Systems & Control Letters 97 (2016) 139–148

Contents lists available at ScienceDirect

Systems & Control Letters

journal homepage: www.elsevier.com/locate/sysconle

A positivity-based approach to delay-dependent stability of systems with large time-varying delays☆

Alexander Domoshnitsky [a,*], Emilia Fridman [b]

[a] Department of Mathematics, Ariel University, Ariel, Israel

[b] Department of Electrical Engineering, Tel-Aviv University, Tel-Aviv, Israel

ARTICLE INFO

Article history:
Received 25 August 2015
Received in revised form
7 August 2016
Accepted 14 September 2016

Keywords:
Time-delay systems
Exponential stability
Positive systems

ABSTRACT

In this paper we propose new explicit tests for positivity and exponential stability of systems with large time-varying delays. Our approach is based on nonoscillation of solutions of the corresponding diagonal scalar delay differential equations. Numerical examples illustrate the efficiency of the results.

1. Introduction

Positive systems appear in various models that are composed of interconnected subsystems, where each subsystem presents a compartment. Compartments exchange variable nonnegative quantities of material with conservation laws describing transfer, accumulation, and outflows between compartments and the environment [1]. Transfers between the compartments have to account time for material, energy, or information in transit between the compartments. This leads to analysis of delay systems of the following form

$$x'(t) + A_0(t)x(t) + \sum_{k=1}^{m} A_k(t)x(t - \theta_k(t)) = 0,$$

$$t \in [0, +\infty), \tag{1.1}$$

$$x(\xi) = \varphi(\xi), \quad \xi < 0, \tag{1.2}$$

where $\varphi : (-\infty, 0) \to \mathbb{R}^n$ is a given continuous n-vector function, defining what can be substituted into the equation for $t - \theta_k(t) < 0, A_k(t) = \left\{a_{ij}^k(t)\right\}_{i,j=1,\ldots,n}$, $k = 0, \ldots, m$, are $n \times n$ matrices with bounded piecewise continuous entries, $x(t) = col\,\{x_1(t), \ldots, x_n(t)\} \in \mathbb{R}^n$ is n-vector with absolutely continuous components, the delays $\theta_k(t)$ are measurable bounded functions for $k = 1, \ldots, m$.

In this paper, we deal with the positivity-based stability analysis of (1.1). This approach was started in [2], and was further developed in [3–7]. For difference and delay differential systems this approach was developed in [8–11,6,1,12–18]. For applications of this approach to additive neural networks see [19,13]. In all the above works that treat (1.1) it is assumed that there is a non-delayed term $A_0(t)x(t)$ with positive terms on the main diagonal of A_0. These diagonal terms should be sufficiently large in order to achieve dominance of the main diagonal of the matrix A_0 over all the other terms (see, for example, the condition (5) of Theorem 3.1 in [1] and condition (iii) of Theorem III. I in [16]). Such an assumption can be interpreted as follows: the diagonal ordinary differential equations describing every compartment, should be exponentially stable, and interconnections between different compartments should be sufficiently weak in order not to destabilize the system (1.1).

The approaches of above papers are not applicable to stabilization of an open-loop unstable system

$$x'(t) + A_0(t)x(t) + \sum_{k=1}^{m} A_k(t)x(t - \theta_k(t)) = u(t),$$

$$t \in [0, +\infty), \tag{1.3}$$

by the delayed feedback $u(t) = -\sum_{k=1}^{m} B(t)x(t - \tau_k(t))$, with $\tau_k(t) > \theta_k(t) > 0$ for $t \in [0, +\infty)$, $k = 1, \ldots, m$. The latter inequality may naturally appear in applications. The presence of

☆ This work was partially supported by Israel Science Foundation (Grant No. 1128/14).

* Corresponding author.
E-mail addresses: adom@ariel.ac.il (A. Domoshnitsky), emilia@eng.tau.ac.il (E. Fridman).

http://dx.doi.org/10.1016/j.sysconle.2016.09.011
0167-6911/

time-delay in the control input may destabilize the closed-loop systems, as pointed out, for example, in [20,21,7]. One of the popular approaches used to cope with delays in the input is the predictor-based approach (see e.g. [22]). Recent developments in this area were presented in [23]. Another way to cope with delays in the input is to reduce systems of the delay differential equations to systems of "integral equations" (see, for example, [24,4] and the references therein). This approach allows to deal with variable delays and coefficients leading to simple stability conditions in a form of inequalities. Based on this approach the positivity-based stability analysis-results were provided in [25–28], where a smallness of delays on the main diagonal was assumed instead of their absence (see, for example, Proposition 2.3). Positivity-based stability of neutral systems with small delays on the main diagonal was considered in [25,29,17] see also the recent paper [30]. Results on stability of systems with distributed delay can be found, for example, in [31,18], where "smallness of delays" on the main diagonal is also assumed.

In the present paper, for the first time, the stability conditions for systems with large time-varying delays are provided under assumption of the closeness of the delays instead of the delays' smallness. Theorems 3.2 and 3.3 present sufficient conditions for the exponential stability in this case. Theorem 3.4 generalizes to systems with large delays the classical theorem about equivalence of the exponential stability, existence of positive solution to a system of linear algebraic inequalities and the fact that a matrix constructed from the coefficients is Hurwitz for system of ordinary differential equations with Metzler matrix (see Definition 2.2 and Proposition 2.2). The presented approach allows to stabilize unstable state-delay systems by feedback with large input delays. The corresponding result is proved under assumption about nonoscillation of the "diagonal" scalar delay equations in Theorem 3.5. A principal possibility to achieve stabilization of system (3.22) (see below) by the feedback control (3.23), where the delays $\tau_{ij}(t)$ are greater than the state delays $\theta_{ij}(t)$ of (3.22), is formulated in Corollaries 3.1 and 3.2. The stability results are formulated in terms of inequalities on the delays and on the coefficients.

The present paper is organized as follows. In Section 2, we discuss positivity-based methods in the stability analysis. In Section 3, we formulate our main results. In Section 4, the proofs of the main results are given.

Notations: Throughout the paper e denotes the Euler number. L_∞ is the space of essentially bounded measurable functions $y : [0,+\infty) \to \mathbb{R}$. For $y \in L_\infty$ denote $y^* = \operatorname{esssup}_{t\geq 0} y(t)$, $y_* = \operatorname{essinf}_{t\geq 0} y(t)$ and for $y^k \in L_\infty (k = 1,\dots,m)$ $-$ $y^+(t) = \max_{k=1,\dots,m}\left\{y^k(t)\right\}$, $y^-(t) = \min_{k=1,\dots,m}\left\{y^k(t)\right\}$.

2. Preliminaries on positivity and stability of time-delay systems

Consider the non-homogeneous system

$$x'(t) - \sum_{k=1}^{m} A_k(t)x(t-\theta_k(t)) = f(t), \quad t \in [0,+\infty), \tag{2.1}$$

$$x(\xi) = 0, \quad \xi < 0, \tag{2.2}$$

where $A_k(t) = \left\{a_{ij}^k(t)\right\}_{i,j=1,\dots,n}$ are $n \times n$ matrices with entries $a_{ij}^k \in L_\infty$, $\theta_k \in L_\infty$ for $k = 1,\dots,m$, $f(t) = col\{f_1(t),\dots,f_n(t)\}$, $f_i \in L_\infty$, for $i = 1,\dots,n$. The components $x_i : [0,+\infty) \to \mathbb{R}$ of the vector $x = col\{x_1,\dots,x_n\}$, are assumed to be absolutely continuous and their derivatives $x_i' \in L_\infty$. A vector-function x is a solution of (2.1) if it satisfies system (2.1) for almost all $t \in [0,+\infty)$.

It was explained in [24] that without loss of generality, the zero initial condition (2.2) can be considered instead of (1.2). The homogeneous system

$$x'(t) - \sum_{k=1}^{m} A_k(t)x(t-\theta_k(t)) = 0, \quad t \in [0,+\infty), \tag{2.3}$$

with initial function defined by (2.2), has n-dimensional space of solutions [24] and this fact is the basis of solutions' representations which will be used below.

Let us define the Cauchy matrix $C(t,s) = \left\{C_{ij}(t,s)\right\}_{i,j=1,\dots,n}$ as follows [24]. For every fixed $s \geq 0$, as a function of the variable t, it satisfies the matrix equation

$$C_t'(t,s) = \sum_{k=1}^{m} A_k(t)C(t-\theta_k(t),s), \quad t \in [s,+\infty), \tag{2.4}$$

where

$$C(\xi,s) = 0, \quad \text{for } \xi < s, \tag{2.5}$$

and

$$C(s,s) = I. \tag{2.6}$$

I is the unit matrix. The general solution of system (2.1), (2.2) can be represented in the form [24]

$$x(t) = \int_0^t C(t,s)f(s)ds + C(t,0)x(0). \tag{2.7}$$

Definition 2.1. The Cauchy matrix $C(t,s)$ is said to satisfy the exponential estimate if there exist positive numbers N and α such that

$$\left|C_{ij}(t,s)\right| \leq N\exp\{-\alpha(t-s)\}, \quad i,j = 1,\dots,n,$$
$$0 \leq s \leq t < +\infty. \tag{2.8}$$

In this case we say that (2.3) is exponentially stable.

