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STABILITY OF SINGULARLY PERTURBED DIFFERENTIAL-DIFFERENCE SYSTEMS: A LMI APPROACH

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Abstract. For linear singularly perturbed system with delay sufficient conditions for stability for all small enough values of singular perturbation parameter ε are obtained in the general case, when delay and ε are independent. The sufficient delay-dependent conditions are given in terms of linear matrix inequalities (LMIs) by applying an appropriate Lyapunov-Krasovskii functional. LMIs are derived by using a descriptor model transformation and Park's inequality for bounding cross terms. A memoryless state-feedback stabilizing controller is obtained. Solution is given also in the case of systems with polytopic parameter uncertainties. Numerical examples illustrate the effectiveness of the new theory.

Keywords. Singular perturbations, time-delay systems, stability, LMI, delay-dependent criteria.

AMS (MOS) subject classification: This is optional. But please supply them whenever possible.

1 Introduction

It is well-known that if the ordinary differential system of equations is asymptotically stable, then this property is robust with respect to small delays (see e.g. [2], [12]). Examples of the systems, where small delays change the stability of the system are given in [13] (see also references therein). All these examples are infinite-dimensional systems, e.g. difference systems, neutral type systems with unstable difference operator or systems of partial differential equations. Another example of a system, sensitive to small delays, is a descriptor system [18]. Recently a new example has been given of a finite dimensional system that may be destabilized by introduction of small delay in the loop [5]. This is a singularly perturbed system. Consider the following simple example:

$$\varepsilon\dot{x}(t) = u(t), \quad u(t) = -x(t-h), \tag{1}$$

where $x(t) \in R$ and $\varepsilon > 0$ is a small parameter. Eq. (1) is stable for $h = 0$, however for small delays $h = \varepsilon g$ with $g > \pi/2$ this system becomes unstable (see e.g.[2]).

Stability of singularly perturbed systems with delays has been studied in two cases: 1) h is proportional to ε and 2) ε and h are independent. The first case, being less general than the second one, is encountered in many publications (see e.g. [3], [10] and references therein). The second case has been studied in the frequency domain in [19], [20] (see also references therein). A Lyapunov-based approach to the problem leading to LMIs has been introduced in [5] for the general case of independent delay and ε. LMI conditions are only sufficient and, thus more conservative. However the method of LMIs is better (than the frequency domain methods) adapted for robust stability of systems with uncertainties and for other control problems (see e.g.[17], [22]).

LMI stability conditions of [5] are based on the conservative model transformation of regular systems with delay used by many authors (see [17], [16] and references therein). The conservatism of [5] , as well as in the regular case (see e.g. [14], [11]) is twofold: the transformed equation is not equivalent to the corresponding differential equation and the bounds placed upon cross terms are wasteful. Recently a new (equivalent to the original equation) model transformation - a descriptor one - has been introduced for stability analysis of regular systems with delay [4]. Moreover, a new bounding of the cross terms and new delay-dependent stability criterion have been obtained in [21].

In the present paper we adopt the methods of [4] and [21] for constructing appropriate Lyapunov-Krasovskii functionals and deriving LMI stability conditions for singularly perturbed systems with delay in the case of independent delay and ε. We show that if a certain ε-independent LMI is feasible than the system is asymptotically stable for all small enough $\varepsilon \geq 0$. Moreover, given $\varepsilon > 0$ we obtain ε-dependent LMI conditions for stability. Thus, by solving the latter LMI for increasing values of ε, one can find an upper bound on ε preserving stability. The stability conditions are obtained also for systems with polytopic uncertainties. We construct an ε-independent state-feedback controller, that stabilizes the system for all small enough $\varepsilon \geq 0$, by solving ε-independent LMI. The latter LMI corresponds to the state-feedback stabilization of the corresponding descriptor system.

Notation: Throughout the paper the superscript 'T' stands for matrix transposition, $\mathcal{R}^n$ denotes the n dimensional Euclidean space with vector norm $|\cdot|$, $\mathcal{R}^{n\times m}$ is the set of all $n\times m$ real matrices, and the notation $P>0$, for $P\in\mathcal{R}^{n\times n}$ means that P is symmetric and positive definite. We also denote $x_t(\theta)=x(t+\theta)$ $(\theta\in[-h,0])$.

2 LMI Stability Conditions

2.1. Delay-dependent conditions for $\varepsilon>0$. Given the following system:

$$E_\varepsilon\dot{x}(t)=A_0x(t)+A_1x(t-h), \tag{2}$$

where $x(t)=col\{x_1(t),x_2(t)\},\ x_1(t)\in\mathcal{R}^{n_1},\ x_2(t)\in\mathcal{R}^{n_2}$ is the system state vector, The matrix E_ε is given by

$$E_\varepsilon=\begin{bmatrix} I_{n_1} & 0\\ 0 & \varepsilon I_{n_2}\end{bmatrix}, \tag{3}$$

where $\varepsilon>0$ is a small parameter. The time delay $h>0$ is assumed to be known. We took for simplicity one delay, but all the results are easily generalized for the case of any finite number of delays.

