

Regional stabilization and H_∞ control of time-delay systems with saturating actuators

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SUMMARY

A linear parameter varying approach is introduced for the design of a constant state-feedback controller that locally stabilizes linear systems with state time-varying delays and saturating actuators and achieves a prescribed performance level for all disturbances with uniformly bounded magnitudes. A polytopic representation is used to describe the saturation behaviour. Delay-dependent sufficient conditions in terms of linear matrix inequalities (LMIs) are obtained for the existence of such a controller. An estimate is made of the domain of attraction for the disturbance-free system. The conditions for the stabilizability and H_∞ performance of the system apply the Lyapunov–Krasovskii functional and the recent descriptor approach to the control of time-delay systems, whereas the conditions for finding an ellipsoid that bounds the set of the states (in the Euclidean space) that are reachable from the origin in finite time are obtained via the Razumikhin approach. The resulting conditions are expressed in terms of linear matrix inequalities, with some tuning parameters, and they apply a different Lyapunov function to each of the vertex points that stem from the polytopic description of the saturation in the actuators. Copyright © 2003 John Wiley & Sons, Ltd.

1. INTRODUCTION

Systems with actuator constraints were extensively studied during the 1960s due to their intimate connection with optimal control. Concurrently, design approaches, such as the describing function method, which dealt specifically with nonlinearities such as saturation were developed. Only very limited research into actuator saturation was carried out during the 1970s and 1980s with the emphasis being placed mostly on the development of the Linear State Space approach and its numerous offshoots. This situation changed during the late 1980s and early 1990s (see Reference [1] for an extensive bibliography of the work carried out during this period), and has continued apace to the present time (see Reference [2] for recent developments). In terms of stabilizability, current research can be classified as: global, semi-global

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(that guarantees that any given compact set of initial conditions, no matter how large, can be included in the domain of attraction of the closed-loop system) and local or regional (that estimates the domain of attraction). The main drawback with the global and the semi-global stabilizability approach lies in the requirement for the open-loop poles to be located in the closed left-half plane (see e.g. References [3, 4]). Relaxing these assumptions has led to investigations into regional stabilization (see e.g. References [5, 6]). The emphasis in this paper, in terms of stabilization in the face of actuator saturation, is therefore on regional stabilization. An effort will however be made to enlarge the estimate of the domain of attraction.

As for linear systems with both bounded controls and *state delays*, some of the previous research effort was concentrated on regional or global stabilization via state feedback and used either matrix measures (see Reference [7]), or the Lyapunov–Razumikhin approach for delayed systems (see Reference [10]). A Lyapunov–Krasovskii approach (which usually leads to less conservative results than Razumikhin approach) was developed for regional stabilization, both in the delay-independent and delay-dependent cases [11–13].

The present paper utilizes the method of References [6] and [13], of transforming a system with actuator saturation non-linearities into a convex polytope of linear systems. The stabilization and H_∞ control of systems with state delay is treated by the Lyapunov–Krasovskii approach via a descriptor model transformation [17, 18], and results in a new system equivalent to the original one which allows for the application of fewer bounds and uses the method introduced by Moon *et al.* [19] for less conservative bounding of cross terms. When uniformly bounded disturbances are present, the issue of finding an ellipsoid that bounds the set of states reachable from the origin in finite time (in Euclidean space) is treated via a Lyapunov–Razumikhin function, along with an S procedure and an application of the first order and parameterizing model transformations (see Reference [20]). Note that when disturbances are present, this seems to be the only approach possible within the Lyapunov framework.

The paper, divided into four sections, begins by formulating the problem in Section 2. A sufficient, delay-dependent state-feedback stabilizing design for the disturbance free situation is presented at the start of Section 3. Both delay-dependent and delay-independent designs which are optimal in the H_∞ sense are then postulated (Theorem 2 and Corollary 1, respectively). The problem with this approach, however, is the over-design due to the quadratic stabilizability inherent in the design procedure (see Reference [16]). In its place, a procedure, allowing for the assignment of a different Lyapunov function for each vertex of the polytope is presented, in order to reduce the conservatism of the former method. Sufficient conditions for H_∞ performance and stabilization at each of the vertices of the polytope are formulated in Theorem 3. Two numerical examples are given which illustrate the effectiveness of the method. The solution procedures are all formulated in terms of LMIs.

Notation: Throughout the paper the superscript ‘T’ stands for matrix transposition, \mathcal{R}^n denotes the n dimensional Euclidean space with vector norm $\|\cdot\|$, $\mathcal{R}^{n \times m}$ is the set of all $n \times m$ real matrices, and the notation $P > 0$, for $P \in \mathcal{R}^{n \times n}$ means that P is symmetric and positive definite. The space of the continuously differentiable vector functions ϕ over $[-h, 0]$ is denoted by $C^1[-h, 0]$. The space of functions in \mathcal{R}^q that are square integrable over $[0, \infty)$ is denoted by $\mathcal{L}_2^q[0, \infty)$ with the norm $\|\cdot\|_{L_2}$ and for a matrix G , G_i denotes the i th row and $\bar{\sigma}(G)$ denotes the largest singular value of G . For any vector $u \in \mathcal{R}^m$ $\text{sat}(u_i, \bar{u}_i) = \text{sign}(u_i) \min(|u_i|, \bar{u}_i)$, $0 < \bar{u}_i$,

$i = 1, \dots, m$. The convex hull of a set \mathcal{X} is the minimal convex set containing \mathcal{X} . For a group of points $x_1, x_2, \dots, x_n \in \mathcal{R}^n$, the convex hull of these points is: $\mathcal{C}o\{x_1, x_2, \dots, x_n\} = \{\sum_{i=1}^n \alpha_i x_i : \sum_{i=1}^n \alpha_i = 1, \alpha_i \geq 0\}$.