Our main results will be based on the following extension of the classical Bohl–Perron theorem:

Proposition 2.1 ([4]). *In the case of bounded delays $\theta_k(t)$ and coefficients in the matrices $A_k(t)$ $(k = 1,\dots,m)$, the fact that for every bounded right-hand side $f(t) = col\{f_1(t),\dots,f_n(t)\}$, the solution $x(t) = col\{x_1(t),\dots,x_n(t)\}$ of system (2.1) is bounded on the semiaxis $[0,+\infty)$ is equivalent to the exponential estimate (2.8) of the Cauchy matrix $C(t,s)$.*

T. Wazewski [5] proved that for system of ordinary differential equations $x'(t) = A(t)x(t)$ the nonnegativity of all off-diagonal elements of $A(t)$

$$a_{ij}(t) \geq 0 \quad \text{for } i \neq j,\ i,j = 1,\dots,n,\ t \in [0,+\infty), \tag{2.9}$$

is necessary and sufficient for the nonnegativity of all entries of the Cauchy matrix $C(t,s) = \left\{C_{ij}(t,s)\right\}_{i,j=1,\dots,n}$ of the system.

Definition 2.2. The matrix A is Metzler if all its off-diagonal elements are nonnegative for $t \geq 0$, i.e. (2.9) is fulfilled.

The fact that all matrices $A_k(t)$ are Metzler together with the smallness of diagonal delays (see condition (2.12)) implies $C_{ij}(t,s) \geq 0$ for $0 \leq s \leq t < +\infty$, $i,j = 1,\dots,n$ [25,26]. In Theorems 3.1 and 3.2 of the present paper, we propose new assumptions on the diagonal delay differential equations (actually, nonoscillation of their solutions), which together with the condition that the matrices $A_k(t)$ are Metzler, imply the nonnegativity of $C(t,s)$.

Consider particular case of (2.3)

$$x'(t) - \sum_{k=1}^{m} A_k x(t - \theta_k(t)) = 0, \quad t \in [0, +\infty),$$
$$x(\xi) = 0, \ \xi < 0, \tag{2.10}$$

where all the matrices $A_k(t)$ are constant, i.e. $A_k(t) \equiv A_k$ for $t \in [0, +\infty)$. In the case of $\theta_k(t) \equiv 0$ for $t \in [0, +\infty)$, we have the autonomous system

$$x'(t) - Ax(t) = 0, \quad t \in [0, +\infty), \tag{2.11}$$

of ordinary differential equations (here $A = \sum_{k=1}^{m} A_k$). System (2.11) is asymptotically stable (and also exponentially stable) if and only if the matrix A is Hurwitz.

Proposition 2.2 (*See, for example, [16,32]*). *If matrix A is Metzler, the following 4 facts are equivalent:*

(A) *A is Hurwitz,*
(B) *there exists a constant-vector $z = col\{z_1, \dots, z_n\}$ with all positive components such that all components of the constant vector Az are negative,*
(C) *the matrix $(-A)^{-1}$ exists and all its entries are nonnegative,*
(D) *the system of ordinary differential equations (2.11) is exponentially stable.*
It is well-known that (2.10) with a Hurwitz matrix $A = \sum_{k=1}^{m} A_k$ can be unstable for sufficiently large delays, i.e. the condition (A) does not imply that for all possible delays the following condition holds:
(E) *the system (2.10) is exponentially stable.*

Remark 2.1. Consider, for example, the scalar delay equation $x'(t) = ax(t - \theta)$, $t \in [0, +\infty)$, where $a < 0$, $|a|\,\theta > \frac{\pi}{2}$. It is clear that the matrix $A = a$ is Metzler and Hurwitz, but this equation is unstable [20,21].

The first result about equivalence of the conditions (B) and (E) for delay systems in the case of Metzler matrices A_k for $k = 1, \dots, m$, was obtained in [26] under the additional assumption on smallness of the products of elements on the main diagonals in A_k, and delays. Taking into account Proposition 2.2, the result of [26] can be presented in the following form:

Proposition 2.3. *Let matrices A_k in system (2.10) be Metzler and the following inequalities be fulfilled*

$$\Theta \sum_{k=1}^{m} \left|a_{ii}^k\right| \le \frac{1}{e}, \quad i = 1, \dots, n, \tag{2.12}$$

where $\Theta = \max_{1 \le k \le m} \operatorname{esssup}_{t \ge 0} \theta_k(t)$. Then for system (2.10) the facts (A), (B), (C) *and* (E) *are equivalent.*

In this paper we extend Propositions 2.2 and 2.3 to systems with large delays.

3. Main results

We study positivity and stability of the following system:

$$x_i'(t) + \sum_{j=1}^{n} \sum_{k=1}^{m} a_{ij}^k(t) x_j(t - \theta_{ij}^k(t)) = 0, \quad t \in [0, +\infty),$$
$$i = 1, \dots, n, \tag{3.1}$$
$$x_i(\xi) = 0, \quad \xi < 0, \ i = 1, \dots, n, \tag{3.2}$$

where $a_{ij}^k \in L_\infty$, $\theta_{ij}^k \in L_\infty$ for $k = 1, \dots, m$.

An important role in analysis of (3.1) is played by the system of n scalar diagonal equations (3.3), (3.2), where

$$x_i'(t) + \sum_{k=1}^{m} a_{ii}^k(t) x_i(t - \theta_{ii}^k(t)) = 0, \quad t \in [0, +\infty),$$
$$i = 1, \dots, n. \tag{3.3}$$

3.1. Positivity of the system

Definition 3.1. The system (3.1) is called positive if all the entries of its Cauchy matrix $C(t, s) = \left\{C_{ij}(t, s)\right\}_{i,j=1,\dots,n}$ are nonnegative in the triangle $0 \le s \le t < \infty$.

In all the existing results, the positivity was obtained under the assumption of smallness of the diagonal delays $\theta_{ii}^k(t)$ (see, for example, the inequality (2.12) and its generalizations on systems with variable delay and coefficients [25] and on equations with distributed delay [31]). In the following assertion, we obtain the nonnegativity of the Cauchy matrix in the case of "large" diagonal delays $\theta_{ii}^k(t)$, assuming a corresponding "compensation" of positive and negative coefficients a_{ii}^k, $k = 1, \dots, m$.

Denote $\Delta_i = \operatorname{esssup}_{t \ge 0}\{\theta_{ii}^+(t) - \theta_{ii}^-(t)\}$.

Theorem 3.1. *Let the following conditions be fulfilled:*

(1) *for every $i = 1, \dots, n$, at least one of the conditions 1(a) or 1(b) be fulfilled:*

1(a) *there exists m_i such that $a_{ii}^k(t) \ge 0$, $a_{ii}^j(t) \le 0$, $\theta_{ii}^k(t) \ge \theta_{ii}^j(t)$ for $k = 1, \dots, m_i, j = m_i + 1, \dots, m$, $\sum_{k=1}^{m_i} a_{ii}^k(t) \ge \frac{1}{e} \sum_{j=m_i+1}^{m} \left|a_{ii}^j(t)\right|$ for $t \in [0, +\infty)$, and*

$$\int_{t-\theta_{ii}^+(t)}^{t} \left\{\sum_{k=1}^{m_i} a_{ii}^k(s) - \frac{1}{e} \sum_{j=m_i+1}^{m} \left|a_{ii}^j(s)\right|\right\} ds \le \frac{1}{e},$$
$$t \in (0, +\infty). \tag{3.4}$$

1(b) *there exists m_i such that $a_{ii}^k(t) \ge 0$ $a_{ii}^j(t) \le 0$, $\theta_{ii}^k(t) \le \theta_{ii}^j(t)$ for $k = 1, \dots, m_i, j = m_i + 1, \dots, m$, $\sum_{k=1}^{m_i} a_{ii}^k(t) \ge \sum_{j=m_i+1}^{m} \left|a_{ii}^j(t)\right|$ for $t \in [0, +\infty)$,*

$$\int_{t-\theta_{ii}^+(t)}^{t} \left\{\sum_{k=1}^{m_i} a_{ii}^k(s) - \sum_{j=m_i+1}^{m} \left|a_{ii}^j(s)\right|\right\} ds \le \frac{1}{e},$$
$$t \in [0, +\infty), \tag{3.5}$$

and

$$\int_{s}^{s+\Delta_i} \sum_{k=1}^{m_i} a_{ii}^k(\xi) d\xi \le \frac{1}{e} \quad \forall s \ge 0. \tag{3.6}$$

(2) *$a_{ij}^k(t) \le 0$ for $i \ne j$, $i, j = 1, \dots, n$, $k = 1, \dots, m$.*
Then system (3.1) is positive.

Remark 3.1. Results about the nonnegativity of the Cauchy matrices obtained in [25,26] (see e.g. Proposition 2.3) follow from Theorem 3.1 in the case of $a_{ii}^j(t) = 0$ for $j = m_i + 1, \dots, m$, $t \in [0, +\infty)$.

