Boundary Constrained Control of Delayed Nonlinear Schrödinger Equation

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***Abstract*—This paper studies regional boundary stabilization of nonlinear Schrödinger equation with state delay and bounded internal disturbance. The boundary constrained control law is designed by using the backstepping method. Regional input-to-state stability of the perturbed system with time-delay is established by a Lyapunov function and a generalized Halanay's inequality. Estimates on the set of initial conditions are found starting from which the solutions are exponentially attracted to a bounded set. A numerical example demonstrates the efficiency of the results.**

***Index Terms*—Boundary control, constrained control law, distributed parameter systems, nonlinear systems, time delay.**

I. Introduction

Schrödinger equation is central to all applications of quantum mechanics. A classical nonlinear Schrödinger equation is governed by

$$\mathrm{i}u_t(x,t) = -u_{xx}(x,t) - \mu|u(x,t)|^2u(x,t) \tag{1}$$

where $\mathrm{i} = \sqrt{-1}$, u is a complex-valued function of time and space, and $\mu \in \mathbb{R}$ is the Landau coefficient. In optics, the nonlinear Schrödinger equation is a model of wave propagation in fiber optics. Its state represents a wave and the nonlinear Schrödinger equation describes the propagation of the wave through a nonlinear medium. For water waves, the nonlinear Schrödinger equation describes the evolution of the envelope of modulated wave groups. Time delay in nonlinear Schrödinger equation may appear due to the Raman effect of nonlinear fiber optics (see, e.g., [1]–[3]). Raman self-frequency shift can down shift the central frequency and delay the pulse arrival time.

Boundary control of linearized Schrödinger equation has been studied in [4]–[6]. Recent results for the Schrödinger equation include the multiplier technique (see, e.g., [4]), the backstepping method (see, e.g., [5]), sliding mode control (see, e.g., [6]), and extension of backstepping to Heat-Schrödinger equation cascade [7] and ODE-Schrödinger equation cascade [8]. Stabilization of linear Schrödinger equation with time delay by boundary or internal feedbacks was dealt with in [9].

Stabilization of systems described by partial differential equations (PDEs) subject to time delay has attracted considerable attention (see, e.g., [10]–[12]). The backstepping method was extended to reaction-diffusion equations with state delay in [13].

Stabilization of finite-dimensional systems in the presence of actuator saturation has been extensively studied (see, e.g., [14]–[17] and the references therein). However, there are only a few works on constrained control for distributed parameter systems. In [18], the internal feedbacks with input constraints of quasi-linear heat equation were designed and the domains of attractions were found via the Galerkin's method. In [19] and [20], global stabilization by distributed state-feedback saturated control of one-dimensional Korteweg-de Vries and wave equations was studied. Global stabilization of linear or semilinear system in the Hilbert space by using constrained control was presented in [21] and [22]. In our recent paper, we have suggested regional boundary stabilization of coupled linear ODE-heat system in the presence of actuator saturation [23]. Our results were based on the backstepping approach [24], and on the direct Lyapunov method for finding domains of attraction of the resulting target systems.

This paper studies regional boundary stabilization of nonlinear Schrödinger equation with state delay and bounded internal disturbance. The boundary constrained control law is designed by using the backstepping method. Note that the regional stabilization of nonlinear Korteweg-de Vries equation was established by backstepping method in [25], where the nonlinear term of the form uu_x allowed to find an estimate on the domain of attraction of the target system. However, direct application of the backstepping method to Schördinger equation (1) with nonlinearity $|u|^2u$ leads to such a target system for which finding of domain of attraction becomes problematic. In this paper, to overcome this difficulty, we first transform the nonlinear system to the one with a new state $v = u_x$, and then apply the backstepping method. We establish regional input-to-state stability (ISS) of the delayed nonlinear target system via a Lyapunov function and a generalized Halanay's inequality [26]. Finally, we investigate the boundary constrained controller. We find an estimate on the set of initial conditions starting from which the state trajectories of the system are exponentially attracted to a bounded set.

Notation: Throughout the paper, $\Re$ (resp. $\Im$) denotes the real (resp. imaginary) part of a complex number. $L^2(0,1)$ stands for the Hilbert space of square integrable scalar functions on $(0,1)$ with inner product $\langle\cdot,\cdot\rangle$ and the corresponding norm $\|\cdot\|_{L^2}$. Given a Hilbert space $\mathcal{H}$, the space of the continuous $\mathcal{H}$-valued functions $z : [a,b] \to \mathcal{H}$ with the induced norm $\|z\|_{C([a,b];\mathcal{H})} = \max_{s\in[a,b]} \|z(s)\|_{\mathcal{H}}$ is denoted by $C([a,b];\mathcal{H})$.

II. Regional Stabilization: No Constraints on Control

A. System Description

We consider the following nonlinear Schrödinger equation with time delay and internal disturbance:

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$$\begin{cases} \mathrm{i}u_t(x,t) &= -u_{xx}(x,t) - \mu u(x,t-\tau(t))|u(x,t)|^2 + f(x,t) \\ u(0,t) &= 0 \\ u_x(1,t) &= U(t) \\ u(x,t) &= \psi(x,t),\ 0 \le x \le 1,\ -h \le t \le 0 \end{cases} \tag{2}$$

where $x \in (0,1)$, i is the imaginary unit, $\mu \in \mathbb{R}$ denotes a Landau coefficient, $u(x,t) \in \mathbb{C}$ is the state of nonlinear Schrödinger equation, $\psi(x,t)$ is the initial state, and $U(t) \in \mathbb{C}$ is the control input. Here, $f(x,t)$ represents an uncertain distributed disturbance source term, $\tau(t)$ is a continuously differentiable bounded delay

$$0 < h_0 \le \tau(t) \le h \tag{3}$$

with some constants h_0 and $h > 0$. The delay $\tau(t)$ may be unknown.

Following [3], we consider the nonlinear delay term "$\mu u(x,t-\tau(t))|u(x,t)|^2$" in system (2). Our results can be easily extended to another kind of nonlinear delay term "$\mu u(x,t)|u(x,t-\tau(t))|^2$" that was studied in [1] and [2] (see Remark 3 below).

We analyze the system (2) in the energy state space $H_L^1(0,1) = \{g \in H^1(0,1) | g(0) = 0\}$ defined in the complex number field $\mathbb{C}$. The space is naturally equipped with the H_L^1-norm given by $\|g\|_{H_L^1}^2 = \int_0^1 |g'(x)|^2\, dx$.

Assumption 1: The disturbances are assumed to be continuously differentiable function of class $C^1([0,\infty); H_L^1(0,1))$ subject to

$$\|f\|_{C^1([0,\infty);H_L^1(0,1))} \le \Delta_f \tag{4}$$

where Δ_f is a given constant upper bound.

