Disturbance Compensation With Finite Spectrum Assignment for Plants With Input Delay

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***Abstract*—This paper presents a method for compensation of unknown bounded smooth disturbances for linear time invariant (LTI) plants with known parameters in the presence of constant and known input delay. The proposed control law is a sum of the classical predictor suggested by Manitius and Olbrot for finite spectrum assignment and a disturbance compensator. The disturbance compensator is a novel control law based on the auxiliary loop for disturbance extraction and on the disturbance prediction. A numerical implementation of the integral terms in the predictor-based control law is studied and sufficient conditions in terms of linear matrix inequalities are provided for an estimate on the maximum delay that preserves the stability. Numerical examples illustrate the efficiency of the method.**

***Index Terms*—Disturbance compensation, input delay, numerical implementation, predictor, stabilization.**

I. Introduction

One of the central problems in the control theory is control of systems affected by unknown disturbances. This problem becomes especially complicated in the presence of input delays that are typical for process control, remote control, chemical technologies, etc. (see, e.g., [2]–[5]). Delay may prevent a designer from using high-gain controllers for disturbance attenuation. The first approach to control of systems with input delay was proposed by Smith for stable plants [6]. For unstable plants, Manitius and Olbrot suggested prediction with finite spectrum assignment in [7]. In the presence of disturbance, the predictor of [7] achieves disturbance attenuation [3], [8]. The above-mentioned papers did not take into account the structure of disturbances. The next step was done in [9], where a method for compensation of a finite number of sinusoidal disturbances was proposed.

In [3] and [7]–[9], integral representations of state predictors were used without considering their numerical implementations. If the prediction horizon h (h is the value of input delay) is too large, the numerical implementation may destabilize the system [11]–[14]. A necessary condition for a bound on h that preserves the stability was provided in [13]. However, sufficient conditions for h preserving the stability under numerical implementations are missing.

In this paper, a more general than in [9] class of $(r+1)$ continuously differentiable disturbances with uniformly bounded $(r+1)$th derivatives is considered. We suggest a control law that is a sum of the classical predictor of [7] and a disturbance compensator. The disturbance compensator is a novel control law based on the auxiliary loop for disturbance extraction and on the disturbance prediction. Note that recently (when this paper was under review), for the same class of disturbances, a similar idea of a control law that predicted disturbances with horizon h and allowed us to compensate their influence on the system was suggested in [10]. The disturbance prediction in [10] was based on the current values of the disturbance and its derivatives till rth order that led to an $(r+1)$ th-order observer for the disturbance and its r derivatives. The numerical implementation issues were not considered in [10].

We propose a disturbance prediction that is based on the current and the delayed values of the disturbance. The latter allows us to design a predictor-based control law that employs a simple scalar observer (the so-called dirty derivative filter as considered, e.g., in [15]). We study the numerical implementation of the predictor-based control law and provide, for the first time, sufficient conditions in terms of linear matrix inequalities (LMIs) for an estimate on the maximum delay that preserves the practical stability (meaning that the solutions of the closed-loop system are ultimately bounded with a small enough bound). The efficiency of the presented method is illustrated by two examples.

II. Problem Formulation

Consider the following system:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t-h) + Bf(t),\ t \geq 0 \\ u(s) &= 0,\ s < 0 \end{aligned} \tag{1}$$

where $x(t) \in R^n$ is the state vector, $u(t) \in R$ is the control, $f(t) \in R$ is an unknown and matched disturbance, $A \in R^{n\times n}$ and $B \in R^n$ are the constant known matrices, and $h > 0$ is known and constant time-delay. Note that our results can be easily extended to the case of multiinputs provided B is full rank (see Remark 2).

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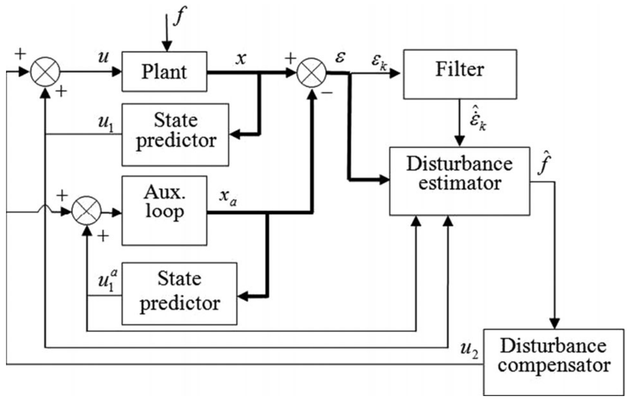

Fig. 1. Control system structure (bold lines denote the vector signals and thin lines denote the scalar ones).

We assume the following.

A1. The function $f : R_+ \in R$ is $(r+1)$ times continuously differentiable. Moreover, the unknown disturbance is uniformly bounded together with its $(r+1)$th derivative.

A2. The pair (A, B) is controllable.

The classical predictor suggested in [7] guarantees the input-to-state stability of (1) leading to ultimate bound

$$\lim_{t\to\infty} \sup_{t\geq 0} |x(t)| \leq \delta \tag{2}$$

where $\delta = \mathrm{O}(\sup_{t\geq 0} |f(t)|)$. Here, $|\cdot|$ is the Euclidean norm of a vector and $\mathrm{O}(\chi)$ for $\chi \in R$ means that $\lim_{\chi\to 0} \frac{\mathrm{O}(\chi)}{\chi} = C$, where C is a constant.