3.2. Positivity-based stability conditions

Theorem 3.2. *Assume the following conditions* (1) *and* (2) *are true:*

(1) *The condition* (1) *of Theorem 3.1 is fulfilled.*
(2) *There exist positive numbers $z_1, \dots, z_n$ such that*

$$\sum_{k=1}^{m} a_{ii}^k(t) z_i - \sum_{j=1,\, j \ne i}^{n} \sum_{k=1}^{m} \left|a_{ij}^k(t)\right| z_j \ge 1, \quad t \in [0, +\infty),$$
$$i = 1, \dots, n. \tag{3.7}$$

Then

(a) *system (3.1) is exponentially stable;*

(b) *the integral estimates*

$$\sup_{t\in[0,\infty)} \int_0^t \sum_{j=1}^{n} \left|C_{ij}(t,s)\right| ds \le z_i, \quad i = 1,\dots,n, \tag{3.8}$$

are true;

(c) *if* $\theta_{ii}^k(t) \le \theta_{ij}^r(t)$ *for* $k, r = 1,\dots,m$, $i \ne j$, $i, j = 1,\dots,n$ *and*

$$\sum_{k=1}^{m} a_{ii}^k(t) - \sum_{j=1,\, j\ne i}^{n} \sum_{k=1}^{m} \left|a_{ij}^k(t)\right| \ge 1, \quad t \in [0,+\infty),$$
$$i = 1,\dots,n, \tag{3.9}$$

the estimates

$$\left|C_{ij}(t,s)\right| \le \left\{ \begin{array}{cc} 1, & t < s+\Theta \\ \exp\{-\beta(t-s)\}, & t \ge s+\Theta \end{array} \right\},$$
$$i, j = 1,\dots,n, \tag{3.10}$$

where $\beta = \min_{1\le i\le n} \operatorname{essinf}_{t\ge 0} \left\{ \sum_{k=1}^{m} a_{ii}^k(t) - \sum_{j=1,\, j\ne i}^{n} \left|a_{ij}^k(t)\right| \right\}$ *and* $\Theta = \max_{1\le i\le n} \operatorname{esssup}_{t\ge 0} \theta_{ii}^k(t)$, *are true.*

Remark 3.2. Existence of positive numbers $z_1,\dots,z_n$ and ε such that

$$\sum_{k=1}^{m} a_{ii}^k(t) z_i - \sum_{j=1,\, j\ne i}^{n} \sum_{k=1}^{m} \left|a_{ij}^k(t)\right| z_j \ge \varepsilon, \quad t \in [0,+\infty),$$
$$i = 1,\dots,n,$$

is equivalent to condition (2) of Theorem 3.2.

Remark 3.3. Let us construct the matrix

$$A = \begin{pmatrix} -\left\{\sum_{k=1}^{m} a_{11}^k(t)\right\}_* & \sum_{k=1}^{m} \left|a_{12}^k(t)\right|^* & \cdots & \sum_{k=1}^{m} \left|a_{1n}^k(t)\right|^* \\ \sum_{k=1}^{m} \left|a_{21}^k(t)\right|^* & -\left\{\sum_{k=1}^{m} a_{22}^k(t)\right\}_* & \cdots & \sum_{k=1}^{m} \left|a_{2n}^k(t)\right|^* \\ \cdots & \cdots & \cdots & \cdots \\ \sum_{k=1}^{m} \left|a_{n1}^k(t)\right|^* & \sum_{k=1}^{m} \left|a_{n2}^k(t)\right|^* & \cdots & -\left\{\sum_{k=1}^{m} a_{nn}^k(t)\right\}_* \end{pmatrix}, \tag{3.11}$$

which is constant and Metzler. The conditions (A) and (B) are equivalent for the matrix A according to Proposition 2.2. Thus, for system (3.1) with constant coefficients and $a_{ij}^k \le 0$ for $i \ne j$, $i, j = 1,\dots,n$, $k = 1,\dots,m$, the condition (2) of Theorem 3.2 is true if and only if the matrix (3.11) is Hurwitz.

Remark 3.4. Estimates (3.8), (3.10) play an important role in the analysis of systems with uncertain coefficients/delays from given intervals [24,4,29].

Theorem 3.3. *Assume that the following conditions are true:*

(1) *the conditions* (1) *and* (2) *of Theorem 3.1 are fulfilled;*

(2) *there exist positive numbers* $Z_1,\dots,Z_n$ *and nonnegative* $\varepsilon_1,\dots,\varepsilon_n$ *such that at least one* ε_{i_1} *is positive,* $\max_{1\le i\le n, i\ne i_1} \operatorname{esssup}_{t\ge 0} \sum_{k=1}^{m} a_{ii_1}^k(t) < 0$, *and*

$$\sum_{j=1}^{n} \sum_{k=1}^{m} a_{ij}^k(t) Z_j \ge \varepsilon_i, \quad t \in [0,+\infty),$$
$$i = 1,\dots,n. \tag{3.12}$$

Then system (3.1) *is exponentially stable.*

3.3. Generalization of Proposition 2.2 on delay systems

Theorem 3.4. *Assume that condition* (1) *of Theorem 3.1 is fulfilled, all the coefficients* a_{ij}^k *in system* (3.1) *are constants, and off-diagonal coefficients are nonpositive, i.e.* $a_{ij}^k \le 0$ *for* $i \ne j$, $i, j = 1,\dots,n$, $k = 1,\dots,m$. *Then system* (3.1) *is exponentially stable if and only if the matrix*

$$A = \begin{pmatrix} -\left\{\sum_{k=1}^{m} a_{11}^k\right\} & \sum_{k=1}^{m} \left|a_{12}^k\right| & \cdots & \sum_{k=1}^{m} \left|a_{1n}^k\right| \\ \sum_{k=1}^{m} \left|a_{21}^k\right| & -\left\{\sum_{k=1}^{m} a_{22}^k\right\} & \cdots & \sum_{k=1}^{m} \left|a_{2n}^k\right| \\ \cdots & \cdots & \cdots & \cdots \\ \sum_{k=1}^{m} \left|a_{n1}^k\right| & \sum_{k=1}^{m} \left|a_{n2}^k\right| & \cdots & -\left\{\sum_{k=1}^{m} a_{nn}^k\right\} \end{pmatrix}, \tag{3.13}$$

is Hurwitz.

Remark 3.5. Under the conditions of Theorem 3.4, the constant matrix of coefficients A in (3.1) is Metzler and system (3.1) is positive according to Theorem 3.1. Theorem 3.4, giving the equivalence of assertions (A) and (E) (see Proposition 2.2, where the matrix A is given by (3.13)), actually implies the equivalence of the assertions (A), (B), (C) and (E) for delay system (3.1).

Remark 3.6. If all conditions of Theorem 3.4 are satisfied and the matrix A defined by (3.13) is Hurwitz, the following equalities hold:

$$\lim_{t\to\infty} \int_0^t \sum_{j=1}^{n} \left|C_{ij}(t,s)\right| ds = z_i, \quad i = 1,\dots,n, \tag{3.14}$$

where $z = col\,\{z_1,\dots,z_n\}$ is the solution of the algebraic system $\sum_{j=1}^{n}\sum_{k=1}^{m} a_{ij}^k z_j = 1$, $i = 1,\dots,n$. Thus, solving the latter algebraic system, we arrive at the best possible result (3.14), i.e. estimates (3.8) on the Cauchy matrix in Theorem 3.2 cannot be improved.

3.4. Stabilization of systems with state-delays by the delayed feedback

Consider the system

$$x_i'(t) - \sum_{j=1}^{n} \sum_{k=1}^{m} a_{ij}^k(t) x_j(t - \theta_{ij}^k(t)) = u(t),$$

where $u(t) = -\sum_{j=1}^{n}\sum_{k=1}^{m} b_{ij}^k(t) x_j(t - \tau_{ij}^k(t))$. The closed-loop system is given by

$$x_i'(t) - \sum_{j=1}^{n} \sum_{k=1}^{m} a_{ij}^k(t) x_j(t - \theta_{ij}^k(t))$$
$$+ \sum_{j=1}^{n} \sum_{k=1}^{m} b_{ij}^k(t) x_j(t - \tau_{ij}^k(t)) = 0,$$
$$t \in [0,+\infty), \quad i = 1,\dots,n, \tag{3.15}$$
$$x_i(\xi) = 0, \quad \xi < 0, \ i = 1,\dots,n, \tag{3.16}$$

where the coefficients a_{ij}^k, b_{ij}^k and the delays τ_{ij}^k are measurable essentially bounded functions.

Remark 3.7. Application of Theorem 3.2 to the stability analysis of (3.15) may lead to hard limitations. Indeed, condition (2) of Theorem 3.2 implies that $\sum_{k=1}^{m} a_{ii}^k(t)$ have to be sufficiently large for $i = 1,\dots,n$, but from conditions 1(a) and 1(b) it follows that they have to be small enough. In the following assertion we avoid this limitation, assuming the smallness of the differences $\tau_{ij}^k(t) - \theta_{ij}^k(t)$ of input and state delays, the corresponding "compensation" of the coefficients $a_{ij}^k(t)$ by $b_{ij}^k(t)$ described by inequalities (3.17) and (3.18), and condition (3.20).

Denote $\Delta_i = \operatorname{esssup}_{t\geq 0}\{\tau_{ii}^+(t) - \theta_{ii}^-(t)\}$.