Our objective is regional stabilization of (2). In order to find a bound on the domain of attraction, we change the state variable $v(x,t) = u_x(x,t)$ (see Remark 2 below for detailed explanation). Then, the new state $v(x,t)$ satisfies the following PDE:

$$\begin{cases} \mathrm{i}v_t(x,t) &= -v_{xx}(x,t) - \mu v(x,t-\tau(t)) \left|\int_0^x v(r,t)dr\right|^2 \\ & \quad -\mu \int_0^x v(r,t-\tau(t))dr \left[\int_0^x v(r,t)dr\overline{v(x,t)} \right. \\ & \quad \left. + v(x,t)\int_0^x \overline{v(r,t)}dr\right] + f_x(x,t) \\ v_x(0,t) &= 0 \\ v(1,t) &= U(t) \\ v(x,t) &= \psi_x(x,t),\ 0 \le x \le 1,\ -h \le t \le 0. \end{cases} \tag{5}$$

B. Backstepping Control

Now, we consider a backstepping transformation

$$z(x,t) = v(x,t) - \int_0^x k(x,y)v(y,t)dy = (I - \mathbb{P})v \tag{6}$$

that transforms the system (5) into the following target Schrödinger equation:

$$\begin{cases} \mathrm{i}z_t(x,t) &= -z_{xx}(x,t) - \mathrm{i}cz(x,t) \\ & \quad +(I-\mathbb{P})[F_1(v,x,t) + f_x(x,t)] \\ z_x(0,t) &= 0 \\ z(1,t) &= U(t) - \int_0^1 k(1,y)v(y,t)dy \\ z(x,t) &= \phi(x,t),\ 0 \le x \le 1,\ -h \le t \le 0 \end{cases} \tag{7}$$

where $c > 0$ is a constant

$$\phi(x,t) = \psi_x(x,t) - \int_0^x k(x,y)\psi_y(y,t)dy \tag{8}$$

and

$$\begin{aligned} F_1(v,x,t) = & -\mu v(x,t-\tau(t)) \left|\int_0^x v(r,t)dr\right|^2 \\ & - \mu \int_0^x v(r,t-\tau(t))dr \left[\int_0^x v(r,t)dr\overline{v(x,t)} \right. \\ & \left. + v(x,t)\int_0^x \overline{v(r,t)}dr\right]. \end{aligned} \tag{9}$$

Note that (6) is a modified transformation of [5] introduced for linear equation given by $v_t = -\mathrm{i}v_{xx}$. Next, we compute the kernel $k(x,y)$. Differentiation of transformation (6) with respect to t yields

$$\begin{aligned} \mathrm{i}z_t(x,t) = & -v_{xx}(x,t) + \int_0^x k(x,y)v_{yy}(y,t)dy \\ & + (I-\mathbb{P})[F_1(v,x,t) + f_x(x,t)]. \end{aligned}$$

Substituting (6) into the resulting equation with some calculations that involve integration by parts, we obtain

$$\begin{aligned} \mathrm{i}z_t(x,t) = & -v_{xx}(x,t) + k(x,x)v_x(x,t) - k_y(x,x)v(x,t) \\ & + k_y(x,0)v(0,t) + \int_0^x k_{yy}(x,y)v(y,t)dy \\ & + (I-\mathbb{P})[F_1(v,x,t) + f_x(x,t)]. \end{aligned}$$

Similarly, the first and second derivatives of $z(x,t)$ with respect to x are given by

$$z_x(x,t) = v_x(x,t) - k(x,x)v(x,t) - \int_0^x k_x(x,y)v(y,t)dy$$

$$\begin{aligned} z_{xx}(x,t) = & v_{xx}(x,t) - \frac{d}{dx}k(x,x)v(x,t) - k(x,x)v_x(x,t) \\ & - k_x(x,x)v(x,t) - \int_0^x k_{xx}(x,y)v(y,t)dy. \end{aligned}$$

Substituting (6) into (5) and comparing with (7), we obtain the following set of conditions on the kernel $k(x,y)$ (cf., [5]):

$$\begin{cases} k_{xx}(x,y) = k_{yy}(x,y) - \mathrm{i}ck(x,y) \\ k_y(x,0) = 0 \\ k(x,x) = \frac{\mathrm{i}c}{2}x. \end{cases} \tag{10}$$

The solution to (10) is given by

$$k(x,y) = \mathrm{i}cx\frac{I_1(\sqrt{\mathrm{i}c(y^2-x^2)})}{\sqrt{\mathrm{i}c(y^2-x^2)}}$$

where $I_1(\cdot)$ denotes the modified Bessel function of the first order: $I_1(x) = \sum_{n=0}^{\infty} \frac{(x/2)^{2n+1}}{n!(n+1)!}$.

The change of variable (6) has an inverse transformation

$$v(x,t) = z(x,t) + \int_0^x l(x,y)z(y,t)dy = (I-\mathbb{P})^{-1}z \tag{11}$$

where

$$l(x,y) = \mathrm{i}cx\frac{J_1(\sqrt{\mathrm{i}c(y^2-x^2)})}{\sqrt{\mathrm{i}c(y^2-x^2)}}$$

and $J_1(\cdot)$ is the Bessel function of the first order: $J_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{2n+1}}{n!(n+1)!}$.

The following functionals are used to simplify the presentation [cf., (6) and (11)] in the upcoming sections:

$$\mathcal{K}[v] = (I - \mathbb{P})v = v(x,t) - \int_0^x k(x,y)v(y,t)dy = z \tag{12}$$

$$\mathcal{L}[z] = (I - \mathbb{P})^{-1}z = z(x,t) + \int_0^x l(x,y)z(y,t)dy = v \tag{13}$$

where both $(I - \mathbb{P})$ and $(I - \mathbb{P})^{-1}$ are bounded on $L^2(0,1)$.

We denote $|v| = |v(x,t)|$ and $|z| = |z(x,t)|$. From (12) and (13), it can be shown that

$$|\mathcal{K}[v]| \le k_1(|v| + \|v\|_{L^1}),\ |\mathcal{L}[z]| \le k_2(|z| + \|z\|_{L^1}) \tag{14}$$

$$\|\mathcal{L}[z]\|_{L^2} \le k_3\|z\|_{L^2},\ \|\mathcal{K}[v]\|_{L^2} \le k_4\|v\|_{L^2} \tag{15}$$

where

$$\begin{aligned} k_1 &= \max\{1, \max_{0\le y\le x\le 1} |k(x,y)|\} \\ k_2 &= \max\{1, \max_{0\le y\le x\le 1} |l(x,y)|\} \\ k_3 &= 1 + \max_{0\le y\le x\le 1} |l(x,y)| \\ k_4 &= 1 + \max_{0\le y\le x\le 1} |k(x,y)|. \end{aligned} \tag{16}$$

