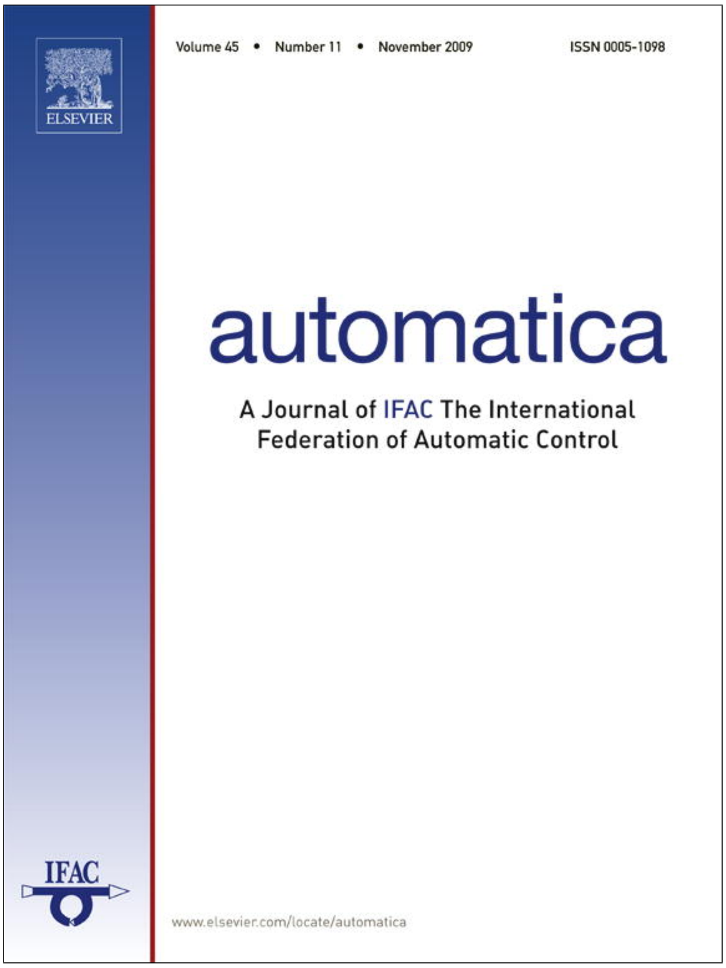
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Technical communique

New conditions for delay-derivative-dependent stability[☆,☆☆]

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ABSTRACT

Two recent Lyapunov-based methods have significantly improved the stability analysis of time-delay systems: the delay-fractioning approach of Gouaisbaut and Peaucelle (2006) for systems with constant delays and the convex analysis of systems with time-varying delays of Park and Ko (2007). In this paper we develop a convex optimization approach to stability analysis of linear systems with interval time-varying delay by using the delay partitioning-based Lyapunov–Krasovskii Functionals (LKFs). *Novel LKFs are introduced with matrices that depend on the time delays*. These functionals allow the derivation of stability conditions that depend on both the upper and *lower bounds on delay derivatives*.

1. Introduction

Over the past decades, much effort has been invested in the analysis and design of systems with time delays (see e.g. Fridman & Shaked, 2002; Hale & Lunel, 1993; He, Wang, Lin, & Wu, 2007; Kolmanovskii & Myshkis, 1999; Niculescu, 2001; Richard, 2003). Among the recent advances in this area, two Lyapunov-based methods should be mentioned that significantly improved the stability analysis: the convex analysis of systems with time-varying delays of Park and Ko (2007) and the delay-fractioning approach of Gouaisbaut and Peaucelle (2006) for systems with constant delays.

These recent methods inspired the present work, where we extend the delay partitioning approach to systems with interval time-varying delay in a convex way. We introduce novel LKFs with matrices that depend on the time delays. This enables us to derive LMI conditions that depend not only on the upper, but also on the lower bound of the delay derivative. The efficiency of the new stability criteria is demonstrated via numerical examples.

2. Stability of systems with time-varying delay

Consider the system

$$\dot{x}(t) = Ax(t) + A_1 x(t-\tau(t)), \tag{1}$$

where $\tau(t) \in [h_a, h_b]$, $h_a \geq 0$ and where A and A_1 are constant matrices. The delay is assumed to be either differentiable with

$$d_1 \leq \dot{\tau}(t) \leq d_2, \tag{2}$$

where d_1 and d_2 are given bounds, or fast-varying (with no restrictions on the delay derivative). The initial condition is given by $x(t_0+\theta) = \phi(\theta)$, $\theta \in [-h_b, 0]$, $\phi \in W$, where W is the space of absolutely continuous functions $\phi : [-h_b, 0] \to R^n$ with the square integrable derivative and with the norm

$$\|\phi\|_W^2 = |\phi(0)|^2 + \int_{-h_b}^{0} [|\phi(s)|^2 + |\dot{\phi}(s)|^2] ds.$$

2.1. A delay partitioning approach to stability

We divide the delay interval $[h_a, h_b]$ into two segments: $[h_1, h_2]$ and $[h_2, h_3]$, where we denote $h_1 = h_a$, $h_3 = h_b$ and $h_2 = (h_a + h_b)/2$. Then, (1) can be represented as

$$\begin{aligned} \dot{x}(t) &= Ax(t) + \chi_{[h_1,h_2]}(\tau(t)) A_1 x(t-\tau(t)) \\ &\quad + [1 - \chi_{[h_1,h_2]}(\tau(t))] A_1 x(t-\tau(t)), \end{aligned} \tag{3}$$

where $\chi_{[h_1,h_2]} : R \to \{0, 1\}$ is the characteristic function of $[h_1, h_2]$

$$\chi_{[h_1,h_2]}(s) = \begin{cases} 1, & \text{if } s \in [h_1, h_2] \\ 0, & \text{otherwise.} \end{cases}$$