Theorem 3.5. *Assume that the following conditions* (1) *and* (2) *are satisfied.*

(1) *For every $i = 1, \ldots, n$, at least one of the conditions* 1(a) *or* 1(b) *holds:*

1(a) $a_{ii}^k(t) \geq 0$ *and* $b_{ii}^k \geq 0$ $(k = 1, \ldots, m)$ *and* $\tau_{ii}^r(t) \geq \theta_{ii}^k(t)$ *for* $k, r = 1, \ldots, m$, $\sum_{k=1}^m b_{ii}^k(t) \geq \frac{1}{e}\sum_{k=1}^m a_{ii}^k(t)$ *for* $t \in [0, +\infty)$, *and*

$$\int_{t-\tau_{ii}^+(t)}^t \left\{\sum_{k=1}^m b_{ii}^k(s) - \frac{1}{e}\sum_{k=1}^m a_{ii}^k(s)\right\} ds \leq \frac{1}{e},$$
$$t \in (0, +\infty), \tag{3.17}$$

where $b_{ii}^k(\xi) = a_{ii}^k(\xi) = 0$ *for* $\xi < 0$.

1(b) $a_{ii}^k(t) \leq 0$ *and* $b_{ii}^k(t) \leq 0$ $(k = 1, \ldots, m)$ *and* $\tau_{ii}^r(t) \geq \theta_{ii}^k(t)$ *for* $k, r = 1, \ldots, m$, $\sum_{k=1}^m (b_{ii}^k(t) - a_{ii}^k(t)) \geq 0$ *for* $t \in [0, +\infty)$,

$$\int_{t-\tau_{ii}^+(t)}^t \left\{\sum_{k=1}^m (b_{ii}^k(s) - a_{ii}^k(s))\right\} ds \leq \frac{1}{e},$$
$$t \in [0, +\infty), \tag{3.18}$$

where $b_{ii}^k(\xi) = a_{ii}^k(\xi) = 0$ *for* $\xi < 0$, *and*

$$\int_s^{s+\Delta_i} \sum_{k=1}^{m_i} \left|a_{ii}^k(\xi)\right| d\xi \leq \frac{1}{e} \quad \forall s \geq 0. \tag{3.19}$$

(2) *There exists a positive ε such that*

$$\sum_{k=1}^m b_{ii}^k(t) - \sum_{k=1}^m a_{ii}^k(t) \geq \varepsilon, \quad t \in [0, +\infty),$$
$$i = 1, \ldots, n \tag{3.20}$$

and

$$\max_{1\leq i\leq n} \sum_{j=1, j\neq i}^n \left\{\sum_{k=1}^m \left|a_{ij}^k\right|^* \left[1 + \frac{1}{\varepsilon}\left[\sum_{k=1}^m \left(\left|a_{ij}^k\right|^* + \left|b_{ij}^k\right|^*\right)\right]\right]\right.$$
$$\left. \times (\tau_{ij}^k(t) - \theta_{ij}^k(t))^* + \frac{1}{\varepsilon}\sum_{k=1}^m \left|a_{ij}^k(t) - b_{ij}^k(t)\right|^*\right\} < 1. \tag{3.21}$$

Then system (3.15) *is exponentially stable.*

One of the goals of this paper is to ensure stabilization of state-delay system

$$x_i'(t) + \sum_{j=1}^n a_{ij}(t) x_j(t - \theta_{ij}(t)) = u_i(t), \quad t \in [0, +\infty),$$
$$i = 1, \ldots, n, \tag{3.22}$$

where $a_{ij} : [0, +\infty) \to (-\infty, +\infty)$, $a_{ii} : [0, +\infty) \to [\varepsilon, +\infty)$, where $\varepsilon > 0$, $\theta_{ij} : [0, +\infty) \to [0, +\infty)$, by the delayed feedback control

$$u_i(t) = -\sum_{j=1}^n b_{ij}(t) x_j(t - \tau_{ij}(t)), \quad t \in [0, +\infty),$$
$$i = 1, \ldots, n, \tag{3.23}$$

where $a_{ij}, b_{ij}, \theta_{ij}, \tau_{ij} \in L_\infty$, and the input delays $\tau_{ij}(t)$ are greater than the state delays $\theta_{ij}(t)$ of this system. Speaking about exponential stabilization, we mean that the corresponding to (3.22), (3.23) closed-loop system is exponentially stable. Theorem 3.5 leads to the following corollaries.

Corollary 3.1. *System* (3.22) *can always be exponentially stabilized by the control* (3.23) *with the delays* $\tau_{ij}(t) > \theta_{ij}(t)$ *for* $i, j = 1, \ldots, n$, $t \in [0, +\infty)$.

The following assertion explains how the coefficients $b_{ij}(t)$ and delays $\tau_{ij}(t)$ could be chosen in (3.23). *Denote* $a = \max_{1\leq i,j\leq n, i\neq j} \left|a_{ij}\right|^*$.

Corollary 3.2. *It is sufficient for the exponential stabilization of system* (3.22) *by the control* (3.23), *to choose* $b_{ij}(t) = a_{ij}(t)$ *for* $i \neq j$, $b_{ii}(t) = a_{ii}(t) - \varepsilon \geq 0$, $\tau_{ij}(t) = \theta_{ij}(t) + \delta$, $\varepsilon > 0$, $\delta > 0$, *for* $i, j = 1, \ldots, n$, *such that*

$$\theta_{ii}(t) + \delta \leq \frac{1}{e\varepsilon}, \int_t^{t+\delta} a_{ii}(\xi) d\xi < \frac{1}{e}, \quad t \in [0, +\infty),$$
$$i = 1, \ldots, n, \tag{3.24}$$

$$(n-1)a\left\{1 + \frac{2}{\varepsilon}a\right\} < \frac{1}{\delta}. \tag{3.25}$$

It is clear that choosing ε small enough, we achieve the first inequality in (3.24), then we can choose δ small enough such that the second inequality in (3.24) and (3.25) is fulfilled.

3.5. Examples

Example 3.1. Consider the following system of delay differential equations with constant coefficients and time-varying delays:

$$\begin{aligned} &x_1'(t) + x_1(t - \theta_{11}(t)) - b_1 x_1(t - \theta_{11}(t) - \varepsilon_1(t)) \\ &\quad = a_{12} x_2(t - \theta_{12}(t)), \\ &x_2'(t) + x_2(t - \theta_{22}(t)) - b_2 x_2(t - \theta_{22}(t) - \varepsilon_2(t)) \\ &\quad = a_{21} x_1(t - \theta_{21}(t)), \end{aligned} \tag{3.26}$$

where $0 \leq \theta_{ii}(t) \leq 1.8$, $0 < \varepsilon_i(t) \leq 0.2$. The coefficients $a_{12} \geq 0$, $a_{21} \geq 0$, $b_1 \geq 0$ and $b_2 \geq 0$ are assumed to be uncertain. For $b_1 = b_2 = a_{12} = a_{21} = 0$ the system may be unstable (e.g. for $\theta_{ii} = const > \frac{\pi}{2}$). To stabilize (3.26) we choose $b_1 = 0.85$ and $b_2 = 0.9$, this leads to the following closed-loop system:

$$\begin{aligned} &x_1'(t) + x_1(t - \theta_{11}(t)) - 0.85 x_1(t - \theta_{11}(t) - \varepsilon_1(t)) \\ &\quad = a_{12} x_2(t - \theta_{12}(t)), \\ &x_2'(t) + x_2(t - \theta_{22}(t)) - 0.9 x_2(t - \theta_{22}(t) - \varepsilon_2(t)) \\ &\quad = a_{21} x_1(t - \theta_{21}(t)). \end{aligned} \tag{3.27}$$

The existing Lyapunov-based methods [20,21] and positivity-based methods for small delays [25,29,26] are not applicable to stability analysis of (3.27) even for $a_{12} = a_{21} = 0$. The results of [33] guarantee positivity and stability of the system with $a_{12} = a_{21} = 0$, but are not applicable if the latter coefficients are nonzero. Theorem 3.1 guarantees positivity for all nonnegative a_{12} and a_{21}. Moreover, Theorem 3.4 gives necessary and sufficient conditions for exponential stability of (3.27) in the form of the following inequality: $a_{12}a_{21} < 0.015$.

Example 3.2. Consider the system of n delay differential equations with bounded time-varying coefficients and bounded large delays

$$x_i'(t) + x_i(t - \theta_{ii}(t)) - b_i(t) x_i(t - \theta_{ii}(t) - \varepsilon_i(t))$$
$$= \sum_{j=1, j\neq i}^n a_{ij}(t) x_j(t - \theta_{ij}(t)), \tag{3.28}$$

where $0 < b_{i*} \leq b_i(t)$, $\varepsilon_i(t) > 0$ for $t \in [0, +\infty)$, $i = 1, \ldots, n$. Note that (3.28) for $n = 2$ coincides with (3.26), where the existing methods [25,29,26,20,21] are not applicable. Theorem 3.2 (see also

