

Wirtinger-like Lyapunov–Krasovskii functionals for discrete-time delay systems

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[Received on 13 July 2016; revised on 5 December 2016; accepted on 11 January 2017]

Time-dependent Lyapunov functionals appeared to be very efficient for sampled-data systems. Recently, new Lyapunov functionals were constructed for sampled-data control in the presence of a constant input delay. The construction of these functionals was based on Wirtinger’s inequality leading to simplified and efficient stability conditions in terms of linear matrix inequalities. In this article, we extend the latter results to the discrete-time sampled-data systems. We show that the proposed approach is less conservative on some examples with a lower number of decision variables.

Keywords: discrete-time delay systems; sampled-data; Lyapunov–Krasovskii functional; Wirtinger inequality.

1. Introduction

Sampled-data systems have been studied extensively over the past decades (see e.g. [Chen & Francis, 1995](#); [Fridman *et al.*, 2004](#); [Mirkin, 2007](#); [Naghshabrizi *et al.*, 2008](#); [Fujioka, 2009](#) and the references therein). Modeling of continuous-time systems with digital control in the form of continuous-time systems with time-varying delay ([Mikheev *et al.*, 1988](#)) and the extension of Krasovskii method to systems with fast varying delays (without any constraints on the delay derivative as in [Fridman & Shaked \(2003\)](#) and to discontinuous delays ([Fridman *et al.*, 2004](#)) have allowed the development of the *time-delay approach* to sampled-data and to network-based control (see Section 7 of [Fridman, 2014](#) for details).

Till [Fridman \(2010\)](#), the conventional time-independent Lyapunov functionals $V(x_t, \dot{x}_t)$ for systems with fast-varying delays were applied to sampled-data systems ([Fridman *et al.*, 2004](#)). These functionals did not take advantage of the sawtooth evolution of the delays induced by sampled-and-hold. The latter drawback was removed in [Fridman \(2010\)](#) and [Seuret \(2012\)](#), where time-dependent Lyapunov functionals (inspired by [Naghshabrizi *et al.*, 2008](#)) were constructed for sampled-data systems. A different time-dependent Lyapunov functional was suggested in [Liu & Fridman \(2012\)](#) which is based on *Wirtinger’s inequality* (see for instance [Kammler, 2007](#), [Liu *et al.*, 2010](#)):

Let $z(t) : (a, b) \rightarrow \mathbb{R}^n$ be absolutely continuous with $\dot{z} \in L_2[a, b]$ and $z(a) = 0$. Then for any $n \times n$ matrix $W > 0$ Wirtinger’s inequality holds:

$$\int_a^b \dot{z}^T(\xi) W \dot{z}(\xi) d\xi \geq \frac{\pi^2}{4(b-a)^2} \int_a^b z^T(\xi) W z(\xi) d\xi.$$

The Wirtinger-based linear matrix inequality (LMI) is a single LMI with fewer decision variables than the LMIs of [Fridman \(2010\)](#) and [Seuret \(2012\)](#). More important, differently from the Lyapunov functionals of [Fridman \(2010\)](#) and [Seuret \(2012\)](#), the extension of the Wirtinger-based Lyapunov functionals to a more general sampled-data system in the presence of a constant input/output delay leads to efficient stability conditions (see e.g. [Liu & Fridman, 2012](#)).

In this article, we aim at extending the results of [Liu & Fridman \(2012\)](#) to discrete-time sampled-data systems. Unlike the continuous-time case, the discrete-time formulation has surprisingly attracted only few attention in the literature even if the formulation represents an efficient way to model the dynamics of discrete-time systems subject to control packet losses. The problem of packet losses indeed appears in many applications of networked control systems (see for instance [Hespanha *et al.*, 2007](#), [Zampieri, 2008](#)). As in the continuous-time case, the Wirtinger-based Lyapunov functionals essentially reduce the numerical complexity of the resulting LMIs leading in some examples to less restrictive conditions. Similarly to the continuous-time case, discrete-time sampled-data can be seen as a discrete-time system subject to a particular time-varying delay, for which there exist many stability conditions (see e.g. [Gao & Chen, 2007](#), [Shao & Han, 2011](#), [Liu & Zhang, 2012](#)). However, such approaches do not account accurately the particularities of the sawtooth delay.

In the continuous-time case, the analysis of this class of functionals is made possible by considering that the functionals do not grow at the sampling instants. A translation of such analysis in the discrete-time framework is not easy and requires a dedicated analysis.

The article is organized as follows. Section 2 describes the problem formulation. Section 3 shows some preliminary summation inequalities including a Wirtinger's and Jensen's inequality as well as a recent summation inequality that includes the Jensen's inequality as a consequence. This last inequality is the counter part of the Wirtinger-based inequality provided in [Seuret & Gouaisbaut \(2013\)](#). Section 4 presents the main results on the stability analysis of discrete-time sampled-data systems. Section 6 shows the efficiency of the proposed method on some examples. Finally Section 7 draws some conclusions.