2. PROBLEM FORMULATION

We consider the following linear system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + A_h x(t - \tau) + B_1 w(t) + B_2 u(t) \\ x(\theta) &= \phi(\theta) \quad \theta \in [-h, 0] \end{aligned} \quad (1)$$

with the objective vector

$$z(t) = Cx(t) + D_{12}u(t) \quad (2)$$

where $x(t) \in \mathcal{R}^n$ is the system state vector, $w(t) \in \mathcal{L}_2^q[0, \infty)$ is the exogenous disturbance signal, $u \in \mathcal{R}^m$ is the control input and $z(t) \in \mathcal{R}^p$ is the state combination (objective function signal) to be attenuated. The matrices A , A_h , B_1 , B_2 , C and D_{12} are constant matrices of appropriate dimensions.

While the time delay τ is not known exactly and may be time-varying, it and its corresponding rate are known to lie within the regions defined by

$$0 \leq \tau \leq h \quad (3a)$$

and

$$0 \leq \dot{\tau}(t) \leq d < 1 \quad (3b)$$

where h and d are given. The theory given below can easily be extended to the case of multiple state delays.

The input vector $u = \text{col}\{u_1, \dots, u_m\}$ is subject to the following amplitude constraints:

$$|u_i(t)| \leq \bar{u}_i, \quad 0 < \bar{u}_i, \quad i = 1, \dots, m \quad (4)$$

and it is assumed that the disturbance vector $w \in \mathcal{W}$ where

$$\mathcal{W} = \{w \in \mathcal{R}^q; w^T w \leq \bar{w}^{-1}, 0 < \bar{w}\} \quad (5)$$

We consider the following state-feedback control law

$$u(t) = Kx(t) \quad (6)$$

where K is a constant gain matrix. We now address two related issues, namely stabilizability and H_∞ control.

Denoting the state trajectory of (1) with the initial condition $x_0 = \phi \in C^1[-h, 0]$ by $x(t, \phi)$, the domain of attraction of the origin of the closed-loop system (1), (6) with $w = 0$ is then the set

$$\mathcal{A} = \left\{ \phi \in C^1[-h, 0]: \lim_{t \rightarrow \infty} x(t, \phi) = 0 \right\}$$

For stabilizability, we seek conditions for the existence of a gain matrix K which leads to an asymptotically stable closed-loop for $w = 0$ and for all τ satisfying (3a) and (3b). Having met these conditions, a simple procedure for finding the gain K shall be presented. Moreover, we obtain an estimate $\mathcal{X}_\delta \subset \mathcal{A}$ of the domain of attraction, where

$$\mathcal{X}_\delta = \left\{ \phi \in C^1[-h, 0]: \max_{[-h,0]} |\phi| \leq \delta_1, \max_{[-h,0]} |\dot{\phi}| \leq \delta_2 \right\} \tag{7}$$

and where $\delta_i > 0, i = 1, 2$ are scalars that will be maximized in the sequel.

For H_∞ control, we seek a gain matrix K in (6) such that, the resulting closed-loop system is internally stable (i.e. asymptotically stable for $w = 0$), and for a prescribed scalar γ , the following holds:

$$J \triangleq \|z\|_2^2 - \gamma^2 \|w\|_2^2 < 0 \quad \forall w \neq 0 \in \mathcal{W}, \quad x_0 = \phi \equiv 0 \tag{8}$$

3. STABILIZATION AND CONTROL

3.1. Preliminaries

Applying the control law of (6) the closed-loop system obtained is

$$\dot{x}(t) = Ax(t) + A_h x(t - \tau) + B_2 \text{sat}(Kx(t), \bar{u}) + B_1 w(t), \quad x(\theta) = \phi(\theta) \quad \theta \in [-h, 0] \tag{9}$$

with the objective vector

$$z(t) = Cx(t) + D_{12} \text{sat}(Kx(t), \bar{u}) \tag{10}$$

Denoting the i th row of K by k_i , we define the polyhedron

$$\mathcal{L}(K, \bar{u}) = \{x \in \mathcal{R}^n: |k_i x| \leq \bar{u}_i, \quad i = 1, \dots, m\}$$

If the control and the disturbance are such that $x \in \mathcal{L}(K, \bar{u})$, then the system (9) admits the following linear representation

$$\dot{x}(t) = Ax(t) + A_h x(t - \tau) + B_2 Kx(t) + B_1 w(t), \quad x(\theta) = \phi(\theta) \quad \theta \in [-h, 0] \tag{11}$$

with the objective vector

$$z(t) = Cx(t) + D_{12} Kx(t) \tag{12}$$

Let Υ be the set of all diagonal matrices in $\mathcal{R}^{m \times m}$ with diagonal elements that are either 1 or 0. For example, if $m = 2$, then

$$\Upsilon = \{D_1, D_2, D_3, D_4\} = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

There are 2^m elements D_i in Υ , and for every $i = 1, \dots, 2^m$ $D_i^- \triangleq I_m - D_i$ is also an element in Υ .

Our goal is to embed $\text{sat}(Kx(t), \bar{u})$ within a convex hull of a group of linear feedbacks. Given two gain matrices $K, H \in \mathcal{R}^{m \times n}$, the matrix set $\{D_i K + D_i^- H, i = 1, \dots, 2^m\}$ is formed by choosing some rows of K and the rest from H . The following lemma establishes the desired result:

Lemma 1 (Cao et al. [8])

Given K and H in $\mathcal{R}^{m \times n}$. Then

$$\text{sat}(Kx(t), \bar{u}) \in \mathcal{Co}\{D_i Kx + D_i^- Hx, \quad i = 1, \dots, 2^m\}$$

for all $x \in \mathcal{R}^n$ that satisfy $|h_i x| \leq \bar{u}_i, \quad i = 1, \dots, m$.