By selecting the feedback controller

$$\begin{aligned} U(t) &= \int_0^1 k(1,y)v(y,t)dy \\ &= k(1,1)u(1,t) - \int_0^1 k_y(1,y)u(y,t)dy \end{aligned} \tag{17}$$

from (7), (9), (12), and (13), one arrives at the target system

$$\begin{cases} \mathrm{i}z_t(x,t) = -z_{xx}(x,t) - \mathrm{i}cz(x,t) + \mathcal{K}[F(z,x,t) + f_x(x,t)] \\ z_x(0,t) = 0 \\ z(1,t) = 0 \\ z(x,t) = \phi(x,t),\ 0 \le x \le 1,\ -h \le t \le 0 \end{cases} \tag{18}$$

where

$$\begin{aligned} F(z,x,t) &= F_1(v,x,t) \\ &= -\mu\mathcal{L}[z(x,t-\tau(t))]\left|\int_0^x \mathcal{L}[z(r,t)]dr\right|^2 \\ &\quad - \mu\overline{\mathcal{L}[z(x,t)]}\int_0^x \mathcal{L}[z(r,t-\tau(t))]dr\int_0^x \mathcal{L}[z(r,t)]dr \\ &\quad - \mu\mathcal{L}[z(x,t)]\int_0^x \mathcal{L}[z(r,t-\tau(t))]dr\int_0^x \overline{\mathcal{L}[z(r,t)]}dr. \end{aligned} \tag{19}$$

C. Halanay's Inequality

We will use the following generalization of Halanay's inequality for ISS analysis.

Lemma 1 (A generalized Halanay's inequality [26]): Given $\Delta_f > 0$, let $V : [t_0 - h, \infty) \to \mathbb{R}^+$ be a locally absolutely continuous function, and $f : [t_0, \infty) \to \mathbb{C}$ be a bounded continuous function satisfying $|f(t)| \le \Delta_f$. If there exist $0 < \delta_1 < \delta_0$ and $\gamma > 0$ such that for all $t \ge t_0$, the following inequality holds:

$$\frac{d}{dt}V(t) + 2\delta_0 V(t) - 2\delta_1 \sup_{-h\le\theta\le 0} V(t+\theta) - \gamma|f(t)|^2 \le 0$$

then we have

$$V(t) \le e^{-2\delta(t-t_0)} \sup_{-h\le\theta\le 0} V(t_0+\theta) + \frac{\gamma}{2\sigma}\Delta_f^2,\ t \ge t_0 \tag{20}$$

where $\sigma = \delta_0 - \delta_1 > 0$, and δ is a unique solution of

$$\delta = \delta_0 - \delta_1 e^{2\delta h}. \tag{21}$$

Remark 1: The result of Lemma 1 follows from [26, Th. 2.4], where $\alpha(t) = -2\delta_0,\ \beta(t) = 2\delta_1,\ \mu^* = 2\delta,\ \gamma(t) = \gamma|f(t)|^2$, and $u(t) = V(t), \tau(t) = h$.

D. Well-Posedness and ISS Analysis

Consider the target system (18). We introduce the Hilbert space $L^2(0,1)$. For the well-posedness, we will use the step method for solution of time-delay systems [10].

Define the system operator $\mathcal{A}_z : D(\mathcal{A}_z) \to L^2(0,1)$ for (18) as follows:

$$\begin{cases} \mathcal{A}_z = \mathrm{i}\dfrac{\partial^2}{\partial x^2} - c \\ D(\mathcal{A}_z) = \{z \in H^2(0,1) | z'(0) = z(1) = 0\}. \end{cases} \tag{22}$$

A simple computation shows that

$$\mathrm{Re}\langle \mathcal{A}_z z, z\rangle = -c\|z\|_{L^2}^2 \le 0,\ \forall z \in D(\mathcal{A}_z).$$

Therefore, the infinitesimal operator $\mathcal{A}_z$ is dissipative. Then from the Lumer–Phillips theorem ([27, Th. 1.4.3]), it follows that the operator $\mathcal{A}_z$ generates an exponentially stable semigroup $e^{\mathcal{A}_z t}$ on $L^2(0,1)$ (see also [5] that claimed this fact).

While being viewed over the time segment $[0, h_0]$, with the operator $\mathcal{A}_z$ at hand, the system (18) can be rewritten as the differential equation in $L^2(0,1)$

$$\frac{d}{dt}\zeta(t) = \mathcal{A}_z\zeta(t) + \hat{F}(\zeta,t) \tag{23}$$

$$\zeta(0) = \phi(\cdot,0) \tag{24}$$

where

$$\zeta(t) = z(\cdot,t),\ \hat{F}(\zeta,t) = -\mathrm{i}(I - \mathbb{P})\bar{F}(\zeta,t) \tag{25}$$

subject to

$$\begin{aligned} \bar{F}(\zeta,t) = &-\mu\mathcal{L}[\phi(t-\tau(t))]\left|\int_0^x \mathcal{L}[\zeta]dr\right|^2 \\ &- \mu\overline{\mathcal{L}[\zeta]}\int_0^x \mathcal{L}[\phi(t-\tau(t))]dr\int_0^x \mathcal{L}[\zeta]dr \\ &- \mu\mathcal{L}[\zeta]\int_0^x \mathcal{L}[\phi(t-\tau(t))]dr\int_0^x \overline{\mathcal{L}[\zeta]}dr + f_x(x,t). \end{aligned}$$

Definition 1: A function $\zeta \in L^1([0,T]; D(\mathcal{A}_z)) \cap C([0,T]; L^2(0,1))$ such that $\dot{\zeta} \in L^1([0,T]; L^2(0,1))$ is called a strong solution of initial value problem (23), (24) if (24) is satisfied and (23) holds a.e. on $[0,T]$.

Our definition of strong solution follows [27, p. 109].

Consider the following initial space for the target system:

$$Z \triangleq C\big([-h,0]; D(\mathcal{A}_z)\big) \cap C^1\big([-h,0]; L^2(0,1)\big) \tag{26}$$

which is endowed with the norm

$$\|\phi\|_Z = \|\phi\|_{C([-h,0];D(\mathcal{A}_z))} + \|\phi\|_{C^1([-h,0];L^2(0,1))}.$$

Lemma 2: Given any $h_0 > 0$ and $\phi \in Z$, let $\hat{F}(\zeta,t)$ be defined by (25), where $t \in [0,h_0]$ and $f \in C^1([0,h_0]; H_L^1(0,1))$. Then, the following statements hold.

(i) $\hat{F}$ is Lipschitz continuous in both variables locally in ζ, i.e., for any $t_1, t_2 \in [0, h_0]$ and any $R > 0$, there exists a positive constant $K(R)$ such that the inequality

$$\|\hat{F}(\zeta_1, t_1) - \hat{F}(\zeta_2, t_2)\|_{L^2} \le K(R)\,(|t_1 - t_2| + \|\zeta_1 - \zeta_2\|_{L^2}) \tag{27}$$

holds for all $\zeta_1, \zeta_2 \in L^2(0,1)$ with $\|\zeta_1\|_{L^2} \le R$, $\|\zeta_2\|_{L^2} \le R$.