Notations: Throughout the article, \mathbb{Z} (\mathbb{N}) denotes the set of (positive) integers, \mathbb{R}^n the n -dimensional Euclidean space with vector norm $|\cdot|$, $\mathbb{R}^{n \times m}$ the set of all $n \times m$ real matrices. For any symmetric matrix $P \in \mathbb{R}^{n \times n}$, the notation $P > 0$ (or $P < 0$) means that P is positive (or negative) definite. The set \mathbb{S}_n^+ refers to the set of symmetric positive definite matrices. For any matrices A, B in $\mathbb{R}^{n \times n}$, the notation $\text{diag}(A, B)$ denotes the block diagonal matrix $\begin{bmatrix} A & 0 \\ * & B \end{bmatrix}$. For any square matrix, $\text{He}(A)$ stands for $A + A^T$. Along the paper, for any real number $a < b$, the notation $[a, b]_{\mathbb{Z}}$ denotes $[a, b] \cap \mathbb{Z}$. The same notations will also hold for open intervals.

2. Problem formulation

Consider a linear discrete-time time-delay system of the form:

$$\begin{cases} x(t+1) = Ax(t) + A_d x(t_k - h) & \forall k \in [k_i, k_{i+1})_{\mathbb{Z}} \\ x(\theta) = x_0(\theta) & \forall \theta \in [-h, 0], \end{cases} \quad (2.1)$$

where $x(t) \in \mathbb{R}^n$ is the state vector, x_0 is the initial condition and $A, A_d \in \mathbb{R}^{n \times n}$ are constant matrices. The delay $h \in \mathbb{N}$ is assumed to be constant and known and the sequence of integers $\{t_k\}_{k \in \mathbb{N}} \subset \mathbb{N}$ satisfies

$$\begin{aligned} \forall k \in \mathbb{N} \quad t_{k+1} - t_k &\in [1, T_{max}]_{\mathbb{Z}}, \\ \lim_{k \rightarrow +\infty} t_k &= +\infty. \end{aligned} \quad (2.2)$$

Following the principles of the input delay approach for discrete-time systems, we can define an input delay function τ given by

$$\begin{aligned} \tau(t) &= t - t_k, \quad \forall t \in [t_k, t_{k+1})_{\mathbb{Z}}, \\ \tau(t+1) &= \begin{cases} \tau(t) + 1 & \text{if } t < t_{k+1} \\ 0 & \text{if } t = t_{k+1}. \end{cases} \end{aligned} \quad (2.3)$$

The system (2.1) can then be rewritten as a discrete-time system subject to a time-varying delay. The delay function can be seen as the discrete-time version of the sawtooth delay function considered in [Fridman *et al.* \(2004\)](#) for continuous-time sampled-data systems.

$$\begin{cases} x(t+1) = Ax(t) + A_d x(t-h-\tau(t)) & \forall t \in [t_k, t_{k+1})_{\mathbb{Z}} \\ x(\theta) = x_0(\theta), \quad \forall \theta \in [-h, 0]_{\mathbb{Z}}, \end{cases} \quad (2.4)$$

where the delay h is constant and where the sampling delay $\tau(t)$ is given in (2.3). In this article, we aim at providing stability conditions for this peculiar class of systems using a method based on discontinuous Lyapunov functionals. This article can be seen as the discrete-time counterpart of the recent article ([Liu & Fridman, 2012](#)). To this end, we will provide a stability analysis of such class of systems, where novel Wirtinger-based terms are added to ‘nominal’ Lyapunov functionals for the stability analysis of the discrete-time systems with the constant delay h .

3. Preliminaries on summation inequalities

3.1. Discrete-time Wirtinger inequality

Wirtinger inequalities are integral inequalities issued from the Fourier analysis. The continuous-time versions of this inequality have already shown their potential for the stability analysis of partial differential equation ([Fridman & Orlov, 2009](#)), sampled-data systems ([Liu & Fridman, 2012](#)) or time-delay systems ([Seuret & Gouaisbaut, 2013](#)). In this article, we aim at showing that this class of inequalities also serves for the stability analysis of discrete-time systems. Indeed a discrete-time version of these inequalities have been extended to the discrete-time framework. It is stated in the following lemma taken from [Ky Fan *et al.* \(1955\)](#).

LEMMA 3.1 For a given $N \in \mathbb{N}_{\geq 0}$, consider a sequence of N real scalars x_0, x_1, \dots, x_N such that $x_0 = 0$. Then, the following inequality holds

$$\sum_{i=0}^{N-1} (x_i - x_{i+1})^2 \geq \lambda_N^2 \sum_{i=0}^{N-1} x_i^2, \quad (3.1)$$

where $\lambda_N = 2 \sin(\pi/(2(2N+1)))$.

A straightforward corollary of this lemma is provided for n -dimensional sequences z and is stated below.

COROLLARY 3.1 For a given $N \in \mathbb{N}_{\geq 0}$, consider a sequence of N real n -dimensional vectors z_0, z_2, \dots, z_N such that $z_0 = 0$. Then, the following inequality holds, for any symmetric positive definite matrix $W \in \mathbb{S}_+^n$.

$$\sum_{i=0}^{N-1} (z_i - z_{i+1})^T W (z_i - z_{i+1}) \geq \lambda_N^2 \sum_{i=0}^{N-1} z_i^T W z_i, \quad (3.2)$$

where $\lambda_N = 2 \sin(\pi/(2(2N+1)))$.