Denote $n\stackrel{\Delta}{=}n_1+n_2$. The matrices A_0 and A_1 are constant $n\times n$ matrices of appropriate dimensions. The matrices in (2) have the following structure:

$$A_i=\begin{bmatrix} A_{i1} & A_{i2}\\ A_{i3} & A_{i4}\end{bmatrix},\ i=0,1. \tag{4}$$

In this section we require A_{04} to be nonsingular.

Consider the fast system

$$\dot{x}_2(t)=A_{04}x_2(t)+A_{14}x_2(t-g),\quad g\in[0,\infty) \tag{5}$$

with characteristic equation

$$\Delta(\lambda)=det(\lambda I-A_{04}-A_{14}e^{-\lambda g}). \tag{6}$$

A necessary condition for robust stability of (2) is given by

Lemma 2.1 *[5] Let (2) is stable for all small enough ε and h. Then for all $g\geq 0$ characteristic equation (6) has no roots with positive real parts.*

According to this lemma we derive criterion for asymptotic stability which is delay-independent in the fast variables and delay-dependent in the slow ones. Following [4] we represent (2) in the equivalent form:

$$\begin{array}{rcl} \dot{x}_1(t) & = & y(t), \\ \begin{bmatrix} \varepsilon\dot{x}_2(t) \\ y(t) \end{bmatrix} & = & \begin{bmatrix} A_{03}+A_{13} & A_{04} \\ A_{01}+A_{11} & A_{02} \end{bmatrix} x(t) + \begin{bmatrix} A_{14} \\ A_{12} \end{bmatrix} x_2(t-h) - \begin{bmatrix} A_{13} \\ A_{11} \end{bmatrix} \int_{-h}^{0} y(t+s)ds. \end{array} \tag{7}$$

The latter system can be represented in the form:

$$\bar{E}_\varepsilon \dot{\bar{x}}(t) = \bar{A}_0 \bar{x}(t) + \bar{A}_1 \bar{x}(t-h) + H \int_{-h}^{0} y(t+s)ds, \tag{8}$$

where

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ y \end{bmatrix}, \ \bar{E}_\varepsilon = \begin{bmatrix} I_{n_1} & 0 & 0 \\ 0 & \varepsilon I_{n_2} & 0 \\ 0 & 0 & 0_{n_1\times n_1} \end{bmatrix}, \ \bar{A}_0 = \begin{bmatrix} 0 & 0 & I_{n_1} \\ A_{03}+A_{13} & A_{04} & 0 \\ A_{01}+A_{11} & A_{02} & -I_{n_1} \end{bmatrix},$$
$$\bar{A}_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & A_{14} & 0 \\ 0 & A_{12} & 0 \end{bmatrix}, \ H = \begin{bmatrix} 0 \\ -A_{13} \\ -A_{11} \end{bmatrix}. \tag{9}$$

A Lyapunov-Krasovskii functional for the system (7) has the form:

$$\begin{array}{l} V(t) = \bar{x}^T(t)\bar{E}_\varepsilon P_\varepsilon \bar{x}(t) + \int_{t-h}^{t} x_1^T(\tau) S x_1(\tau) d\tau + \int_{t-h}^{t} x_2^T(\tau) U x_2(\tau) d\tau \\ + \int_{-h}^{0}\int_{t+\theta}^{t} y^T(s) [A_{13}^T \ A_{11}^T] R_3 \begin{bmatrix} A_{13} \\ A_{11} \end{bmatrix} y(s) ds d\theta \end{array} \tag{10}$$

where P_ε has the structure of

$$P_\varepsilon = \begin{bmatrix} P_{1\varepsilon} & 0 \\ P_2 & P_3 \end{bmatrix}, \quad P_{1\varepsilon} = \begin{bmatrix} P_{11} & \varepsilon P_{12}^T \\ P_{12} & P_{13} \end{bmatrix} \tag{11}$$

with $P_{11} \in \mathcal{R}^{n_1\times n_1}$, $P_{13} \in \mathcal{R}^{n_2\times n_2}$, $P_3 \in \mathcal{R}^{n_1\times n_1}$ and

$$0 < S \in \mathcal{R}^{n_1\times n_1}, \quad 0 < U \in \mathcal{R}^{n_2\times n_2}, \quad 0 < R_3 \in \mathcal{R}^{n\times n}.$$