In this paper, our objective is to design a controller that decreases δ achieving $\delta = \mathrm{O}(h^{r+1} \sup_{t\geq 0} |f^{(r+1)}(t)|)$ for the class of disturbances with small enough $h^{r+1} \sup_{t\geq 0} |f^{(r+1)}(t)| << \sup_{t\geq 0} |f(t)|$. Sufficient conditions for this objective are given below in Theorems 1 (under integral predictor-based laws) and 2 (under numerical implementations of the integral terms of the predictors). The proofs of Theorems 1 and 2 are given in Appendixes A and B, respectively. The novel control law that we propose is based on the disturbance extraction and its prediction.

III. Predictive Disturbance Compensation Control Scheme

We choose a vector $K^T \in R^n$ such that the matrix $A + BK$ is Hurwitz and suggest the control law in the form of the sum:

$$u(t) = u_1(t) + u_2(t) \tag{3}$$

where

$$u_1(t) = K\left[e^{Ah}x(t) + \int_{t-h}^{t} e^{A(t-\theta)}Bu_1(\theta)d\theta\right] \tag{4}$$

is a classical predictor for finite spectrum assignment [7]. The novel control law u_2 that will be designed below is aimed for disturbances compensation. We illustrate the design procedure in Fig. 1 (see "State predictor" and "Disturbance compensator").

In order to extract the disturbance f from the closed-loop system (1), (3), we use the method described in [16]. We introduce an auxiliary loop in the form

$$\begin{aligned} &\dot{x}_a(t) = Ax_a(t) + Bu_1^a(t-h) + Bu_2(t-h) \\ &x_a(0) = 0 \\ &u_1^a(t) = K\left[e^{Ah}x_a(t) + \int_{t-h}^{t} e^{A(t-\theta)}Bu_1^a(\theta)d\theta\right]. \end{aligned} \tag{5}$$

Defining the error function $\varepsilon = x - x_a$, from (1), (3), and (5) we arrive at the error equation

$$\dot{\varepsilon}(t) = A\varepsilon(t) + B\left(u_1(t-h) - u_1^a(t-h)\right) + Bf(t). \tag{6}$$

Denote $B = \mathrm{col}\{b_1, b_2, ..., b_n\}$. Choosing any k $(k = 1, \ldots, n)$ with $b_k \neq 0$, we rewrite the kth equation of system (6) in the form

$$\begin{aligned} \dot{\varepsilon}_k(t) &= a_k^T\varepsilon(t) \\ &\quad + b_k\left(u_1(t-h) - u_1^a(t-h)\right) + b_k f(t) \end{aligned} \tag{7}$$

where a_k is the kth row of the matrix A. From (7), we obtain

$$\begin{aligned} f(t) = b_k^{-1}\big[&\dot{\varepsilon}_k(t) - a_k^T\varepsilon(t) \\ &- b_k\left(u_1(t-h) - u_1^a(t-h)\right)\big]. \end{aligned} \tag{8}$$

Note that the signal $\dot{\varepsilon}_k$ is not available in (8). In order to find its estimate $\hat{\dot{\varepsilon}}_k$, we can use any existing observer (see, e.g., [15]–[18]). We suggest the following simple dirty derivative filter [15] (see "Filter" in Fig. 1):

$$\mu\dot{\hat{\dot{\varepsilon}}}_k(t) + \hat{\dot{\varepsilon}}_k(t) = \dot{\varepsilon}_k(t), \quad \hat{\varepsilon}_k(0) = 0 \tag{9}$$

where $\mu > 0$ is a small enough number. Thus, the resulting estimate $\hat{f}$ of f (see "Disturbance estimator" in Fig. 1) has a form

$$\begin{aligned} \hat{f}(t) = b_k^{-1}\big[&\hat{\dot{\varepsilon}}_k(t) - a_k^T\varepsilon(t) \\ &- b_k\left(u_1(t-h) - u_1^a(t-h)\right)\big]. \end{aligned} \tag{10}$$

In order to construct the disturbance compensator u_2, we approximate $\hat{f}(t)$ by its past values $\hat{f}(t-h), \ldots, \hat{f}(t-(r+1)h)$ via the mean value theorem [19]:

$$\hat{f}(t) = \sum_{j=1}^{r+1}(-1)^{j-1}C_{r+1}^{j}\hat{f}(t-jh) + \hat{E}(t). \tag{11}$$

Here, the remainder $\hat{E}(t)$ is given by

$$\hat{E}(t) = h^{r+1}\hat{f}^{(r+1)}(t-(r+1)\theta h), \; 0 < \theta < 1. \tag{12}$$

Approximation of unknown signals via the mean value theorem was suggested in [20]. From (10) and (11), we find

$$f(t) = \hat{f}(t) + b_k^{-1}\eta(t) \tag{13}$$

where

$$\eta(t) = \dot{\varepsilon}_k(t) - \hat{\dot{\varepsilon}}_k(t). \tag{14}$$

Substitution of $u(t) = u_1(t) + u_2(t)$ and (13) into (1) leads to

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu_1(t-h) + Bu_2(t-h) \\ &\quad + B\sum_{j=1}^{r+1}(-1)^{j-1}C_{r+1}^{j}\hat{f}(t-jh) + B\lambda(t) \end{aligned} \tag{15}$$