Proof. Since $W > 0$, there exists an orthogonal matrix $U = [U_1^T \ U_2^T \ \dots \ U_n^T]^T$ and a positive definite diagonal matrix $\Delta = \text{diag}(\Delta_1, \Delta_2, \dots, \Delta_n)$ such that $W = U^T \Delta U$. It holds

$$\begin{aligned} \lambda_N^2 \sum_{i=0}^{N-1} z_i^T W z_i &= \lambda_N^2 \sum_{i=0}^{N-1} \sum_{j=1}^n (U_j z_i)^T \Delta_j (U_j z_i) \\ &= \sum_{j=1}^n \Delta_j \lambda_N^2 \sum_{i=0}^{N-1} (U_j z_i)^2. \end{aligned} \quad (3.3)$$

Following the same procedure, we also have

$$\sum_{i=0}^{N-1} (z_i - z_{i+1})^T W (z_i - z_{i+1}) = \sum_{j=1}^n \Delta_j \sum_{i=0}^{N-1} (U_j (z_i - z_{i+1}))^2. \quad (3.4)$$

Since the vector $z_0 = 0$, the scalar $U_j z_0$ is zero for all $j = 1, \dots, n$. Hence the Wirtinger inequality in Lemma 3.1 ensures that

$$\sum_{i=0}^{N-1} (U_j (z_i - z_{i+1}))^2 \geq \lambda_N^2 \sum_{i=0}^{N-1} (U_j z_i)^2, \quad j = 1, \dots, n.$$

Finally computing the sum over $j = 1, \dots, n$ of the previous inequality weighted by Δ_j leads to

$$\sum_{j=1}^n \Delta_j \sum_{i=0}^{N-1} (U_j (z_i - z_{i+1}))^2 \geq \sum_{j=1}^n \Delta_j \lambda_N^2 \sum_{i=0}^{N-1} (U_j z_i)^2$$

which is equivalent to (3.2) thanks to (3.3) and (3.4). \square

3.2. Summation inequalities

In this section, two summation inequalities are recalled. The first one is the Jensen inequality and is stated here.

LEMMA 3.2 For a given symmetric positive definite matrix $Z \in \mathbb{S}_+^n$, any sequence of discrete-time variable x in $[-h, 0]_{\mathbb{Z}} \rightarrow \mathbb{R}^n$, where $h \geq 1$, the following inequality holds:

$$\sum_{i=-h+1}^0 y^T(i)Zy(i) \geq \frac{1}{h}\Theta_0^T Z \Theta_0, \quad (3.5)$$

where $y(i) = x(i) - x(i-1)$ and $\Theta_0 = x(0) - x(-h)$.

The second lemma is a recent inequality proposed in [Seuret *et al.* \(2015\)](#) that extends the Wirtinger-based integral inequality (see [Seuret & Gouaisbaut, 2013](#)) to the discrete-time case.

LEMMA 3.3 For a given symmetric positive definite matrix $Z \in \mathbb{S}_+^n$, any sequence of discrete-time variable x in $[-h, 0]_{\mathbb{Z}} \rightarrow \mathbb{R}^n$, where $h \geq 1$, the inequality

$$\sum_{i=-h+1}^0 y^T(i)Zy(i) \geq \frac{1}{h} \begin{bmatrix} \Theta_0 \\ \Theta_1 \end{bmatrix}^T \begin{bmatrix} Z & 0 \\ 0 & 3\left(\frac{h+1}{h-1}\right)Z \end{bmatrix} \begin{bmatrix} \Theta_0 \\ \Theta_1 \end{bmatrix} \quad (3.6)$$

holds where

$$\begin{aligned} y(i) &= x(i) - x(i-1), \\ \Theta_0 &= x(0) - x(-h), \\ \Theta_1 &= x(0) + x(-h) - \frac{2}{h+1} \sum_{i=-h}^0 x(i). \end{aligned}$$

Proof. The proof is provided in [Seuret *et al.* \(2015\)](#) and is therefore omitted. \square

REMARK 3.1 The inequality provided in Lemma 3.3 implies

$$\sum_{i=-h+1}^0 y^T(i)Zy(i) \geq \frac{1}{h}\Theta_0^T Z \Theta_0,$$

which is exactly the Jensen summation inequality. Therefore, Lemma 3.3 is less conservative than the celebrated Jensen inequality since a positive quantity is added in the right-hand side of the inequalities.

4. Stability analysis

4.1. Wirtinger-based functional

In this section, we aim at proposing a new functional to deal with the discrete-time sampled-data system (2.4) by an appropriate use of the discrete time Wirtinger inequality resumed in Lemma 3.1. This contribution is proposed in the following lemma.

LEMMA 4.1 Consider the following Lyapunov functional, for a given matrix $W \in \mathbb{S}_+^n$, a given $k \in \mathbb{N}$ and for all $t \in [t_k, t_{k+1})_{\mathbb{Z}}$

$$V_W(x_t) = \sum_{i=t_k-h}^{t-1} y^T(i)Wy(i) - \lambda_T^2 \sigma(t, x_t), \quad (4.1)$$

where

$$\begin{aligned} y(i) &= x(i+1) - x(i) \quad \forall i \in \mathcal{N} \\ \lambda_T &= 2 \sin \left(\frac{\pi}{2(2T_{\max} + 1)} \right) \\ \sigma(t, x_t) &= \begin{cases} \sum_{i=t_k}^{t-1} v(i)Wv(i) & t \in [t_k + 1, t_{k+1} - 1] \\ 0 & t = t_k, \end{cases} \\ v(i) &= x(i-h) - x(t_k - h), \quad i \in [t_k, t_{k+1} - 1]. \end{aligned} \quad (4.2)$$

Then, the forward difference of the functional V_W satisfies the inequality

$$\Delta V_W(x_t) \leq (x(t+1) - x(t))^T W (x(t+1) - x(t)) - \lambda_T^2 v^T(t)Wv(t) \quad (4.3)$$

holds, for all $t \in [t_k, t_{k+1})_{\mathbb{Z}}$, and for any sampling satisfying (2.2).

