On Delay-Dependent Passivity

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***Abstract*—Sufficient conditions for passivity (positive realness) are obtained for continuous-time, linear, retarded, and neutral-type systems. A delay-dependent solution is given in terms of linear matrix inequalities (LMIs) by using a descriptor model transformation of the system and by applying Park's inequality for bounding cross terms. A memoryless state-feedback solution is derived. Numerical examples are given which illustrate the effectiveness of the new theory.**

***Index Terms*—Delay-dependent criteria, linear matrix inequalities (LMIs), positive-real lemma, time-delay systems.**

I. Introduction

Positive realness (passivity) theory plays an important role in both electrical network and control systems (see, e.g., [1], [2]) and it has roots in circuit theory ([3], [4]). For systems with delay of retarded-type, positive realness has been studied by [5]–[7]. In [5], delay-independent sufficient conditions in terms of LMIs have been derived. In [6] necessary and sufficient conditions are given in terms of positivity of some kernel matrix constructed via transition matrix. In [7] frequency domain approach is applied and sufficient conditions are obtained. For infinite-dimensional systems, a positive-real lemma has been obtained in terms of Riccati operator equations (see [2] and the references therein).

In the present note we give delay-dependent sufficient conditions for passivity of neutral type systems. We apply descriptor-type Lyapunov–Krasovskii functionals that were recently introduced in [10]–[12] for delay-dependent stability and control and Park's inequality for bounding cross terms [13]. We also present a memoryless state-feedback controller via LMIs, such that the resulting closed-loop system is passive.

Notation: Throughout the note, the superscript T stands for matrix transposition, $\mathcal{R}^n$ denotes the n dimensional Euclidean space, $\mathcal{R}^{n\times m}$ is the set of all $n \times m$ real matrices, and the notation $P > 0$, for $P \in \mathcal{R}^{n\times n}$ means that P is symmetric and positive–definite. Denote $x_t(\theta) = x(t+\theta)(\theta \in [-h, 0])$.

II. Passivity and Positive Realness for Linear Time-Delay Systems

Given the following system

$$\begin{aligned}\dot{x}(t) - F\dot{x}(t-g) &= \sum_{i=0}^{2} A_i x(t-h_i) + B_1 w(t)\\ z(t) &= Cx(t) + Dw(t)\end{aligned} \qquad \text{(1a-b)}$$

where $x(t) \in \mathcal{R}^n$ is the system state vector, $w(t) \in \mathcal{R}^q$ is the exogenous input, which can be either a control input or a reference signal and $z(t) \in \mathcal{R}^q$ is the output of the system. The time delays $0 = h_0$, $0 < h_i$, $i = 1, 2$ and $g > 0$ are assumed to be known. The matrices A_i, $i = 0, \ldots, 2$, F, B_1 and C are constant matrices of appropriate dimensions. Denote $h = \max\{h_1, h_2\}$. For simplicity only we consider a single delay g and two delays h_1 and h_2. The results of this note can be easily applied to the case of multiple delays $g_1, \ldots, g_m$, $h_1, \ldots, h_m$ and a distributed delay.

Equation (1a) describes a system of neutral type since it contains a derivative with delay. In the case of $F = 0$ (1a) is a retarded type system (see, e.g., [8]). Neutral systems are encountered in modeling of lossless transmission lines, or in dynamical processes including steam and water pipes (see, e.g., [8] and the references therein). Unlike retarded systems, linear neutral systems may be destabilized by small changes of the delay and may be unstable even in the case when all the roots of the characteristic equation have negative real parts [8].

We are looking for a criterion for passivity that depends on the delays h_i and does not depend on g. Delay-independence with respect to g guarantees that small changes in g do not destabilize the system [8]. To guarantee robustness of the results with respect to small changes of delay, we assume that the difference equation $\mathcal{D}x_t = x(t) - Fx(t-g) = 0$ is asymptotically stable for all values of g or, equivalentaly, that

A1 F is a Schur–Cohn stable matrix, i.e., all the eigenvalues of F are inside the unit circle.

The transfer function of (1a-b) from w to z is given by

$$G(s) = C\left[s\left(I - Fe^{-sg}\right) - \sum_{i=0}^{2} A_i e^{-sh_i}\right]^{-1} B_1 + D.$$

Definition 1: [1] The system (1a-b) is called passive if

$$2\int_0^{t_1} w^T(t)z(t)dt \geq 0 \qquad (2)$$

for all $t_1 \geq 0$ and for all solution of (1a-b) with $x_0 = 0$.

Another less restrictive definition of passivity is given by [14].

Definition 2: [14] The system (1a-b) is called passive if there exists $\gamma \geq 0$ such that

$$2\int_0^{t_1} w^T(t)z(t)dt \geq -\gamma\int_0^{t_1} w^T(s)w(s)ds. \qquad (3)$$

for all $t_1 \geq 0$ and for all solution of (1a-b) with $x_0 = 0$.