3.2. Transformation of the nonlinear system to a linear system with polytopic type uncertainty

Having transformed the saturation nonlinearity into a convex hull of linear feedbacks, we can proceed to establish a convex polytope whose vertices consist of the closed loop system matrix pairs $[A_j \ C_j], \quad j = 1, \dots, 2^m$ to be defined in the sequel (see Lemma 2). Furthermore in order to reduce the notational complexity, we resort to using \bar{A} and \bar{C} , both of which are an arbitrarily chosen matrix pair from within the convex polytope. The following stems from Lemma 1.

Lemma 2

Given any convex compact set $\mathcal{S}_c \in \mathcal{R}^n$, assume that there exists H in $\mathcal{R}^{m \times n}$ such that $|h_i x| \leq \bar{u}_i$ for all $x(t) \in \mathcal{S}_c$. Then for $x(t) \in \mathcal{S}_c$ the system (9) and (10) admits the following representation:

$$\dot{x}(t) = \sum_{j=1}^{2^m} \lambda_j(t) A_j x(t) + A_h x(t - \tau) + B_1 w(t) \tag{13a}$$

$$z(t) = \sum_{j=1}^{2^m} \lambda_j(t) C_j x(t) \tag{13b}$$

where

$$A_j = A + B_2(D_j K + D_j^- H) \tag{14a}$$

$$C_j = C + D_{12}(D_j K + D_j^- H), \quad j = 1, \dots, 2^m \tag{14b}$$

$$\sum_{j=1}^{2^m} \lambda_j(t) = 1, \quad 0 \leq \lambda_j(t), \quad \forall 0 < t \tag{14c}$$

We denote the polytope by

$$\Omega_\alpha = \sum_{j=1}^{2^m} \lambda_j \Omega_j \quad \text{for all } 0 \leq \lambda_j \leq 1, \quad \sum_{j=1}^{2^m} \lambda_j = 1 \tag{15}$$

where its vertices are described by

$$\Omega_j = [A_j \ C_j], \quad j = 1, \dots, 2^m$$

The problem becomes one of finding \mathcal{S}_c and a corresponding H such that the state of the system

$$\dot{x}(t) = \bar{A}(t)x(t) + A_h x(t - \tau) + B_1 w(t) \tag{16a}$$

$$z(t) = \bar{C}(t)x(t) \tag{16b}$$

is in \mathcal{S}_c for $w(t) \in \mathcal{W}$, with delay τ satisfying (3a) and (3b), $|h_i x| \leq \bar{u}_i$, $i = 1, \dots, m$ and the control requirements are satisfied for an \bar{A} & \bar{C} residing within Ω_x .

3.3. Stabilization

Applying the method of References [17] and [18], the system of (16a) may be represented in the following equivalent descriptor form (for $\phi \in C^1[-h, 0]$):

$$\begin{aligned} \dot{x}(t) &= y(t), & 0 &= -y(t) + (\bar{A} + A_h)x(t) - A_h \int_{t-\tau(t)}^t y(s) ds + B_1 w(t) \\ x(s) &= \phi(s), & y(s) &= \dot{\phi}(s), & s &\in [-h, 0] \end{aligned} \quad (17)$$

Application of the Lyapunov–Krasovskii functional of the form:

$$V(t) = \bar{x}^T(t) E \tilde{P} \bar{x}(t) + V_2 + V_3 \quad (18)$$

where

$$\bar{x}(t) = \text{col}\{x(t), y(t)\} \quad (19a)$$

$$E = \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix} \quad (19b)$$

$$\tilde{P} = \begin{bmatrix} P_1 & 0 \\ P_2 & P_3 \end{bmatrix} \quad (19c)$$

$$P_1 = P_1^T > 0 \quad (19d)$$

and

$$V_2 = \int_{-h}^0 \int_{t+\theta}^t y^T(s) R^{-1} y(s) ds d\theta \quad (19e)$$

$$V_3 = \int_{t-\tau(t)}^t x^T(s) S^{-1} x(s) ds \quad (19f)$$

and where for a positive scalar β , we choose \mathcal{S}_c of Section 3.2 to be an ellipsoid of the form:

$$\mathcal{X}_{P_1, \beta} = \{x(t): x^T(t) P_1 x(t) \leq \beta^{-1}\} \quad (20)$$

We obtain the following result by adopting the method of References [11, 12, 8] and [21]:

Theorem 1

When $w(t) \equiv 0$, the system (9) with the delay τ and its rate $\dot{\tau}$ satisfying (3a) and (3b) is asymptotically stable with \mathcal{X}_δ inside the domain of attraction if, for some positive scalar ε , there

exist $0 < Q_1, S, Q_2, Q_3, Z_1, Z_2, Z_3, R \in \mathcal{R}^{n \times n}, Y, G \in \mathcal{R}^{m \times n}$ and $\beta \in \mathcal{R}^1$ that satisfy the following set of inequalities:

$$\begin{bmatrix} Q_2 + Q_2^T + hZ_1 & \sum_j & hQ_2 & 0 & Q_1 \\ * & -Q_3 - Q_3^T + hZ_3 & hQ_3 & (\varepsilon - 1)A_h S & 0 \\ * & * & -hR & 0 & 0 \\ * & * & * & -(1 - d)S & 0 \\ * & * & * & * & -S \end{bmatrix} < 0, \quad j = 1, \dots, 2^m \quad (21a)$$

$$\begin{bmatrix} R & 0 & \varepsilon R A_h^T \\ * & Z_1 & Z_2 \\ * & * & Z_3 \end{bmatrix} \geq 0 \quad (21b)$$

$$\begin{bmatrix} \beta & g_i \\ * & \bar{u}_i^2 Q_1 \end{bmatrix} \geq 0, \quad i = 1, \dots, m \quad (22)$$

and

$$\delta_1^2 [\bar{\sigma}(Q_1^{-1}) + h\bar{\sigma}(S^{-1})] + \frac{h^2}{2} \delta_2^2 \bar{\sigma}(R^{-1}) \leq \beta^{-1} \quad (23)$$

where

$$\sum_j = Q_3 - Q_2^T + Q_1(A^T + \varepsilon A_h^T) + (Y^T D_j + G^T D_j^-) B_2^T + hZ_2 \quad (24)$$

and where g_i denote the i th row of G . The feedback gain matrix which stabilizes the system is given by

$$K = YQ_1^{-1} \quad (25)$$

Proof

Conditions are sought to ensure that

$$\dot{V} < 0 \quad (26)$$

for any $x(t) \in \mathcal{X}_{P_1, \beta}$ where $\mathcal{X}_{P_1, \beta}$ is defined in (20).