Consider the following Lyapunov functional:

$$V(t, x_t, \dot{x}_t) = x^T(t) P(\tau(t)) x(t) + \int_{t-\tau(t)}^{t-h_1} x^T(s) Q x(s) ds + \int_{t-h_1}^{t} x^T(s) S_0 x(s) ds + \int_{t-h_2}^{t-h_1} \xi^T(s) \begin{bmatrix} S_{11} & S_{12} \\ * & S_{13} \end{bmatrix} \xi(s) ds$$

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$$+\sum_{i=0}^{2}(h_{i+1}-h_i)\int_{-h_{i+1}}^{-h_i}\int_{t+\theta}^{t}\dot{x}^{\mathrm{T}}(s)R_i\dot{x}(s)\mathrm{d}s\mathrm{d}\theta, \qquad (4)$$

where $h_0 = 0, \xi(s) = \mathrm{col}\{x(s), x(s-(h_2-h_1))\}$,

$$Q \geq 0, \quad R_i > 0,\ S_0 > 0, \begin{bmatrix} S_{11} & S_{12} \\ * & S_{13} \end{bmatrix} > 0. \qquad (5)$$

We seek $P(\tau(t)) \triangleq \bar{P}(t)$ of the form

$$P(\tau(t)) = \chi_{[h_1,h_2]}(\tau(t))\left[\frac{\tau(t)-h_1}{h_2-h_1}P^1 + \frac{h_2-\tau(t)}{h_2-h_1}P^2\right]$$
$$+[1-\chi_{[h_1,h_2]}(\tau(t))]\left[\frac{\tau(t)-h_2}{h_3-h_2}P^3 + \frac{h_3-\tau(t)}{h_3-h_2}P^1\right], \qquad (6)$$

i.e.

$$P(\tau(t)) = \begin{cases} \dfrac{\tau(t)-h_1}{h_2-h_1}P^1 + \dfrac{h_2-\tau(t)}{h_2-h_1}P^2, & h_1 \leq \tau(t) \leq h_2, \\ \dfrac{\tau(t)-h_2}{h_3-h_2}P^3 + \dfrac{h_3-\tau(t)}{h_3-h_2}P^1, & h_2 < \tau(t) \leq h_3, \end{cases}$$

where $P^k > 0, k = 1, 2, 3$. Note that the function $\bar{P}(t)$ is continuous in t, since $\lim_{\tau(t)\to h_2} P(\tau(t)) = P^1$.

Following (Hale & Lunel, 1993), we define

$$\dot{V}(t, x_t, \dot{x}_t) = \lim\sup_{s\to 0^+}\frac{1}{s}[V(t+s, x_{t+s}, \dot{x}_{t+s})$$
$$- V(t, x_t, \dot{x}_t)]. \qquad (7)$$

We are seeking for conditions guaranteeing that

$$\dot{V} \leq -\alpha|x(t)|^2 \qquad (8)$$

for some scalar $\alpha > 0$. We first consider $\tau \neq h_2$. We have

$$\dot{\bar{P}}(t)_{|\tau\neq h_2} = \dot{\tau}(t)\left[\frac{\chi(P^1-P^2)}{h_2-h_1} + \frac{(1-\chi)(P^3-P^1)}{h_3-h_2}\right]. \qquad (9)$$

Denoting $V_0 = x^{\mathrm{T}}(t)P(\tau(t))x(t)$, we find

$$\dot{V}_{0|\tau\neq h_2} = x^{\mathrm{T}}(t)\dot{\bar{P}}(t)x(t) + 2\dot{x}^{\mathrm{T}}(t)\left[\chi\left[\frac{\tau(t)-h_1}{h_2-h_1}P^1 + \frac{h_2-\tau(t)}{h_2-h_1}P^2\right] + (1-\chi)\right.$$
$$\left.\times\left[\frac{\tau(t)-h_2}{h_3-h_2}P^3 + \frac{h_3-\tau(t)}{h_3-h_2}P^1\right]\right]x(t). \qquad (10)$$

Moreover,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\sum_{i=0}^{2}(h_{i+1}-h_i)\int_{-h_{i+1}}^{-h_i}\int_{t+\theta}^{t}\dot{x}^{\mathrm{T}}(s)R_i\dot{x}(s)\mathrm{d}s\mathrm{d}\theta\right]$$
$$= \dot{x}^{\mathrm{T}}(t)\left[\sum_{i=0}^{2}(h_{i+1}-h_i)^2R_i\right]\dot{x}(t)$$
$$- \sum_{i=0}^{2}(h_{i+1}-h_i)\int_{t-h_{i+1}}^{t-h_i}\dot{x}^{\mathrm{T}}(s)R_i\dot{x}(s)\mathrm{d}s. \qquad (11)$$