Different model transformations were used in the past for delay-dependent stability (see, e.g., [9] and [13]). Recently, a new (descriptor) model transformation has been introduced [10]. Unlike previous transformations, the descriptor model leads to a system which is equivalent to the original one, it does not depend on additional assumptions for the stability of the transformed system and it requires bounding of fewer crossterms. It was shown in [10] and [12] that the latter transformation leads to less conservative conditions for stability and H_∞ control.

Following [10], we represent (1a-b) in the equivalent descriptor form

$$\dot{x}(t) = y(t), \quad y(t) = Fy(t-g) + \sum_{i=0}^{2} A_i x(t-h_i) + B_1 w(t). \qquad (4)$$

The latter is equivalent to the following descriptor system with discrete and distributed delay in the variable y:

$$\begin{aligned}\dot{x}(t) &= y(t)\\ y(t) &= Fy(t-g) + \left(\sum_{i=0}^{2} A_i\right)x(t)\\ &\quad - \sum_{i=1}^{2} A_i \int_{t-h_i}^{t} y(\tau)d\tau + B_1 w(t).\end{aligned} \qquad (5)$$

Manuscript received May 9, 2001; revised September 24, 2001. Recommended by Associate Editor V. Balakrishnan. This work was supported by the Ministry of Absorption of Israel, and also by the C & M Maus Chair at Tel Aviv University.

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Publisher Item Identifier S 0018-9286(02)03748-0.

A Lyapunov–Krasovskii functional for the system (5) has the form

$$V(x_t, y_t) = \left[x^T(t) y^T(t)\right] EP \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \sum_{i=1}^{2} \int_{t-h_i}^{t} x^T(\tau) S_i x(\tau) d\tau + \int_{t-g}^{t} y^T(\tau) U y(\tau) d\tau + \sum_{i=1}^{2} \int_{-h_i}^{0} \int_{t+\theta}^{t} y^T(s) A_i^T R_{i3} A_i y(s) d\tau d\theta \tag{6}$$

where

$$E = \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix} \quad P = \begin{bmatrix} P_1 & 0 \\ P_2 & P_3 \end{bmatrix} \quad P_1 > 0,\ U > 0,\ S_i > 0. \tag{7a-b}$$

The first term of (6) corresponds to the descriptor system, the third—to the delay-independent conditions with respect to the discrete delays of y, while the second and the fourth terms correspond to the delay-dependent conditions with respect to the distributed delays (with respect to x).

We obtain the following.

Theorem 1: Assume A1. Consider the system of (1a-b). Let there exist $n \times n$*-matrices* $0 < P_1, P_2, P_3, S_i = S_i^T, U = U^T, W_{i1}, W_{i2}, W_{i3}, W_{i4}, R_{i1} = R_{i1}^T, R_{i2}, R_{i3} = R_{i3}^T, i = 1, 2$ and $\gamma \geq 0$ *that satisfy the linear matrix inequality (LMI)*, as shown in (8) at the bottom of the page, *where*

$$\begin{aligned} \Psi_1 &= \left(\sum_{i=0}^{2} A_i^T\right) P_2 + P_2^T \left(\sum_{i=0}^{2} A_i\right) + \sum_{i=1}^{2} \left(W_{i3}^T A_i + A_i^T W_{i3}\right) + \sum_{i=1}^{2} S_i \\ \Psi_2 &= P_1 - P_2^T + \left(\sum_{i=0}^{2} A_i^T\right) P_3 + \sum_{i=1}^{2} A_i^T W_{i4} \\ \Psi_3 &= -P_3 - P_3^T + \sum_{i=1}^{2} \left(U_i + h_i A_i^T R_{i3} A_i\right) \\ \Phi_{i1} &= [\, W_{i1}^T + P_1 \quad W_{i3}^T + P_2^T \,] \\ \Phi_{i2} &= [\, W_{i2}^T \quad W_{i4}^T + P_3^T \,] \\ R_i &= \begin{bmatrix} R_{i1} & R_{i2} \\ R_{i2}^T & R_{i3} \end{bmatrix}. \end{aligned} \tag{9}$$

Then, the following holds.