Proof. For a given $k \in \mathbb{N}$, consider first $t \in [t_k, t_{k+1} - 2]_{\mathbb{Z}}$. Then the computation of ΔV_W straightforwardly leads, for all $t \in [t_k, t_{k+1} - 2]_{\mathbb{Z}}$, to

$$\begin{aligned} \Delta V_W(x_t) &= \sum_{i=t_k-h}^t y^T(i)Wy(i) - \sum_{i=t_k-h}^{t-1} y^T(i)Wy(i) - \lambda_T^2 (\sigma(t+1, x_{t+1}) - \sigma(t, x_t)) \\ &= y^T(t)Wy(t) - \lambda_T^2 (\sigma(t+1, x_{t+1}) - \sigma(t, x_t)). \end{aligned} \quad (4.4)$$

From the definition of $\sigma(t, x_t)$, it is easy to see that, if $t \neq t_k$, we have

$$\sigma(t+1, x_{t+1}) - \sigma(t, x_t) = \sum_{i=t_k}^{t_k} v(i)Wv(i) - 0 = v^T(t)Wv(t)$$

and if $t = t_k$

$$\sigma(t_k+1, x_{t_k+1}) - \sigma(t_k, x_{t_k}) = \sum_{i=t_k}^{t_k} v(i)Wv(i) - \sum_{i=t_k}^{t_k-1} v(i)Wv(i) = v^T(t_k)Wv(t_k).$$

This ensures that, for all $t \in [t_k, t_{k+1} - 2]_{\mathbb{Z}}$, the following equality holds

$$\Delta V_W(x_t) = (x(t+1) - x(t))^T W(x(t+1) - x(t)) - \lambda_T^2 v^T(t) W v(t). \quad (4.5)$$

Consider now the remaining case $t = t_{k+1} - 1$. The computation of ΔV_W leads to

$$\begin{aligned} \Delta V_W(x_t) &= \sum_{i=t_{k+1}-h}^{t_{k+1}-1} y^T(i) W y(i) - \sum_{i=t_k-h}^{t_{k+1}-2} y^T(i) W y(i) - \lambda_T^2 \left(0 - \sum_{i=t_k-h}^{t_{k+1}-h-2} v^T(i) W v(i) \right) \\ &= y^T(t) W y(t) - \lambda_T^2 v^T(t) W v(t) - \psi, \end{aligned}$$

where

$$\psi = \sum_{i=t_k-h}^{t_{k+1}-h-1} y^T(i) W y(i) - \lambda_T^2 \sum_{i=t_k-h}^{t_{k+1}-h-1} v^T(i) W v(i).$$

By noting that

$$\begin{aligned} v(t_k) &= 0, \quad \forall k \in \mathbb{N} \\ v(i+1) - v(i) &= y(i) \quad \forall i \in [t_k, t_{k+1})_{\mathbb{Z}}, \\ t_{k+1} - t_k &\leq T_{max} \quad \forall k \in \mathbb{N}, \end{aligned}$$

the assumptions of the Wirtinger inequality in Corollary 3.1 are satisfied, which guarantees that $\psi \geq 0$. It thus holds that, for $t = t_{k+1} - 1$

$$\Delta V_W(x_t) \leq (x(t+1) - x(t))^T W(x(t+1) - x(t)) - \lambda_T^2 v^T(t) W v(t). \quad (4.6)$$

Then, combining (4.4) and (4.6) proves the result. \square

Note that inequality (4.6) is actually an equality when $t \neq t_{k+1} - 1$ and is an inequality only when $t = t_{k+1} - 1$. The computation of this inequality only relies on the computation of the forward increment of functional V_W and the use of the Wirtinger inequality.

The objective in the remainder of this article is to include this functional in the stability analysis of discrete sampled-data systems. Next, we will propose two stability theorems which rely on the use of the Jensen inequality and on Lemma 3.3.

4.2. Jensen-based theorem

The following theorem holds

THEOREM 4.1 For given h and T_{max} in \mathbb{N} , assume that there exist $n \times n$ matrices P, Q, Z and $W \in \mathbb{S}_+^n$ such that the LMI condition

$$\Pi(T_{max}, h) < 0 \quad (4.7)$$

holds where

$$\begin{aligned} \Pi(T_{max}, h) &= \begin{bmatrix} Q & 0 & 0 \\ * & -Q & 0 \\ * & * & -\lambda_T^2 W \end{bmatrix} + M_1^T P M_1 - M_2^T P M_2 + M_0^T (W + h^2 Z) M_0 - M_3^T Z M_3, \\ M_0 &= [A - I \quad A_d \quad -A_d], \quad M_1 = [A \quad A_d \quad -A_d], \\ M_2 &= [I \quad 0 \quad 0], \quad M_3 = [I \quad -I \quad 0], \\ \lambda_T &= 2 \sin \left(\frac{\pi}{2(2T_{max} + 1)} \right). \end{aligned} \quad (4.8)$$

Then system (2.4) is asymptotically stable for the constant delay h and any aperiodic sampling satisfying (2.2).