where

$$\lambda(t) = \hat{E}(t) + b_k^{-1}\eta(t). \tag{16}$$

Choosing in (15) the control law u_2 as

$$u_2(t) = -\sum_{j=1}^{r+1}(-1)^{j-1}C_{r+1}^{j}\hat{f}(t-(j-1)h) \tag{17}$$

(see "Disturbance compensator" in Fig. 1), we arrive at

$$\dot{x}(t) = Ax(t) + Bu_1(t-h) + B\lambda(t). \tag{18}$$

It will be shown in Appendix A that the solutions of (18), (4) are ultimately bounded and their ultimate bound is of the same order as the ultimate bound $\Delta(\mu) := \lim_{t\to\infty} \sup_{t\geq 0} |\lambda(t)|$ of λ and that

$$\lim_{\mu\to 0} \Delta(\mu) = h^{r+1} \sup_{t\geq 0} \left| f^{(r+1)}(t) \right|. \tag{19}$$

Thus, the proposed control law allows us to decrease the influence of the disturbance on the solutions of the closed-loop system if $h^{r+1} \sup_{t\geq 0} |f^{(r+1)}(t)| \ll \sup_{t\geq 0} |f(t)|$. This is different from [3], [7], [8], where the closed-loop systems and the corresponding bounds directly depend on f.

We are in a position to formulate the main result of this section.

Theorem 1: Given $K^{\mathrm{T}} \in R^n$ such that the matrix $A_0 := A + BK$ is Hurwitz and given a scalar $\alpha > 0$, let there exist a constant $\beta > 0$ and an $n \times n$ matrix $P > 0$ that satisfy the following LMI:

$$Q := \begin{pmatrix} A_0^T P + PA_0 + 2\alpha P & Pe^{Ah}B \\ * & -\beta \end{pmatrix} < 0. \tag{20}$$

Then, for all small enough $\mu > 0$ there exists $\Delta(\mu) > 0$ such that the solutions of (1) under the control law (3)–(5), (9), (10), (17) are ultimately bounded and (2) holds with $\delta = \mathrm{O}(\Delta(\mu))$, where $\Delta(\mu)$ satisfies (19).

LMI (20) is always feasible for $\alpha < \max Re(\sigma(A_0))$ (here $\sigma(A_0)$ denotes an eigenvalue of A_0) and for large enough β.

Remark 1: In [3], the influence of the disturbance f is attenuated by the control law $u = u_1$ only. In [9], the results are confined to sinusoidal signals f whereas the control law u_2 is needed for the identification of parameters of sinusoidal signals and for their compensation. The proposed control law allows us to compensate a wider than in [9] class of disturbances and employs a simple scalar observer (9) (in [10], an $(r+1)$ th-order observer is used for the disturbance preditctor).

Remark 2: Our results can be easily extended to (1) with several inputs $u(t) \in R^m$ if B is full rank. In this case, there always exist m linearly independent rows of B. Then, similarly to (8), f can be found from (6) by employing the corresponding to these rows equations of (6).

IV. Numerical Implementation of the Predictive Control Scheme

Note that the integral terms in control laws (4) and (5) are supposed to be implemented numerically. For numerical implementation of these terms, a cubature formula can be used:

$$u_1(t) = K\Bigg[e^{Ah}x(t) + \sum_{p=0}^{q} m_p e^{q^{-1}phA} Bu_1(t - q^{-1}ph)\Bigg] \tag{21}$$

$$u_1^a(t) = K\Bigg[e^{Ah}x_a(t) + \sum_{p=0}^{q} m_p e^{q^{-1}phA} Bu_1^a(t - q^{-1}ph)\Bigg] \tag{22}$$

where the values of m_p depend on the chosen numerical scheme, the integer q determines the approximation precision. In our consideration, we assume that the values of m_p are small enough for large enough q. This is the case, e.g., in the trapezoidal rule.

We will present below LMI-based sufficient conditions for finding h that preserves ultimate boundedness of solutions to (1) under the predictor-based control law with u_1 and u_2 given by (21) and (22). The idea of the Lyapunov-based analysis of the resulting closed-loop system is the following. From (18), we find

$$\begin{aligned} Bu_1(t - q^{-1}h) &= \dot{x}(t - (q^{-1}p - 1)h) \\ &- Ax(t - (q^{-1}p - 1)h) - B\lambda(t - q^{-1}ph) \\ &- B\lambda(t - (q^{-1}p - 1)h). \end{aligned} \tag{23}$$

Substitution of the right-hand side of (23) into (21) leads to

$$\begin{aligned} \dot{x}(t) = Ax(t) + BK\Bigg[&e^{Ah}x(t-h) \\ &+ \sum_{p=0}^{q} m_p e^{q^{-1}phA} \\ &\times \left(\dot{x}(t - q^{-1}ph) - Ax(t - q^{-1}ph)\right)\Bigg] \\ &- BK\sum_{p=0}^{q} m_p e^{q^{-1}phA}\lambda(t - q^{-1}ph) + B\lambda(t). \end{aligned} \tag{24}$$

Solving (23) with respect to $\dot{x}$, we arrive at the neutral-type system with the input that is given by a linear combination of $\lambda(t)$ and its delayed values. By using a simple Lyapunov functional for the resulting neutral-type system, we will derive in Appendix B sufficient LMI-based conditions for its input-to-state stability. Then, the estimate on the ultimate bound of the solutions to the closed-loop system under (21) will follow from the ultimate bound $\Delta(\mu)$ of λ and from relation (19).