i) *The system (1a-b) is passive in the sense of Definition 2.*

ii) *In the case of* $\gamma = 0$ *for all* $\omega \in R$ *with*

$$\det\left[i\omega\left(I - Fe^{-i\omega g}\right) - \sum_{i=0}^{2} A_i e^{-i\omega h_i}\right] \neq 0 \tag{10}$$

the transfer matrix G *of (1a-b) is positive real, i.e.,*

$$G(i\omega)^* + G(i\omega) \geq 0.$$

Proof: (i) We note that

$$\left[x^T y^T\right] EP \begin{bmatrix} x \\ y \end{bmatrix} = x^T P_1 x$$

and, hence, differentiating the first term of (6) with respect to t we have

$$\frac{d}{dt}\left\{\left[x^T(t) y^T(t)\right] EP \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}\right\} = 2x^T(t) P_1 \dot{x}(t) = 2\left[x^T(t) y^T(t)\right] P^T \begin{bmatrix} \dot{x}(t) \\ 0 \end{bmatrix}. \tag{11}$$

Substituting (5) into (11), we obtain

$$\begin{aligned} \frac{dV(x_t, y_t)}{dt} - 2z^T w - \gamma w^T w &= \xi^T \begin{bmatrix} \Psi & P^T \begin{bmatrix} 0 \\ B_1 \end{bmatrix} - \begin{bmatrix} C^T \\ 0 \end{bmatrix} & P^T \begin{bmatrix} 0 \\ F \end{bmatrix} \\ * & -\gamma I - D - D^T & 0 \\ * & * & -U \end{bmatrix} \xi \\ &\quad - \sum_{i=1}^{2} \left[x^T(t - h_i) S_i x(t - h_i) + \int_{t-h_i}^{t} y^T(\tau) A_i^T R_{i3} A_i y(\tau) d\tau - \eta_i\right] \end{aligned} \tag{12}$$

where $\xi \triangleq col\{x(t), y(t),\ w(t), y(t-g)\}$ and

$$\begin{aligned} \Psi &\triangleq P^T \begin{bmatrix} 0 & I \\ \left(\sum_{i=0}^{2} A_i\right) & -I \end{bmatrix} + \begin{bmatrix} 0 & \left(\sum_{i=0}^{2} A_i^T\right) \\ I & -I \end{bmatrix} P + \begin{bmatrix} \sum_{i=1}^{2} S_i & 0 \\ 0 & \sum_{i=1}^{2} \left(U_i + h_i A_i^T R_{i3} A_i\right) \end{bmatrix} \\ \eta_i(t) &\triangleq -2 \int_{t-h_i}^{t} \left[x^T(t) y^T(t)\right] P^T \begin{bmatrix} 0 \\ A_i \end{bmatrix} y(s) ds. \end{aligned} \tag{13}$$

For any $2n \times 2n$-matrices $R_i > 0$ and M_i, the following inequality holds [13]:

$$-2 \int_{t-h_i}^{t} b^T(s) a(s) ds \leq \int_{t-h_i}^{t} \begin{bmatrix} a(s) \\ b(s) \end{bmatrix}^T \begin{bmatrix} R_i & R_i M_i \\ M_i^T R_i & (2,2) \end{bmatrix} \begin{bmatrix} a(s) \\ b(s) \end{bmatrix} ds \tag{14}$$

for $a(s) \in R^{2n}$, $b(s) \in R^{2n}$ given for $s \in [t - h_i, t]$. Here, $(2,2) = (M_i^T R_i + I) R_i^{-1} (R_i M_i + I)$.

$$\begin{bmatrix} \Psi_1 & \Psi_2 & P_2^T B_1 - C^T & h_1 \Phi_{11} & h_2 \Phi_{21} & -W_{13}^T A_1 & -W_{23}^T A_2 & P_2^T F \\ * & \Psi_3 & P_3^T B_1 & h_1 \Phi_{12} & h_2 \Phi_{22} & -W_{14}^T A_1 & -W_{24}^T A_2 & P_3^T F \\ * & * & -\gamma I - D - D^T & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -h_1 R_1 & 0 & 0 & 0 & 0 \\ * & * & * & * & -h_2 R_2 & 0 & 0 & 0 \\ * & * & * & * & * & -S_1 & 0 & 0 \\ * & * & \cdot & * & * & * & -S_2 & 0 \\ * & \cdot & \cdot & * & * & * & * & -U \end{bmatrix} \leq 0 \tag{8}$$

Using this inequality for $a(s) = col\{0 \;\; A_i\}y(s)$ and $b = Pcol\{x(t)\; y(t)\}$, we obtain

$$\begin{aligned}\eta_i \leq & h_i \begin{bmatrix} x^T & y^T \end{bmatrix} P^T \left(M_i^T R_i + I\right) R_i^{-1} \left(R_i M_i + I\right) P \begin{bmatrix} x \\ y \end{bmatrix} \\ & + 2\left(x^T(t) - x^T(t-h_i)\right) \begin{bmatrix} 0 & A_i^T \end{bmatrix} R_i M_i P \begin{bmatrix} x \\ y \end{bmatrix} \\ & + \int_{t-h_i}^{t} y^T(s) \begin{bmatrix} 0 & A_i^T \end{bmatrix} R_i \begin{bmatrix} 0 \\ A_i \end{bmatrix} y(s)ds. \end{aligned} \tag{15}$$