We start with the case of $\tau \in [h_1, h_2)$, where $\chi = 1$. Using the fact that $\int_{t-h_{j+1}}^{t-h_j} f_j(s)\mathrm{d}s = \int_{t-h_{j+1}}^{t-\tau(t)} f_j(s)\mathrm{d}s + \int_{t-\tau(t)}^{t-h_j} f_j(s)\mathrm{d}s$,where $f_j(s) = \dot{x}^{\mathrm{T}}(s)R_j\dot{x}(s)$, we apply Jensen's inequality (Gu, Kharitonov, & Chen, 2003)

$$\int_{t-h_{i+1}}^{t-h_i}[(h_{i+1}-h_i)f_i(s)]\mathrm{d}s \geq \int_{t-h_{i+1}}^{t-h_i}\dot{x}^{\mathrm{T}}(s)\mathrm{d}sR_i\int_{t-h_{i+1}}^{t-h_i}\dot{x}(s)\mathrm{d}s,$$
$$\int_{t-\tau(t)}^{t-h_j}[(h_{j+1}-h_j)f_j(s)]\mathrm{d}s \geq (\tau(t)-h_j)(h_{j+1}-h_j)v_{j1}^{\mathrm{T}}R_jv_{j1},$$
$$\int_{t-h_{j+1}}^{t-\tau(t)}[(h_{j+1}-h_j)f_j(s)]\mathrm{d}s \geq (h_{j+1}-\tau(t))(h_{j+1}-h_j)v_{j2}^{\mathrm{T}}R_jv_{j2},$$

where

$$v_{j1} = \frac{1}{\tau(t)-h_j}\int_{t-\tau(t)}^{t-h_j}\dot{x}(s)\mathrm{d}s, \quad v_{j2} = \frac{1}{h_{j+1}-\tau(t)}$$
$$\times\int_{t-h_{j+1}}^{t-\tau(t)}\dot{x}(s)\mathrm{d}s, \qquad (12)$$

and $j = 1, i = 0, 2$. Here, for $\tau(t) \to h_j$, we have

$$\lim_{\tau(t)\to h_j}\frac{1}{\tau(t)-h_j}\int_{t-\tau(t)}^{t-h_j}\dot{x}(s)\mathrm{d}s = \dot{x}(t-h_j).$$

For $h_{j+1} - \tau(t) \to 0$ the vector $\frac{1}{h_{j+1}-\tau(t)}\int_{t-h_{j+1}}^{t-\tau(t)}\dot{x}(s)\mathrm{d}s$ is defined similarly as $\dot{x}(t-h_{j+1})$. We thus find

$$\dot{V}|_{\tau<h_2} \leq \dot{V}_{0|\tau<h_2} + \dot{x}^{\mathrm{T}}(t)\left[\sum_{i=0}^{2}(h_{i+1}-h_i)^2R_i\right]\dot{x}(t)$$
$$+ x^{\mathrm{T}}(t)S_0x(t) - x^{\mathrm{T}}(t-h_1)S_0x(t-h_1)$$
$$+ \begin{bmatrix} x(t-h_1) \\ x(t-h_2) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} S_{11} & S_{12} \\ * & S_{13} \end{bmatrix} \begin{bmatrix} x(t-h_1) \\ x(t-h_2) \end{bmatrix}$$
$$- \begin{bmatrix} x(t-h_2) \\ x(t-h_3) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} S_{11} & S_{12} \\ * & S_{13} \end{bmatrix} \begin{bmatrix} x(t-h_2) \\ x(t-h_3) \end{bmatrix}$$
$$- [x(t)-x(t-h_1)]^{\mathrm{T}}R_0[x(t)-x(t-h_1)]$$
$$- [x(t-h_2)-x(t-h_3)]^{\mathrm{T}}R_2[x(t-h_2)-x(t-h_3)]$$
$$- (\tau(t)-h_j)(h_{j+1}-h_j)v_{j1}^{\mathrm{T}}R_jv_{j1}$$
$$- (h_{j+1}-\tau(t))(h_{j+1}-h_j)v_{j2}^{\mathrm{T}}R_jv_{j2}$$
$$+ x^{\mathrm{T}}(t-h_1)Qx(t-h_1)$$
$$- (1-\dot{\tau}(t))x^{\mathrm{T}}(t-\tau(t))Qx(t-\tau(t)). \qquad (13)$$

where $j = 1, i = 0, 2$. We insert free-weighting $n \times n$-matrices $T_1, W_3, Y_{k1}, Z_{k1} (k = 1, 2)$ by adding the following expressions to $\dot{V}(t)|_{\tau<h_2}$:

$$0 = 2[x^{\mathrm{T}}(t)Y_{11}^{\mathrm{T}} + \dot{x}^{\mathrm{T}}(t)Y_{21}^{\mathrm{T}} + x^{\mathrm{T}}(t-\tau(t))T_1^{\mathrm{T}}]$$
$$[-x(t-h_1) + x(t-\tau(t)) + (\tau(t)-h_1)v_{11}],$$
$$0 = 2[x^{\mathrm{T}}(t)Z_{11}^{\mathrm{T}} + \dot{x}^{\mathrm{T}}(t)Z_{21}^{\mathrm{T}} + x^{\mathrm{T}}(t-h_3)W_3^{\mathrm{T}}]$$
$$[x(t-h_2) + (h_2-\tau(t))v_{22} - x(t-\tau(t))]. \qquad (14)$$

We further use the descriptor method, where the right-hand side of the expression

$$0 = 2[x^{\mathrm{T}}(t)P_{2j}^{\mathrm{T}} + \dot{x}^{\mathrm{T}}(t)P_{3j}^{\mathrm{T}}][Ax(t) + A_1x(t-\tau(t)) - \dot{x}(t)], \qquad (15)$$

with some $n \times n$-matrices P_{2j}, P_{3j} is added into the right-hand side of (13).