To formulate the main result of this section, we will use the following notations:

$$\begin{aligned} &M := I - BKm_0 \\ &F_p := M^{-1}BKm_p e^{q^{-1}phA} \ (p = 1, \dots, q) \\ &D_j := -M^{-1}BKm_j e^{q^{-1}jhA}A \ (j = 1, \dots, q-1) \\ &D_q := M^{-1}BK\left(e^{Ah} - m_q e^{Ah}A\right), A_s := A + \sum_{p=1}^{q} D_p. \end{aligned} \tag{25}$$

Theorem 2: Let the matrix $M = I - BKm_0$ be nonsingular. Given $h > 0$ and a scalar $\alpha > 0$, let there exist a constant $\beta > 0$ and $n \times n$ matrices $P > 0$, P_2, P_3, $S_p > 0$, $R_p > 0$, $Q_p > 0$, and $p = 1, 2, \dots, q$ that satisfy the following LMI:

$$\Psi := \begin{bmatrix} \Psi_{11} & \Psi_{12} & \Psi_{13} & \Psi_{14} \\ * & \Psi_{22} & 0 & 0 \\ * & * & \Psi_{33} & 0 \\ * & * & * & -\beta \end{bmatrix} < 0 \tag{26}$$

with notations given by (25), where

$$\Psi_{11}^{11} = P_2^T A_s + A_s^T P_2 + \sum_{p=1}^{q} S_p$$

$$\Psi_{11}^{12} = P - P_2^T + A_s^T P_3$$

$$\Psi_{11}^{22} = -P_3 - P_3^T + \sum_{p=1}^{q} pq^{-1}hR_p + \sum_{p=1}^{q} Q_p$$

$$\Psi_{11} = \begin{bmatrix} \Psi_{11}^{11} & \Psi_{11}^{12} \\ * & \Psi_{11}^{22} \end{bmatrix}$$

$$\Psi_{12} = -\frac{h}{q}\begin{bmatrix} P_2^T D_1 & 2P_2^T D_2 & \dots & qP_2^T D_q \\ P_3^T D_1 & 2P_3^T D_2 & \dots & qP_3^T D_q \end{bmatrix}$$

$$\Psi_{13} = \begin{bmatrix} P_2^T F_1 & \dots & P_2^T F_q \\ P_3^T F_1 & \dots & P_3^T F_q \end{bmatrix}$$

$$\Psi_{14} = \begin{bmatrix} P_2^T B \\ P_3^T B \end{bmatrix}$$

$$\Psi_{22} = -h\mathrm{diag}\{q^{-1}e^{-2\alpha q^{-1}h}R_1, \ \dots \\ e^{-2\alpha(q-1)q^{-1}h}R_{q-1}, \ e^{-2\alpha h}R_q\}$$

$$\Psi_{33} = -\mathrm{diag}\{e^{-\alpha q^{-1}h}Q_1, \ \dots \\ e^{-\alpha(q-1)q^{-1}h}Q_{q-1}, \ e^{-\alpha h}Q_q\}.$$

Then for all small enough $\mu > 0$ there exists $\Delta(\mu) > 0$ that satisfies (19) and such that solutions of (1) under the control law (3), (5), (9), (10), (17), (21), (22) are ultimately bounded and (2) holds with $\delta = \mathrm{O}(\Delta(\mu))$.

LMI (26) is always feasible for $\alpha < \max Re(\sigma(A_0))$ and small enough m_p and h.

V. Examples

Example 1: Consider (1) with parameters from [9], where

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \ B = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \ h = 0.45.$$

Choose $\mu = 0.01$ in filter (9), where $k = 2$. Use $r = 3$ in the disturbance compensation control law (17) where $\hat{f} = \hat{\dot{\varepsilon}}_2 - [1\,0]^T \varepsilon$. Control laws (21) and (22) used for numerical implementation with $K = [-3 \ -3]$ and $q = 5$ are defined as follows

$$u_1(t) = -[3\ 3]\Bigg[e^{0.45A}x(t) + 0.09\Bigg(0.5u_1(t) + \sum_{p=1}^{4} e^{0.45q^{-1}pA}Bu_1(t - 0.45q^{-1}p) + 0.5e^{0.45A}Bu_1(t - 0.45)\Bigg)\Bigg]$$

$$u_1^a(t) = -[3\ 3]\Bigg[e^{0.45A}x_a(t) + 0.09\Bigg(0.5u_1^a(t) + \sum_{p=1}^{4} e^{0.45q^{-1}pA}Bu_1^a(t - 0.45q^{-1}p) + 0.5e^{0.45A}Bu_1^a(t - 0.45)\Bigg)\Bigg].$$

Here, the trapezoidal rule is used for the approximation of the integral term. For numerical simulations, we choose $x(0) = [1\ 2]^T$. Note, that the control laws u_1 and u_1^a are verified for $q = 10, 30, 50$. However, enlarging q does not affect on the stability and quality of transients in the closed-loop system [11], [12], [14].

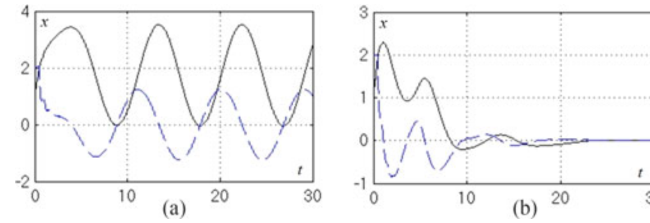

Fig. 2. Plots of the state under the control law of (a) [3], [7], [8] with $u = u_1$ and (b) [9].