(ii) There exists a unique strong solution of (23) on some interval $[0, t'] \subset [0, h_0]$ (here, $t' > 0$ depends on ϕ). Moreover, if this solution admits a priori estimate

$$\|\zeta\|_{L^2} \le C(\phi)$$

where $C(\phi)$ is constant depending on initial state ϕ, then the solution exists on the entire interval $[0, h_0]$. This solution is Lipschitz continuous on $[0, h_0]$ meaning that $\|\zeta(t_1) - \zeta(t_2)\|_{L^2} \le L(\phi)|t_1 - t_2|$ (here, $L(\phi)$ depends on ϕ).

Proof:

(i) Since $\phi \in Z$, τ is continuously differentiable, and $\mathcal{L}$ is bounded [cf., (15)], we have

$$\begin{aligned}
&\|\mathcal{L}[\phi(t_1 - \tau(t_1)) - \phi(t_2 - \tau(t_2))]\|_{L^2}\\
&\le \|\mathcal{L}[\phi(t_1 - \tau(t_1)) - \phi(t_1 - \tau(t_2))]\|_{L^2}\\
&\quad + \|\mathcal{L}[\phi(t_1 - \tau(t_2)) - \phi(t_2 - \tau(t_2))]\|_{L^2}\\
&\le k_3\|\phi\|_Z \left(1 + \max_{0 \le s \le h_0} |\dot{\tau}(s)|\right)|t_1 - t_2|,\ \forall t_1, t_2 \in [0, h_0].
\end{aligned} \tag{28}$$

Here, k_3 and k_4 are given by (16).

Since $I - \mathbb{P}$ is bounded [cf., (15)], application of the Minkowski's and Cauchy–Schwarz's inequalities leads to

$$\begin{aligned}
&\|\hat{F}(\zeta_1, t_1) - \hat{F}(\zeta_2, t_2)\|_{L^2}\\
&\le \|(I - \mathbb{P})\| \cdot \|\bar{F}(\zeta_1, t_1) - \bar{F}(\zeta_2, t_2)\|_{L^2}\\
&\le k_4\|\bar{F}(\zeta_1, t_1) - \bar{F}(\zeta_2, t_2)\|_{L^2}\\
&\le k_4\|\bar{F}(\zeta_1, t_1) - \bar{F}(\zeta_1, t_2)\|_{L^2} + k_4\|\bar{F}(\zeta_1, t_2) - \bar{F}(\zeta_2, t_2)\|_{L^2}\\
&\le 3k_4\mu\|\mathcal{L}[\phi(t_1 - \tau(t_1)) - \phi(t_2 - \tau(t_2))]\|_{L^2} \cdot \|\mathcal{L}[\zeta_1]\|^2_{L^2}\\
&\quad + 3k_4\mu\|\mathcal{L}[\phi(t_2 - \tau(t_2))]\|_{L^2} \cdot \|\mathcal{L}[\zeta_1 - \zeta_2]\|\\
&\quad \times [\|\mathcal{L}[\zeta_1]\|_{L_2} + \mathcal{L}[\zeta_2]\|_{L_2}] + k_4\|f(\cdot, t_1) - f(\cdot, t_2)\|_{H^1_L}\\
&\quad \times \forall t_1, t_2 \in [0, h_0],\ \forall \zeta_1, \zeta_2 \in L^2(0,1).
\end{aligned} \tag{29}$$

The fact that $f \in C^1([0, h_0]; H^1_L(0,1))$ implies the local Lipschitz condition holds, i.e., $\|f(\cdot, t_1) - f(\cdot, t_2)\|_{H^1_L} \le K_1|t_1 - t_2|, \forall t_1, t_2 \in [0, h_0]$ for some $K_1 > 0$.

Taking into account (28) and the latter relation, we conclude that there exists a positive constant $K(R)$ such that (27) holds for all $\zeta_1, \zeta_2 \in L^2(0,1)$ with $\|\zeta_1\|_{L^2} \le R$, $\|\zeta_2\|_{L^2} \le R$.

(ii) The existence and uniqueness of local strong solution of (23) on some interval $[0, t'] \subset [0, h_0]$ follow now from [27, Ths. 6.1.4 and 6.1.6].

Continuation of this solution under *a priori* bound to entire interval $[0, h_0]$ follows from [28, Th. 6.23.5]. The fact that solution is Lipschitz follows from the proof of [27, Th. 6.1.6] [cf., (6.1.20)]. ■

Definition 2:

(ii) By strong solution of the initial value problem for the target system (18) with the initial function $\phi \in Z$, we understand a function $z \in L^1([0, \infty); D(\mathcal{A}_z)) \cap C([0, \infty); L^2(0,1))$ such that $\dot{z} \in L^1([0, \infty); L^2(0,1))$ and the corresponding evolution equation holds a.e. on $[0, \infty)$.

Now, we are in a position to formulate the conditions that guarantee the existence of strong solution of (18) for all $t \ge 0$ with small enough initial data $\max_{-h \le \theta \le 0} \|\phi(\cdot, \theta)\|_{L^2}$ as well as for regional ISS of (18).

Proposition 1: Consider the target system (18) under Assumption 1. Given $h_0 > 0$, $c > 0$, $\Delta_f > 0$, $C_1 > 0$, and $\mu \in \mathbb{R}$, assume that there exist $0 < \delta_1 < \delta_0$ and $\gamma > 0$ such that the following linear matrix inequalities (LMIs) hold:

$$(\delta_0 - \delta_1)C_1 > \gamma\Delta_f^2 \tag{30}$$

$$\Theta_0 = \begin{bmatrix} -c + \delta_0 & 6k_1k_2k_3^2|\mu|C_1 & k_1\\ * & -\delta_1 & 0\\ * & * & -\gamma \end{bmatrix} < 0 \tag{31}$$

where $k_i > 0$ $(i = 1, 2, 3)$ are defined by (16). Then, for any initial condition ϕ from the set

$$\mathcal{X}_\phi = \left\{\phi \in Z : \max_{-h \le \theta \le 0} \|\phi(\cdot, \theta)\|^2_{L^2} \le C_1 - \frac{\gamma}{\sigma}\Delta_f^2\right\} \tag{32}$$

where $\sigma = \delta_0 - \delta_1$, the following statements holds.

(i) *Well-Posedness:* There exists a unique strong solution of the initial value problem for the target system (18) for all $t \ge 0$.

(ii) *Regional ISS:* The strong solution of (18) satisfies

$$\int_0^1 |z(x,t)|^2 dx \le e^{-2\delta t} \max_{\theta \in [-h, 0]} \int_0^1 |z(x,\theta)|^2 dx + \frac{\gamma}{\sigma}\Delta_f^2 \tag{33}$$

for all $t \ge 0$, where δ is a unique solution of (21).

(iii) Given $C_1 > 0$ and any $\delta > 0$, the decay rate δ in the inequality (33) can be achieved by appropriate choice of the design parameter c.