Setting $\eta_j(t) = \mathrm{col}\{x(t), \dot{x}(t), x(t-h_1), x(t-h_2), v_{j1}, v_{j2}, x(t-\tau(t)), x(t-h_3)\}$, we obtain that along (3)

$$\dot{V}|_{\tau<h_2} \leq \eta_1^{\mathrm{T}}(t)\Psi_{|\tau<h_2}\eta_1(t) \leq -\alpha_1|x(t)|^2, \qquad (16)$$

for some scalar $\alpha_1 > 0$ if

$$\Psi_{|\tau<h_2} \triangleq \left[\begin{matrix} \Omega_{11}^1 + \dfrac{\dot{\tau}(t)(P^1-P^2)}{h_2-h_1} & \Omega_{12}^1 & R_0 - Y_{11}^{\mathrm{T}} & Z_{11}^{\mathrm{T}} & (\tau(t)-h_1)Y_{11}^{\mathrm{T}} \\ * & \Omega_{22}^1 & -Y_{21}^{\mathrm{T}} & Z_{21}^{\mathrm{T}} & (\tau(t)-h_1)Y_{21}^{\mathrm{T}} \\ * & * & \phi_3^{(1)} & S_{12} & 0 \\ * & * & * & \phi_4^{(1)} & 0 \\ * & * & * & * & \phi_5^{(1)} \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{matrix}\right.$$

$$\begin{matrix} (h_2-\tau(t))Z_{11}^{\mathrm{T}} & \Omega_{17}^{1} & 0 \\ (h_2-\tau(t))Z_{21}^{\mathrm{T}} & \Omega_{27}^{1} & 0 \\ 0 & -T_1 & 0 \\ 0 & 0 & R_2+W_3-S_{12} \\ 0 & (\tau(t)-h_1)T_1 & 0 \\ \phi_6^{(1)} & 0 & (h_2-\tau(t))W_3 \\ * & -(1-\dot{\tau}(t))Q+T_1+T_1^{\mathrm{T}} & -W_3 \\ * & * & -(S_{13}+R_2) \end{matrix} \Bigg] < 0, \tag{17}$$

where

$$\begin{aligned} \Omega_{11}^{j} &= A^{\mathrm{T}}P_{2j} + P_{2j}^{\mathrm{T}}A + S_0 - R_0, \\ \Omega_{12}^{1} &= \left[\frac{\tau(t)-h_1}{h_2-h_1}P^1 + \frac{h_2-\tau(t)}{h_2-h_1}P^2\right] - P_{21}^{\mathrm{T}} + A^{\mathrm{T}}P_{31}, \\ \Omega_{22}^{j} &= -P_{3j} - P_{3j}^{\mathrm{T}} + \sum_{i=0}^{2}(h_{i+1}-h_i)^2R_i, \\ \Omega_{17}^{j} &= Y_{1j}^{\mathrm{T}} - Z_{1j}^{\mathrm{T}} + P_{2j}^{\mathrm{T}}A_1, \\ \Omega_{27}^{j} &= Y_{2j}^{\mathrm{T}} - Z_{2j}^{\mathrm{T}} + P_{3j}^{\mathrm{T}}A_1, \\ \phi_3^{(1)} &= -(S_0 + R_0 - S_{11} - Q), \\ \phi_4^{(1)} &= -(S_{11} + R_2 - S_{13}), \\ \phi_5^{(1)} &= -(h_2-h_1)(\tau(t)-h_1)R_1, \\ \phi_6^{(1)} &= -(h_2-h_1)(h_2-\tau(t))R_1, i, \quad j = 1, 2. \end{aligned} \tag{18}$$

The latter LMI leads for $\tau(t) \to h_1$ and for $\tau(t) \to h_2$ to the following LMIs:

$$\Psi_1 = \begin{bmatrix} \Omega_{11}^{1} + \frac{\dot{\tau}(t)(P^1-P^2)}{h_2-h_1} & \Omega_{12|\tau(t)=h_1}^{1} & R_0 - Y_{11}^{\mathrm{T}} & Z_{11}^{\mathrm{T}} & (h_2-h_1)Z_{11}^{\mathrm{T}} & \Omega_{17}^{1} & 0 \\ * & \Omega_{22}^{1} & -Y_{21}^{\mathrm{T}} & Z_{21}^{\mathrm{T}} & (h_2-h_1)Z_{21}^{\mathrm{T}} & \Omega_{27}^{1} & 0 \\ * & * & \phi_3^{(1)} & S_{12} & 0 & -T_1 & 0 \\ * & * & * & \phi_4^{(1)} & 0 & 0 & R_2+W_3-S_{12} \\ * & * & * & * & \phi_6^{(1)}{}_{|\tau(t)=h_1} & 0 & (h_2-h_1)W_3 \\ * & * & * & * & * & -(1-\dot{\tau}(t))Q+T_1+T_1^{\mathrm{T}} & -W_3 \\ * & * & * & * & * & * & -(S_{13}+R_2) \end{bmatrix} < 0, \tag{19}$$

and

$$\Psi_2 = \begin{bmatrix} \Omega_{11}^{1} + \frac{\dot{\tau}(t)(P^1-P^2)}{h_2-h_1} & \Omega_{12|\tau(t)=h_2}^{1} & R_0 - Y_{11}^{\mathrm{T}} & Z_{11}^{\mathrm{T}} & (h_2-h_1)Y_{11}^{\mathrm{T}} & \Omega_{17}^{1} & 0 \\ * & \Omega_{22}^{1} & -Y_{21}^{\mathrm{T}} & Z_{21}^{\mathrm{T}} & (h_2-h_1)Y_{21}^{\mathrm{T}} & \Omega_{27}^{1} & 0 \\ * & * & \phi_3^{(1)} & S_{12} & 0 & -T_1 & 0 \\ * & * & * & \phi_4^{(1)} & 0 & 0 & R_2+W_3-S_{12} \\ * & * & * & * & \phi_5^{(1)}{}_{|\tau(t)=h_2} & (h_2-h_1)T_1 & 0 \\ * & * & * & * & * & -(1-\dot{\tau}(t))Q+T_1+T_1^{\mathrm{T}} & -W_3 \\ * & * & * & * & * & * & -(S_{13}+R_2) \end{bmatrix} < 0. \tag{20}$$

