and $N > 0$, the system (27) is stabilizable and achieves H∞-norm less than γ if for some tuning scalar parameter δ there exist $n \times n$ matrices $0 < \bar{P}_1, \ \bar{P}, \ \bar{R}_a, \bar{U}, \ \bar{S}_p = \bar{S}_p^T, \ \bar{Q}_p, \ \bar{R}_{pq} = \bar{R}_{qp}^T$, $p = 0, 1, ..., N, \ q = 0, 1, ..., N, \ \bar{W}_{kj}, j = 1, ..., N, \ k = j, ..., N$ and $m \times n$-matrices Y_0, Y_1, Y_{dp} such that satisfy (19) and the following LMIs:*

$$\begin{bmatrix} \bar{P}_1 & \tilde{Q} \\ * & \tilde{R} + \tilde{S} \end{bmatrix} > 0, \qquad (32)$$

$$
\begin{bmatrix}
\hat{\Xi} & D^s + Y^s & D^a & \begin{matrix} F\bar{P} \\ \delta F\bar{P} \\ 0 \end{matrix} & \begin{matrix} 0 \\ \bar{U} \\ 0 \end{matrix} & \begin{matrix} B_1 \\ \delta B_1 \\ 0 \end{matrix} & \begin{matrix} \bar{P}^T C_0^T + Y_0^T D^T \\ 0 \\ \bar{P}^T C_1^T + Y_1^T D^T \end{matrix} \\
* & -R_{ds} - S_d + W & 0 & 0 & 0 & 0 & \begin{bmatrix} h_1 Y_{d1}^T D^T \\ ... \\ h_N Y_{dN}^T D^T \end{bmatrix} \\
* & * & -3(S_d - W) & 0 & 0 & 0 & 0 \\
* & * & * & -\bar{U} & 0 & 0 & 0 \\
* & * & * & * & -\bar{U} & 0 & 0 \\
* & * & * & * & * & -\gamma^2 I_q & 0 \\
* & * & * & * & * & * & -I_k
\end{bmatrix} < 0, \tag{33}
$$

where

$$
\hat{\Xi} = \begin{bmatrix}
A_0\bar{P} + \bar{P}^T A_0^T + BY_0 + Y_0^T B^T + \bar{Q}_0 + \bar{Q}_0^T + \bar{S}_0 & \bar{P}_1 - \bar{P} + \delta\bar{P}^T A_0^T + \delta Y_0^T B^T & A_1\bar{P} + BY_1 - \bar{Q}_N \\
* & -\delta\bar{P} - \delta\bar{P}^T & \delta(A_1\bar{P} + BY_1) \\
* & * & -\bar{S}_N
\end{bmatrix},
$$

$$
Y^s = [Y_1^s ... Y_N^s], \quad Y_p^s = \begin{bmatrix} hBY_{dp} \\ \delta hBY_{dp} \\ 0 \end{bmatrix}, \quad W = \begin{bmatrix} \bar{W}_{11} & ... & \bar{W}_{1N} \\ ... & ... & ... \\ \bar{W}_{1N}^T & ... & \bar{W}_{NN} \end{bmatrix} \tag{34}
$$

and where $\tilde{R}, \tilde{Q}, \tilde{S}$ and $D^s, D^a, R_{ds}, R_{da}, S_d$ are given by (13) and (15) correspondingly with bars over R_{pq}, Q_p, S_p, $p = 1, ..., N$, $q = 1, ..., N$.

The gains of state-feedback (28) are given by $K_0 = Y_0\bar{P}^{-1}$, $K_1 = Y_1\bar{P}^{-1}$, $K_{dp} = Y_{dp}\bar{P}^{-1}$.

Remark 3.1: *Our design method is based on the assumption $P_3 = \delta P_2, \delta \in R$, which may be restrictive. We note that since P_2 and P_3 are slack variables, the above assumption is not too much conservative. An alternative method seems to be the iterative one. The iterative method based on the discretized Lyapunov functional is not desirable, because each iteration may need a lot of computational time.*

Remark 3.2: *Consider (27) with $A_0, A_1, A_{dp}, F, B_1, B_2$, C_0, C_1, C_{dp} $(p = 1, ..., N)$ and D from the uncertainty polytope given by (24), where*

$$
\Omega_j = \begin{bmatrix} A_0^{(j)} A_1^{(j)} & A_{dp}^{(j)} & F^{(j)} & B_1^{(j)} & B_2^{(j)} & C_0^{(j)} C_1^{(j)} & C_{dp}^{(j)} & D^{(j)} \end{bmatrix}.
$$

To design a state-feedback $H\infty$ control law for the system inside the polytope one have to solve LMIs (32), (33) and (19) simultaneously for all the M vertices, applying the same matrices $\bar{P}$ and Y_0, Y_1, Y_{dp}.