Proof. Consider the functional

$$V(x_t) = V_1(x_t) + V_W(x_t)$$

where the functional V_W have been defined previously. The functional V_1 is built to assess stability of the delayed term $A_1 x(t-h)$ which appears in equation (2.4). Indeed a classical functional for discrete-time delay system is given by

$$V_1(x_t) = x^T(t) P x(t) + \sum_{i=t-h}^{t-1} x^T(i) Q x(i) + h \sum_{i=-h+1}^0 \sum_{j=t+i-1}^{t-1} y^T(j) Z y(j), \quad (4.9)$$

where $y(i) = x(i+1) - x(i)$. Define the increment of the Lyapunov–Krasovskii functional as follows $\Delta V(x_t) = V(x_{t+1}) - V(x_t)$. From Lemma 4.1, we show that

$$\begin{aligned} \Delta V(x_t) &= x^T(t+1) P x(t+1) - x^T(t) P x(t) + x^T(t) Q x(t) - x^T(t-h) Q x(t-h) \\ &\quad + h^2 (x(t+1) - x(t))^T Z (x(t+1) - x(t)) + (x(t+1) - x(t))^T W (x(t+1) - x(t)) \\ &\quad - \lambda_T^2 v^T(t) W v(t) - h \sum_{j=t-h}^{t-1} y^T(j) Z y(j). \end{aligned}$$

Applying Jensen's inequality to the summation term ensures that

$$\begin{aligned} \Delta V(x_t) &\leq x^T(t+1) P x(t+1) - x^T(t) P x(t) + x^T(t) Q x(t) - x^T(t-h) Q x(t-h) \\ &\quad + h^2 (x(t+1) - x(t))^T Z (x(t+1) - x(t)) - (x(t) - x(t-h))^T Z (x(t) - x(t-h)) \\ &\quad + (x(t+1) - x(t))^T W (x(t+1) - x(t)) - \lambda_T^2 v^T(t) W v(t). \end{aligned}$$

It follows from the previous calculations that

$$\Delta V(x_t) \leq \begin{bmatrix} x(t) \\ x(t-h) \\ v(t) \end{bmatrix}^T \Pi(T_{max}, h) \begin{bmatrix} x(t) \\ x(t-h) \\ v(t) \end{bmatrix}.$$

Then asymptotic stability results from the condition $\Pi(T, h) < 0$, which concludes the proof. \square

4.3. Improved stability Theorem

As it was noticed in [Seuret et al. \(2015\)](#), the conservatism induces by the Jensen inequality can be notably reduced by considering the refined summation inequality provided in [Lemma 3.3](#). The resulting analysis leads to the following theorem.

THEOREM 4.2 For given h and T_{max} in \mathbb{N} , assume that there exist a $2n \times 2n$ matrix $P > 0$ and $n \times n$ matrices $Q > 0$, $Z > 0$ and $W > 0$ such that the LMI condition

$$\Phi(T_{max}, h) < 0 \tag{4.10}$$

holds where

$$\begin{aligned} \Phi(T_{max}, h) &= \begin{bmatrix} Q & 0 & 0 & 0 \\ 0 & -Q & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\lambda_T^2 W \end{bmatrix} + N_1^T P N_1 - N_2^T P N_2 + N_0^T (W + h^2 Z) N_0 - N_3^T \tilde{Z} N_3 \\ N_0 &= \begin{bmatrix} A - I & A_d & 0 & -A_d \end{bmatrix} \\ N_1 &= \begin{bmatrix} A & A_d & 0 & -A_d \\ 0 & -I & (h+1)I & 0 \end{bmatrix} \\ N_2 &= \begin{bmatrix} I & 0 & 0 & 0 \\ -I & 0 & (h+1)I & 0 \end{bmatrix} \\ N_3 &= \begin{bmatrix} I & -I & 0 & 0 \\ I & I & -2I & 0 \end{bmatrix} \\ \tilde{Z} &= \begin{bmatrix} Z & 0 \\ 0 & 3\frac{h+1}{h-1}Z \end{bmatrix} \\ \lambda_T &= 2\sin\left(\frac{\pi}{2(2T_{max} + 1)}\right). \end{aligned} \tag{4.11}$$

Then system (2.4) is asymptotically stable for the constant delay h and any aperiodic sampling satisfying (2.2).

Proof. Consider the functional

$$V(x_t) = V_2(x_t) + V_W(x_t),$$

where we use the same definition for the functional V_W as in Theorem 4.1. In order to fully take advantages of the summation inequality provided in Lemma 3.3, we select the following functional V_2 given by

$$\begin{aligned} V_2(x_t) = & \begin{bmatrix} x(t) \\ \sum_{i=t-h}^{t-1} x(i) \end{bmatrix}^T P \begin{bmatrix} x(t) \\ \sum_{i=t-h}^{t-1} x(i) \end{bmatrix} + \sum_{i=t-h}^{t-1} x^T(i) Q x(i) \\ & + h \sum_{i=-h+1}^0 \sum_{j=t+i-1}^{t-1} y^T(j) Z y(j), \end{aligned} \quad (4.12)$$

where $y(i) = x(i+1) - x(i)$. This functional has been build according to the method provided in [Seuret et al. \(2015\)](#). The forward difference of the Lyapunov–Krasovskii functional yields