Remark 3.2, where $z_1 = \cdots = z_n = 1$) implies the following sufficient condition for the exponential stability of system (3.28):

$$(1 - b_{i*})(\theta_{ii}(t) + \varepsilon_i(t)) \le \frac{1}{e}, \quad \varepsilon_i(t) \le \frac{1}{e}, \tag{3.29}$$

and there exists $\varepsilon > 0$ such that $b_i(t) + \sum_{j=1, j\neq i}^{n} |a_{ij}(t)| + \varepsilon < 1$, $t \in [0, +\infty)$, $i = 1, \dots, n$. By Theorem 3.1 inequalities (3.29) guarantee positivity of system (3.28) for $a_{ij}(t) \ge 0$. In the case of constant coefficients $b_i(t) = b_i$, we obtain that the inequalities $\varepsilon_i(t) \le \frac{1}{e}$ and existence of a positive ε such that $\sum_{j=1, j\neq i}^{n} |a_{ij}(t)| + \varepsilon < 1 - b_i \le \frac{1}{e(\theta_{ii}(t)+\varepsilon_i(t))}$, $t \in [0, +\infty)$, $i = 1, \dots, n$, imply the exponential stability of (3.28). If all $a_{ij}(t) = a_{ij}$ are nonnegative constants and

$$\theta_{ii}(t) \le 5.9, \qquad 0 < \varepsilon_i(t) \le 0.1, \qquad 0.95 \le b_i < 1, \tag{3.30}$$

then (3.29) is fulfilled and, according to Theorem 3.4, (3.28) is exponentially stable if and only if $\frac{\Delta^i}{\Delta} > 0$ for all $i = 1, \dots, n$. Here

$$\Delta = \begin{vmatrix} 1 - b_1 & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & 1 - b_2 & \cdots & -a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ -a_{n1} & -a_{n2} & \cdots & 1 - b_n \end{vmatrix},$$

and Δ^i is obtained from Δ by setting $col\{1, \dots, 1\}$ instead of the ith column. Let us consider (3.28) for $n = 3$ under condition (3.30) and $a_{12} = 0.03$, $a_{13} = 0.02$, $a_{21} = 0.01$, $a_{23} = 0.02$, $a_{31} = 0.02$, $a_{32} = 0.01$, $b_2 = 0.96$, $b_3 = 0.95$. In this case system (3.28) is exponentially stable if and only if $b_1 < 0.975$.

Example 3.3. In [32] (see Example 2), a model which could, for example, be used to describe a formation of four vehicles is considered. Following [34] we consider a more realistic situation, where the distance between the vehicles and the position of a certain vehicle are measured with delays $\theta(t)$ and $\tau(t)$ respectively. Introducing delays in drivers' reactions, we can present this model in the following form:

$$\begin{aligned} x_1'(t) &= a_{13}[x_3(t - \theta(t)) - x_1(t - \theta(t))] - b_{11}x_1(t - \tau(t)), \\ x_2'(t) &= a_{23}[x_3(t - \theta(t)) - x_2(t - \theta(t))] \\ &\quad + a_{21}[x_1(t - \theta(t)) - x_2(t - \theta(t))] - b_{22}x_2(t - \tau(t)), \\ x_3'(t) &= a_{34}[x_4(t - \theta(t)) - x_3(t - \theta(t))] \\ &\quad + a_{32}[x_2(t - \theta(t)) - x_3(t - \theta(t))] + b_{33}x_3(t - \tau(t)), \\ x_4'(t) &= a_{43}[x_3(t - \theta(t)) - x_4(t - \theta(t))] - b_{44}(t)x_4(t - \tau(t)). \end{aligned} \tag{3.31}$$

In the case of $\theta = \tau = 0$, $b_{22} = b_{33} = 0$, $b_{11} = 1$, $b_{44} = 4$ we get the system of Example 2 in [32]. The parameters $a_{ij} \ge 0$ represent position adjustments based on distance measurements between the vehicles. The situation analyzed in [32], reflects the case when the first and fourth vehicles can maintain stable positions on their own, but the second and third vehicles rely on the distance measurements for stabilization. Our objective is to find conditions on the coefficients and delays in system (3.31) that guarantee its exponential stability. Note that the exponential stabilization in the case of $\theta = \tau = 0$ can be achieved also by the direct control only on third vehicle. Theorem 3.3 implies the following:

Proposition 3.1. *Let $\theta = \tau = 0$, $b_{11} = b_{22} = b_{44} = 0$. Then system (3.31) is exponentially stable for every positive number b_{33}.*

Theorems 3.2 and 3.1 imply the exponential stability and positivity of system (3.31) respectively:

Proposition 3.2. *If $\tau(t) > \theta(t)$, $b_{ii} > 0$ for $i = 1, \dots, 4$, $(a_{13} + b_{11})\tau(t) \le \frac{1}{e}$, $(a_{23}+a_{21}+b_{22})\tau(t) \le \frac{1}{e}$, $(a_{34}+a_{32}+b_{33})\tau(t) \le \frac{1}{e}$, $(a_{43} + b_{44})\tau(t) \le \frac{1}{e}$ for $t \in [0, +\infty)$, then system (3.31) is positive and exponentially stable.*

Consider the case where the control input enters only the third vehicle: $b_{33} > 0$, $b_{11} = b_{22} = b_{44} = 0$. Here Theorems 3.3 and 3.1 imply the exponential stability and nonnegativity of the Cauchy matrix:

Proposition 3.3. *If $\tau(t) > \theta(t)$, $b_{33} > 0$, $b_{11} = b_{22} = b_{44} = 0$, $a_{13} > 0$, $a_{23} > 0$, $a_{43} > 0$, $a_{13}\tau(t) \le \frac{1}{e}$, $(a_{23} + a_{21})\tau(t) \le \frac{1}{e}$, $(a_{34} + a_{32} + b_{33})\tau(t) \le \frac{1}{e}$, $a_{43}\tau(t) \le \frac{1}{e}$ for $t \in [0, +\infty)$, then system (3.31) is positive and exponentially stable.*

4. Proofs

Lemma 4.1. *Let the condition (1) of Theorem 3.1 be fulfilled, then the Cauchy functions of all scalar diagonal equations (3.3) are positive in the triangle $0 \le s \le t < +\infty$.*

Proof. The proof is a straightforward extension of the proof of Theorem 9.5 and Corollary 9.2 [33] from the case of one pair to m pairs of delays.

Proof of Theorem 3.1. The initial value problem

$$x_i'(t) + \sum_{j=1}^{n}\sum_{k=1}^{m} a_{ij}^k(t)x_j(t - \theta_{ij}^k(t)) = f_i(t), \qquad x_i(\xi) = 0,$$
$$\xi < 0,\ x_i(0) = 0,\ i = 1, \dots, n \tag{4.1}$$

is equivalent to the system of integral equations

$$\begin{aligned} x_i(t) &= -\int_0^t C_i(t,s) \sum_{j=1, j\neq i}^{n}\sum_{k=1}^{m} a_{ij}^k(s)x_j(s - \theta_{ij}^k(s))ds \\ &\quad + \int_0^t C_i(t,s)f_i(s)ds, \quad i = 1, \dots, n. \end{aligned} \tag{4.2}$$

System (4.2) can be presented in the vector form

$x(t) = (Tx)(t) + \psi(t)$,

where $x = col\{x_1, \dots, x_n\}$, with the operator

$$(Tx)(t) = -col\left[\int_0^t C_i(t,s) \sum_{j=1, j\neq i}^{n}\sum_{k=1}^{m} a_{ij}^k(s)x_j(s - \theta_{ij}^k(s))ds\right]_{i=1}^{n}, \tag{4.3}$$

$$x_i(\xi) = 0, \quad \xi < 0,\ i = 1, \dots, n, \tag{4.4}$$

and the vector-function

$$\psi(t) = col\left[\int_0^t C_i(t,s)f_i(s)ds\right]_{i=1}^{n}. \tag{4.5}$$

Lemma 4.1 implies the positivity of the Cauchy functions $C_i(t,s)$ of all n scalar diagonal equations (3.3) in the triangle $0 \le s \le t < +\infty$. Together with the assumption $a_{ij}^k(t) \le 0$ for $i \neq j$, $i, j = 1, \dots, n$, $k = 1, \dots, m$, this leads to the positivity of the operator T. On every finite interval $[0, \omega]$, the spectral radius of the operator $T : C^n[0, \omega] \to C^n[0, \omega]$, where $C^n[0, \omega]$ is the space of continuous vector-functions $y : [0, \omega] \to \mathbb{R}^n$, is zero [24]. There exists a bounded operator $(I - T)^{-1} = I + T + T^2 + T^3 + \cdots$, which is positive. Now it is clear that for every vector $f = col\{f_1, \dots, f_n\}$ with nonnegative components $f_1, \dots, f_n$, all the components of the solution-vector $x = col\{x_1, \dots, x_n\}$ of system (4.2) will be nonnegative. We can conclude from solution representation (2.7) that all elements of the Cauchy matrix $C(t,s)$ are nonnegative in the triangle $0 \le s \le t < +\infty$.

Proof of Theorem 3.2. Assume for simplicity that $t-\theta_{ij}^k(t)\geq 0$ for $i,j=1,\dots,n$, $k=1,\dots,m$, $t\geq 0$. Consider the initial value problems for the differential equations

$$x_i'(t)+\sum_{j=1}^{n}\sum_{k=1}^{m}a_{ij}^k(t)x_j(t-\theta_{ij}^k(t))=f_i(t),\quad t\in[0,+\infty),\quad i=1,\dots,n, \tag{4.6}$$

and

$$y_i'(t)+\sum_{k=1}^{m}a_{ii}^k(t)y_i(t-\theta_{ii}^k(t))-\sum_{j=1,j\neq i}^{m}\sum_{k=1}^{m}\left|a_{ij}^k(t)\right|y_j(t-\theta_{ij}^k(t))=f_i(t),\quad t\in[0,+\infty),\ i=1,\dots,n \tag{4.7}$$

with the same initial conditions

$$x_i(0)=\beta_i,\qquad y_i(0)=\beta_i,\quad i=1,\dots,n. \tag{4.8}$$

These problems are equivalent to the systems of the integral equations

$$x_i(t)=-\int_0^t C_i(t,s)\sum_{j=1,j\neq i}^{m}\sum_{k=1}^{m}a_{ij}^k(s)x_j(s-\theta_{ij}^k(s))ds+\int_0^t C_i(t,s)f_i(s)ds+C_i(t,0)\beta_i,\quad i=1,\dots,n, \tag{4.9}$$

$$y_i(t)=\int_0^t C_i(t,s)\sum_{j=1,j\neq i}^{m}\sum_{k=1}^{m}\left|a_{ij}^k(s)\right|y_j(s-\theta_{ij}^k(s))ds+\int_0^t C_i(t,s)f_i(s)ds+C_i(t,0)\beta_i,\quad i=1,\dots,n, \tag{4.10}$$

where $C_i(t,s)$ are the Cauchy functions of the diagonal equations (3.3), respectively.