The inequalities (22) guarantee that $|h_i x| \leq \bar{u}_i, \forall x \in \mathcal{X}_{P_1, \beta}, i = 1, \dots, m$. This results from the fact that when $x \in \mathcal{X}_{P_1, \beta}$, the following inequalities

$$2\bar{u}_i \geq \bar{u}_i(1 + \beta x^T P_1 x) \geq 2|h_i x|, \quad i = 1, \dots, m$$

imply that $|h_i x| \leq \bar{u}_i, i = 1, \dots, m$. The latter inequality, which can be written as

$$[1 \pm x^T] \begin{bmatrix} \bar{u}_i & h_i \\ * & \beta \bar{u}_i P_1 \end{bmatrix} \begin{bmatrix} 1 \\ \pm x \end{bmatrix} \geq 0, \quad i = 1, \dots, m$$

is satisfied by (22), where $g_i \triangleq h_i Q_1$, $i = 1, \dots, m$ and $Q_1 \triangleq P_1^{-1}$, and the polytopic system representation of (16) is thus valid. Similar results have been obtained in Reference [9]. In what follows, we shall show that (21)–(22) guarantee that $\dot{V} < 0$ and that the bound (inequality (23)) on the initial condition \mathcal{X}_δ leads to $x(t)$ remaining within the ellipsoid defined by (20).

Differentiating (18) with respect to t and using a similar line of reasoning as in Reference [21], we find that (26) holds if the following inequalities in \tilde{P} and $\tilde{Z} \in \mathcal{R}^{2n \times 2n}$, $\tilde{W} \in \mathcal{R}^{n \times 2n}$ and $R \in \mathcal{R}^{n \times n}$ are feasible:

$$\begin{bmatrix} \Psi & -\tilde{P}^T \begin{bmatrix} 0 \\ A_h \end{bmatrix} - \tilde{W}^T \\ * & -(1-d)S^{-1} \end{bmatrix} < 0 \quad (27a)$$

$$\begin{bmatrix} R^{-1} & \tilde{W} \\ * & \tilde{Z} \end{bmatrix} \geq 0 \quad (27b)$$

where

$$\begin{aligned} \Psi = & \tilde{P}^T \begin{bmatrix} 0 & I \\ A_j & -I \end{bmatrix} + \begin{bmatrix} 0 & A_j^T \\ I & -I \end{bmatrix} \tilde{P} + \begin{bmatrix} S^{-1} & 0 \\ 0 & hR^{-1} \end{bmatrix} + h\tilde{Z} \\ & + \begin{bmatrix} \tilde{W} \\ 0 \end{bmatrix} + [\tilde{W}^T \quad 0] \end{aligned} \quad (27c)$$

From the requirement that $0 < P_1$ (inequality (19d)), and the fact that in (27) $-(P_3 + P_3^T)$ must be negative definite, it follows that \tilde{P} is non-singular. Choosing

$$\tilde{W} = \varepsilon A_h^T [P_2 \quad P_3] \quad (28)$$

we define

$$\tilde{P}^{-1} = Q = \begin{bmatrix} Q_1 & 0 \\ Q_2 & Q_3 \end{bmatrix}, \quad Z = \begin{bmatrix} Z_1 & Z_2 \\ Z_2^T & Z_3 \end{bmatrix} = Q^T \tilde{Z} Q \quad (29a)$$

and

$$\Delta = \text{diag}\{Q, I\} \quad (29b)$$

and then multiply (27a) by Δ^T and Δ , on the left and on the right, respectively. Finally, applying Schur's formula to the quadratic term in Q in the resulting inequality and denoting KQ_1 by Y and HQ_1 by G , (21) is obtained. Similarly, (21b) is obtained by multiplying (27b), from the left and the right, by $\text{diag}\{R, Q^T\}$ and $\text{diag}\{R, Q\}$, respectively, and using (28). Since (22) guarantees the validity of the polytopic system representation of (16), the LMIs (21) and (22) imply that $\dot{V} < 0$.

From $\dot{V} < 0$ it follows that $V(t) < V(0)$ and therefore

$$\begin{aligned} x^T(t) P_1 x(t) & \leq V(t) < V(0) \\ & \leq \max_{\theta \in [-h, 0]} |\phi(\theta)|^2 [\bar{\sigma}(Q_1^{-1}) + h\bar{\sigma}(S^{-1})] + \max_{\theta \in [-h, 0]} |\dot{\phi}(\theta)|^2 \frac{h^2}{2} \bar{\sigma}(R^{-1}) \leq \beta^{-1}. \end{aligned} \quad (30)$$

Inequality (23) then guarantees that for all initial functions $\phi \in \mathcal{X}_\delta$, the trajectories of $x(t)$ remain within $\mathcal{X}_{P_1, \beta}$, and the polytopic system representation (16) is valid. Thus $x(t)$ is a trajectory of the linear system (16) and $\dot{V} < 0$ along the trajectories of the latter system which implies that $\lim_{t \rightarrow \infty} x(t) = 0$. \square