The first term of (10) corresponds to the descriptor system, the second and the fourth terms - to the delay-dependent conditions with respect to x_1 and the third - to the delay-independent conditions with respect to x_2. For $\varepsilon = 0$ Lyapunov-Krasovskii functional of (10) corresponds to descriptor system of (8) with $\varepsilon = 0$ [6]. We obtain the following:

Theorem 2.2 (i) *Given* $\varepsilon > 0,\ h > 0$, *the system (2) is asymptotically stable if there exist matrices* $P_\varepsilon \in \mathcal{R}^{(n_1+n)\times(n_1+n)}$ *of (11)* $0 < P_{11} \in \mathcal{R}^{n_1\times n_1}$, $0 < P_{13} \in \mathcal{R}^{n_2\times n_2}$, $P_2 \in \mathcal{R}^{n_1\times n}$, $P_3 \in \mathcal{R}^{n_1\times n_1}$ *such that* $E_\varepsilon P_{1\varepsilon} > 0$ *and matrices* $S = S^T \in \mathcal{R}^{n_1\times n_1}$, $U = U^T \in \mathcal{R}^{n_2\times n_2}, W \in \mathcal{R}^{(n_1+n)\times(n_1+n)}$ *and* $R = R^T \in \mathcal{R}^{(n_1+n)\times(n_1+n)}$, *that satisfy the following LMI:*

$$\begin{bmatrix} \bar{\Psi}_\varepsilon & hX & -W^T\begin{bmatrix} 0 \\ A_{13} \\ A_{11}\end{bmatrix} & P_\varepsilon^T\begin{bmatrix} 0 \\ A_{14} \\ A_{12}\end{bmatrix} \\ * & -hR & 0 & 0 \\ * & * & -S & 0 \\ * & * & * & -U \end{bmatrix} < 0, \tag{12}$$

where

$$X = W^T + P_\varepsilon^T,$$

$$\bar{\Psi}_\varepsilon = \Psi_\varepsilon + W^T\begin{bmatrix} 0 & 0 & 0 \\ A_{13} & 0 & 0 \\ A_{11} & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & A_{13}^T & A_{11}^T \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} W$$

and

$$\Psi_\varepsilon \triangleq P_\varepsilon^T\begin{bmatrix} 0 & 0 & I_{n_1} \\ A_{03}+A_{13} & A_{04} & 0 \\ A_{01}+A_{11} & A_{02} & -I_{n_1}\end{bmatrix} + \begin{bmatrix} 0 & 0 & I_{n_1} \\ A_{03}+A_{13} & A_{04} & 0 \\ A_{01}+A_{11} & A_{02} & -I_{n_1}\end{bmatrix}^T P_\varepsilon + \begin{bmatrix} S & 0 & 0 \\ 0 & U & 0 \\ 0 & 0 & h[0\ A_{13}^T\ A_{11}^T]R\begin{bmatrix} 0 \\ A_{13} \\ A_{11}\end{bmatrix}\end{bmatrix}. \tag{13}$$

(ii) *Given* $h > 0$, *if there exists* P_0 *of (11)* $0 < P_{11} \in \mathcal{R}^{n_1\times n_1}$, $0 < P_{13} \in \mathcal{R}^{n_2\times n_2}$, $P_2 \in \mathcal{R}^{n_1\times n}$, $P_3 \in \mathcal{R}^{n_1\times n_1}$ *and matrices* $S = S^T \in \mathcal{R}^{n_1\times n_1}$, $U = U^T \in \mathcal{R}^{n_2\times n_2}, W \in \mathcal{R}^{(n_1+n)\times(n_1+n)}$ *and* $R = R^T \in \mathcal{R}^{(n_1+n)\times(n_1+n)}$, *such that (12) is feasible for* $\varepsilon = 0$ *then (2) is asymptotically stable for all small enough* $\varepsilon > 0$ *and* $0 \le \bar{h} \le h$.

Proof: (i) Differentiating the first term of (10) with respect to t we have:

$$\frac{d}{dt}\bar{x}^T(t)\bar{E}_\varepsilon P_\varepsilon \bar{x}(t) = 2\bar{x}^T(t)P_\varepsilon \bar{E}_\varepsilon \dot{\bar{x}}(t). \tag{14}$$