$$\begin{aligned} \Delta V(x_t) = & \begin{bmatrix} x(t+1) \\ \sum_{i=t-h+1}^t x(i) \end{bmatrix}^T P \begin{bmatrix} x(t+1) \\ \sum_{i=t-h+1}^t x(i) \end{bmatrix} - \begin{bmatrix} x(t) \\ \sum_{i=t-h}^{t-1} x(i) \end{bmatrix}^T P \begin{bmatrix} x(t) \\ \sum_{i=t-h}^{t-1} x(i) \end{bmatrix} \\ & + x^T(t) Q x(t) - x^T(t-h) Q x(t-h) + h^2 (x(t+1) - x(t))^T Z (x(t+1) - x(t)) \\ & - h \sum_{j=t-h}^{t-1} y^T(j) Z y(j). \end{aligned}$$

Define the $\xi(t) = \frac{1}{h+1} \sum_{i=t-h}^t x(i)$ and applying the summation provided in Lemma 3.3 to the last term ensures that

$$\begin{aligned} \Delta V(x_t) \leq & \begin{bmatrix} x(t+1) \\ (h+1)\xi(t) - x(t-h) \end{bmatrix}^T P \begin{bmatrix} x(t+1) \\ (h+1)\xi(t) - x(t) \end{bmatrix} \\ & - \begin{bmatrix} x(t) \\ (h+1)\xi(t) - x(t) \end{bmatrix}^T P \begin{bmatrix} x(t) \\ (h+1)\xi(t) - x(t) \end{bmatrix} \\ & + h^2 (x(t+1) - x(t))^T Z (x(t+1) - x(t)) - (x(t) - x(t-h))^T Z (x(t) - x(t-h)) \\ & - 3 \left(\frac{h+1}{h-1} \right) (x(t) + x(t-h) - 2\xi(t))^T Z (x(t) + x(t-h) - 2\xi(t)). \end{aligned}$$

It follows from the previous calculations that

$$\Delta V(x_t) \leq \begin{bmatrix} x(t) \\ x(t-h) \\ \xi(t) \\ v(t) \end{bmatrix}^T \Phi(T_{max}, h) \begin{bmatrix} x(t) \\ x(t-h) \\ \xi(t) \\ v(t) \end{bmatrix}.$$

Then asymptotic stability results from the condition $\Phi(T_{max}, h) < 0$, which concludes the proof. \square

REMARK 4.1 In the previous developments, we only focussed on the case of discrete-time delay systems with a single delay and a single sampling. However, the methodology can be extended to the case of multiple delays and multiple sampling by introducing additional functional terms. For the sake of consistency, this problem is not addressed in this article.

4.4. Comparison with approaches from the literature

In this article, we consider functionals of the form

$$V(x_t) = V_1(x_t) + V_w(x_t),$$

where the functional V_1 (or V_2) aims at assessing the stability of system (2.1) without sampling and where the functional $V_w(x_t)$ aims at ensuring the robustness with respect to the sampling. In [Seuret et al. \(2015\)](#), the functional can also be split into two parts where the first one is again the same V_1 (or V_2) but the second part is related to the time-varying delay case. Therefore the conditions provided in [Seuret et al. \(2015\)](#) only address the stability of the system driven by

$$x(t+1) = Ax(t) + A_d x(t-h(k)),$$

where the delay $h(k)$ can take any values between h and $h+T$, without respecting the constraint imposed in this article on the sawtooth form of the delay. Therefore, the conditions provided in this article and the one provided in [Seuret et al. \(2015\)](#) does not treat the same problem. However, it is correct to say that the conditions of [Seuret et al. \(2015\)](#) guarantee stability of the sampled-data system (2.1) but also to a larger class of delay systems.

The idea of this article is to propose a dedicated construction of the functional to cope with the stability analysis of sampled and delayed closed-loop system driven by (2.1).

4.5. Example 1

Consider the continuous time sampled-data system linear driven for all $\mathbf{t} \in [kT_0, (k+1)T_0]_{\mathbb{Z}}$ by

$$\dot{x}(\mathbf{t}) = A_c x(\mathbf{t}) + B_{cd} K x((k-h)T_0), \quad (4.13)$$

where \mathbf{t} represents the continuous time and where

$$A_c = \begin{bmatrix} 0 & 1 \\ 0 & -0.1 \end{bmatrix}, B_{cd} = \begin{bmatrix} 0 & 0 \\ -0.375 & -1.15 \end{bmatrix}, K = [-0.375 \quad -1.15]$$

and T_0 is the discretization period, h is the delay and k is a positive integer. The associated discretized system is given by discrete-time systems with delay given in (2.4) with the matrices

$$A = e^{A_c T_0}, \quad A_d = \int_0^{T_0} e^{A_c(T_0-s)} ds B_{cd} K.$$

The stability conditions provided in this article and from the literature are tested on this system for several values of the discretization period $T_0 = 0.1$ and $T_0 = 0.01$. The results and a comparison with existing results from the literature are presented in Tables 2 and 3.