We substitute (15) into (12) and integrate the resulting inequality in t from 0 to t_1. We obtain (by Schur complements) that (3) holds if the LMI, as shown in (16) at the bottom of the page, is feasible, where for $i = 1, 2$

$$\begin{aligned} W_i =& R_i M_i P, \quad W_i = \begin{bmatrix} W_{i1} & W_{i2} \\ W_{i3} & W_{i4} \end{bmatrix}, \\ \Phi_i =& W_i^T + P^T, \quad \Phi_i = \begin{bmatrix} \Phi_{i1} & \Phi_{i2} \end{bmatrix}, \\ \bar{\Psi} =& \Psi + \sum_{i=1}^{2} W_i^T \begin{bmatrix} 0 & 0 \\ A_i & 0 \end{bmatrix} + \sum_{i=1}^{2} \begin{bmatrix} 0 & A_i^T \\ 0 & 0 \end{bmatrix} W_i. \end{aligned}$$

LMI (8) results from the latter LMI by expansion of the block matrices.

(ii) Let ω be such that (10) holds and consider $w(t) = e^{i\omega t} w_0$, $w_0 \in R^q$. Define

$$x(t) = e^{i\omega t} \left(i\omega \left(I - F e^{-i\omega g} \right) - \sum_{i=0}^{2} A_i e^{-i\omega h_i} \right)^{-1} B_1 \omega_0$$

and $z(t) = Cx(t) + Dw(t)$. Then $z(t) = e^{i\omega t} G(i\omega) w_0$, the triple $(w,\ x,\ z)$ satisfies (1a-b) and

$$2w^T(t)z(t) = w_0^T \left[G^*(i\omega) + G(i\omega)\right] w_0.$$

From (2), it follows that for all $t_1 \geq 0$:

$$2\int_0^{t_1} w^T(t)z(t)dt = t_1 w_0^* \left(G^*(i\omega) + G(i\omega)\right) w_0 \geq 0.$$

Since w_0 is arbitrary, this yields (ii). □

Remark 1: For $\gamma = 0$ and $D = 0$ LMI (8) implies that

$$P^T \begin{bmatrix} 0 \\ B_1 \end{bmatrix} = \begin{bmatrix} C^T \\ 0 \end{bmatrix}. \tag{17}$$

For

$$W_i = -P, \; R_i = \frac{\varepsilon I_{2n}}{h_i}, \; i = 1, \ldots, m \tag{18}$$

LMI (8) implies for $\varepsilon \to 0^+$ the delay-independent LMI shown in (19) at the bottom of the page. If LMI (19) is strictly feasible (i.e., holds with strict inequality) then (8) is feasible for a small enough $\varepsilon > 0$ and for R_i and W_i that are given by (18). Thus, from Theorem 1 the following corollary holds.

Corollary 1: Items (i) and (ii) of Theorem 1 hold if there exist $0 < P_1 = P_1^T,\ P_2,\ P_3, U = U^T$ and $S_i = S_i^T,\ i = 1, 2$ such that (19) is strictly feasible.

Remark 2: As we have seen above, the delay-dependent conditions of Theorem 1 [with strict LMI (8)] are most powerful in the sense that they provide sufficient conditions for both the delay-dependent and the delay-independent cases (where (19) is strictly feasible). In the latter case, (8) is feasible for $h_i \to \infty,\ i = 1, 2$. Moreover, strict LMI (8) yields the following LMI:

$$\begin{bmatrix} -P_3 - P_3^T + U & P_3^T F \\ * & -U \end{bmatrix} < 0 \tag{20}$$

and, thus, implies A1 [10].

Remark 3: In the case of system (1a-b), with distributed delay

$$\begin{aligned}\dot{x}(t) - F\dot{x}(t-g_i) = & \sum_{i=0}^{2} A_i x(t-h_i) \\ & + \int_{-h}^{0} A_d(s) x(t+s) ds + B_1 w(t) \end{aligned}$$

and exponential matrix $A_d(s) = A_{d1} \exp\{-A_{d0} s\}$, Theorem 1 can be applied to the following augmented system with discrete delays:

$$\begin{aligned} \dot{v}(t) =& x(t) - e^{A_{d0} d} x(t-d) + A_{d0} v(t), \\ \dot{x}(t) - F\dot{x}(t-g) =& \sum_{i=0}^{2} A_i x(t-h_i) \\ & + A_{d1} v(t) + B_1 w(t) \end{aligned} \tag{21}$$

where $v(t) = \int_{t-h}^{t} e^{A_{d0}(t-s)} x(s) ds$.