Substituting (7) into (14) we obtain:

$$\frac{dV(x_t)}{dt} = \xi^T \begin{bmatrix} \Psi_\varepsilon & P_\varepsilon^T \begin{bmatrix} 0 \\ A_{14} \\ A_{12} \end{bmatrix} \\ * & -U \end{bmatrix} \xi + \eta$$
$$- \left[x_1^T(t-h)Sx_1(t-h) + x_2^T(t-h)Ux_2(t-h) + \int_{t-h}^t y^T(s)[A_{13}^T \ A_{11}^T]R_3 \begin{bmatrix} A_{13} \\ A_{11} \end{bmatrix} y(s)ds \right], \tag{15}$$

where $\xi \triangleq col\{\bar{x}(t) \ , x_2(t-h)\}$, Ψ_ε is defined by (13) and

$$\eta(t) \triangleq -2 \int_{t-h}^t \bar{x}^T(t) P_\varepsilon^T \begin{bmatrix} 0 \\ A_{13} \\ A_{11} \end{bmatrix} y(s)ds.$$

For any $(n_1+n) \times (n_1+n)$-matrices $R > 0$ and M the following inequality holds [21]:

$$-2 \int_{t-h}^t b^T(s)a(s)ds \le \int_{t-h}^t \begin{bmatrix} a(s) \\ b(s) \end{bmatrix}^T \begin{bmatrix} R & RM \\ M^T R & (2,2) \end{bmatrix} \begin{bmatrix} a(s) \\ b(s) \end{bmatrix} ds \tag{16}$$

for $a(s) \in \mathcal{R}^{n_1+n}$, $b(s) \in \mathcal{R}^{n_1+n}$ given for $s \in [t-h, t]$. Here $(2,2) = (M^T R + I)R^{-1}(RM + I)$.

Denoting $W = RMP_\varepsilon$ and using this inequality for $a(s) = col\{0 \ A_{11} \ A_{13}\}y(s)$ and $b = P_\varepsilon \bar{x}(t)$ we obtain

$$\eta(t) \le h\bar{x}^T(t)(W^T + P_\varepsilon^T)R^{-1}(W + P_\varepsilon)\bar{x}(t) + 2(x_1^T(t) - x_1^T(t-h)) \begin{bmatrix} 0 & A_{13}^T & A_{11}^T \end{bmatrix} W\bar{x}(t)$$
$$+ \int_{t-h}^t y^T(s)[0 \ A_{13}^T \ A_{11}^T]R_3 \begin{bmatrix} 0 \\ A_{13} \\ A_{11} \end{bmatrix} y(s)ds. \tag{17}$$

We substitute (17) into (15). Hence, if (12) holds then $dV/dt < 0$ and (2) is internally stable.

(ii) If (12) is feasible for $\varepsilon = 0$, then it is feasible for all small enough $\varepsilon > 0$ and thus due to (i) (2) is asymptotically stable for these values of $\varepsilon > 0$. LMI (12) is convex with respect to h. Hence, if it is feasible for some h then it is feasible for all $0 \le \bar{h} < h$.

2.2. Delay-dependent stability of the descriptor system. We will show that (12) for $\varepsilon = 0$ guarantees asymptotic stability of the descriptor system (2), where $\varepsilon = 0$. The following lemma will be useful:

Lemma 2.3 *[6] Assume that the difference equation*

$$\mathcal{D}x_t = x(t) + A_{04}^{-1}A_{14}x(t-g) = 0$$

is asymptotically stable, or equivalently [12] assume that all the eigenvalues of $A_{04}^{-1}A_{14}$ *are inside a unit circle. Then if there exist positive numbers* α, β, γ *and a continuous functional* $V : C_{n+n_1}[-h, 0] \to \mathcal{R}$ *such that*

$$\beta|\phi_1(0)|^2 \le V(\phi) \le \gamma|\phi|^2, \quad \dot{V}(\phi) \le -\alpha|\phi(0)|^2, \tag{18}$$

and the function $\bar{V}(t) = V(\bar{x}_t)$ *is absolutely continuous for* $\bar{x}_t$ *satisfying (7) with* $\varepsilon = 0$*, then (7) (and thus (2) with* $\varepsilon = 0$*) is asymptotically stable.*

Consider the descriptor system (2) with $\varepsilon = 0$. If (12) holds for $\varepsilon = 0$, then the Lyapunov-Krasovskii functional of (10) with $\varepsilon = 0$ is nonnegative and has a negative-definite derivative. By Lemma 2.3 the latter guarantees the asymptotic stability of the descriptor system provided that all the eigenvalues of $A_{04}^{-1}A_{14}$ are inside a unit circle. We show next that (12) with $\varepsilon = 0$ yields the following inequality:

$$\begin{bmatrix} A_{04}^T P_{13} + P_{13}A_{04} + U & P_{13}A_{14} \\ A_{14}^T P_{13} & -U \end{bmatrix} < 0, \tag{19}$$

that guarantees the stability of the fast system (5) for all delays $g \ge 0$. Hence , A_{04} is Hurwitz and all the eigenvalues of $A_{04}^{-1}A_{14}$ are inside a unit circle [6].