TABLE 1 *Complexity of the LMI conditions tested in the example*

Theorems	Complexity
Liu & Zhang (2012)	$(3.5n^2 + 2.5n) \times 4n$
Seuret <i>et al.</i> (2015)	$(10n^2 + 3.5n) \times 12n$
Theorem 4.1	$(2n^2 + 2n) \times 3n$
Theorem 4.2	$(3.5n^2 + 2.5n) \times 4n$

TABLE 2 *Results derived from various theorems showing the maximal admissible constant delay h for several values of the sampling period for system with $T_0 = 0.1$*

T_{max}	1	2	3	4	5	6	7	8	9	10	11
Liu & Zhang (2012)	9	8	7	6	5	4	3	2	1	-	-
Seuret <i>et al.</i> (2015)	9	8	7	6	5	4	3	2	1	-	-
Theorem 4.1	9	8	7	6	5	4	4	3	2	1	1
Theorem 4.2	9	8	7	6	6	5	4	3	2	1	1

TABLE 3 *Results derived from various theorems showing the maximal admissible constant delay h for several values of the sampling period for system with $T_0 = 0.01$*

T_{max}	5	10	25	50	75	100	125	133
Liu & Zhang (2012)	101	96	81	56	31	4	—	—
Seuret <i>et al.</i> (2015)	109	103	86	58	33	7	—	—
Theorem 4.1	102	98	85	65	45	25	7	1
Theorem 4.2	110	106	91	69	47	26	7	1

The stability conditions from Liu & Zhang (2012) and Seuret *et al.* (2015) address the problem of stability analysis of discrete-time systems subject to an unknown time-varying delay but which belongs to the interval $[h, h + T]$. To the best of our knowledge, these results are the most efficient conditions for the stability analysis of discrete systems with interval time-varying delays. The sawtooth delay addressed in this article is only a particular case of this more general class of time-varying delays.

Table 1 compares the complexity of Theorem 4.1 and 4.2 to theorems taken from Liu & Zhang (2012) and Seuret *et al.* (2015). Tables 2 and 3 show that our theorems essentially reduce the complexity of the conditions provided in Liu & Zhang (2012) and Seuret *et al.* (2015) leading to less conservative results.

Finally, Fig. 1 depicts the solutions of system (4.13) taken with a sampling period of the continuous time systems $T_0 = 0.1$ and the input delay $h = 6$. From Table 2, the maximal length between two

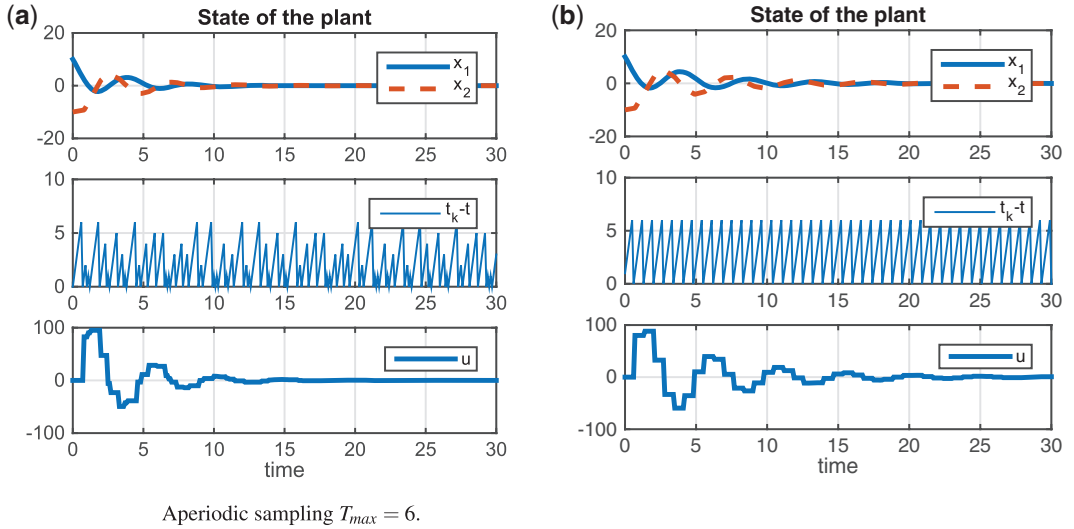


FIG. 1. Example 1. Simulation results representing the state x (top), the timer $\tau = t - t_k$ (middle), and the control inputs $u = Kx(t_k - h)$ (bottom), of System (4.13) with $T_0 = 0.1$ with $h = 5$, $T_{max} = 6$ and aperiodic (a) and periodic (b) samplings.

successive control update $t_{k+1} - t_k$ is upper bounded by 6. Figure 1 shows two simulations of a periodic and an aperiodic implementation of the control input $u = Kx(t_k - h)$ where it can be seen that the solutions of the systems remain stable in both cases. It is also worth noting that the system remains stable with the periodic implementation up to $T = 12$, which means that the stability conditions resulting from the functional term related to the Wirtinger inequality are still conservative even if they already improve the condition issued from the time-varying delay case, which, again, allows assessing stability of the system with a larger class of delay functions than sawtooth delays.

5. Model reduction and predictor control

5.1. Definitions

Consider the linear discrete-time system driven by

$$\begin{cases} x(t+1) = Ax(t) + Bu(t_k - h) & \forall t \in [t_k, t_{k+1}) \cap \mathbb{N} \\ x(\theta) = x_0(\theta) & \forall \theta \in [-h, 0], \end{cases} \quad (5.1)$$

where $x(t) \in \mathbb{R}^n$ is the state vector, x_0 is the initial condition and $A, A_d \in \mathbb{R}^{n \times n}$ are constant matrices. with the initial condition x_0 . The prediction-based approach consists in considering the following control law

$$\begin{aligned} u(t) &= K\bar{x}(t+h) \\ \bar{x}(t+h) &= A^h x(t) + \sum_{i=0}^{h-1} A^{h-i-1} Bu(t-h+i). \end{aligned} \quad (5.2)$$

In this formulation, the vector \bar{x} is the h -step ahead state prediction.