It is easy to see that Ψ_1 results from $\Psi_{|\tau<h_2,\tau=h_1}$, where we have deleted the zero row and the zero column. Denoting:

$$\begin{aligned} \eta_{1i}(t) = \mathrm{col}\{&x(t), \dot{x}(t), x(t-h_1), x(t-h_2), v_{ji}, \\ &x(t-\tau(t)), x(t-h_3)\}, \quad i = 1, 2, \end{aligned}$$

the latter two LMIs imply (16) because

$$\begin{aligned} &\frac{h_2-\tau(t)}{h_2-h_1}\eta_{12}^{\mathrm{T}}(t)\Psi_1\eta_{12}(t) + \frac{\tau(t)-h_1}{h_2-h_1}\eta_{11}^{\mathrm{T}}(t)\Psi_2\eta_{11}(t) \\ &= \eta_1^{\mathrm{T}}(t)\Psi_{|\tau<h_2}\eta_1(t) \le -\alpha_1|x(t)|^2 \end{aligned}$$

and $\Psi_{|\tau<h_2}$ is thus convex in $\tau(t) \in [h_1, h_2)$.

LMI (19) leads for $\dot{\tau}(t) = d_i$, $i = 1, 2$ to the following:

$$\Psi_{1i} = \Psi_{1|\dot{\tau}(t)=d_i} < 0, \quad i = 1, 2. \tag{21}$$

The two LMIs (21) imply (19) because

$$\frac{d_2-\dot{\tau}(t)}{d_2-d_1}\Psi_{11} + \frac{\dot{\tau}(t)-d_1}{d_2-d_1}\Psi_{12} = \Psi_1 < 0.$$

and Ψ_1 is thus convex in $\dot{\tau}(t) \in [d_1, d_2]$. Similarly, we can obtain that Ψ_2 is also convex in $\dot{\tau}(t) \in [d_1, d_2]$.

For $\tau(t) \in (h_2, h_3]$, where $\chi = 0$, we apply the above arguments and representations with $j = 2$ and $i = 0, 1$. Similarly to (14), we insert free-weighting $n \times n$-matrices $T_2, W_1, Y_{k2}, Z_{k2} (k = 1, 2)$ by adding the following expressions to $\dot{V}$:

$$\begin{aligned} 0 \;=\; & 2[x^{\mathrm{T}}(t)Y_{12}^{\mathrm{T}} + \dot{x}^{\mathrm{T}}(t)Y_{22}^{\mathrm{T}} + x^{\mathrm{T}}(t-\tau(t))T_2^{\mathrm{T}} + x^{\mathrm{T}}(t-h_1)W_1^{\mathrm{T}}] \\ & [-x(t-h_2) + x(t-\tau(t)) + (\tau(t)-h_2)v_{21}], \\ 0 \;=\; & 2[x^{\mathrm{T}}(t)Z_{12}^{\mathrm{T}} + \dot{x}^{\mathrm{T}}(t)Z_{22}^{\mathrm{T}}] \\ & [x(t-h_3) + (h_3-\tau(t))v_{22} - x(t-\tau(t))]. \end{aligned}$$

Similarly to (15), the expression with $j = 2$ is added to $\dot{V}$. We then arrive at the following:

$$\dot{V}|_{\tau>h_2} \le \eta_2^{\mathrm{T}}(t)\Psi_{|\tau>h_2}\eta_2(t),$$

where

$$\Psi_{|\tau>h_2} \triangleq \begin{bmatrix} \Omega_{11}^{2} + \frac{\dot{\tau}(t)(P^3-P^1)}{h_3-h_2} & \Omega_{12}^{2} & R_0 & -Y_{12}^{\mathrm{T}} & (\tau(t)-h_2)Y_{12}^{\mathrm{T}} & (h_3-\tau(t))Z_{12}^{\mathrm{T}} & \Omega_{17}^{2} & Z_{12}^{\mathrm{T}} \\ * & \Omega_{22}^{2} & 0 & -Y_{22}^{\mathrm{T}} & (\tau(t)-h_2)Y_{22}^{\mathrm{T}} & (h_3-\tau(t))Z_{22}^{\mathrm{T}} & \Omega_{27}^{2} & Z_{22}^{\mathrm{T}} \\ * & * & \phi_3^{(2)} & R_1 - W_1^{\mathrm{T}} + S_{12} & (\tau(t)-h_2)W_1^{\mathrm{T}} & 0 & W_1^{\mathrm{T}} & 0 \\ * & * & * & \phi_4^{(2)} & 0 & 0 & -T_2 & -S_{12} \\ * & * & * & * & \phi_5^{(2)} & 0 & (\tau(t)-h_2)T_2 & 0 \\ * & * & * & * & * & \phi_6^{(2)} & 0 & 0 \\ * & * & * & * & * & * & -(1-\dot{\tau}(t))Q+T_2+T_2^{\mathrm{T}} & 0 \\ * & * & * & * & * & * & * & -S_{13} \end{bmatrix} < 0, \tag{22}$$

and where

$$\begin{aligned} \Omega_{12}^{2} &= \left[\frac{\tau(t)-h_2}{h_3-h_2}P^3 + \frac{h_3-\tau(t)}{h_3-h_2}P^1\right] - P_{22}^{\mathrm{T}} + A^{\mathrm{T}}P_{32}, \\ \phi_3^{(2)} &= -(S_0 + R_0 + R_1 - S_{11} - Q), \\ \phi_4^{(2)} &= -(S_{11} + R_1 - S_{13}), \\ \phi_5^{(2)} &= -(h_3-h_2)(\tau(t)-h_2)R_2, \\ \phi_6^{(2)} &= -(h_3-h_2)(h_3-\tau(t))R_2. \end{aligned} \tag{23}$$