Lemma 2.4 *If (12) with* $\varepsilon = 0$ *is feasible, then (19) is feasible, the fast system (5) is asymptotically stable for all delays* $g \ge 0$, A_{04} *is Hurwitz and all the eigenvalues of* $A_{04}^{-1}A_{14}$ *are inside a unit circle.*

Proof. It is obvious from the requirement of $0 < P_{11},\ 0 < P_{13}$, and the fact that in (12) $-P_3 - P_3^T$ must be negative definite, that P_0 is nonsingular. Defining

$$P_0^{-1} = Q_0 = \begin{bmatrix} Q_1 & 0 \\ Q_2 & Q_3 \end{bmatrix}, \quad Q_1 = \begin{bmatrix} Q_{11} & 0 \\ Q_{12} & Q_{13} \end{bmatrix}, \ Q_{11} \in \mathcal{R}^{n_1 \times n_1},\ Q_{13} \in \mathcal{R}^{n_2 \times n_2},\ Q_3 \in \mathcal{R}^{n_1 \times n_1}, \tag{20}$$

and $\Delta = diag\{Q,\ I_{2n+n_1}\}$ we multiply (12) by Δ^T and Δ, on the left and on the right, respectively. Since the term (2,2) of the matrix is equal to zero, the latter inequality implies

$$\begin{bmatrix} Q_{13}A_{04}^T + A_{04}Q_{13} + Q_{13}UQ_{13} & A_{14} \\ Q_{13}A_{14}^T & -U \end{bmatrix} < 0. \tag{21}$$

Multiplying (21) by $diag\{P_{13},\ I_{n_2}\}$ from the left and the right we obtain (19). From (19) it follows that A_{04} is Hurwitz and all the eigenvalues of $A_{04}^{-1}A_{14}$ are inside a unit circle [6].

From Theorem 2.2, Lemmas 2.3 and 2.4 we obtain

Corollary 2.5 *Given $h>0$, if there exists P_0 of (11) $0<P_{11}\in\mathcal{R}^{n_1\times n_1}$, $0<P_{13}\in\mathcal{R}^{n_2\times n_2}$, $P_2\in\mathcal{R}^{n_1\times n}$, $P_3\in\mathcal{R}^{n_1\times n_1}$ and matrices $S=S^T\in\mathcal{R}^{n_1\times n_1}$, $U=U^T\in\mathcal{R}^{n_2\times n_2}, W\in\mathcal{R}^{(n_1+n)\times(n_1+n)}$ and $R=R^T\in\mathcal{R}^{(n_1+n)\times(n_1+n)}$, such that (12) is feasible for $\varepsilon=0$ then (2) is asymptotically stable for all small enough $\varepsilon\geq 0$ and $0\leq\bar{h}\leq h$.*

Remark 1 *For stability of descriptor system (2) with $\varepsilon=0$ it is sufficient to require feasibility of (12) for $\varepsilon=0$ with $P_{11}>0$, whereas P_{13} may be non-symmetric. Positivity of P_{13} guarantees stability of (2) for small enough $\varepsilon>0$.*

Example 1 [5]. Consider the system

$$\dot{x}_1=x_2(t)+x_1(t-h),\quad \varepsilon\dot{x}_2=-x_2(t)+0.5x_2(t-h)-2x_1(t). \tag{22}$$

For $h=0$ this system is asymptotically stable for all small enough ε since A1 and A2 hold. It is well-known (see e.g. [12]) that the fast system $\dot{x}_2(t)=-x_2(t)+0.5x_2(t-g)$ is asymptotically stable for all g. Thus necessary condition for robust stability with respect to small ε is satisfied. It was shown in [5] that the system is robustly asymptotically stable with respect to small ε and h and for $\varepsilon=0.5,\ h=0$ the system is unstable. The conditions of [5] are conservative. Thus for $\varepsilon=0$ (22) is delay-independently stable [6], while LMI of [5] for $\varepsilon=0$ is feasible only for $h\leq 0.144$.

Applying Theorem 2.2 we find that for $0\leq\varepsilon\leq 0.3$ the system is asymptotically stable for all delays, while for $\varepsilon=0.4$ the system is asymptotically stable for $0\leq h\leq 0.0048$ (compare with $0\leq h\leq 0.001$ obtained in [5]). For $\varepsilon=0.5$ LMI (12) is not feasible for $h\to 0$ since the system is unstable for $h=0$. We see that the results of the present paper are essentially less conservative than those of [5]. This is due to new (descriptor) model transformation of the system and Park's bounds of the cross terms.