LEMMA 5.1 The closed-looped system (5.1) with the control scheme (5.2) can be expressed as

$$z(t+1) = (A + BK)z(t) - A^h BK[z(t-h) - z(t_k - h)]. \quad (5.3)$$

Proof. Define the new state $z(t) = \bar{x}(t+h)$, where \bar{x} is given in (5.2). It holds

$$\begin{aligned} z(t+1) &= A^h(Ax(t) + Bu(t_k - h)) + \sum_{i=0}^{h-1} A^{h-i-1} Bu(t+1-h+i) \\ &= A^h(Ax(t) + Bu(t_k - h)) + A \sum_{i=0}^{h-1} A^{h-i-1} Bu(t-h+i) + Bu(t) - A^h Bu(t-h) \\ &= A[A^h x(t) + \sum_{i=0}^{h-1} A^{h-i-1} Bu(t-h+i)] + Bu(t) + A^h Bu(t_k - h) - A^h Bu(t-h) \\ &= Az(t) + Bu(t) + A^h B[u(t_k - h) - u(t-h)]. \end{aligned}$$

Finally, reinjecting the definition of $u = Kz$ in the previous equation leads to the result. \square

5.2. Stability conditions

The following theorem holds

THEOREM 5.1 For a given controller gain K and a given delay h , assume that there exists two $n \times n$ matrices $P > 0$ and $W > 0$ such that the LMI condition

$$\begin{bmatrix} -P & 0 & (A + BK - I)^T W & (A + BK)^T P \\ * & -\lambda_T^2 W & -(A^h BK)^T W & -(A^h BK)^T P \\ * & * & -W & 0 \\ * & * & * & -P \end{bmatrix} < 0 \quad (5.4)$$

holds where $\lambda_T = 2\sin\left(\frac{\pi}{2(2T+1)}\right)$.

Proof. Consider the functional

$$V_i(x_i) = x^T(t)Px(t) + V_W(x_i),$$

where the functionals V and V_W have been defined previously. It follows from the previous calculations that

$$\Delta V_i(x_i) \leq \begin{bmatrix} x(t) \\ v(t) \end{bmatrix}^T \Psi_2 \begin{bmatrix} x(t) \\ v(t) \end{bmatrix}$$

TABLE 4 *Evolution of the maximal admissible sampling period T for several values of the input delay h*

T	1	2	3	4	5	6	7	8	9	10	Number of decision variables
Theorem 5.1	3	7	10	13	15	16	17	19	20	21	$3.5n^2 + 2.5n$

where

$$\begin{aligned} \Psi_2 = & \begin{bmatrix} -P & 0 \\ * & -\lambda_T^2 W \end{bmatrix} + \begin{bmatrix} (A+BK)^T - I \\ -(A^h BK)^T \end{bmatrix} W \begin{bmatrix} (A+BK)^T - I \\ -(A^h BK)^T \end{bmatrix}^T \\ & + \begin{bmatrix} (A+BK)^T \\ -(A^h BK)^T \end{bmatrix} P \begin{bmatrix} (A+BK)^T \\ -(A^h BK)^T \end{bmatrix}^T. \end{aligned} \quad (5.5)$$

Then asymptotic stability results from the condition $\Psi_2 < 0$, which is equivalent to (5.4) by application of the Schur complement. \square

5.3. Example 2

Consider the linear discrete-time systems with delay given in (2.4) with the matrices taken from Gao *et al.* (2004)

$$A = \begin{bmatrix} 0.8 & 0 \\ 0.05 & 0.9 \end{bmatrix}, \quad A_d = \begin{bmatrix} -0.1 & 0 \\ -0.2 & -0.1 \end{bmatrix}.$$

The results are presented in Table 4. One can see from this example that the robustness of the predictor control is reinforced for large delays. This means that the more the delay, the more the maximal allowable sampling period is obtained. A possible interpretation for such behavior is that the matrix A_0 is Schur stable. Therefore the matrix A_0^h in the LMI conditions becomes smaller when the delay increases so that the contribution of λ_T becomes sufficiently great to ensure robustness with respect to the sampling period.

A counter part of this numerical results is that the performances of the closed loop systems may be affected. This means that increasing the delay T for large values of h would lead to power performances. In order to measure the performance degradation, one may look at \mathcal{L}_2 performance criteria or exponential stability criteria with guaranteed decay rate. For the latter solution, one would need to lightly modify the Wirtinger-based functional to account for exponential stability. This can be achieved following the idea developed for the continuous-time case in Lemma 1 of Selivanov & Fridman (2016).

6. Conclusions

This article addresses the stability analysis of discrete time sampled-data systems. The approach developed in this article can be interpreted as the counterpart of the recent result on continuous-time systems from Liu & Fridman (2012). Two stability theorems have been provided and are tested on a simple example showing the efficiency of the method.

Funding

This work was supported by the ANR project SCIDiS contract number 15-CE23-0014.

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