Proof: (i) and (ii): Step 1: From (ii) of Lemma 2, it follows that there exists a strong solution of (23) initialized with $\phi \in \mathcal{X}_\phi (\subset Z)$ on some interval $[0, t'] \subset [0, h_0]$. Therefore, we first employ the Lyapunov functional for $[0, t']$

$$V(t) = \frac{1}{2}\int_0^1 |z(x,t)|^2 dx. \tag{34}$$

Taking the time derivative of the Lyapunov function along (18), we obtain

$$\begin{aligned}
&\dot{V}(t) + 2\delta_0 V(t) - 2\delta_1 \sup_{-h \le \theta \le 0} V(t + \theta) - \gamma\|f(\cdot, t)\|^2_{H^1_L}\\
&\le -c\int_0^1 |z(x,t)|^2 dx + \delta_0 \int_0^1 |z(x,t)|^2 dx\\
&\quad - \delta_1 \int_0^1 |z(x, t - \tau(t))|^2 dx - \gamma\|f(x,t)\|^2_{H^1_L}\\
&\quad + \int_0^1 |z(x,t)| \cdot |\mathcal{K}[F(z,x,t) + f_x(x,t)]| dx.
\end{aligned} \tag{35}$$

From (14) and (15), we have

$$\int_0^1 |z(x,t)|\cdot|\mathcal{K}[F(z,x,t)]|dx$$
$$\le k_1\int_0^1 |z(x,t)|[|F(z,x,t)|+\|F(z,x,t)\|_{L^1}]dx$$
$$\int_0^1 |\mathcal{L}[z(x,t)]|^2dx \le k_3^2\int_0^1 |z(x,t)|^2dx. \quad (36)$$

By the Cauchy–Schwartz inequality and (36), we obtain

$$\int_0^1 |z(x,t)|\cdot|F(z,x,t)|dx$$
$$\le |\mu|\int_0^1 |z(x,t)||\mathcal{L}[z(x,t-\tau(t))]|\left|\int_0^x \mathcal{L}[z(r,t)]dr\right|^2 dx$$
$$+2|\mu|\int_0^1 |z(x,t)||\mathcal{L}[z(x,t)]|\int_0^x|\mathcal{L}[z(r,t-\tau(t))]|\,dr$$
$$\times\int_0^x|\mathcal{L}[z(r,t)]|drdx$$
$$\le 6|\mu|k_2k_3^2\|z(\cdot,t)\|_{L^2}^2[\|z(\cdot,t-\tau(t))\|_{L^2}\|z(\cdot,t)\|_{L^2}]. \quad (37)$$

Similarly, by the Minkowski inequality, we find

$$\int_0^1 |z(x,t)|\|F(z,x,t)\|_{L^1}dx$$
$$\le 6|\mu|k_2k_3^2\|z(\cdot,t)\|_{L^2}^2[\|z(\cdot,t-\tau(t))\|_{L^2}\|z(\cdot,t)\|_{L^2}].$$

Set $\eta(t)=\mathrm{col}\left\{\|z(\cdot,t)\|_{L^2},\|z(\cdot,t-\tau(t))\|_{L^2},\|f(\cdot,t)\|_{H_L^1}\right\}$. If (30) and (31) are feasible, then for any initial state $\phi\in\mathcal{X}_\phi$, we obtain

$$\dot V(t)+2\delta_0V(t)-2\delta_1\sup_{-h\le\theta\le0}V(t+\theta)-\gamma\|f(\cdot,t)\|_{H_L^1}^2$$
$$\le\eta(t)^\top\Theta_0\eta(t)\le0.$$

By the generalized Halanay inequality, (33) holds on $[0,t']$.

Step 2: Given $\phi\in\mathcal{X}_\phi$. The solution of (18) initialized with $\phi\in\mathcal{X}_\phi$ is uniformly bounded whenever it exists meaning that

$$\|z(\cdot,t)\|_{L^2}\le e^{-2\delta t}\max_{\theta\in[-h,0]}\|\phi(\cdot,\theta)\|_{L^2}^2+\frac{\gamma}{\sigma}\Delta_f^2\le C_1$$

where δ is the solution of (21).

Thus, from (ii) of Lemma 2, we obtain that the strong solution of (18) initialized with $\phi\in\mathcal{X}_\phi$ exists for all $t\in[0,h_0]$.

Step 3: We apply the same arguments step-by-step for $[h_0,2h_0]$, $[2h_0,3h_0],\ldots$. For $t\in[h_0,2h_0]$, the system (18) can be also rewritten as an evolution equation (23) with $\zeta(h_0)\in D(\mathcal{A}_z)$, where $\hat F(\zeta,t)$ is given by (25) with a modified ϕ

$$\phi(\theta)=\begin{cases}\phi(\theta), & \theta\le0\\ \zeta(\theta), & \theta\in[0,h_0].\end{cases}$$

Since the modified ϕ is Lipschitz, inequalities (28) and (29), and thus, (27) hold for any $t_1,t_2\in[h_0,2h_0]$, any $R>0$ and for all $\zeta_1,\zeta_2\in L^2(0,1)$ such that $\|\zeta_1\|_{L^2}\le R$, $\|\zeta_2\|_{L^2}\le R$. Therefore, by applying arguments of Steps 1 and 2, we conclude that there exists a strong solution for $t\in[h_0,2h_0]$. Then by step method, the strong solution exists for all $t\ge0$. Furthermore, the generalized Halanay's inequality implies (33) for all $t\ge0$. The proof of (i) and (ii) is completed.

(iii) The LMIs (30) and (31) are always feasible for appropriate choice of c. Indeed, applying Schur complement theorem, we have

$$\Theta_0<0\Longleftrightarrow -c+\delta_0+k_1^2\gamma^{-1}-(6k_1k_2k_3^2|\mu|C_1)^2\delta_1^{-1}<0. \quad (38)$$

For given $C_1>0$, $\delta_1>0$ and large enough $\gamma>0$, we can always choose $\delta_0>\delta_1$ as large as we want that satisfy (30), (38) by appropriate choice of c. The latter means that the decay rate bound can be improved by appropriate choice of design parameter c. ∎

Returning to the original system (2) with the control law (17), from the definition of $D(\mathcal{A}_z)$ [cf., (22)], $v=u_x$ and the backstepping transformation (12), we define

$$D(\mathcal{A}_u)=\Big\{u\in H^3(0,1)|u(0)=0,u'(1)=k(1,1)u(1)$$
$$-\int_0^1 k_y(1,y)u(y)dy\Big\}$$

and initial space for the original system

$$\Omega=C([-h,0];D(\mathcal{A}_u))\cap C^1([-h,0];H_L^1(0,1)). \quad (39)$$

Definition 3: By strong solution of the initial value problem for the closed-loop system (2), (17) with the initial function $\psi\in\Omega$, we understand a function

$$u\in L^1([0,\infty);D(\mathcal{A}_u))\cap C([0,\infty);H_L^1(0,1))$$

such that $\dot u\in L^1([0,\infty);H_L^1(0,1))$ and the corresponding evolution equation holds a.e. on $[0,\infty)$.