Define the operators T and $|T|:C^n[0,\infty)\to C^n[0,\infty)$, where $C^n[0,\infty)$ is the space of measurable essentially bounded functions by the equalities (4.3) and

$$(|T|y)(t)=col\left[\int_0^t C_i(t,s)\sum_{j=1,j\neq i}^{n}\sum_{k=1}^{m}\left|a_{ij}^k(s)\right|y_j(s-\theta_{ij}(s))ds\right]_{i=1}^{n}, \tag{4.11}$$

respectively.

The condition (1), according to Lemma 4.1, implies that the Cauchy functions $C_i(t,s)$ of the diagonal scalar equations (3.3) satisfy the inequalities $C_i(t,s)>0$ in the triangle $0\leq s\leq t<+\infty$. The operator $|T|:C^n[0,\infty)\to C^n[0,\infty)$ is a positive Volterra operator. On every finite interval $[0,\omega]$, the spectral radius of the operator $|T|:C^n[0,\omega]\to C^n[0,\omega]$ is zero (where $C^n[0,\omega]$ is the space of continuous vector-functions $y:[0,\omega]\to R^n$) [24]. Then there exists a bounded operator $(I-|T|)^{-1}=I+|T|+|T|^2+|T|^3+\cdots$, which is positive. Since $y=(I-|T|)^{-1}(\psi+r)$, where ψ is defined by (4.5) and $r(t)=col\{r_1(t),\dots,r_n(t)\}, r_i(t)=C_i(t,0)\beta_i$, then for every nonnegative f_i and β_i, $i=1,\dots,n$, all the components y_i of the solution-vector to integral equation (4.10) and to problem (4.7), (4.8) are nonnegative. It means that all elements $\left\{C_{ij}^0(t,s)\right\}_{i,j=1,\dots,n}$ of the Cauchy matrix $C^0(t,s)$ of system (4.7) are nonnegative.

Substituting $y_i(t)=z_i$, $i=1,\dots,n$, into the left-hand side of (4.7), we obtain there $\sum_{k=1}^m a_{ii}^k(t)z_i-\sum_{j=1,j\neq i}^n\sum_{k=1}^m\left|a_{ij}^k(t)\right|z_j$. Now it is clear that $y_i(t)=z_i$ satisfies the following initial value problem:

$$y_i'(t)+\sum_{k=1}^{m}a_{ii}^k(t)y_i(t-\theta_{ii}^k(t))-\sum_{j=1,j\neq i}^{m}\sum_{k=1}^{m}\left|a_{ij}^k(t)\right|y_j(t-\theta_{ij}^k(t))=\psi_i(t),\quad t\in[0,+\infty),\ y_i(0)=z_i,\ i=1,\dots,n \tag{4.12}$$

where

$$\psi_i(t)=\sum_{k=1}^{m}a_{ii}^k(t)z_i-\sum_{j=1,j\neq i}^{n}\sum_{k=1}^{m}\left|a_{ij}^k(t)\right|z_j,\quad t\in[0,+\infty),\ i=1,\dots,n. \tag{4.13}$$

According to condition (2) of Theorem 3.2, $\psi_i(t)\geq 1>0$. It is clear that in the case $\beta_i=z_i$, $i=1,\dots,n$, this vector function $z=col\{z_1,\dots,z_n\}$ satisfies also the system of integral equations (4.10). We have

$$z_i=\int_0^t C_i(t,s)\sum_{j=1,j\neq i}^{n}\left|a_{ij}(s)\right|z_jds+\int_0^t C_i(t,s)\psi_i(s)ds+C_i(t,0)z_i, \tag{4.14}$$

for $i=1,\dots,n$, leading to

$$z=(|T|z)(t)+\Phi(t), \tag{4.15}$$

where the vector $\Phi(t)$ is defined by the equality

$$\Phi(t)=col\left\{\int_0^t C_i(t,s)\psi_i(s)ds+C_i(t,0)z_i\right\}_{i=1}^{n}. \tag{4.16}$$

From the inequalities $\psi_i(s)\geq 1>0$ for $t\in[0,+\infty)$, $i=1,\dots,n$, it follows that every component of the vector $\Phi(t)$ is greater than a positive constant.

The solution $y=col\{y_1,\dots,y_n\}$ of the initial value problem (4.12) can be written in the form

$$y(t)=\int_0^t C^0(t,s)\psi(s)ds+C^0(t,0)z, \tag{4.17}$$

where $C^0(t,s)=\left\{C_{ij}^0(t,s)\right\}_{i,j=1}^{n}$ is the Cauchy matrix of system (4.7) and $\psi=col\{\psi_1,\dots,\psi_n\}$.

By virtue of the theorem about the integral inequality (see Theorem 5.6, in paragraph 2, Chapter 2 of the book [35]), the spectral radius of the completely continuous operator $|T|:C^n[0,\infty)\to C^n[0,\infty)$ is less than one. It follows from this fact and nonnegativity of the Cauchy matrix $C^0(t,s)$ that the solution $y=col\{y_1,\dots,y_n\}$ of system (4.7) is bounded for every bounded right hand side $f=col\{f_1,\dots,f_n\}$. Now the Bohl–Perron theorem (see Proposition 2.1) implies the exponential estimate (2.8), i.e. there exist positive numbers N and α such that

$$\left|C_{ij}^0(t,s)\right|\leq N\exp\{-\alpha(t-s)\},\quad i,j=1,\dots,n,\quad \leq s\leq t<+\infty. \tag{4.18}$$

Using representation (4.17) for solution of initial value problem (4.12), we obtain

$$z_i=\int_0^t\sum_{j=1}^{n}C_{ij}^0(t,s)\psi_i(s)ds+\sum_{j=1}^{n}C_{ij}^0(t,0)z_i,\quad i=1,\dots,n. \tag{4.19}$$

Nonnegativity of all elements $C^0_{ij}(t,s)$ of the Cauchy matrix $C^0(t,s)$ implies the inequality

$$\int_0^t \sum_{j=1}^n C^0_{ij}(t,s)ds \le z_i, \quad i=1,\dots,n. \tag{4.20}$$

By virtue of Theorem 5.3 in Chapter 2 of the book [35], the following inequality for the spectral radii of the operators T and $|T|: C^n[0,\infty) \to C^n[0,\infty)$ is true: $\rho(T) \le \rho(|T|) < 1$.

Comparing now solutions x and y of problems (4.6)–(4.8) and (4.8) respectively, we obtain $|x_i(t)| \le |y_i(t)|$ for $t \in [0,+\infty)$, $i=1,\dots,n$. Using the representations of solutions, we obtain $\left|C_{ij}(t,s)\right| \le C^0_{ij}(t,s)$ for $i,j=1,\dots,n$, in the triangle $0 \le s \le t < +\infty$ and consequently exponential estimate (2.8) is true for the Cauchy matrix $C(t,s)$ of system (3.1).

Assume now that the inequalities (or at least one of them) $t-\theta^k_{ij}(t) \ge 0$ for $t \ge 0$ are not true. In this case we extend all the coefficients on the interval $[-\Theta,0)$ such that $\theta_{ij}(t) \equiv 0$ for $t \in [-\Theta,0)$, the coefficients $a_{ij}(t)$ can be extended such that condition (1) of Theorem 3.1 is fulfilled for $t \in [-\Theta,+\infty)$.

It is clear that the Cauchy matrices of system (3.1) and new extended on $[-\Theta,+\infty)$ system coincide in the triangle $0 \le s \le t < +\infty$. Repeating all the proofs on the interval $[-\Theta,+\infty)$, we obtain the proof of the assertions (a) and (b) of Theorem 3.2 without the assumption $t-\theta_{ij}(t) \ge 0$ for $t \ge 0$.