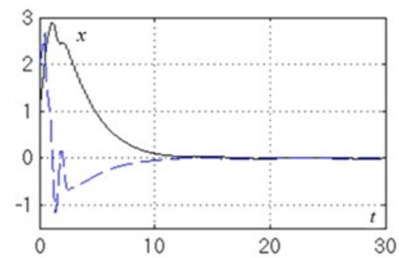

Fig. 3. Plots of the state of the proposed control law.

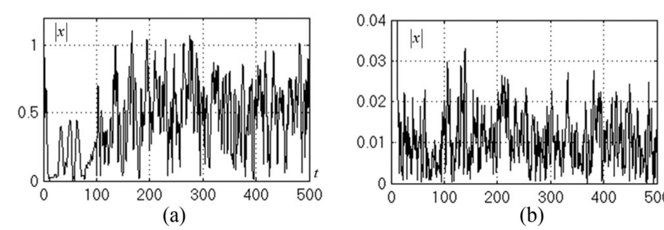

Fig. 4. Plot of x under the control law of (a) [9] and (b) the proposed one.

Consider first the following disturbance: $f = 1 + \sin 0.2t$. In Figs. 2 and 3, the plots of the state are presented for the control law [3], [7], [8] (for $u = u_1$), the control law from [9] and the proposed one, respectively. In figures, the solid curve corresponds to x_1 and the dashed curve corresponds to x_2. The simulations show that the control law of [3], [7], [8] does not compensate this disturbance. The control law of [9] ensures the exact compensation of the disturbance. Moreover, the proposed control law compensates the disturbance with the accuracy $\delta = 0.02$.

Now consider the case of a nonsinusoidal disturbance $f = 1 + \sin 0.2t + \omega$, where ω is defined as a solution of the following differential equation:

$$15\omega^{(4)} + 38\omega^{(3)} + 32\omega^{(2)} + 10\dot{\omega} + \omega = \chi\mathrm{sat}(g(t)) \\ \omega^{(i)}(0) = 0, \ i = 0, \dots, 3 \quad (27)$$

where $\chi = 200$, sat$(\cdot)$ is a saturation function, g is a piecewise-constant signal (which is constant on intervals $[0, 0.1), [0.1, 0.2), \dots$) with normally distributed random values and with the zero mean and the variance equal to $1/16$. In Fig. 4, the plots of x are presented for the control law of [9] and the proposed one, respectively. It follows from the simulations that the control law of [9] ensures the accuracy $\delta = 1.1$. The proposed control law ensures the accuracy $\delta = 0.033$. The simulations show that the controller $u = Kx$ cannot stabilize the system

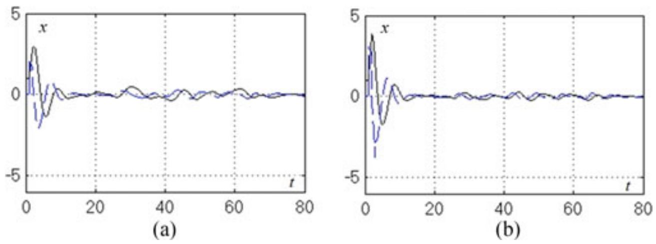

Fig. 5. Plots of the state under the proposed control law for (a) $r = 0$ and (b) $r = 2$ in (17).

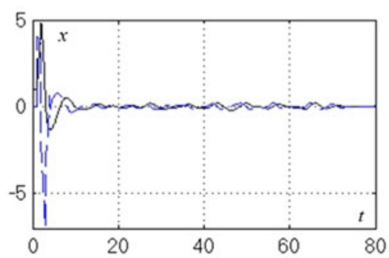

Fig. 6. Plots of the state under the proposed control law for $r = 3$ in (17).

with $h > 0.23$ and $f = 0$. LMI (26) is feasible for $h \leq 0.3$. Moreover, for $h = 0.3$ our results are favorably compared with [3], [7]–[9]. The system under the proposed control law and the control laws of [3], [7]–[9] loses the stability for $h > 0.48$, meaning that the LMI-based bound for h is rather efficient.

Example 2: Consider the model of the dc motor [21] in the form

$$I\ddot{\varphi}(t) = k\Phi i(t-h) - M(t) \tag{28}$$

where φ is a rotation angle of the motor shaft, I is an inertia moment of the motor rotating part, i is a current in the armature circuit, k is a constructive constant, Φ is a magnetic flux, M is a resistance moment depending on unknown load, and $h = 0.66$ is a time-delay caused by remote control [21]. Denote $x_1 = \varphi$, $x_2 = \dot{\varphi}$, $u = (k\Phi/I)i$, and $\omega = (1/I)M$. Then, the model (28) can be represented in the form of (1), where $A = \left[\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right]$ and $B = [0\,1]^T$. The goal is to design such a control law that the motor shaft angle rotates to zero and stops with some accuracy.

Let $K = [-3\ \ -3]$ and $q = 5$ in (21) and (22), where the trapezoidal rule is used. Choose $\mu = 0.01$ and $k = 2$ in filter (9). The disturbance ω is simulated by the solution of (27), where $\chi = 40$. The simulations show that the control law $u = Kx$ cannot stabilize the system with $h > 0.41$ and $f = 0$, whereas the proposed numerically implemented control law cannot stabilize the system for $h > 0.74$. LMI (26) is feasible for $h \leq 0.66$ (which is not far from 0.73 that follows from simulations).