The following theorem states the well-posedness as well as regional ISS for the closed-loop system (2), (17).

Theorem 1: Consider the original system (2) with the control law (17) under Assumption 1. Given $h_0>0$, $c>0$, $\Delta_f>0$, $C_1>0$, and $\mu\in\mathbb{R}$, assume that there exist $0<\delta_1<\delta_0$ and $\gamma>0$ such that LMIs (30) and (31) hold, where $k_i>0$ $(i=1,2,3,4)$ are defined by (16). The initial state ψ is assumed to be in the set

$$\mathcal{X}_\psi=\left\{\psi\in\Omega: k_4\max_{-h\le\theta\le0}\|\psi(\cdot,\theta)\|_{H_L^1}\le\sqrt{C_1-\frac{\gamma}{\sigma}\Delta_f^2}\right\} \quad (40)$$

where $\sigma=\delta_0-\delta_1$ and Ω is defined by (39). Then the following statements hold.

(i) *Well-Posedness:* A strong solution of the closed-loop system exists for all $t\ge0$.

(ii) *Regional ISS:* The closed-loop system is exponentially ISS

$$\|u(\cdot,t)\|_{H_L^1}^2\le k_3^2k_4^2e^{-2\delta t}\max_{\theta\in[-h,0]}\|u(\cdot,\theta)\|_{H_L^1}^2+\frac{\gamma k_3^2}{\sigma}\Delta_f^2,\forall t\ge0. \quad (41)$$

Proof: By employing (12) and (15) with $v=u_x$, we have

$$\|z(\cdot,t)\|_{L^2}=\|\mathcal{K}[u_x(\cdot,t)]\|_{L^2}\le k_4\|u_x(\cdot,t)\|_{L^2}.$$

Hence, if the initial state ψ of (2), (17) starts from the set $\mathcal{X}_\psi$, then the initial state ϕ of the target system (18) belongs to the set $\mathcal{X}_\phi$, where $\mathcal{X}_\phi$ is given by (32), that is to say

$$\max_{-h\le\theta\le0}\|\phi(\cdot,\theta)\|_{L^2}^2\le k_4^2\max_{-h\le\theta\le0}\|\psi(\cdot,\theta)\|_{H_L^1}^2\le C_1-\frac{\gamma}{\sigma}\Delta_f^2. \quad (42)$$

Therefore, from Proposition 1 and the model transformation formulas (12) and (13) with bounded operators $(I-\mathbb{P})$ and its inverse, we obtain that if the LMIs (30) and (31) are feasible, then the strong solution of (2), (17) with initial condition $\psi\in\mathcal{X}_\psi$ exists. Moreover, the strong solution of target system (18) initialized with $\phi\in\mathcal{X}_\phi$ satisfies (33).

By using (15) and (33), together with (42), we arrive at (41)

$$\|u(\cdot,t)\|^2_{H^1_L} = \|u_x(\cdot,t)\|^2_{L^2} = \|\mathcal{L}[z(\cdot,t)]\|^2_{L^2}$$
$$\le k_3^2\|z(\cdot,t)\|^2_{L^2} \le k_3^2\left[e^{-2\delta t}\max_{\theta\in[-h,0]}\|\phi(\cdot,\theta)\|^2_{L^2} + \frac{\gamma}{\sigma}\Delta_f^2\right]$$
$$\le k_3^2\left[k_4^2 e^{-2\delta t}\max_{\theta\in[-h,0]}\|\psi(\cdot,\theta)\|^2_{H^1_L} + \frac{\gamma}{\sigma}\Delta_f^2\right], \quad \forall t\ge 0.$$

■

Remark 2: Here, we explain why we introduce the change of variable: $v(x,t) = u_x(x,t)$. If we use the backstepping transformation directly, without this change of variable, then by applying arguments of Proposition 1, instead of (37) we arrive at the following inequality:

$$\int_0^1 |z(x,t)|\cdot|F(z,x,t)|dx$$
$$\le 3|\mu|\int_0^1 |z(x,t)||\mathcal{L}[z(x,t-\tau(t))]|\mathcal{L}[z(x,t)]|^2 dx$$

where $\mathcal{L}$ is a bounded linear operator which is related to the backstepping transformation. It is seen that the right-hand side of the latter inequality cannot be bounded in terms of $\|z\|_{L^2}$.

Consider the system (18) without the disturbance. Then for the unperturbed system, we obtain the following result.

Corollary 1: Consider the system (18) with $f(x,t)\equiv 0$. Given scalars $h_0 > 0$, $C_1 > 0$ and $\mu\in\mathbb{R}$, let $c > 0$ be a scalar such that the following inequality

$$c > 12k_1k_2k_3^2|\mu|C_1 \tag{43}$$

holds, where $k_i > 0$ $(i = 1,2,3)$ are defined by (16). Then, the solutions of unperturbed system starting from the initial set $\mathcal{X}_\phi$ given by (32) with $\Delta_f = 0$ converge to zero with a small enough decay rate.

Proof: From (31), we conclude that the unperturbed system is exponentially stable if the LMI

$$\begin{bmatrix} -c+\delta_0 & 6k_1k_2k_3^2|\mu|C_1 \\ * & -\delta_1\end{bmatrix} < 0 \tag{44}$$

is satisfied. The feasibility of strict LMI (44) with $\delta_0 = \delta_1 > 0$ implies its feasibility for $\bar\delta_0$ and $\bar\delta_1$ given by $\bar\delta_0 = \delta_0 + \varepsilon > \delta_0 = \bar\delta_1$ for small enough $\varepsilon > 0$. Since Halanay's inequality holds with $\bar\delta_0$ and $\bar\delta_1$, the system is exponentially stable with a small enough decay rate.

By Schur complements, we show next that (43) guarantees the feasibility of (44) with $\delta_0 = \delta_1 > 0$. Indeed, the LMI (44) is feasible with $\delta_0 = \delta_1 = \bar\delta$ iff

$$-(c-\bar\delta)\bar\delta + [6k_1k_2k_3^2|\mu|C_1]^2 < 0. \tag{45}$$

It is easily seen that $\bar\delta = \frac{c}{2}$ minimizes the left-hand side of (45) in $\bar\delta$. Substituting $\bar\delta = \frac{c}{2}$ into (45), we obtain

$$-\frac{c^2}{4} + [6k_1k_2k_3^2|\mu|C_1]^2 < 0$$

i.e., (43) guarantees exponential stability. ■

Remark 3: As in [1] and [2], we can consider the nonlinear delay term "$\mu u(x,t)|u(x,t-\tau(t))|^2$." In this case, we can apply the same transformation $v = u_x$ and further the same backstepping transformation that leads to the target system (18) with a correspondingly modified nonlinearity F

$$F(z,x,t) = -\mu\mathcal{L}[z(x,t)]\left|\int_0^x \mathcal{L}[z(r,t-\tau(t))]dr\right|^2$$
$$-\mu\overline{\mathcal{L}[z(x,t-\tau(t))]}\int_0^x \mathcal{L}[z(r,t)]dr$$
$$\times\int_0^x \mathcal{L}[z(r,t-\tau(t))]dr - \mu\mathcal{L}[z(x,t-\tau(t))]$$
$$\times\int_0^x \mathcal{L}[z(r,t)]dr\int_0^x \overline{\mathcal{L}[z(r,t-\tau(t))]}dr.$$