Example 1 [5]: We consider the following system:

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-h) + B_1 w \quad z(t) = Cx(t) \tag{22}$$

$$\begin{bmatrix} \bar{\Psi} & P^T \begin{bmatrix} 0 \\ B_1 \end{bmatrix} - \begin{bmatrix} C^T \\ 0 \end{bmatrix} & h_1 \Phi_1 & h_2 \Phi_2 & -W_1^T \begin{bmatrix} 0 \\ A_1 \end{bmatrix} & -W_2^T \begin{bmatrix} 0 \\ A_2 \end{bmatrix} & P^T \begin{bmatrix} 0 \\ F \end{bmatrix} \\ * & -\gamma I - D - D^T & 0 & 0 & 0 & 0 & 0 \\ * & * & -h_1 R_1 & 0 & 0 & 0 & 0 \\ * & * & * & -h_2 R_2 & 0 & 0 & 0 \\ * & * & * & * & -S_1 & 0 & 0 \\ * & * & * & * & * & -S_2 & 0 \\ * & * & * & * & * & * & -U \end{bmatrix} \leq 0 \tag{16}$$

$$\begin{bmatrix} A_0^T P_2 + P_2^T A_0 + \sum_{i=1}^{2} S_i & P_1 - P_2^T + A_0^T P_3 & P_2^T B_1 - C^T & P_2^T A_1 & P_2^T A_2 & P_2^T F \\ * & -P_3 - P_3^T + U & P_3^T B_1 & P_3^T A_1 & P_3^T A_2 & P_3^T F \\ * & * & -\gamma I - D - D^T & 0 & 0 & 0 \\ * & * & * & -S_1 & 0 & 0 \\ * & * & \cdot & \cdot & -S_2 & 0 \\ * & * & \cdot & \cdot & * & -U \end{bmatrix} \leq 0. \tag{19}$$

where

$$A_0 = \begin{bmatrix} -a_1 & k \\ -k & -a_2 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 0 \\ -c & 0 \end{bmatrix}$$
$$B_1 = [0 \quad 1]^T, \quad \text{and} \quad C = [0 \quad 1].$$

In [5], it is shown that the system is passive with $\gamma = 0$ for

$$a_1 > 0 \quad a_2 > 0 \quad c^2 < a_2^2.$$

By Theorem 1, choosing $a_1 = 1$, $a_2 = 2$, $c = 2.3$ and $k = 2.2$, which do not satisfy the conditions of [5], we find that the system is delay independently passive with $\gamma = 0$. Increasing c and taking $c = 2.7$ we obtain that the system is passive for $0 \le h < 9.8$.

The polar plot of the transfer function from w to z is depicted in Fig. 1. It resides entirely in the right half of the complex plane.

We note that in the latter example we utilized (17) and the fact that $P_3^T B_1 = 0$. Since $B_1 = [0 \quad 1]^T$ and $C = B_1^T P_2$, we had that

$$P_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \bar{P}_2 + \begin{bmatrix} 0 \\ 1 \end{bmatrix} C \text{ and } P_3 = \begin{bmatrix} \bar{P}_3 \\ 0 \end{bmatrix}$$

where $\bar{P}_2,\ \bar{P}_3 \in \mathcal{R}^{1\times 2}$. Thus, we solved (8) for P_1, $\bar{P}_2$, $\bar{P}_3$, S, R_i, $i = 1, 2, 3$, and W_j, $j = 1 \ldots 4$.

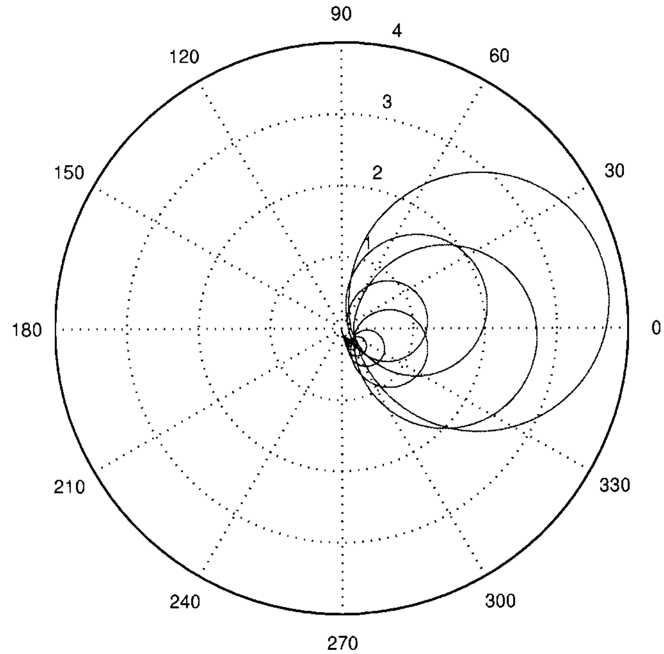

Fig. 1. The polar plot of the transfer function from w to z in Example 1 for $h = 9.7$.