3.4. An ellipsoid bound on the set of the states

The situation when $w \neq 0$ is treated next. Conditions are sought such that the trajectory $x(t)$ of the closed-loop system remains within the ellipsoid $\mathcal{X}_{P, \beta}$ defined by (20), when the initial function ϕ is zero. With Razumikhin’s approach to the stability of time-delay systems (see e.g. Reference [22]), it has been shown in Reference [15] that by defining the function

$$V(\xi) = \xi^T P \xi \tag{31}$$

it is sufficient that, for some positive scalars λ_1 and λ_2

$$\frac{d}{dt} V + \lambda_1(V - \beta^{-1}) + \lambda_2(\bar{w} - w^T w) \leq 0 \tag{32}$$

along trajectories satisfying

$$x^T(t + \theta) P x(t + \theta) \leq x^T(t) P x(t), \quad \forall \theta \in [-h, 0] \tag{33}$$

in order to guarantee that the trajectory $x(t)$ remains within $\mathcal{X}_{P, \beta}$. Requiring that

$$(\lambda_2 + \gamma_3 h) \bar{w} \beta - \lambda_1 \leq 0 \tag{34}$$

for some positive scalar γ_3 , conditions were derived in Reference [15] for satisfying inequality (32) by solving for

$$\frac{d}{dt} V + \lambda_1 V - \lambda_2 w^T w - h \gamma_3 \bar{w} \leq 0 \tag{35}$$

for all $x(t)$ satisfying inequality (33). The following result was obtained in Reference [15] via the first order and the parameterizing model transformations [20]:

Lemma 3 (Fridman and Shaked [15])

Consider the system (16) with a zero initial function ϕ , a given \bar{A} and also $0 < P \in \mathcal{R}^{n \times n}$ and $0 < \beta \in \mathcal{R}^1$. The trajectories of $x(t)$ remain within the ellipsoid $\mathcal{X}_{P, \beta}$ of (20) for all $w(t) \in \mathcal{W}$ and delays τ satisfying (3a), if, for some positive scalars $\gamma_k, k = 0, \dots, 3, \lambda_1$ and λ_2 that satisfy (34), there exists a W in $\mathcal{R}^{n \times n}$ that satisfies the following inequality:

$$\begin{bmatrix} \Psi & PB_1 & PA_h - W & hW\bar{A} & hWA_h & hWB_1 \\ * & -\lambda_2 I & 0 & 0 & 0 & 0 \\ * & * & -\gamma_0 P & 0 & 0 & 0 \\ * & * & * & -h\gamma_1 P & 0 & 0 \\ * & * & * & * & -h\gamma_2 P & 0 \\ * & * & * & * & * & -h\gamma_3 I \end{bmatrix} \leq 0 \tag{36a}$$

where

$$\Psi = W + W^T + P\bar{A} + \bar{A}^T P + (\lambda_1 + \gamma_0 + h\gamma_1 + h\gamma_2)P \tag{36b}$$

Remark 1

It follows from Equation (36b) that the smaller λ_1 becomes, the less restrictive is inequality (36a). One may therefore change inequality (34) into an equality.

The conditions of Lemma 3 have been derived for a given \bar{A} . In order to ensure that $x(t) \in \mathcal{X}_{P,\beta}$ for all \bar{A} in Ω_a and that we exploit the convexity properties of Lemma 2, additional conditions should be added to (36) which will guarantee that, for any $x(t)$ satisfying (20) the system representation (16) is valid. As in inequalities (22), in order to guarantee that $|h_i x| \leq \bar{u}_i, \forall x \in \mathcal{X}_{P,\beta}, i = 1, \dots, m$ we require the following:

$$\begin{bmatrix} \beta & g_i P \\ * & \bar{u}_i^2 P \end{bmatrix} \geq 0, \quad i = 1, \dots, m \tag{37}$$

The latter inequalities, if satisfied, yield a valid polytopic representation of (16). Applying Lemma 2 and using the convexity property of \bar{A} in (36), the following is thus obtained:

Lemma 4

Consider the system (9). For given $0 < P \in \mathcal{R}^{n \times n}, K \in \mathcal{R}^{m \times n}$ and positive scalars β and \bar{w} , the trajectories of $x(t)$ remain within the ellipsoid $\mathcal{X}_{P,\beta}$ of (20) for all $w(t) \in \mathcal{W}, \phi(\theta) \equiv 0$ and delays τ that satisfy (3a) if there exist $W \in \mathcal{R}^{n \times n}, H \in \mathcal{R}^{m \times n}$ and positive scalars $\gamma_k, k = 0, \dots, 3$ and λ that satisfy (37) and the following set of inequalities,

$$\begin{bmatrix} \Psi & PB_1 & PA_h - W & hW(A + B_2(D_j K + D_j^- H)) & hWA_h & hWB_1 \\ * & -\lambda I & 0 & 0 & 0 & 0 \\ * & * & -\gamma_0 P & 0 & 0 & 0 \\ * & * & * & -h\gamma_1 P & 0 & 0 \\ * & * & * & * & -h\gamma_2 P & 0 \\ * & * & * & * & * & -h\gamma_3 I \end{bmatrix} \leq 0, \quad j = 1, \dots, 2^m \tag{38a}$$

where

$$\begin{aligned} \Psi = & W + W^T + P(A + B_2(D_j K + D_j^- H)) + (A + B_2(D_j K + D_j^- H))^T P \\ & + ((\lambda + h\gamma_3)\bar{w}\beta + \gamma_0 + h(\gamma_1 + \gamma_2))P. \end{aligned} \tag{38b}$$

Remark 2

In the statement of Lemma 4 it is assumed that both P and K are given. The inequalities obtained are nevertheless nonlinear due to the products of H by W . These non-linearities can be resolved if W is chosen a priori in the form of $W = PF$, where F is given.