We note that $\Psi_{|\tau>h_2}$ is convex in $\tau(t) \in (h_2, h_3]$ and, thus, for the feasibility of LMI (22), it is sufficient to verify this LMI for $\tau(t) \to h_2$ and for $\tau(t) \to h_3$. Denote the resulting LMIs by

$\Psi_{02} < 0$ and $\Psi_{03} < 0$, respectively, where the zero columns and rows are deleted from Ψ_{02} and Ψ_{03}. Clearly Ψ_{02} and Ψ_{03} are also convex in $\dot\tau(t) \in [d_1, d_2]$. Along the system (3), we therefore have

$$\dot V_{|\tau\neq h_2} \le \chi_{[h_1,h_2]}(\tau(t))\eta_1^{\mathrm T}(t)\Psi_{|\tau<h_2}\eta_1(t) + [1-\chi_{[h_1,h_2]}(\tau(t))]\eta_2^{\mathrm T}(t)\Psi_{|\tau>h_2}\eta_2(t) \le -\alpha|x(t)|^2 \quad (24)$$

for some positive scalar $\alpha > 0$.

For $\tau = h_2$, taking into account the definition (7) of $\dot V$, we find

$$\dot V_{|\tau=h_2} \le \max\{\eta_1^{\mathrm T}(t)\Psi_{|\tau<h_2}\eta_1(t), \eta_2^{\mathrm T}(t)\Psi_{|\tau>h_2}\eta_2(t)\} \le -\alpha|x(t)|^2. \quad (25)$$

By using Theorem 8.1.6 of (Kolmanovskii & Myshkis, 1999), we finally obtain the following:

Theorem 1. *Let there exist $n \times n$-matrices $Q, R_i (i = 0,1,2)$, $S_0, S_{11}, S_{12}, S_{13}$, satisfying (5), and $n \times n$-matrices $P^k > 0, k = 1,2,3$, $W_1, W_3, P_{2j}, P_{3j}, Y_{1j}, Y_{2j}, T_j, Z_{1j}$ and $Z_{2j}, j = 1,2$ such that the eight LMIs: (17), for $\tau(t) \to h_1$ and $\tau(t) \to h_2$, and (22), for $\tau(t) \to h_2$ and $\tau(t) \to h_3$, where $\dot\tau(t) = d_1, d_2$, with notations given in (18) and (23), are feasible. Then (1) is asymptotically stable for all differentiable delays $\tau(t) \in [h_1, h_3]$ with $d_1 \le \dot\tau(t) \le d_2$.*

For unknown d_1, by substituting $P = P^1 = P^2 = P^3$ and $\dot\tau(t) = d_2$ into (17) and (22), we arrive at the following:

Corollary 1. *Let there exist $n \times n$-matrices $Q, R_i (i = 0,1,2)$, $S_0, S_{11}, S_{12}, S_{13}$, satisfying (5), and $n\times n$-matrices $P > 0, W_1, W_3, P_{2j}, P_{3j}, Y_{1j}, Y_{2j}, T_j, Z_{1j}$ and $Z_{2j}, j = 1,2$ such that the four LMIs: (17), for $\tau(t) \to h_1$ and $\tau(t) \to h_2$, and (22), for $\tau(t) \to h_2$ and $\tau(t) \to h_3$, where $\dot\tau(t) = d_2$, with notations given in (18) and (23), are feasible. Then (1) is asymptotically stable for all differentiable delays $\tau(t) \in [h_1, h_3]$ with $\dot\tau(t) \le d_2$. Moreover, if the above LMIs are feasible with $Q = 0$, then (1) is asymptotically stable for all fast-varying delays in $[h_1, h_3]$.*

2.2. On other possibilities for delay partitioning

If h_a is big enough, delay partitioning of $[0, h_a]$ can improve the result. For simplicity we combine delay partitioning of $[0, h_a]$ with a non-partitioned $[h_a, h_b]$, where $h_a = h_1$ and $h_b = h_2$. We apply

$$\begin{aligned} V(t, x_t, \dot x_t) &= x^{\mathrm T}(t)P(\tau(t))x(t) \\ &+ \int_{t-\frac{h_1}{2}}^{t} \begin{bmatrix} x(s) \\ x\left(s-\frac{h_1}{2}\right)\end{bmatrix}^{\mathrm T} \begin{bmatrix} S_{01} & S_{02} \\ * & S_{03}\end{bmatrix} \begin{bmatrix} x(s) \\ x\left(s-\frac{h_1}{2}\right)\end{bmatrix} \mathrm{d}s \\ &+ \frac{h_1}{2}\int_{-\frac{h_1}{2}}^{0}\int_{t+\theta}^{t} \dot x^{\mathrm T}(s)R_0\dot x(s)\mathrm{d}s\mathrm{d}\theta \\ &+ \int_{t-h_2}^{t-h_1} x^{\mathrm T}(s)S_1x(s)\mathrm{d}s + \int_{t-\tau(t)}^{t-h_1} x^{\mathrm T}(s)Qx(s)\mathrm{d}s \\ &+ (h_2-h_1)\int_{-h_2}^{-h_1}\int_{t+\theta}^{t} \dot x^{\mathrm T}(s)R_1\dot x(s)\mathrm{d}s\mathrm{d}\theta, \end{aligned} \quad (26)$$

where $\tau(t) \in [h_1, h_2]$ and where

$$P(\tau(t)) = \frac{\tau(t)-h_1}{h_2-h_1}P^1 + \frac{h_2-\tau(t)}{h_2-h_1}P^2, \quad P^1 > 0, P^2 > 0, \quad (27)$$

$$Q \ge 0, R_0 > 0, R_1 > 0, S_1 > 0, \begin{bmatrix} S_{01} & S_{02} \\ * & S_{03}\end{bmatrix} > 0. \quad (28)$$