For numerical simulations, we choose $x(0) = [1\,0]^T$. In Figs. 5 and 6, the plots of x_1 and x_2 are presented for the proposed control law with $r = 0, r = 2$, and $r = 3$ in (17). The simulations show that disturbances are compensated by the proposed control law with the accuracy δ equal to $0.6, 0.3$, and 0.1 for r equal to $0, 2$ and 3, respectively.

VI. Conclusion

In this paper, the new control law has been suggested for the compensation of unknown bounded smooth disturbances acting on the LTI plant with input delay. The proposed control law is the sum of the classical predictor and of a novel disturbance compensation loop. For the first time, the stability under the numerically implemented predictor-based controllers have been analyzed via Lyapunov–Krasovskii method. Efficient sufficient LMI-based conditions are provided for the maximum value of the delay that preserves the stability. Further improvements may be achieved by using other Lyapunov functionals. An extension of the presented method to uncertain plants may be a topic for future research.

Appendix

A. Proof of Theorem 1

The proof consists of five steps.

Step 1: Ultimate boundedness of $\varepsilon^{(r+2)}$. By using the reduction approach [1], consider the following change of the state in (6):

$$z_0(t) = e^{Ah}\varepsilon(t) + \int_{t-h}^{t} e^{A(t-\theta)} B\left[u_1(\theta) - u_1^a(\theta)\right] d\theta.$$

Then, $u_1(\theta) - u_1^a(\theta) = Kz_0(\theta)$, and we arrive at

$$z_0(t) = e^{Ah}\varepsilon(t) + \int_{t-h}^{t} e^{A(t-\theta)} BKz_0(\theta)d\theta. \tag{29}$$

Differentiating (29) and taking into account (4), (5), and (6), we find

$$\dot{z}_0(t) = A_0 z_0(t) + e^{Ah}Bf(t), \quad A_0 := A + BK. \tag{30}$$

Differentiating $(r+1)$ times (30), we obtain

$$\dot{z}_0^{(r+2)}(t) = A_0 z_0^{(r+1)}(t) + e^{Ah}Bf^{(r+1)}(t). \tag{31}$$

Consider the Lyapunov function $V = z_0^{(r+1)}(t)^T(t)Pz_0^{(r+1)}(t)$ and differentiate it along (36). We have

$$\begin{aligned} W &= \dot{V} + 2\alpha V - \beta(f^{(r+1)})^2 \\ &= \left[(z_0^{(r+1)})^T\ f^{(r+1)}\right] Q \left[(z_0^{(r+1)})^T\ f^{(r+1)}(t)\right]^T \leq 0 \end{aligned}$$

where the latter inequality follows from (20). Then, (36) is input-to-state stable, and the uniform boundedness of $f^{(r+1)}$ implies the ultimate boundedness of $z_0^{(r+1)}$. Hence, $z_0^{(r+2)}$ defined by the right-hand side of (31) is also ultimately bounded. Further, from the equation that results from the differentiation $(r+2)$ times of (29), we conclude that $\varepsilon^{(r+2)}$ is ultimately bounded.

Step 2: The feasibility of (20). Since A_0 is Hurwitz, the Lyapunov inequality $A_0^T P + PA_0 + 2\alpha P < 0$ is always feasible for $\alpha < \max Re(\sigma(A_0))$. Then, by the Schur complement, (20) is feasible for large enough β.

Step 3: Ultimate bound on $\eta^{(r+1)}$. Differentiating (14) and substituting from (9) $\dot{\hat{\varepsilon}}_k(t) = \mu^{-1}\eta(t)$, we obtain

$$\mu\dot{\eta}(t) = -\eta(t) + \mu\ddot{\varepsilon}_k(t). \tag{32}$$

Differentiating (32) $(r+1)$ times, we have

$$\mu\eta^{(r+2)}(t) = -\eta^{(r+1)}(t) + \mu\varepsilon^{(r+2)}(t).$$

Since $\varepsilon^{(r+2)}$ is ultimately bounded, then $\eta^{(r+1)}$ is ultimately bounded and

$$\lim_{t\to\infty}\sup_{t\geq 0}\left|\eta^{(r+1)}(t)\right| = \mathrm{O}(\mu). \tag{33}$$

Step 4: Ultimate bound on λ. Differentiating $(r+1)$ times (13), we find $\hat{f}^{(r+1)}(t) = f^{(r+1)}(t) - b_k^{-1}\eta^{(r+1)}(t)$. Then, (12) can be presented as

$$\hat{E}(t) = -h^{r+1}\Big[f^{(r+1)}(t-(r+1)\theta h) - b_k^{-1}\eta^{(r+1)}(t-(r+1)\theta h)\Big],\ 0<\theta<1. \quad (34)$$

Due to (33) and (34) and the fact that $f^{(r+1)}$ is uniformly bounded, it follows from (16) that $\lambda(t)$ is ultimately bounded $\Delta(\mu) := \lim_{t\to\infty} \sup_{t\geq 0}|\lambda(t)| < \infty$ and (19) is satisfied.