2.2. Delay-independent conditions

Theorem 2.6 . *Given $\varepsilon\geq 0$ the system (2) is asymptotically stable for all $h\geq 0$ if there exist $n\times n$-matrix P_ε of the form*

$$P_\varepsilon=\begin{bmatrix} P_1 & \varepsilon P_2^T \\ P_2 & P_3 \end{bmatrix}$$

with $n_1 \times n_1$-matrix $P_1 > 0$ and $n_2 \times n_2$-matrix P_3 and $n \times n$ matrices $U = U^T$, $R = R^T$ that satisfy the following LMI:

$$\begin{bmatrix} P_\varepsilon^T A_0 + A_0^T P_\varepsilon + Q & P_\varepsilon^T A_1 \\ * & -U \end{bmatrix} < 0. \tag{23}$$

If (23) is feasible for $\varepsilon = 0$, then system (2) is delay-independently asymptotically stable for all small enough $\varepsilon \geq 0$.

Proof is obtained by similar to Theorem 2.2 arguments by using Lyapunov-Krasovskii functional of the form

$$V(t) = x^T(t)E_\varepsilon P_\varepsilon x(t) + \int_{t-h}^{t} x^T(\tau)Ux(\tau)d\tau$$

.

Another delay-independent condition follows from Theorem 2.2. For

$$W = -P_\varepsilon, \; R = \frac{\delta I_{2n}}{h}, \tag{24}$$

LMI (12) implies for $\delta \to 0^+$ the following delay-independent LMI:

$$\begin{bmatrix} \bar{\Psi}_\varepsilon & P_\varepsilon^T \begin{bmatrix} 0 \\ A_{13} \\ A_{11} \end{bmatrix} & P_\varepsilon^T \begin{bmatrix} 0 \\ A_{14} \\ A_{12} \end{bmatrix} \\ * & -S & 0 \\ * & * & -U \end{bmatrix} < 0, \tag{25}$$

where

$$\bar{\Psi}_\varepsilon = P_\varepsilon^T \begin{bmatrix} 0 & 0 & I_{n_1} \\ A_{03} & A_{04} & 0 \\ A_{01} & A_{02} & -I_{n_1} \end{bmatrix} + \begin{bmatrix} 0 & 0 & I_{n_1} \\ A_{03} & A_{04} & 0 \\ A_{01} & A_{02} & -I_{n_1} \end{bmatrix}^T P_\varepsilon + \begin{bmatrix} S & 0 & 0 \\ 0 & U & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

If LMI (25) is feasible then (12) is feasible for small enough $\varepsilon > 0$ and W and R given by (24). Thus, from Theorem 2.2 the following corollary follows:

Corollary 2.7 *Given $\varepsilon > 0$, system (2) is asymptotically stable for all $g \geq 0, h \geq 0$, if there exist $0 < P_1 = P_1^T$, P_2, P_3, and $Q = Q^T$, $S = \tilde{S}^T$, that satisfy (25).*

2.5. Stability of singularly perturbed systems with polytopic uncertainties. Stability criteria of this section were derived for the system (2) where the system matrices A_i, $i = 0, 1$ are known. However, since the LMIs of these criteria are affine in the system matrices, the theorems can be used to derive criteria that will guarantee stability in the case where the system matrices are not exactly known and they reside within a given polytope.

Denoting $\Omega = \begin{bmatrix} A_0 & A_1 \end{bmatrix}$ we assume that $\Omega \in \mathcal{C}o\{\Omega_j,\ j = 1,...N\}$, namely,

$$\Omega = \sum_{j=1}^{N} f_j \Omega_j \quad \text{for some} \quad 0 \leq f_j \leq 1,\ \sum_{j=1}^{N} f_j = 1$$

where the N vertices of the polytope are described by

$$\Omega_j = \begin{bmatrix} A_i^{(j)} & i = 0, 1 \end{bmatrix}.$$

Then e.g. from Corollary 2.5 we readily obtain the following:

Corollary 2.8 *Consider the system of (2), where the system matrices reside within the polytope* Ω*. This system is asymptotically stable for all small enough* $\varepsilon \geq 0$ *if there exist* P_0 *of (11) with* $0 < P_{11}^{(j)} \in \mathcal{R}^{n_1 \times n_1}$, $P_{12} \in \mathcal{R}^{n_2 \times n_2}$, $0 < P_{13} \in \mathcal{R}^{n_2 \times n_2}$, $P_2 \in \mathcal{R}^{n_1 \times n}$, $P_3 \in \mathcal{R}^{n_1 \times n_1}$ *and* $W^{(j)} \in \mathcal{R}^{(n+n_1)\times(n+n_1)}$ $j = 1, ..., N$, $0 < R^{(j)} \in \mathcal{R}^{(n+n_1)\times(n+n_1)}$, $0 < U^{(j)} \in \mathcal{R}^{n_2 \times n_2}$, $0 < S^{(j)} \in \mathcal{R}^{n_1 \times n_1}$ $j = 1, ..., N$ *that satisfy (12) for* $\varepsilon = 0$ *and* $j = 1, ..., N$*, where the matrices*