We have:

$$\frac{\mathrm d}{\mathrm{d}t}[x^{\mathrm T}(t)P(\tau(t))x(t)] = x^{\mathrm T}(t)\frac{\dot\tau(t)(P^1-P^2)}{h_2-h_1}x(t) + 2\dot x^{\mathrm T}(t)\Big[\frac{\tau(t)-h_1}{h_2-h_1}P^1 + \frac{h_2-\tau(t)}{h_2-h_1}P^2\Big]x(t).$$

By using arguments of Theorem 1 (without partitioning of $[h_a, h_b]$), we obtain that for some $\alpha > 0$

$$\dot V \le \eta^{\mathrm T}(t)\Phi\eta(t) \le -\alpha|x(t)|^2, \quad (29)$$

where $\eta(t) = \mathrm{col}\{x(t), \dot x(t), x(t-h_1), x(t-h_2), v_1, v_2, x(t-\tau(t)), x(t-\frac{h_1}{2})\}$, if the LMI

$$\Phi = \begin{bmatrix} \Phi_{11} + \frac{\dot\tau(t)(P^1-P^2)}{h_2-h_1} & \Phi_{12} & -Y_1^{\mathrm T} & Z_1^{\mathrm T} & (\tau(t)-h_1)Y_1^{\mathrm T} & (h_2-\tau(t))Z_1^{\mathrm T} & Y_1^{\mathrm T}-Z_1^{\mathrm T}+P_2^{\mathrm T}A_1 & S_{02}+R_0 \\ * & \Phi_{22} & -Y_2^{\mathrm T} & Z_2^{\mathrm T} & (\tau(t)-h_1)Y_2^{\mathrm T} & (h_2-\tau(t))Z_2^{\mathrm T} & Y_2^{\mathrm T}-Z_2^{\mathrm T}+P_3^{\mathrm T}A_1 & 0 \\ * & * & S_1-S_{03}+Q & 0 & 0 & 0 & -T & -S_{02}^{\mathrm T} \\ * & * & * & -S_1 & 0 & 0 & 0 & 0 \\ * & * & * & * & \Phi_5 & 0 & (\tau(t)-h_1)T & 0 \\ * & * & * & * & * & \Phi_6 & 0 & 0 \\ * & * & * & * & * & * & -(1-\dot\tau(t))Q+T+T^{\mathrm T} & 0 \\ * & * & * & * & * & * & * & S_{03}-S_{01}-R_0 \end{bmatrix} < 0 \quad (30)$$

holds, where

$$\begin{aligned} \Phi_{11} &= A^{\mathrm T}P_2 + P_2^{\mathrm T}A + S_{01} - R_0, \\ \Phi_{12} &= \Big[\frac{\tau(t)-h_1}{h_2-h_1}P^1 + \frac{h_2-\tau(t)}{h_2-h_1}P^2\Big] - P_2^{\mathrm T} + A^{\mathrm T}P_3, \\ \Phi_{22} &= -P_3 - P_3^{\mathrm T} + \frac{1}{4}h_1^2R_0 + (h_2-h_1)^2R_1, \\ \Phi_5 &= -(h_2-h_1)(\tau(t)-h_1)R_1, \\ \Phi_6 &= -(h_2-h_1)(h_2-\tau(t))R_1. \end{aligned} \quad (31)$$

The following result is then obtained.

Theorem 2. *Let there exist $n\times n$-matrices $Q, R_0, S_{01}, S_{02}, S_{03}, R_1, S_1$, satisfying (28), and $n \times n$-matrices $P^1 > 0, P^2 > 0, P_2, P_3 T, Y_1, Y_2, Z_1, Z_2$ such that the four LMIs: (30), for $\tau(t) \to h_1$ and $\tau(t) \to h_2$, where $\dot\tau(t) = d_1, d_2$, with notations given in (31), are feasible. Then (1) is asymptotically stable for all differentiable delays $h_1 \le \tau(t) \le h_2$ satisfying $d_1 \le \dot\tau(t) \le d_2$.*

When d_1 is unknown, by setting $P = P^1 = P^2, \dot\tau(t) = d_2$ in (30), we obtain the following

Corollary 2. *Let there exist $n\times n$-matrices $Q, R_0, S_{01}, S_{02}, S_{03}, R_1, S_1$, satisfying (28), and $n\times n$-matrices $P > 0, P_2, P_3 T, Y_1, Y_2, Z_1, Z_2$ such that the two LMIs: (30), for $\tau(t) \to h_1$ and $\tau(t) \to h_2$, where $\dot\tau(t) = d_2$, with notations given in (31), are feasible. Then (1) is asymptotically stable for all differentiable delays $h_1 \le \tau(t) \le h_2$ satisfying $\dot\tau(t) \le d_2$. Moreover, if the above LMIs are feasible with $Q = 0$, then (1) is asymptotically stable for all fast-varying delays in $[h_1, h_2]$.*

Remark 1. The examples below illustrate that for big enough h_a (for big enough $h_b - h_a$) the delay partitioning of $[0, h_a]$ (of $[h_a, h_b]$) improve the result. Thus, in Example 2, the results by Corollary 2 are worse for $h_a \le 1$ and better for $h_a \ge 2$ than the ones by Corollary 1. Partitioning of the above intervals into $n > 2$ subintervals may lead to further improvements. In (Jiang & Han, 2008) another delay partitioning was introduced, which corresponded to the partitioning into two subintervals of $[0, h_a]$ and of $[0, h_b]$. Example 2 below shows that our approach yields less conservative results.