The result of Lemma 4 can be used to determine whether a given state-feedback control law leads to state trajectories $x(t)$ that remain within $\mathcal{X}_{P,\beta}$ for all $w(t) \in \mathcal{W}$. It cannot readily be applied in synthesis where the state-feedback gain matrix that achieves the required boundedness is sought. Therefore, choosing $W = \varepsilon PA_h$ and denoting $Q_1 = P^{-1}$, (38a) can be multiplied on both sides by $\text{diag}\{Q_1, I\}$ and the following is obtained.

Lemma 5

Consider the system (9) with the feedback control law (6). Given positive scalars β and \bar{w} , the trajectories of $x(t)$ remain within the ellipsoid $\mathcal{X}_{P,\beta}$ of (20), for some $0 < P \in \mathcal{R}^{n \times n}$ and for all $w(t) \in \mathcal{W}$, $\phi(\theta) \equiv 0$ and delays τ that satisfy (3a) if there exist $0 < Q_1 \in \mathcal{R}^{n \times n}$, Y and $G \in \mathcal{R}^{m \times n}$ and positive scalars γ_k , $k = 0, \dots, 3$ and λ that satisfy (22) and the following set of inequalities:

$$\begin{bmatrix} \Psi_{1j} & B_1 & (1 - \varepsilon)A_h & \varepsilon h(AQ_1 + B_2(D_j Y + D_j^- G)) & \varepsilon h A_h Q_1 & \varepsilon h B_1 \\ * & -\lambda I & 0 & 0 & 0 & 0 \\ * & * & -\gamma_0 Q_1 & 0 & 0 & 0 \\ * & * & * & -h\gamma_1 Q_1 & 0 & 0 \\ * & * & * & * & -h\gamma_2 Q_1 & 0 \\ * & * & * & * & * & -h\gamma_3 I \end{bmatrix} \leq 0, \quad j = 1, \dots, 2^m \tag{39a}$$

where

$$\begin{aligned} \Psi_{1j} = & (A + \varepsilon A_h)Q_1 + Q_1(A^T + \varepsilon A_h^T) + B_2(D_j Y + D_j^- G) + (Y^T D_j + G^T D_j^-)B_2^T \\ & + ((\lambda + h\gamma_3)\bar{w}\beta + \gamma_0 + h(\gamma_1 + \gamma_2))Q_1 \end{aligned} \tag{39b}$$

The matrix P is then given by $P = Q_1^{-1}$ and the feedback gain matrix which leads to $x(t) \in \mathcal{X}_{P,\beta}$ is given by $K = YQ_1^{-1}$.

3.5. H_∞ control

The H_∞ performance is achieved if

$$\frac{d}{dt} V + z^T(t)z(t) - \gamma^2 w^T(t)w(t) < 0$$

where V is given by (18). Similarly to Theorem 1, we obtain the following.

Lemma 6

The inequality (8) holds for a given $K \in \mathcal{R}^{m \times n}$ if the conditions of Lemma 5 are satisfied and there exist \tilde{P} , of the form of (19), and $\tilde{Z} \in \mathcal{R}^{2n \times 2n}$, $S, R \in \mathcal{R}^{n \times n}$, $\bar{W} \in \mathcal{R}^{n \times 2n}$ and $H \in \mathcal{R}^{m \times n}$ that satisfy the following.

$$\begin{bmatrix} \tilde{\Psi}_j & \tilde{P}^T \begin{bmatrix} 0 \\ B_1 \end{bmatrix} & \tilde{P}^T \begin{bmatrix} 0 \\ A_h \end{bmatrix} - \bar{W}^T & \begin{bmatrix} C_j^T \\ 0 \end{bmatrix} \\ * & -\gamma^2 I & 0 & 0 \\ * & * & -(1 - d)S^{-1} & 0 \\ * & * & * & -1 \end{bmatrix} < 0, \quad j = 1, \dots, 2^m \tag{40a}$$

$$\begin{bmatrix} R^{-1} & \bar{W} \\ * & \tilde{Z} \end{bmatrix} \geq 0 \tag{40b}$$

where

$$\begin{aligned} \tilde{\Psi}_j = \tilde{P}^T & \begin{bmatrix} 0 & I \\ A_j + B_2(D_j K + D_j^- H) & -I \end{bmatrix} + \begin{bmatrix} 0 & A_j^T + (K^T D_j + H^T D_j^-) B_2^T \\ I & -I \end{bmatrix} \tilde{P} \\ & + \begin{bmatrix} S^{-1} & 0 \\ 0 & hR^{-1} \end{bmatrix} + [\tilde{W}^T \quad 0] + \begin{bmatrix} \tilde{W} \\ 0 \end{bmatrix} \end{aligned}$$

In order to obtain a tractable set of inequalities that can be used for both, boundedness of the states and H_∞ performance synthesis purposes, the following is further assumed:

$$\tilde{W} = (\bar{\varepsilon} - 1)[P_2 \ P_3] \tag{41}$$

where $\bar{\varepsilon}$ is a tuning parameter. Next taking Q of (29a), we obtain from Lemma 6 that $J < 0$ if the conditions of Lemma 5 are satisfied and if for some scalar $\bar{\varepsilon}$ there exist $0 < Q_1, S, Q_2, Q_3, R, Z_1, Z_2, Z_3 \in \mathcal{R}^{n \times n}$, Y and $G \in \mathcal{R}^{m \times n}$ that satisfy (22) and the following inequality:

$$\begin{bmatrix} Q_2 + Q_2^T + hZ_1 & \sum_j & Q_1 & hQ_2^T & Q_1 C^T + (Y^T D_j + G^T D_j^-) D_{12}^T & 0 & 0 \\ * & -Q_3 - Q_3^T + hZ_3 & 0 & hQ_3^T & 0 & B_1 & (\bar{\varepsilon} - 1)A_h S \\ * & * & -S & 0 & 0 & 0 & 0 \\ * & * & * & -hR & 0 & 0 & 0 \\ * & * & * & * & -I & 0 & 0 \\ * & * & * & * & * & -\gamma^2 I_q & 0 \\ * & * & * & * & * & * & -(1-d)S \end{bmatrix} < 0 \quad j = 1, \dots, 2^m \tag{42a}$$

$$\begin{bmatrix} R & 0 & \bar{\varepsilon} R A_h^T \\ * & Z_1 & Z_2 \\ * & * & Z_3 \end{bmatrix} \geq 0 \tag{42b}$$

where \sum_j is defined in (24) with ε replaced by $\bar{\varepsilon}$ and where Y and G are the same decision variables that appear in the conditions of Lemma 5. The next result thus follows from Lemma 5 and (42).