$$A_i, P_{11}, W,\ R,\ S$$

are taken with the upper index j*.*

3 Delay-Dependent Robust Stabilization by Memoryless State-Feedback

We apply the results of the previous section to the stabilization problem. Given the system

$$E_\varepsilon \dot{x}(t) = A_0 x(t) + A_1 x(t-h) + B_2 u(t), \tag{26}$$

where E_ε is defined by (3). In this section we do not assume that A_{04} is nonsingular. Similarly to case without delay (with A_1=0), we call such a system as a *non-standard* singularly perturbed system. In the case of singular A_{04} open-loop system (26) with $\varepsilon = o$ and without delay, i.e. with $A_1 = 0$, have index more than one (see e.g. [1]). Hence, index of system (26) with $u = 0$ and with delay, which is defined in [6] to be equal to the index of (26) with $A_1 = 0$, is higher than one. Such a system have an impulse solution [6]. The non-singularity of A_{04} guarantees the existence and the uniqueness of solution to initial value problem for (26) with $u = 0$ [9].

We look for the state-feedback ε-independent gain matrix K which, via the control law

$$u(t) = Kx(t), \quad K = [K_1, K_2] \tag{27}$$

stabilizes system (26) for all small enough ε. We derive delay-dependent conditions since they are less conservative. Substituting (27) into (26), we obtain the structure of (2) with $A_0 + B_2K$ instead of A_0 . Applying (ii) of Theorem 3.1 to the above matrices, results in a nonlinear matrix inequality because of the terms $P_2^T B_2 K$ and $P_3^T B_2 K$. We therefore consider another version of the Theorem 2.1 which is derived from (12).

In order to obtain an LMI we have to restrict ourselves to the case of $W_0 = \delta P_0$, where $\delta \in R$ is a scalar parameter. Note that for $\delta = -1$ (12) yields the delay-independent condition of Corollary 3.6. As it was mentioned in the proof of Lemma 2.3, P_0 is nonsingular. Defining $P_0^{-1} = Q_0$ by (20) and $\Delta = diag\{Q,\ I_{2n+n_1}\}$ we multiply (12) by Δ^T and Δ, on the left and on the right, respectively. Applying the Schur formula to the quadratic term in Q, we obtain the following inequality:

$$\begin{bmatrix} \Xi_1 + \bar{\Xi}_2 & h(\delta+1)I_{n+n_1} & \delta\begin{bmatrix} 0 \\ A_{13} \\ A_{11} \end{bmatrix} & \begin{bmatrix} 0 \\ A_{14} \\ A_{12} \end{bmatrix} & Q^T\begin{bmatrix} I_{n_1} \\ 0 \end{bmatrix} & Q^T\begin{bmatrix} 0 \\ I_{n_2} \\ 0 \end{bmatrix} & hQ^T\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & A_{13}^T & A_{11}^T \end{bmatrix} \\ * & -hR & 0 & 0 & 0 & 0 & 0 \\ * & * & -S & 0 & 0 & 0 & 0 \\ * & * & * & -U & 0 & 0 & 0 \\ * & * & * & * & -S^{-1} & 0 & 0 \\ * & * & * & * & * & -U^{-1} & 0 \\ * & * & * & * & * & * & -hR^{-1} \end{bmatrix} < 0, \tag{28}$$

where

$$\Xi_1 = \begin{bmatrix} 0 & 0 & I_{n_1} \\ A_{03} + (1+\delta)A_{13} & A_{04} & 0 \\ A_{01} + (1+\delta)A_{11} & A_{02} & -I_{n_1} \end{bmatrix} Q + Q^T \begin{bmatrix} 0 & 0 & I_{n_1} \\ A_{03} + (1+\delta)A_{13} & A_{04} & 0 \\ A_{01} + (1+\delta)A_{11} & A_{02} & -I_{n_1} \end{bmatrix}^T \tag{29}$$

$$\bar{\Xi}_2 = \begin{bmatrix} 0 \\ B_2 \end{bmatrix} [K\ 0_{n_1}]Q + Q^T \begin{bmatrix} K^T \\ 0_{n_1} \end{bmatrix} [0\ B_2^T].$$

Denoting $KQ_1 = Y$ we obtain the following:

Theorem 3.1 *Consider the system of (26), (3). The state-feedback law of (27) asymptotically stabilizes (26), (3) for all small enough* $\varepsilon \geq 0$ *if for some prescribed scalar* $\delta \in R$*, there exist* $0 < Q_1 \in \mathcal{R}^{n\times n}$, $0 < \bar{S} = S^{-1} \in \mathcal{R}^{n_1\times n_1}$, $0 < \bar{U} = U^{-1} \in \mathcal{R}^{n_2\times n_2}$, $Q_2 \in \mathcal{R}^{n_1\times n}$ *and* $Q_3 \in \mathcal{R}^{n_1\times n_1}$ *of (20) ,* $0 < \bar{R} = R^{-1} \in \mathcal{R}^{(n+n_1)\times(n+n_1)}$ *,* $Y \in \mathcal{R}^{\ell\times n}$ *that satisfy*

$$\begin{bmatrix} \Xi_1+\Xi_2 & h(\delta+1)\bar{R} & \delta\begin{bmatrix}0\\A_{13}\\A_{11}\end{bmatrix}\bar{S} & \begin{bmatrix}0\\A_{14}\\A_{12}\end{bmatrix}\bar{U} & Q^T\begin{bmatrix}I_{n_1}\\0\end{bmatrix} & Q^T\begin{bmatrix}0\\I_{n_2}\\0\end{bmatrix} & hQ^T\begin{bmatrix}0&0&0\\0&0&0\\0&A_{13}^T&A_{11}^T\end{bmatrix} \\ * & -h\bar{R} & 0 & 0 & 0 & 0 & 0 \\ * & * & -\bar{S} & 0 & 0 & 0 & 0 \\ * & * & * & -\bar{U} & 0 & 0 & 0 \\ * & * & * & * & -\bar{S} & 0 & 0 \\ * & * & * & * & * & -\bar{U} & 0 \\ * & * & * & * & * & * & -h\bar{R} \end{bmatrix} < 0, \qquad (30)$$

where

$$\Xi_2 = \begin{bmatrix}0\\B_2\end{bmatrix}[Y\ 0_{n_1}] + \begin{bmatrix}Y^T\\0_{n_1}\end{bmatrix}[0\ B_2^T].$$

The state-feedback gain is then given by $K = YQ_1^{-1}$*.*

Example 2: We consider the system

$$E_\varepsilon\dot{x}(t) = A_1x(t-h) + Bu(t), \qquad (31)$$

where

$$E_\varepsilon = \begin{bmatrix}1&0\\0&\varepsilon\end{bmatrix}, \quad A_1 = \begin{bmatrix}-1&0\\1&-1\end{bmatrix}, \quad B = \begin{bmatrix}-0.5\\1\end{bmatrix}.$$

Note that in this example $A_{04} = 0$. Applying Theorem 3.1 for e.g. $h = 1$ we find the stabilizing state-feedback $u = Kx$, where $K = [42.4\ \ -1940.1]$. Applying next Theorem 2.2 to the closed loop system (31), $u = Kx$, we verify that the closed-loop system is asymptotically stable for $h \leq 1.39$ and all $\varepsilon \geq 0$. For $h = 1.4$ LMIs of these theorems are not feasible for all values of $\varepsilon \geq 0$.

The LMI in Theorem 3.1 is affine in the system matrices. It can thus be applied also to the case where these matrices are uncertain and are known to reside within a given polytope.

4 Conclusions

A LMI solution is proposed for the problem of stability and robust state-feedback stabilization of linear time-invariant singularly perturbed systems with delay. Sufficient conditions for asymptotic stability of system for small enough values of ε are given in terms of ε-independent LMI, that guarantees stability of the corresponding descriptor system. State-feedback ε-independent stabilizing controller is derived then from this LMI for non-standard singularly perturbed system. The controller stabilizes the descriptor system and singularly perturbed system for all small enough $\varepsilon \geq 0$. One additional advantage of the new method that, unlike conventional singularly perturbed methods (see e.g. [15]), it gives also sufficient conditions for stability for prechosen $\varepsilon > 0$ in terms of ε-dependent LMI. By solving the latter LMI for increasing values of ε one can find an upper bound on the values of ε for which the system preserves asymptotic stability.

The method develops LMI approach to stability of singularly perturbed systems with delay introduced in [5]. In this paper a new less conservative criterion is derived. It is based on the new Lyapunov function approach to systems with delay introduced in [5], [6] and [8]. The LMI sufficient conditions that are obtained allow solutions to the stabilization problem in the uncertain case where the system parameters lie within an uncertainty polytope.

One question that often arises when solving control problems for systems with time-delay is whether the solution obtained for certain delays h will satisfy the design requirements for all delays $\bar{h} \leq h$. The answer is the affirmative since the LMIs in Theorems are convex in the time delays.

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6 References

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