Step 5: Ultimate bound of x. Consider next the following change of the state $x(t)$ in (18):

$$z_1(t) = e^{Ah}x(t) + \int_{t-h}^{t} e^{A(t-\theta)}Bu_1(\theta)d\theta.$$

It follows from (4) that $u_1(\theta) = Kz_1(\theta)$ and that

$$x(t) = e^{-Ah}\left[z_1(t) - \int_{-h}^{0} e^{As}BKz_1(s+t)ds\right]. \quad (35)$$

Differentiating (35) and taking into account (18), we obtain

$$\dot{z}_1(t) = A_0 z_1(t) + e^{Ah}B\lambda(t). \quad (36)$$

Under (20), the system (36) is input-to-state stable and

$$\sigma_{\min}(P)\lim_{t\to\infty}\sup_{t\geq 0}|z_1(t)|^2 \leq \lim_{t\to\infty}\sup_{t\geq 0}\left(z_1^T(t)Pz_1(t)\right) \leq 0.5\alpha^{-1}\beta\Delta^2(\mu). \quad (37)$$

From (35), we find

$$|x(t)| \leq \left\|e^{-Ah}\right\|\Bigg[|z_1(t)| + \max_{s\in[-h,0]}\|e^{As}\|\,|BK|\int_{-h}^{0}|z_1(s+t)|\,ds\Bigg]. \quad (38)$$

Inequalities (37), (38) and (33) imply that (2) holds with $\delta = \mathrm{O}(\mu)$.

B. Proof of Theorem 2

The proof consists of three steps.

Step 1: Boundedness of $\varepsilon^{(r+2)}$. From (1) with $u = u_1 + u_2$ and (5), we have

$$\begin{aligned} Bu_1(t-q^{-1}h) &= \dot{x}(t-(q^{-1}p-1)h) - Ax(t-(q^{-1}p-1)h) - Bu_2(t-q^{-1}ph) - Bf(t-(q^{-1}p-1)h)\\ Bu_1(t-q^{-1}h) &= \dot{x}_a(t-(q^{-1}p-1)h) - Ax_a(t-(q^{-1}p-1)h) - Bu_2(t-q^{-1}ph). \end{aligned}$$

Substituting the right-hand sides of the latter equations into (21) and (22), we obtain

$$\begin{aligned} u_1(t) &= K\Bigg[e^{Ah}x(t) + \sum_{p=0}^{q} m_p e^{q^{-1}phA} \times [\dot{x}(t-(q^{-1}p-1)h) - Ax(t-(q^{-1}p-1)h) - Bu_2(t-q^{-1}ph) - Bf(t-(q^{-1}p-1)h)]\Bigg]\\ u_1^a(t) &= K\Bigg[e^{Ah}x_a(t) + \sum_{p=0}^{q} m_p e^{q^{-1}phA} \times [\dot{x}_a(t-(q^{-1}p-1)h) - Ax_a(t-(q^{-1}p-1)h) - Bu_2(t-q^{-1}ph)]\Bigg]. \end{aligned} \quad (39)$$

Plugging (39) into (6), we arrive at

$$\dot{\varepsilon}(t) = A\varepsilon(t) + BK\Bigg[e^{Ah}\varepsilon(t-h) + \sum_{p=0}^{q} m_p e^{q^{-1}phA} \times \left(\dot{\varepsilon}(t-q^{-1}ph) - A\varepsilon(t-q^{-1}ph)\right)\Bigg] - BK\sum_{p=0}^{q} m_p e^{q^{-1}phA} f(t-q^{-1}ph) + Bf(t). \quad (40)$$

Solving (40) with respect to $\dot{\varepsilon}(t)$, we obtain a neutral-type system

$$\dot{\varepsilon}(t) = A_s\varepsilon(t) + \sum_{p=1}^{q}\left[F_p\dot{\varepsilon}(t-pq^{-1}h) - D_i\int_{t-pq^{-1}h}^{t}\dot{\varepsilon}(s)ds\right] + Bw(f) \quad (41)$$

with notations (25), where

$$w(f) := M^{-1}\Bigg[f(t) - K\sum_{p=0}^{q} m_p e^{q^{-1}phA} f(t-q^{-1}ph)\Bigg]. \quad (42)$$

Differentiation of (41) $(r+1)$ times leads to

$$\dot{\zeta}(t) = A_s\zeta(t) + \sum_{p=1}^{q}\left[F_p\dot{\zeta}(t-pq^{-1}h) - D_i\int_{t-pq^{-1}h}^{t}\dot{\zeta}(s)ds\right] + Bw(f^{(r+1)}) \quad (43)$$

where $\zeta(t) = \varepsilon^{(r+1)}(t)$. For the input-to-state stability analysis of (41), consider the following simple Lyapunov functional [22]–[24]:

$$\begin{aligned} V &= \sum_{i=1}^{3} V_i \\ V_1 &= \zeta^T(t)P\zeta(t) \\ V_2 &= \sum_{p=1}^{q} \int_{t-pq^{-1}h}^{t} e^{2\alpha(s-t)}\dot{\zeta}^T(s)Q_p\dot{\zeta}(s)ds \\ V_3 &= \sum_{p=1}^{q} \left[\int_{t-pq^{-1}h}^{t} e^{2\alpha(s-t)}\zeta^T(s)S_p\zeta(s)ds \right. \\ &\quad \left. + \int_{-pq^{-1}h}^{0} \int_{t+\theta}^{t} e^{2\alpha(s-t)}\dot{\zeta}^T(s)R_p\dot{\zeta}(s)dsd\theta \right] \end{aligned} \tag{44}$$