Table 1
Example 1: Max. value of h_b achieved for $h_a = 0$.

$d_2 \setminus d_1$		0	−0.1	−0.3	−0.5	−0.7	−1	unkn
0.1	Theorem 2	5.63	5.57	5.56	5.56	5.56	5.56	5.55
	Theorem 1	6.39	6.38	6.37	6.37	6.37	6.37	6.28
0.3	Theorem 2	2.60	2.50	2.38	2.37	2.37	2.37	2.35
	Theorem 1	2.69	2.58	2.49	2.47	2.46	2.46	2.41
0.5	Theorem 2	1.66	1.57	1.44	1.35	1.31	1.30	1.26
	Theorem 1	1.79	1.70	1.56	1.48	1.44	1.40	1.27
0.7	Theorem 2	1.27	1.22	1.15	1.09	1.06	1.06	1.06
	Theorem 1	1.56	1.50	1.37	1.29	1.25	1.20	1.12
1	Theorem 2	1.22	1.18	1.12	1.07	1.06	1.06	1.06
	Theorem 1	1.50	1.43	1.31	1.24	1.20	1.18	1.12

Table 2
Example 1: Max. value of h_b for $h_a = 1$.

$d_2 \setminus$ Method	He07	Shao09	Corollary 2	Corollary 1
0.3	2.21	2.24	2.42	2.42
unknown	1.51	1.61	1.76	1.79

Table 3
Example 1: Max. value of h_b achieved for $h_a = 1$.

$d_2 \setminus d_1$		0	−0.1	−0.3	−0.5
0.3	Theorem 2	2.68	2.57	2.46	2.45
	Theorem 1	2.71	2.60	2.50	2.47
1	Theorem 2	1.80	1.77	1.76	1.76
	Theorem 1	1.88	1.85	1.82	1.80

2.3. Examples

Example 1. Consider (1) with

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \quad \text{and} \quad A_1 = \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix}.$$

For $h_a = 0$, choosing d_1 and d_2 as in Table 1 and applying Theorems 1 and 2, we find the maximum values of h_b for which the system remains asymptotically stable (see Table 1). For unknown d_1, our results coincide with those of the corresponding Corollaries 1 and 2. The result of Corollary 2 coincides with the one of Park and Ko (2007) (the latter are less conservative than those of He et al. (2007) and Shao (2009)), but the LMIs of Corollary 2 possess a fewer number of decision variables. It is seen that the value of h_b depends on $d_1 \le 0$ and it grows for $d_1 \to 0$.

For $h_a = 1$, d_1 unknown and $d_2 = 0.3$ or unknown, the results obtained by various methods in the literature for the admissible upper-bounds h_b, which guarantee the stability of the system (1) are listed in Table 2. For $h_a = 1$, choosing d_1 and d_2 and applying Theorems 1 and 2, we obtain the results given in Table 3.

Example 2. Consider (1) with

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix} \quad \text{and} \quad A_1 = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}.$$

For $h_a = 0$ we find, by applying Corollary 2, that the system is asymptotically stable for all fast-varying delays $\tau(t) \in [0, 1.868]$, which coincides with the result of Park and Ko (2007) (the latter is less conservative than those of He (2007), Jiang and Han (2008) and Shao (2009)). Corollary 1 leads to a bigger interval $\tau(t) \in [0, 2.118]$.

Table 4
Example 2: Max. value of h_b for different h_a.

h_a	1	2	3	4	5
Jiang and Han (2008)	1.804	2.521	3.331	4.188	5.072
Shao (2009)	1.873	2.504	3.259	4.074	–
Corollary 2	2.120	2.724	3.458	4.257	5.097
Corollary 1	2.169	2.646	3.321	4.090	–

Table 5
Example 2: Max. value of h_b achieved for $d_2 = 1$.

$h_a \setminus d_1$		0	−0.3	−0.7	−1	unknown
0	Theorem 1	2.204	2.194	2.180	2.171	2.118
1	Theorem 1	2.196	2.189	2.182	2.179	2.169

For $h_a = 1, 2, 3, 4, 5$ and fast-varying delays we obtain, by applying Corollaries 1 and 2, the maximum values of h_b given in Table 4. These results are favorably compared with the existing ones.

For $d_2 = 1$, choosing d_1 and h_a as in Table 5 we apply Theorems 1 and 2. The results by Theorem 1 are d_1-dependent (see Table 5), where the value of h_b grows for $d_1 \to 0$. For unknown d_1 the results coincide with those of Corollary 1. The results by Theorem 2 are d_1-independent and coincide with the ones by Corollary 2.

3. Conclusions

In this paper new LKFs with matrices depending on the time delay are introduced. These delay partitioning-based LKFs lead to stability conditions that depend on both the upper and *lower bounds on the delay derivative*. Two examples illustrate the efficiency of the new method and the improvement that can be achieved by using the lower bound on the delay derivative.

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