To prove estimate (3.10) consider the problem

$$\begin{aligned}(\pounds^s_i v)(t,s) \equiv\ & x_i'(t) + \sum_{k=1}^m a^k_{ii}(t)x_i(t-\theta^k_{ii}(t)) \\ & - \sum_{j=1,j\ne i}^n \sum_{k=1}^m \left|a^k_{ij}(t)\right| x_j(t-\theta^k_{ij}(t)) = 0, \\ & t \in [s,+\infty), \quad i=1,\dots,n, \end{aligned} \tag{4.21}$$

$$x_i(\xi) = 0, \quad \xi < s,\ i=1,\dots,n. \tag{4.22}$$

It is clear from the definition of the Cauchy matrix that for every fixed s, the columns of the $C^0(t,s)$ satisfy the problem (4.21), (4.22) with the initial condition $C^0(s,s)=I$. The matrix-function $v(t,s) = col\{v_1(t,s),\dots,v_n(t,s)\}$, where the components $v_{ij}(t,s) (i=1,\dots,n)$ of the vector $v_j(t,s)$ are defined by the equalities

$$v_{ij}(t,s) = \left\{\begin{matrix} 1, & t<s+\Theta \\ \exp\{-\beta(t-s)\}, & t \ge s+\Theta \end{matrix}\right\}, \quad i=1,\dots,n,$$

satisfies the inequalities

$$(\pounds^s_i v)(t,s) \ge 0, \quad t \in [s,+\infty),\ i=1,\dots,n. \tag{4.23}$$

$$v_i(\xi)=0, \quad \xi<s,\ i=1,\dots,n. \tag{4.24}$$

All entries of the Cauchy matrix $C^0(t,s)$ of system (4.7) are nonnegative in the triangle $0 \le s \le t < +\infty$ according to Theorem 3.1 and, consequently. This implies $v(t,s) \ge C^0(t,s)$ and, using the inequality $C^0(t,s) \ge |C(t,s)|$, we obtain that $v(t,s) \ge C^0(t,s) \ge |C(t,s)|$ in the triangle $0 \le s \le t < +\infty$.

Proof of Theorem 3.3. Let us demonstrate that choosing sufficiently small δ we can obtain the vector $z_{i_1} = Z_{i_1}$, $z_i = Z_i(1+\delta)$ for $i=1,\dots,n$, $i \ne i_1$, satisfying the condition (2) of Theorem 3.2.

Denoting $\varepsilon = -\max_{1\le i\le n} \operatorname{esssup}_{t\ge 0} \sum_{k=1,i\ne i_0}^m a^k_{ii_1}(t)$, and, using the assumption that $\varepsilon > 0$ in the condition (2), we get for all $i \ne i_1$ the following

$$\begin{aligned}\sum_{j=1}^n\sum_{k=1}^m a^k_{ij}(t)z_j &= \sum_{j=1}^n\sum_{k=1}^m a^k_{ij}(t)Z_j + \delta\sum_{j=1}^n\sum_{k=1}^m a^k_{ij}(t)Z_j \\ &- \delta\sum_{k=1}^m a^k_{ii_1}(t)Z_{i_1} \ge \varepsilon_i + \delta\varepsilon_i + \varepsilon \ge \varepsilon > 0, \quad t \in [0,+\infty).\end{aligned} \tag{4.25}$$

For $i=i_1$ we can choose sufficiently small δ such that $\varepsilon_0 \equiv \delta\sum_{k=1}^m a^k_{i_1i_1}(t)Z_{i_1} < \varepsilon_{i_1}$ and get the existence of a positive ε such that

$$\begin{aligned}\sum_{j=1}^n\sum_{k=1}^m a^k_{i_1j}(t)z_j &= \sum_{j=1}^n\sum_{k=1}^m a^k_{i_1j}(t)Z_j + \delta\sum_{j=1}^n\sum_{k=1}^m a^k_{i_1j}(t)Z_j \\ &- \delta\sum_{k=1}^m a^k_{i_1i_1}(t)Z_{i_1} \ge \varepsilon_{i_1} + \delta\varepsilon_{i_1} - \varepsilon_0 \ge \varepsilon > 0, \\ & t \in [0,+\infty).\end{aligned} \tag{4.26}$$

Thus, the condition (2) of Theorem 3.2 is fulfilled. Reference to Theorem 3.2 completes the proof.

Proof of Theorem 3.4. In order to prove sufficiency we note that conditions (A) and (B) are equivalent for the Metzler matrix A (see Proposition 2.2). The condition (B) coincides with the condition (2) of Theorem 3.2. Then all the conditions of Theorem 3.2 are fulfilled and, according to Theorem 3.2, we obtain the exponential stability of system (3.1).

To prove necessity, let us consider initial value problem (4.6), (4.8), where $\beta_i = z_i, f_i(t) \equiv 1$ for $t \ge \Theta, i = 1,\dots,n$, $\Theta = \max_{1\le i,j\le n}\max_{1\le k\le m} \operatorname{esssup}_{t\ge 0}\theta^k_{ij}(t)$. The constant vector $z = col\{z_1,\dots,z_n\}$ satisfies this system. The representation of solutions (2.7) leads to the equalities

$$\begin{aligned} z_i &= \int_0^t \sum_{j=1}^n C_{ij}(t,s)f_i(s)ds + \sum_{j=1}^n C_{ij}(t,0)z_i \\ &= \int_0^\Theta \sum_{j=1}^n C_{ij}(t,s)f_i(s)ds + \int_\Theta^t \sum_{j=1}^n C_{ij}(t,s)ds \\ &+ \sum_{j=1}^n C_{ij}(t,0)z_i, \quad i=1,\dots,n. \end{aligned} \tag{4.27}$$

The exponential estimate (2.8) of the Cauchy matrix implies that

$$\int_0^\Theta \sum_{j=1}^n C_{ij}(t,s)f_i(s)ds \to 0, \quad C_{ij}(t,0) \to 0 \text{ for } t \to +\infty, \quad i=1,\dots,n. \tag{4.28}$$

The condition $C_{ii}(s,s) = 1$ leads to existence of the interval $[s,s+\delta]$ where $C_{ii}(t,s) > 0$. This and nonnegativity of $C_{ij}(t,s)$ lead to the conclusion that all components of the constant vector z are positive. We have proven that the exponential estimate (2.8) implies assertion (B) for system (3.1). Equivalence of (A) and (B) (see Proposition 2.2) completes the proof.

Proof of Theorem 3.5. Assume for simplicity that $t-\theta^k_{ij}(t) \ge 0$ and $t-\tau^k_{ij}(t) \ge 0$ and use the transform

$$x_i(t) = \int_0^t C_i(t,s)u_i(s)ds, \quad t \in [0,+\infty),\ i=1,\dots,n, \tag{4.29}$$

where $C_i(t,s)$ are the Cauchy functions of the diagonal scalar equations

$$x_i'(t) - \sum_{k=1}^m a^k_{ii}(t)x_i(t-\theta^k_{ii}(t)) + \sum_{k=1}^m b^k_{ii}(t)x_i(t-\tau^k_{ii}(t)) = 0, \quad t \in [0,+\infty), \tag{4.30}$$

where $x_i(\xi)=0$, $\xi<0$, and $u_i \in L_\infty$, $i=1,\dots,n$.

After substitution (4.29) into the system

$$x_i'(t)-\sum_{j=1}^{n}\sum_{k=1}^{m}a_{ij}^k(t)x_j(t-\theta_{ij}^k(t))+\sum_{j=1}^{n}\sum_{k=1}^{m}b_{ij}^k(t)x_j(t-\tau_{ij}^k(t))=f_i(t), \quad (4.31)$$

$i=1,\dots,n$, with the zero initial functions, we get the following system

$$u_i(t)-\sum_{j=1,j\neq i}^{n}\sum_{k=1}^{m}a_{ij}^k(t)\int_{t-\tau_{ij}^k(t)}^{t-\theta_{ij}^k(t)}\left\{u_j(s)+\int_0^s\frac{\partial}{\partial s}C_i(s,\xi)u_j(\xi)d\xi\right\}ds+\sum_{j=1,j\neq i}^{n}\sum_{k=1}^{m}[b_{ij}^k(t)-a_{ij}^k(t)]\int_0^{t-\tau_{ij}^k(t)}C_j(t,s)u_j(s)ds=f_i(t), \quad (4.32)$$

for $t\in[0,+\infty),\ i=1,\dots,n$, for $u=col\{u_1,\dots,u_n\}, u_i\in L_\infty^n$.

According to the definition of the Cauchy function, $C_i(t,s)$ as a function of t for fixed s satisfies the equation

$$\frac{\partial}{\partial t}C_i(t,s)=\sum_{k=1}^{m}a_{ii}^k(t)C_i(t-\theta_{ii}^k(t),s)-\sum_{k=1}^{m}b_{ii}^k(t)C_i(t-\tau_{ii}^k(t),s), \quad (4.33)$$

where $C_i(t,s)=0$ for $t<s$. According to Lemma 4.1, the condition (1) implies that $C_i(t,s)\geq 0$ in the triangle $0\leq s\leq t<+\infty$. From this and the condition (2) it follows that

$$\int_0^t C_i(t,s)ds\leq\frac{1}{\varepsilon},\quad i=1,\dots,n.$$

Define the operator $\Omega: L_\infty^n\to L_\infty^n$ by the formula

$$(\Omega u)(t)=col\left\{\sum_{j=1,j\neq i}^{n}\sum_{k=1}^{m}a_{ij}^k(t)\int_{t-\tau_{ij}^k(t)}^{t-\theta_{ij}^k(t)}\left\{u_j(s)+\int_0^s\frac{\partial}{\partial s}C_j(s,\xi)u_j(\xi)d\xi\right\}ds-\sum_{j=1,j\neq i}^{n}\sum_{k=1}^{m}[b_{ij}^k(t)-a_{ij}^k(t)]\int_0^{t-\tau_{ij}^k(t)}C_j(t-\tau_{ij}^k(t),s)u_j(s)ds\right\}_{i=1}^{n}. \quad (4.34)$$

Estimating its norm, we obtain

$$\|\Omega\|\leq\max_{1\leq i\leq n}\sum_{j=1,j\neq i}^{n}\left\{\sum_{k=1}^{m}\left|a_{ij}^k\right|^*\left[1+\frac{1}{\varepsilon}\sum_{k=1}^{m}\left(\left|a_{ij}^k\right|^*+\left|b_{ij}^k\right|^*\right)\right]\cdot(\tau_{ij}^k(t)-\theta_{ij}^k(t))+\frac{1}{\varepsilon}\sum_{k=1}^{m}\left|b_{ij}^k(t)-a_{ij}^k(t)\right|\right\}. \quad (4.35)$$

Now it is clear that inequality (3.21) implies $\|\Omega\|<1$. It follows from this fact that solution $x=col\{x_1,\dots,x_n\}$ of system (3.15) is bounded for every bounded right hand side $f=col\{f_1,\dots,f_n\}$. The Bohl–Perron theorem (see Proposition 2.1) claims that this fact implies the exponential estimates of the Cauchy matrix of system (3.15).