III. State-Feedback Control

We apply the results of the previous section to the state-feedback design of passive systems. Given the system

$$\begin{aligned} \dot{x}(t) - \bar{F}\dot{x}(t-g) =& \bar{A}_0 x(t) + \bar{A}_1 x(t-h_1) \\ &+ \bar{A}_2 x(t-h_2) + B_1 w(t) + B_2 u(t) \\ z(t) =& \bar{C}x(t) + D_{12}u(t) + Dw(t) \end{aligned} \tag{23}$$

where x and w are defined in Section II, $u \in \mathcal{R}^{\ell}$ is the part of the control input that is used for feedback, $\bar{F}$, $\bar{A}_0$, $\bar{A}_1$, $\bar{A}_2$, B_1, B_2 are constant matrices of appropriate dimension, z is the objective vector, $\bar{C} \in \mathcal{R}^{q\times n}$, $D_{12} \in \mathcal{R}^{q\times \ell}$ and $D \in \mathcal{R}^{q\times q}$.

We look for the state-feedback gain matrix K which, via the control law

$$u(t) = Kx(t) \tag{24}$$

achieves passivity with $\gamma \ge 0$ of the closed-loop system.

$$\left[\begin{array}{cccccccccccc}
\Xi & \begin{bmatrix} 0 \\ B_1 \end{bmatrix} + Q^T \begin{bmatrix} C^T \\ 0 \end{bmatrix} & h_1(\varepsilon_1 I + I) & h_2(\varepsilon_2 I + I) & \varepsilon_1 \begin{bmatrix} 0 \\ A_1 \end{bmatrix} & \varepsilon_2 \begin{bmatrix} 0 \\ A_2 \end{bmatrix} & \begin{bmatrix} 0 \\ \bar{F} \end{bmatrix} & Q^T \begin{bmatrix} I_n \\ 0 \end{bmatrix} & Q^T \begin{bmatrix} I_n \\ 0 \end{bmatrix} & Q^T \begin{bmatrix} 0 \\ I_n \end{bmatrix} & h_1 Q^T \begin{bmatrix} 0 & 0 \\ 0 & A_1^T \end{bmatrix} & h_2 Q^T \begin{bmatrix} 0 & 0 \\ 0 & A_2^T \end{bmatrix} \\
* & -\gamma I - D - D^T & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & -h_1 R_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & -h_2 R_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & -S_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & -S_2 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & -U_1 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & -S_1^{-1} & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & -S_2^{-1} & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & -U_1^{-1} & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & -h_1 R_1^{-1} & 0 \\
* & * & * & * & * & * & * & * & * & * & * & -h_2 R_2^{-1}
\end{array}\right] \le 0 \tag{27}$$

Remark 4: The case where $B_1 = B_2$ in the system description of (23) corresponds to the standard case where w is the external control command and the actual input to the plant is $w + Kx$. The case where $B_1 \neq B_2$ describes the situation where only a part of the control inputs is used for feedback.

Substituting (24) into (23), we obtain the structure of (1a-b) with

$$A_0 = \bar{A}_0 + B_2 K,\ A_i = \bar{A}_i,\ i = 1, 2,\ C = \bar{C} + D_{12} K. \tag{25}$$

Applying Theorem 1 to the above matrices, results in a nonlinear matrix inequality because of the terms $P_2^T B_2 K$ and $P_3^T B_2 K$. We therefore consider another version of LMI condition which is derived from (16).

In order to obtain an LMI, we have to restrict ourselves to the case of $W_i = \varepsilon_i P$, $i = 1, 2$, where $\varepsilon_i \in R$ is a scalar parameter. Note that for $\varepsilon_i = 0$ (8) implies the delay-dependent conditions of [10] (for $\bar{F} = 0$), while for $\varepsilon_i = -1$ (8) yields the delay-independent condition of Corollary 1. It is obvious, from the requirement of $0 < P_1$ and the fact that in (8) the term $-(P_3 + P_3^T)$ must be negative–definite, that P is nonsingular. Defining

$$P^{-1} = Q = \begin{bmatrix} Q_1 & 0 \\ Q_2 & Q_3 \end{bmatrix} \quad \text{and} \quad \Delta = diag\{Q,\ I\} \tag{26a-b}$$

we multiply (16) by Δ^T and Δ, on the left and on the right, respectively. Applying the Schur formula to the quadratic term in Q, we obtain the inequality, as shown in (27) at the bottom of the previous page, where

$$\Xi = \begin{bmatrix} 0 & I \\ \sum_{i=0}^{2} A_i & -I \end{bmatrix} Q + Q^T \begin{bmatrix} 0 & \sum_{i=0}^{2} A_i^T \\ I & -I \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \sum_{i=1}^{2} \varepsilon_i A_i & 0 \end{bmatrix} Q + Q^T \begin{bmatrix} 0 & \sum_{i=1}^{2} \varepsilon_i A_i^T \\ 0 & -I \end{bmatrix}.$$

We substitute (25) into (27), denote KQ_1 by Y and obtain the following.