Theorem 2

For given positive scalars γ and \bar{w} , there exists a state-feedback gain K that internally stabilizes (9) and leads to (8) for all delays τ that satisfy (3), if for some tuning parameters ε and $\bar{\varepsilon}$ there exist $0 < Q_1, S, Q_2, Q_3, R \in \mathcal{R}^{n \times n}$, $Y \in \mathcal{R}^{m \times n}$ and positive scalars $\beta, \gamma_k, k = 0, 1, \dots, 3$ and λ that satisfy (42), (22) and (39).

If a solution exists, the feedback gain matrix that achieves the required performance is given by (25).

The conditions of Theorem 2 depend on the upper-bound of the delay length h . Corresponding delay-independent conditions that provide a result that is valid for all $0 < h$ and $\tau \leq d$ are readily obtained from Theorem 2 by letting $\varepsilon = \bar{\varepsilon} = 0$, $R = \rho^{-1}I_n$, $Z = 0$ and $\rho, \gamma_1, \gamma_2, \gamma_3 \rightarrow 0$. The following result is obtained.

Corollary 1

For given positive scalars γ and \bar{w} , there exists a state-feedback gain K which for zero initial conditions leads to (8) for all $w(t) \in \mathcal{W}$, independently of the delay length and for $\tau \leq d$, if there exist if for some scalar ε there exist $0 < Q_1, S, Q_2, Q_3, R \in \mathcal{R}^{m \times n}$, Y and $G \in \mathcal{R}^{m \times n}$ and positive scalars β, γ_0 and λ that satisfy (22) and the following set of inequalities:

$$\begin{bmatrix} Q_2 + Q_2^T & \sum_j & 0 & 0 & Q_1 & Q_1 C^T + (Y^T D_j + G^T D_j^-) D_{12}^T \\ * & -Q_3 - Q_3^T & B_1 & A_h S & 0 & 0 \\ * & * & -\gamma^2 I_q & 0 & 0 & 0 \\ * & * & * & -(1-d)S & 0 & 0 \\ * & * & * & * & -S & 0 \\ * & * & * & * & * & -I \end{bmatrix} < 0 \quad (43a)$$

and

$$\begin{bmatrix} A Q_1 + Q_1 A^T + B_2 (D_j Y + D_j^- G) + (Y^T D_j + G^T D_j^-) B_2^T + (\lambda \bar{w} + \gamma_0) Q_1 & B_1 & A_h Q_1 \\ & * & -\lambda I \\ & * & * & -\gamma_0 Q_1 \end{bmatrix} \leq 0, \quad j = 1, \dots, 2^m \quad (43b)$$

$$(\lambda \bar{w} + \gamma_0) \leq 0 \quad (43c)$$

where \sum_j is defined in (24), with ε replaced by zero.

If a solution exists, the feedback gain matrix that achieves the required performance is given by (25).

Remark 3

The results of Theorem 2 assume that except for the time-delay, which satisfies (3), the matrices of the system model of (9) are all known. Treating the scalar parameters in the inequalities of the theorem as tuning parameters, these inequalities become LMIs that are affine in the system's matrices. These LMIs can thus be used to solve the H_∞ control problem in the case when uncertainty is encountered in the matrices of the system. Assuming that this uncertainty is of the polytopic type [16], a solution to the control problem is obtained by solving the inequalities of Theorem 2 at each vertex of the uncertainly polytope.

3.6. Parameter varying solution

In (42a) the same Q_1, Q_2 and Q_3 appear in all of the 2^m inequalities. The solution to all of the inequalities in Theorem 2, if it exists, will lead to a local quadratic stability condition. The

requirement for quadratic stability imposes a serious constraint on the solution which will now be alleviated.

It is readily verified that (42a) is equivalent to

$$\begin{bmatrix} \hat{Q}^T \hat{A}_j^T + \hat{A}_j \hat{Q} + \text{diag}\{hZ, 0\} & Q^T \begin{bmatrix} I & 0 \\ 0 & hI \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ B_1 & (\bar{\varepsilon} - 1)A_h S \end{bmatrix} \\ * & - \begin{bmatrix} S & 0 \\ * & hR \end{bmatrix} & 0 \\ * & * & - \begin{bmatrix} \gamma^2 I_q & 0 \\ * & (1 - d)S \end{bmatrix} \end{bmatrix} \leq 0, \quad j = 1, \dots, 2^m \tag{44a}$$

where

$$\hat{Q} = \text{diag}\{Q, \frac{1}{2}I_p\} \tag{44b}$$

$$\hat{A}_j = \hat{A}_0 + \begin{bmatrix} 0 \\ B_2 \\ D_{12} \end{bmatrix} (D_j K + D_j^- H) [I_n \ 0] \tag{44c}$$

$$\hat{A}_0 = \begin{bmatrix} 0 & I_n & 0 \\ A + \varepsilon A_h & -I_n & 0 \\ C & 0 & -I_p \end{bmatrix} \tag{44d}$$