Let us assume now that at least one of the inequalities $t-\theta_{ij}^k(t)\geq 0$ or $t-\tau_{ij}^k(t)\geq 0$ for $t\geq 0, i,j=1,\dots,n, k=1,\dots,m$, is not true. In this case we extend all the coefficients on the interval $[-\Theta,0)$, where $\Theta=\max_{1\leq i,j\leq n}\max_{1\leq k\leq m}\left\{\text{esssup}_{t\geq 0}\left\{\theta_{ij}^k(t),\tau_{ij}^k(t)\right\}\right\}$, such that $\theta_{ij}^k(t)\equiv 0$ and $\tau_{ij}^k(t)\equiv 0$ for $t\in[-\Theta,0)$. The coefficients $a_{ij}^k(t)$ and $b_{ii}^k(t)$ can be extended such that the conditions (1) and (2) of Theorem 3.5 are fulfilled for $t\in[-\Theta,+\infty)$. This completes the proof of Theorem 3.5.

Proof of Corollary 3.1. Under the conditions of Theorem 3.5, we have: $m=1, m_i=1, \tau_{ij}^1(t)=\tau_{ij}(t), \theta_{ij}^1(t)=\theta_{ij}(t), \tau_{ii}^+(t)=\tau_{ii}(t), \theta_{ii}^-(t)=\theta_{ii}(t), b_{ii}^1(t)=b_{ii}(t), a_{ij}^1(t)=-a_{ij}(t)$. Choosing $b_{ii}(t)$ such that $b_{ii}(t)-a_{ii}(t)$ is small enough, we can guarantee the feasibility of inequality (3.18). Choosing $\tau_{ii}(t)$ close to $\theta_{ii}(t)$, we can achieve that (3.19) is valid. Thus we can guarantee that the condition 1(b) is satisfied. It is clear also that, choosing $b_{ij}(t)=a_{ij}(t)$ for all off-diagonal coefficients ($i\neq j$) and $\tau_{ij}(t)$ close enough to $\theta_{ij}(t)$, we can guarantee that the inequality (3.21) is valid.

Proof of Corollary 3.2. To prove it, we only note that conditions (3.24) and (3.25) imply inequalities (3.18), (3.19) and (3.21).

References

[1] W.M. Haddad, V. Chellaboina, Stability theory for nonnegative and compartmental dynamical systems with time delay, Systems Control Lett. 51 (5) (2004) 355–361.
[2] S.A. Tchaplygin, New Method of Approximate Integration of Differential Equations, GTTI, Moscow-Leningrad, 1932.
[3] V. Lakshmikantham, S. Leela, Differential and Integral Inequalities, Academic Press, 1969.
[4] N.V. Azbelev, P.M. Simonov, Stability of Differential Equations with Aftereffect, in: Stability and Control: Theory, Methods and Applications, vol. 20, Taylor & Francis, London, 2003.
[5] T. Wazewski, Systemes des equations et des inegalites differentielled aux deuxieme membres et leurs applications, Ann. Polon. Math. 23 (1950) 112–166.
[6] W.M. Haddad, V. Chellaboina, Q. Hui, Nonnegative and Compartmental Dynamical Systems, Princeton University Press, 2010.
[7] S.-I. Niculescu, Delay Effects on Stability: A Robust Control Approach, in: LNCIS, vol. 269, Springer-Verlag, Heidelberg, 2001.
[8] C. Briat, Robust stability and stabilization of uncertain linear positive systems via integral linear constraints- L_l and L_∞-gains characterizations, Internat. J. Robust Nonlinear Control 23 (17) (2013) 1932–1954.
[9] M. Buslowicz, Robust stability of positive continuous time linear systems with delays, Int. J. Appl. Math. Comput. Sci. 20 (4) (2010) 665–670.
[10] F. Cacace, A. Germani, C. Manes, R. Setola, A new approach to the internally positive representation of linear MIMO systems, IEEE Trans. Automat. Control 57 (1) (2012) 119–134.
[11] I. Gyori, Interaction between oscillation and global asymptotic stability in delay differential equations, Differential Integral Equations 3 (1990) 181–200.
[12] J. Hofbauer, J.W.-H. So, Diagonal dominance and harmless off-diagonal delays, Proc. Amer. Math. Soc. 128 (2000) 2675–2682.
[13] H.R. Feyzmahdavian, T. Charalambous, M. Johansson, Exponential stability of homogeneous positive systems of degree one with time-varying delays, IEEE Trans. Automat. Control 59 (6) (2014) 1594–1599.
[14] T. Kaczorek, Stability of positive continuous-time linear systems with delays, Bull. Pol. Acad. Sci. Tech. Sci. 57 (4) (2009) 395–398.
[15] X. Liu, W. Yu, L. Wang, Stability analysis for continuous-time positive systems with time-varying delays, IEEE Trans. Automat. Control 55 (4) (2010) 1024–1028.
[16] P.H.A. Ngoc, Stability of positive differential systems with delay, IEEE Trans. Automat. Control 58 (1) (2013) 203–209.
[17] A.P. Tchangani, M. Dambrine, J.P. Richard, Stability, attraction domains, and ultimate boundedness for nonlinear neutral systems, Math. Comput. Simul. 45 (1998) 2991–2998.
[18] A.P. Tchangani, M. Dambrine, J.P. Richard, V.B. Kolmanovskii, Stability of nonlinear differential equations with distributed delay, Nonlinear Anal. 34 (1998) 1081–1095.
[19] S.A. Campbell, Delay independent stability for additive neural networks, Differential Equations Dynam. Systems 9 (3–4) (2001) 115–138.
[20] E. Fridman, Introduction to Time-Delay Systems: Analysis and Control, Springer, 2014.
[21] K. Gu, V. Kharitonov, J. Chen, Stability of Time-delay Systems, Birkhauser, Boston, 2003.
[22] Z. Artstein, Linear systems with delayed controls: a reduction, IEEE Trans. Automat. Control AC-27 (1982) 869–879.
[23] F. Mazenc, S.-I. Niculescu, Generating positive and stable solutions through delayed state feedback, Automatica 47 (3) (2011) 525–533.
[24] N.V. Azbelev, V.P. Maksimov, L.F. Rakhmatullina, Introduction to the Theory of Functional Differential Equations, in: Advanced Series in Math. Science and Engineering, vol. 3, World Federation Publisher Company, Atlanta, GA, 1995.
[25] D. Bainov, A. Domoshnitsky, Nonnegativity of the Cauchy matrix and exponential stability of a neutral type system of functional differential equations, Extracta Math. 8 (1992) 75–82.
[26] A. Domoshnitsky, M.V. Sheina, Nonnegativity of Cauchy matrix and stability of systems with delay, Differ. Uravn. 25 (1989) 201–208.
[27] I. Gyori, F. Hartung, Fundamental solution and asymptotic stability of linear delay differential equations, Dyn. Contin. Discrete Impuls. Syst. 13 (2) (2006) 261–287.
[28] J. Zhu, J. Chen, Stability of systems with time-varying delays: An $\mathcal{L}_1$ small-gain perspective, Automatica 52 (2015) 260–265.

[29] A. Domoshnitsky, M. Gitman, R. Shklyar, Stability and estimate of solution to uncertain neutral delay systems, Bound. Value Probl. 2014 (2014) 55. http://dx.doi.org/10.1186/1687-2770-2014-55.
[30] F. Mazenc, Stability analysis of time-varying neutral time-delay systems, IEEE Trans. Automat. Control 60 (2) (2015) 540–546.
[31] L. Berezansky, E. Braverman, On nonoscillation and stability for systems of differential equations with a distributed delay, Automatica 48 (2012) 612–618.
[32] A. Rantzer, Distributed control of positive systems, 14 May, 2014. arXiv:1203.0047v3 [math OC].
[33] A. Domoshnitsky, Maximum principles and nonoscillation intervals for first order Volterra functional differential equations, Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal. 15 (2008) 769–814.
[34] R. Sipahi, S.I. Niculescu, C.T. Abdallad, W. Michiels, K. Gu, Stability and stabilization of systems with time delay: Limitations and opportunities, IEEE Control Syst. Mag. 31 (1) (2011) 38–65.
[35] M.A. Krasnosel'skii, G.M. Vainikko, P.P. Zabreiko, Ja.B. Rutitskii, V. Ja. Stezenko, Approximate Solutions of Operator Equations, Wolters-Noordhoff Publishing, Groningen, 1972.