Lemma 2 for the corresponding target system holds. We can apply the same V to the target system. Here, we do not need Halanay's inequality since the delayed state is treated as bounded in L^2-norm. The LMI of Theorem 1 here has a form

$$\Theta_0 = \begin{bmatrix} -c+\delta+12k_1k_2k_3^2|\mu|C_1 & k_1 \\ * & -\gamma\end{bmatrix} < 0. \tag{46}$$

Let $\mathcal{X}_\psi$ be defined by (40), where σ is replaced by δ. Then, solution of the closed-loop system starting from any initial state $\psi\in\mathcal{X}_\psi$ exponentially converges in the sense that (41) holds, which implies that the closed-loop system is ISS.

Remark 4: In Corollary 1, we have presented an explicit condition (43) in terms of system parameters that guarantees the feasibility of (31) and (46) with a small enough δ and large enough γ. The delay-independent stability condition (43) [and thus, (31) and (46)] are not conservative because the same condition (43) follows from the direct Lyapunov analysis of the nondelayed target system [cf., (46)].

III. Constrained Control Law

In this section, we consider the system (2) with the control law, which is subject to the following amplitude constraint:

$$|U(t)| \le \bar u.$$

We design the state feedback controller in the following form:

$$U_{\text{sat}}(t) = \text{sat}_{\mathbb{C}}(U(t),\bar u) \tag{47}$$

where $U(t)$ is given by (17), and the saturation function is defined by

$$\text{sat}_{\mathbb{C}}(s,\bar u) := \begin{cases} s, & |s|\le\bar u \\ \dfrac{s}{|s|}\bar u, & |s| > \bar u.\end{cases}$$

Applying the latter control law (47), we represent the saturated closed-loop system as (7) with the following boundary condition:

$$z(1,t) = \text{sat}_{\mathbb{C}}(U(t),\bar u) - U(t). \tag{48}$$

We now provide sufficient conditions for saturation avoidance.

Proposition 2: Consider the target system (7) with the constrained control law (47) under Assumption 1. Given scalars $h_0 > 0$, $\Delta_f > 0$, $c > 0$, $C_1 > 0$, and $\mu\in\mathbb{R}$, assume that there exist $0 < \delta_1 < \delta_0$ and $\gamma > 0$ such that LMIs (30) and (31) hold, where $k_i > 0$ $(i = 1,2,3)$ are given by (16). Let $\beta > 0$ satisfy the following inequalities:

$$\frac{\gamma}{\sigma}\Delta_f^2 < \beta \le C_1,\ \sigma = \delta_0 - \delta_1 \tag{49}$$

$$\bar u^2 \ge \beta\left[\max_{0\le y\le 1}|l(1,y)|\right]^2. \tag{50}$$

Consider the set of initial states ϕ

$$\mathcal{X}_\phi^\beta = \left\{\phi\in Z : \max_{-h\le\theta\le 0}\|\phi(\cdot,\theta)\|^2_{L^2} \le \beta - \frac{\gamma}{\sigma}\Delta_f^2\right\}$$

which is a subset of $\mathcal{X}_\phi$ given by (32). Then all strong solutions of (7), (47) starting from $\mathcal{X}_\phi^\beta$ satisfy (33).

Proof: From the backstepping transformations (6) and (11), $U(t)$ admits the following representation:

$$U(t) = \int_0^1 k(1,y)v(y,t)dy = \int_0^1 l(1,y)z(y,t)dy.$$

Due to (11), $l(x,y)$ is a continuous function bounded on any compact. Then, Jensen's inequality implies

$$|U(t)| \le \max_{0\le y\le 1} |l(1,y)| \cdot \|z(\cdot,t)\|_{L^2}. \tag{51}$$

Denote $K \triangleq \max_{0\le y\le 1} |l(1,y)|$. Given $\bar{u} > 0$, we define the following set:

$$\mathcal{J}(K,\bar{u}) = \{z(\cdot,t) \in L^2(0,1) : K\|z(\cdot,t)\|_{L^2} \le \bar{u}\}. \tag{52}$$

Thus, we can obtain that if $z \in \mathcal{J}(K,\bar{u})$, then $|U(t)| \le \bar{u}$, and the saturation is avoided. Thus, the system (7) subject to (48) admits the linear representation (18).

By Proposition 1, we find that if the LMIs (30), (31) and the inequality (49) are satisfied, then we have

$$\|z(\cdot,t)\|_{L^2} \le e^{-2\delta t} \max_{\theta\in[-h,0]} \|\phi(\cdot,\theta)\|_{L^2}^2 + \frac{\gamma}{\sigma}\Delta_f^2, \quad \forall t \ge 0. \tag{53}$$

Moreover, the inequality (53) guarantees that the trajectories $z(x,t)$ starting from $\phi \in \mathcal{X}_\phi^\beta$ remain within the reachable set $\mathcal{X}_\beta$, where

$$\mathcal{X}_\beta = \left\{ z(\cdot,t) \in L^2(0,1) : \int_0^1 |z(x,t)|^2 dx \le \beta \right\}.$$

The "ball" $\mathcal{X}_\beta$ is contained in $\mathcal{J}(K,\bar{u})$, if the following implication holds:

$$\|z(\cdot,t)\|_{L^2}^2 \le \beta \Rightarrow K\|z(\cdot,t)\|_{L^2} \le \bar{u}$$

for all $z(x,t)$, i.e., if

$$K^2\|z(\cdot,t)\|_{L^2}^2 \le \beta^{-1}\bar{u}^2\|z(\cdot,t)\|_{L^2}^2.$$

The latter inequality is guaranteed if the inequality (50) holds. Thus, the inequality (50) guarantees the saturation avoidance, and together with Proposition 1 imply (33). ■

From Proposition 2, by arguments of Theorem 1, we arrive at the following theorem.

Theorem 2: Consider the original system (2) with the constrained control law (47) under Assumption 1. Given scalars $h_0 > 0$, $\Delta_f > 0$, $c > 0$, $C_1 > 0$, and $\mu \in \mathbb{R}$, assume that there exist $0 < \delta_1 < \delta_0$ and $\gamma > 0$ such that LMIs (30) and (31) hold, where $k_i > 0$ $(i = 1,2,3,4)$ are given by (16). Let $\beta > 0$ satisfy the inequalities (49) and (50). Then, all the strong solutions of the closed-loop system (2), (47) starting from the initial set

$$\mathcal{X}_\psi^\beta = \left\{ \psi \in \Omega : k_4 \max_{-h\le\theta\le 0} \|\psi(\cdot,\theta)\|_{H_L^1} \le \sqrt{\beta - \frac{\gamma}{\sigma}\Delta_f^2} \right\}$$

satisfy (41).