Example 3.1: *The following model of combustion $H\infty$ control in rocket motor chambers has been considered in [26]: Eq. (27), where*

$$
A_0 = \begin{bmatrix}
\Delta\rho & 0 & 0 & 0 \\
0 & 0 & 0 & -5 \\
-0.5556 & 0 & -0.5556 & 0.5556 \\
0 & 1 & -1 & 0
\end{bmatrix},
$$

$$
A_d = \begin{bmatrix}
-\frac{1-\Delta\rho}{r} & 0 & \frac{1}{r} & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix},
$$

$|\Delta\rho| \le \Delta\rho_{max}$, $A_1 = 0$,

$B_2 = [0\ 5\ 0\ 0]^T$, $r = 1$, $F = 0$,

$B_1 = [0\ 0\ 1\ 0]^T$, $C_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$,

$C_1 = 0$, $C_d = 0$, $D = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

The above model was derived in [2], [3]. Robust stabilization of this model (described as a system with norm-bounded uncertainty) has been studied in [28] via

a simple LKF, where it has been found that the system is stabilizable by memoryless state-feedback $u(t) = K_0 x(t)$ *for* $\Delta\rho_{max} = 0.16$. *Simulations in [28] showed that the system is stabilizable for greater value of* $\Delta\rho_{max} = 0.5$. *Representing the above system as a polytopic system with two vertices reached by* $\Delta\rho = \pm\Delta\rho_{max}$ *and applying Remark 3.2 for* $\Delta\rho_{max} = 0.5$ *with* $\delta = 1$ *and* $N = 1$ *we find that the system is internally stabilizable by memoryless state-feedback and has* $H\infty$-*norm less than* $\gamma = 3.5262$.

Consider next $\Delta\rho_{max} = 0.15$. *It has been found in [26] by using a simple descriptor Lyapunov functional and the polytopic representation that the* $H\infty$ *control problem is solvable for the following minimum values of* γ:

$$\gamma = 14 \text{ for } u(t) = K_0 x(t), \ \gamma = 22 \text{ for } u(t)$$
$$= K_0 x(t) + K_d \int_{-r}^{0} x(t + \theta) d\theta.$$

By applying Remark 3.2 with uniform mesh for $N = 1, 2, 3, 5, 10$ *and choosing (for simplicity)* $\delta = 1$, *we achieve essentially smaller values of* γ *(see Table 3) for either memoryless control* u_0 *or delayed control* u_1 *or distributed control* u_d *given by*

$$u_0(t) = K_0 x(t), \ u_1(t) = K_0 x(t) + K_1 x(t - r),$$
$$u_d(t) = K_0 x(t) + \int_{-r}^{0} K_d(\theta) x(t + \theta) d\theta,$$

where K_d *is defined by (29) (only values of* K_{d1} *are given in Table 3). The values of* γ *become smaller for greater values of N. The distributed control law leads to a better performance than the other controllers. In Table 3 we give also the computational time (in minutes) and the number of the scalar variables in*

the LMIs (including γ*). We see that the improvement is achieved at the expence of the computational time.*

Though the improvement of the $H\infty$ *performance by distributed control law was discussed in the existing literature (see e.g. [6], [18]), the presented method seems to be the first LMI method that shows this improvement explicitly in the numerical example (compare e.g. with the result (35) of [26]).*

4. On Numerical Complexity and Further Improvements

Alternative Lyapunov-based methods for $H\infty$-norm analysis of system (2) are the methods which apply *simple Lyapunov functionals*. Thus, in the case of constant A_d, C_d the following simple LKF can be chosen (see [14], [26])

$$V_s = [x^T(t) \ \dot{x}^T(t)] \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} P_1 & 0 \\ P_2 & P_3 \end{bmatrix} \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}$$
$$+ \int_{t-r}^{t} x^T(s) S x(s) ds + \int_{t-g}^{t} \dot{x}^T(\xi) U \dot{x}(\xi) d\xi$$
$$+ \int_{-r}^{0} \int_{t+\theta}^{t} \dot{x}^T(s) R \dot{x}(s) ds d\theta$$
$$+ \int_{-r}^{0} \int_{t+\theta}^{t} x^T(s) R_d x(s) ds d\theta, \quad (36)$$

with positive constant matrices P_1, U, R, R_d, S. It is well-known that the LMIs derived *via* simple LKFs are convex in r: if these LMIs are feasible for $r = r_0$, then they are feasible for all $0 \le r \le r_0$. Hence, the resulting value of achievable $\gamma(r_0)$ for $r = r_0$ is valid for all $r \in [0, r_0]$ and therefore $\gamma(r_0) \ge \sup_{r \in [0, r_0]} \gamma(r)$,

Table 3. Example 3.1.