Theorem 2: Assume **A1**. *Consider the system of (23). The state-feedback law of (24) achieves passivity of the closed-loop system with $\gamma \geq 0$ if for some prescribed scalars ε_1, $\varepsilon_2 \in R$, there exist $Q_1 > 0$, $0 < \bar{S}_1 = S_1^{-1}$, $0 < \bar{S}_2 = S_2^{-1}$, $0 < \bar{U} = U^{-1}$, $Q_2,\ Q_3, \in \mathcal{R}^{n \times n}$, $0 < \bar{R}_1 = R_1^{-1}$, $0 < \bar{R}_2 = R_2^{-1} \in \mathcal{R}^{2n \times 2n}$, $K \in \mathcal{R}^{\ell \times n}$ and $Y \in \mathcal{R}^{\ell \times n}$ that satisfy the LMI* shown in (28) at the bottom of the page, *where $\bar{R}_{i1}$, $\bar{R}_{i2}$ and $\bar{R}_{i3}$ are the (1,1), (1,2) and (2,2) blocks of $\bar{R}_i$, $i = 1, 2$, and where*

$$\Xi_1 = Q_3 - Q_2^T + Q_1 \left(\sum_{i=0}^{2} \bar{A}_i^T + \sum_{i=1}^{2} \varepsilon_i \bar{A}_i^T \right) + Y^T B_2^T.$$

The state-feedback gain is then given by (24), where

$$K = Y Q_1^{-1}. \tag{29}$$

$$\left[\begin{array}{ccccccc}
Q_2 + Q_2^T & \Xi_1 & Q_1 \bar{C}^T + Y^T D_{12}^T & h_1(\varepsilon_1+1)\bar{R}_{11} & h_1(\varepsilon_1+1)\bar{R}_{12} & h_2(\varepsilon_2+1)\bar{R}_{21} & h_1(\varepsilon_2+1)\bar{R}_{22} \\
* & -Q_3 - Q_3^T & B_1 & h_1(\varepsilon_1+1)\bar{R}_{12}^T & h_1(\varepsilon_1+1)\bar{R}_{13} & h_2(\varepsilon_2+1)\bar{R}_{22}^T & h_2(\varepsilon_2+1)\bar{R}_{23} \\
* & * & -\gamma I - D - D^T & 0 & 0 & 0 & 0 \\
* & * & 0 & -h_1\bar{R}_{11} & -h_1\bar{R}_{12} & 0 & 0 \\
* & * & * & * & -h_1\bar{R}_{13} & 0 & 0 \\
* & * & * & * & * & -h_2\bar{R}_{21} & -h_2\bar{R}_{22} \\
* & * & * & * & * & * & -h_2\bar{R}_{23} \\
* & * & * & * & * & * & * \\
* & * & * & * & * & * & * \\
* & * & * & * & * & * & * \\
* & * & * & * & * & * & * \\
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* & * & * & * & * & * & * \\
* & * & * & * & * & * & * \\
* & * & * & * & * & * & * \\
* & * & * & * & * & * & *
\end{array}\right.$$
$$\left.\begin{array}{cccccccccc}
0 & 0 & Q_1 & Q_1 & 0 & Q_2^T & 0 & h_1 Q_2^T A_1^T & 0 & h_2 Q_2^T A_2^T \\
\varepsilon_1 A_1 \bar{S}_1 & \varepsilon_2 A_2 \bar{S}_2 & 0 & 0 & \bar{F}\bar{U} & Q_3^T & 0 & h_1 Q_3^T A_1^T & 0 & h_2 Q_3^T A_2^T \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\bar{S}_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & -\bar{S}_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & -\bar{S}_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & -\bar{S}_2 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & -\bar{U} & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & -\bar{U} & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & -h_1\bar{R}_{11} & -h_1\bar{R}_{12} & 0 & 0 \\
* & * & * & * & * & * & * & -h_1\bar{R}_{13} & 0 & 0 \\
* & * & * & * & * & * & * & * & -h_2\bar{R}_{21} & -h_2\bar{R}_{22} \\
* & * & * & * & * & * & * & * & * & -h_2\bar{R}_{23}
\end{array}\right] \leq 0 \tag{28}$$