where P, Q_p, S_p and R_p are positive matrices. We use the descriptor method with free matrices P_2 and P_3 (see [22]), where $\dot{\zeta}$ is not substituted by the right-hand side of (43). Differentiating (44) along (43), we have

$$\begin{aligned} \dot{V}_1 + 2\alpha V_1 &= 2\zeta^T(t)P\dot{\zeta}(t) + 2\alpha\zeta^T(t)P\zeta(t) \\ &\quad + 2[\zeta^T(t)P_2^T + \dot{\zeta}^T(t)P_3^T][-\dot{\zeta}(t) + A_s\zeta(t) \\ &\quad - \sum_{p=1}^{q} D_p \int_{t-pq^{-1}h}^{t} \dot{\zeta}(s)ds \\ &\quad + \sum_{p=1}^{q} F_p\dot{\zeta}(t-pq^{-1}h) + Bw(f^{(r+1)})] \\ \dot{V}_2 + 2\alpha V_2 &= \dot{\zeta}^T(t)\sum_{p=1}^{q} Q_p\dot{\zeta}(t) \\ &\quad - \sum_{p=1}^{q} e^{-2\alpha pq^{-1}h}\dot{\zeta}^T(t-pq^{-1}h)Q_p\dot{\zeta}(t-pq^{-1}h) \\ \dot{V}_3 + 2\alpha V_3 &= \zeta^T(t)\sum_{p=1}^{q} S_p\zeta(t) \\ &\quad - \sum_{p=1}^{q} e^{-2\alpha pq^{-1}h}\zeta^T(t-pq^{-1}h)S_p\zeta(t-pq^{-1}h) \\ &\quad + \sum_{p=1}^{q} iq^{-1}h\dot{\zeta}^T(t)R_p\dot{\zeta}(t) \\ &\quad - \sum_{p=1}^{q} \int_{t-pq^{-1}h}^{t} e^{2\alpha(s-t)}\dot{\zeta}^T(s)R_p\dot{\zeta}(s)ds. \end{aligned} \tag{45}$$

By Jensen's inequality [25]

$$\begin{aligned} &-\int_{t-pq^{-1}h}^{t} e^{2\alpha(s-t)}\dot{\zeta}^T(s)R_p\dot{\zeta}(s)ds \\ &\leq -e^{-2\alpha pq^{-1}h}\int_{t-pq^{-1}h}^{t} \dot{\zeta}^T(s)dsR_p\int_{t-pq^{-1}h}^{t} \dot{\zeta}(s)ds. \end{aligned}$$

Denote

$$\begin{aligned} \xi_1 &= \text{col}\{\zeta, \dot{\zeta}\} \\ \xi_2 &= h^{-1}\text{col}\left\{ q\int_{t-q^{-1}h}^{t} \dot{\zeta}(s)ds \right. \\ &\qquad \left. 0.5q\int_{t-2q^{-1}h}^{t} \dot{\zeta}(s)ds, \cdots, \int_{t-h}^{t} \dot{\zeta}(s)ds \right\} \\ \xi_3 &= \text{col}\{\dot{\zeta}(t-q^{-1}h), \dot{\zeta}(t-2q^{-1}h), \ldots, \dot{\zeta}(t-h)\} \\ \xi &= \text{col}\{\xi_1, \xi_2, \xi_3, w(f^{(r+1)})\}. \end{aligned}$$

From (44)–(45) we arrive at $\dot{V} + 2\alpha V - \beta(w(f^{(r+1)}))^2 \leq \xi^T\Psi\xi \leq 0$, where the last inequality follows from (26). Then, by comparison principle,

$$\lim_{t\to\infty} \sup_{t\geq 0}(\zeta^T(t)P\zeta(t)) \leq 0.5\alpha^{-1}\beta\max_{t\geq 0}(w(f^{(r+1)}))^2$$

i.e., $\zeta(t)$ is ultimately bounded. LMI (26) guarantees also the stability of the difference equation $\zeta(t) = \sum_{p=1}^{q} F_p\zeta(t-pq^{-1}h)$ [22]. Consider now (43) as a difference equation with respect to $\dot{\zeta}(t)$, where the nonhomogeneous term is defined by $\zeta(t)$ and $w(f^{(r+1)})$. Then, by using (iii) of [4, Lemma 3.1], we conclude that $\dot{\zeta} = \varepsilon^{(r+2)}$ is ultimately bounded due to ultimate boundedness of ζ and uniform boundedness of $w(f^{(r+1)})$.

Step 2: Ultimate bound on x. Ultimate boundedness of λ and relation (19) follow from Steps 3 and 4 of the proof of Theorem 1.

Since the structure of (24) is similar to the one of (40), we conclude that LMI (26) imply $\lim_{t\to\infty}\sup_{t\geq 0}(x^T(t)Px(t)) \leq 0.5\alpha^{-1}\Delta^2(\mu)\beta$. The latter yields (2) with $\delta = O(\Delta(\mu))$.

Step 3: The feasibility of LMI (26). Since m_0 is small enough, the matrix $(I - BKm_0)$ is invertible. Additionally, since m_p ($p = 1, \ldots, q$) are small enough, the matrices F_p defined by (25) are small enough. The latter guarantees the stability of the difference equation $\zeta(t) = \sum_{p=1}^{q} F_p\zeta(t-pq^{-1}h)$. The stability of the difference equation and the fact that A_s given by (25) is Hurwitz guarantee the feasibility of LMI (26) for small enough m_p and h [4], [22].

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