IV. Example

Consider the system (2) and (7) with the parameters $c = 8$, $\mu = \pm 0.01$, $\bar{u} = 10$, and $\Delta_f = 0.01$. Here, $k_1 = k_2 = 4.6259$ and $k_3 = k_4 = 5.6259$. In order to increase an initial ball $\mathcal{X}_\psi^\beta$ inside of domain of initial set, we minimize γ (i.e., maximize $M_0 \triangleq \beta - \gamma\Delta_f^2/\sigma$) subject to the LMIs of Theorem 2. Fixing $C_1 = 0.064$, we find that the LMIs of Theorem 2 are feasible with $\delta_0 = 2$, $\delta_1 = 1.5$ and $\gamma = 14.3570$ leading to $\max M_0 = 0.0612$. Hence, we obtain the following ball inside of initial set for original system:

$$\mathcal{X}_\psi^\beta = \left\{ \psi \in \Omega : 5.6259 \max_{\theta\in[-h,0]} \|\psi(\cdot,\theta)\|_{H_L^1} \le 0.247 \right\}.$$

Next, a finite difference method is applied to compute the state of the closed-loop perturbed system (2) with the constrained control law (47). The steps of space and time are taken as 0.1 and 0.00001, respectively. Here, we choose the time-varying delay $\tau(t) = 0.2(\sin t + 1)$, the space-and time-varying perturbation $f(x,t) = 0.01\sin(x)\sin(t)$, and the following initial conditions:

$$u(x,\theta) = \psi(x,\theta) = 0.88 \sin\left(\frac{x}{10}\right), \ \theta \in [-0.4, 0]. \tag{54}$$

Hence, $5.6259 \max_{\theta\in[-0.4,0]} \|\psi(\cdot,\theta)\|_{H_L^1} = 0.245 < 0.247$. Fig. 1 demonstrates the state of the closed-loop system of (2) with constrained control (47). It is seen that the solution of the closed-loop system converges to a neighborhood of the origin as the time increases.

In Fig. 2, we choose the constant delay $\tau(t) \equiv 0.2$ and the initial state outside $\mathcal{X}_\psi^\beta$: $u(x,\theta) = \psi(x,\theta) = 15\sinh x + 5ix$, $\theta \in [-0.2, 0]$. Here, $\max_{\theta\in[-0.2,0]} \|\psi(\cdot,\theta)\|_{H_L^1} = 18.47$. It is seen that the state of nonlinear Schrödinger equation is vibrating. The simulations of the solutions confirm the theoretical results.

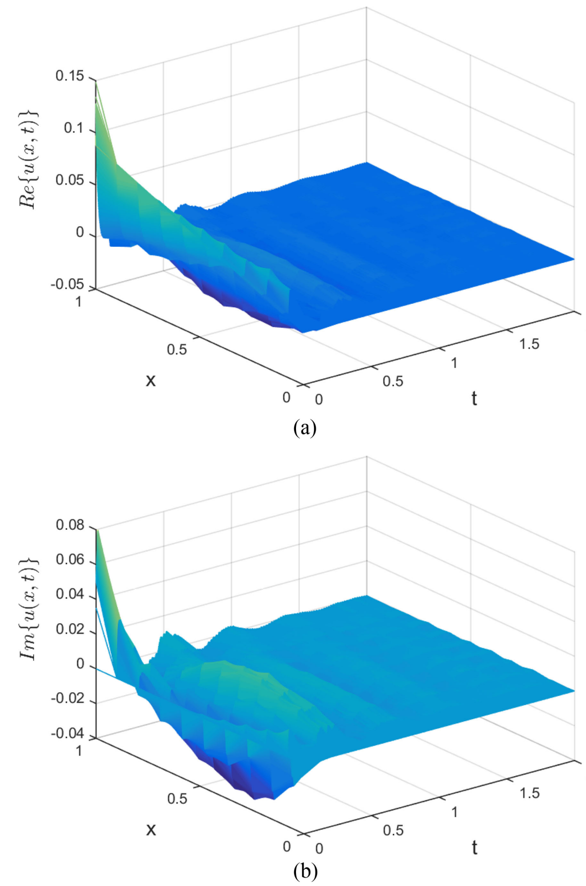
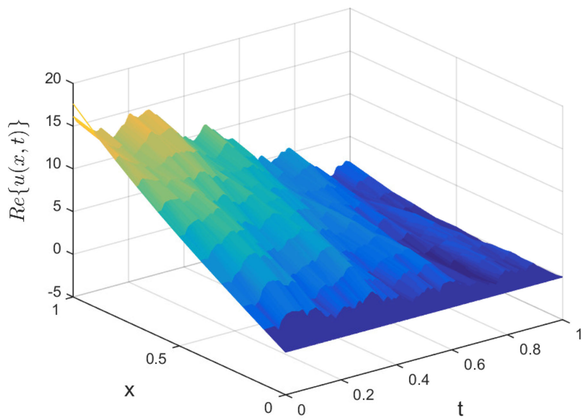
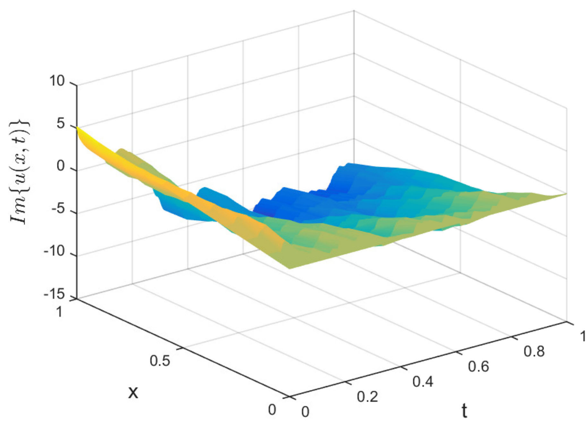


Fig. 1. Input-to-state stability of the perturbed system with time delay. (a) State $\Re\{u(x,t)\}$. (b) State $\Im\{u(x,t)\}$.



(a)



(b)

Fig. 2. Instability when the initial state is chosen outside the ball $\mathcal{X}_{\psi}^{\beta}$. (a) State $\Re\{u(x,t)\}$. (b) State $\Im\{u(x,t)\}$.

V. Conclusion

In this note, we have extended the backstepping approach to nonlinear Schrödinger equation with state delay and internal disturbance. By using a state transformation and applying further the backstepping method, we have found LMI conditions for regional ISS of the nonlinear target and original systems. We have also provided LMI conditions for an estimate on the initial set and on the attractive set in the case of constrained controller. The LMIs for ISS of the delayed target system were derived via generalized Halanay's inequality. Observer-based boundary control of nonlinear PDEs may be a topic for future research.

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