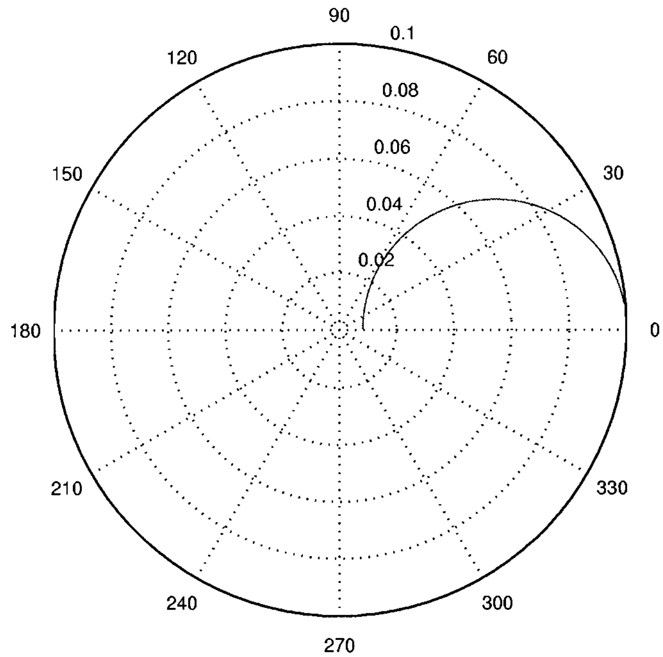

Fig. 2. The polar plot of the closed-loop transfer function from w to z for $D = 0.1$ in Example 2.

The LMIs in Theorems 1 and 2 are affine in the system matrices. Thus, it can be applied also to the case where these matrices are uncertain and are known to reside within a given polytope.

Example 2: We consider the system

$$\begin{aligned}\dot{x}(t) - \bar{F}\dot{x}(t-g) =& \bar{A}_0 x(t) + \bar{A}_1 x(t-h) \\ &+ B_1 w(t) + B_2 u(t) \\ z(t) =& \bar{C}x(t) + D_{12}u(t) + Dw(t)\end{aligned} \tag{30}$$

where

$$\begin{aligned}\bar{A}_0 &= \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \ \bar{A}_1 = \begin{bmatrix} -1 & 0 \\ -3 & 0 \end{bmatrix} \\ B_1 &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} \ B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \ \bar{C} = [\,0 \quad 1\,] \\ D_{12} &= 0.1, \text{ and } \bar{F} = 0.\end{aligned} \tag{31}$$

This system describes a case where the external input w and the feedback input u are applied via the same input matrix $B_1 = B_2$. Considering $D = 0$, we seek a state-feedback gain matrix K that will result in a passive closed-loop system. Applying Theorem 2 we obtained that for $\gamma = 0.2$, $h = 1.26$ and $\epsilon = -.3$, the closed-loop system (30) with

$$u = Kx(t) = [\,.0143 \quad -99.4224\,]\,x(t)$$

is passive. The same state-feedback gain matrix makes the system positive real in the case of $\gamma = 0$ and $D = 0.1$. The polar plot of the resulting closed-loop transfer function, from the control input w to z (with $D = .1$) is depicted in Fig. 2. It is clearly seen that this plot resides entirely in the right half of the complex plane and its "distance" from the imaginary axis may be considered as the overdesign that stems from the fact that the condition provided by Theorem 2 is only sufficient.

The above results refer to the case where $\bar{F} = 0$. For $\bar{F} = diag\{-.1, -.2\}$ we obtained, applying $\epsilon = -.33$ and $\gamma = 0.2$, that the state feedback gain

$$K = [\,.0015 \quad -99.8821\,]$$

yields a passive system $\forall\ h \in [0\ 1.145]$.

Remark 5: Note that the products $\epsilon_i A_i \bar{S}_i$ and $\epsilon_i R_{ij}$ ($i = 1, 2$, $j = 1, 2, 3$) in (28) are nonlinear in the unknown parameters. However, since ϵ_i are scalars, we solve (28) for different values of ϵ_i that lead to a minimum γ. For example, in Example 2 we calculated the minimum achievable γ for different values of ϵ. We have found there that the function $\epsilon(\gamma)$ is convex and this is how we obtained the optimal value of $\epsilon = -0.33$.

IV. Conclusion

A delay-dependent solution is proposed for the problem of passive state feedback control of linear time-invariant neutral and retarded type systems. The solution provides sufficient conditions in the form of LMIs. Although these conditions are not necessary, the overdesign entailed is minimal since it is based on an equivalent (descriptor) model transformation, which leads to the bounding of a smallest number of cross terms and since a new less conservative bounding is applied.

One question that often arises when solving control problems for systems with time-delay is whether the solution obtained for certain delays h_i will satisfy the design requirements for all delays $\bar{h}_i \leq h_i$. The answer is the affirmative, since the LMI in Theorem 2 is affine in the time delays.

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