N	u	γ	K_0	K_1/K_{d1}	Time	nb vars
1	u_0	1.53	18.76 -10.50 5.16 -24.53	0	0.15	255
1	u_1	1.52	19.81 -5.41 2.70 -12.26	0.07 0.01 -0.11 -0.003	0.15	259
1	u_d	1.47	42.09 -17.53 3.21 -51.70	-19.74 7.74 -0.75 24.05	0.15	259
2	u_0	1.12	5.44 -3.97 1.06 -8.67	0	0.3	487
2	u_1	1.11	8.25 -5.44 0.93 -13.07	-0.26 0.15 -0.06 0.32	0.3	491
2	u_d	1.10	30.28 -16.54 -1.44 -48.42	-25.51 13.37 1.85 40.98	0.3	491
3	u_0	1.065	2.28 -2.49 -0.36 -4.25	0	1	815
3	u_1	1.063	1.75 -2.04 -0.21 -3.11	-0.25 0.23 0.09 0.42	1	819
3	u_d	1.054	7.157 -7.2064 -1.84 -14.10	-9.04 8.50 2.83 18.25	1	819
5	u_0	1.045	12.39 -7.48 -0.30 -20.38	0	6	1759
5	u_1	1.047	0.003 -0.058 -0.002 0.012	-0.001 0.003 -0.000 -0.001	6	1763
5	u_d	1.040	24.85 -12.96 -3.47 -39.16	-18.85 9.22 3.26 29.61	6	1763
10	u_0	1.043	12.02 -7.11 -0.55 -19.61	0	70	5799
10	u_1	1.039	11.64 -6.64 -0.81 -18.41	-1.03 0.49 0.06 1.37	70	5803
10	u_d	1.035	35.94 -18.58 -5.35 -57.14	-30.57 15.14 5.29 48.62	70	5803

i.e. $\gamma(r_0) \geq \sup_{r \in [0, r_0]} \gamma^*(r)$. We note that $\gamma^*(r)$ is not necessarily a monotonically increasing function of r (see e.g. Example 2.1 and also Example 1 from [6]). Thus, in Example 2.1 the simple LKFs-based methods cannot achieve $\gamma < 0.4545$ for $r = 0.6$, whereas the discretized LKF with $N = 1$ leads to $\gamma = 0.3132$. Of course, the improvement is achieved at the account of numerical complexity.

LMIs for stability analysis of neutral systems *via* the simple LKF (36) involve $4.5n^2 + 2.5n$ scalar variables, whereas the descriptor discretized LKF-based LMIs for $N = 1$ involve $7.5n^2 + 2.5n$ scalar variables (of symmetric matrices $P_1, R_{11}, R_{00}, S_0, S_1$ and non-symmetric $P_2, P_3, Q_0, Q_1, R_{01}$). The difference of $3n^2$ scalar variables may become essential for large n.

If the same numerical result can be achieved by a discretized method and by a simple LKF-based one, then the number of scalar variables in the LMIs by the discretized Lyapunov functional method may be less. Thus, the stability in the well-known example

$$\dot{x}(t) = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix} x(t) + \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix} x(t - r)$$

has been recently analyzed by delay fractioning method and a corresponding simple LKF in [8]. The stability for $r = 6.05$ in this example was proved by using 42 scalar variables in LMIs of [8], whereas the discretized methods for $N = 1$ lead to the same result and use less variables (27 by method of Gu [9] and 35 by the descriptor discretized LKF). We note, that the result by simple LKF is stronger in the sense that the stability is proved for all $0 \leq r \leq 6.05$.

For arbitrary N and uniform mesh, the number of scalar variables for stability analysis by the descriptor discretized method is $3n^2 + n + (2N + 1)\frac{n(n+1)}{2} + N(3N + 3)\frac{n^2}{2}$. The descriptor discretized Lyapunov method uses additional matrices P_2 and P_3 and thus involves $2n^2$ more scalar variables than the method of Gu [9]. These additional matrices can lead to a slower convergence in the stability analysis than by Gu's method (see Example 2.2). However, as we mentioned in Introduction, the main advantages of the descriptor method are in the simplified form of BRL and in the application to the design problems. We note that in the design we choose $P_2 = \delta P_3$ with a scalar δ and thus the difference in the number of scalar variables becomes n^2. However, the only possible design procedure *via* the discretized method of Gu seems to be the iterative one, which is not desirable since each iteration may need a lot of computational time.

The reduction of the number of variables in the LMI conditions *via* complete LKF is important

direction for the future research. Some results in this direction were presented recently in [23], where LKF with special forms of Q and R in (1) led to LMIs with a fewer variables, but with worse numerical results.

Finally the results for neutral systems may be further improved by combining the augmented Lyapunov functional [17] with the complete one. However, such improvements lead to further computational complexity.

5. Conclusions

Descriptor discretized Lyapunov functional method is extended to state-feedback $H\infty$ control of systems with both, discrete and distributed delays. The new method leads to simplified BRL conditions for systems with distributed delays and, for the first time, treats both, discrete and distributed delays, *via* discretized Lyapunov functional method. In three numerical examples considered for the retarded-type systems, the resulting values of $H\infty$-norm converge to the exact ones. The presented method seems to be *the first LMI method* that in some numerical examples leads to values of H_∞-norm close to analytical ones for retarded type systems.

A numerical example shows that the distributed delay term in the state feedback improves the $H\infty$ performance. The new method essentially improves the existing $H\infty$ control results even for small values of N. Moreover, it provides new tools for the important design problems, such as $H\infty$ control of systems, which are not stabilizable without delay.

The presented method, as other discretized Lyapunov functional methods, is encountered with heavy computations. The reduction of the number of variables in the LMI conditions as well as further improvements (especially for neutral systems) may be the topics for the future research.

Acknowledgments

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