Similarly, (39a) is equivalent to

$$\begin{bmatrix} \hat{\hat{A}}_j \tilde{Q} + \tilde{Q} \hat{\hat{A}}_j^T & \begin{bmatrix} I_n \\ 0 \end{bmatrix} [B_1 (1 - \varepsilon)A_h \quad \varepsilon h A_h Q_1 \quad \varepsilon h B_1] \\ * & -\text{diag}\{\lambda I_q, \gamma_0 Q_1, h \gamma_2 Q_1, \gamma_3 I_q\} \end{bmatrix} \leq 0, \quad j = 1, \dots, 2^m \tag{45a}$$

where

$$\tilde{Q} = \text{diag}\{Q_1, Q_1\} \tag{45b}$$

$$\hat{\hat{A}}_j = \hat{\hat{A}}_0 + \begin{bmatrix} I \\ 0 \end{bmatrix} B_2 (D_j K + D_j^- H) [I_n \ I_n] \tag{45c}$$

$$\hat{\hat{A}}_0 = \begin{bmatrix} A + \varepsilon A_h + \phi_1 I_n & \varepsilon h A \\ 0 & -\frac{h}{2} \gamma_1 I_n \end{bmatrix} \tag{45d}$$

and

$$\phi_1 = \frac{1}{2}((\lambda + h \gamma_3) \bar{w} \beta + \gamma_0 + h(\gamma_1 + \gamma_2)) \tag{45e}$$

The following is applied to (44) and (45).

Lemma 7

The inequalities (44a) and (45a) are equivalent to the following.

$$\begin{bmatrix} \hat{G}^T \hat{A}_j^T + \hat{A}_j \hat{G} + \text{diag}\{hZ, 0\} & -\hat{Q}^T + \hat{G}^T - \hat{A}_j \hat{H} & Q^T \begin{bmatrix} I & 0 \\ 0 & hI \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ B_1 & (\bar{\varepsilon} - 1)A_h S \end{bmatrix} \\ * & -\hat{H} - \hat{H}^T & 0 & 0 \\ * & * & -\begin{bmatrix} S & 0 \\ * & hR \end{bmatrix} & 0 \\ * & * & * & -\begin{bmatrix} \gamma^2 I_q & 0 \\ * & (1-d)S \end{bmatrix} \end{bmatrix} \leq 0 \quad (46)$$

and

$$\begin{bmatrix} \hat{A}_j \tilde{G} + \tilde{G} \hat{A}_j^T & -\tilde{Q} + \tilde{G} - \hat{A}_j \tilde{H} & \begin{bmatrix} I_n \\ 0 \end{bmatrix} \begin{bmatrix} B_1 & (1-\varepsilon)A_h & \varepsilon h A_h Q_1 & \varepsilon h B_1 \end{bmatrix} \\ * & -\tilde{H} - \tilde{H}^T & 0 \\ * & * & -\text{diag}\{\lambda I_q, \gamma_0 Q_1, h\gamma_2 Q_1, \gamma_3 I_q\} \end{bmatrix} \leq 0, \quad j = 1, \dots, 2^m \quad (47)$$

respectively, where \hat{H} , \tilde{H} , \hat{G} and \tilde{G} are matrices of appropriate dimensions.

Proof

The proof applies arguments similar to those used in Reference [23] for verifying robust stability of systems without delay. Denoting the left hand side of (44a) by \hat{W}_j , then if there exists a solution to $\hat{W}_j < 0$, for a specific $j \in [0, 2^m]$, it is readily found, using Schur's formula, that for $\hat{G} = \hat{Q}$ and $\hat{H} = \rho I_{2n+p}$, where ρ is a positive scalar, (46) holds for any ρ that satisfies

$$\rho [\hat{A}^{(j)T} \ 0 \ 0]^T [\hat{A}^{(j)T} \ 0 \ 0] < -2\hat{W}_j$$

On the other hand, if (46) possesses a solution one can multiply (46) by Γ and Γ^T , on the right and on the left, respectively, where:

$$\Gamma = \begin{bmatrix} I_n & 0 & 0 & 0 \\ -\hat{A}^{(j)T} & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}$$

The resulting LMI is then

$$\begin{bmatrix} \hat{W}_j & \begin{bmatrix} -\hat{Q} + \hat{G}^T + \hat{A}^{(j)}\hat{H}^T \\ 0 \end{bmatrix} \\ * & -\hat{H} - \hat{H}^T \end{bmatrix} < 0 \tag{48}$$

and (44a) clearly follows. Similar arguments are applied to (47) and (45). \square

The inequalities of Lemma 7 are now ready to be applied to the uncertain case. In the case where the matrices of the system are not exactly known, we assume that they belong to $\Omega \in \mathcal{Co}\{\Omega_j, j = 1, \dots, \bar{N}\}$, where,

$$\Omega = \sum_{j=1}^{\bar{N}} f_j \bar{\Omega}_j \quad \text{for some } 0 \leq f_j \leq 1, \quad \sum_{j=1}^{\bar{N}} f_j = 1 \tag{49}$$

where the $\bar{N} (\bar{N} > 2^m)$ vertices of the polytope are described by

$$\Omega_j = [A^{(j)} \quad A_h^{(j)} \quad B_1^{(j)} \quad C^{(j)} \quad D_j]$$

It is assumed, for simplicity, that the matrices B_2 and D_{12} in (9) and (10) are known.

Defining the structures:

$$\hat{G}_j = \begin{bmatrix} G_1 & 0 \\ G_2^{(j)} & G_3^{(j)} \end{bmatrix} \tag{50a}$$

$$\hat{H}_j = \begin{bmatrix} \alpha_1 G_1 & 0 \\ H_2^{(j)} & H_3^{(j)} \end{bmatrix} \tag{50b}$$

$$\tilde{G}_j = \begin{bmatrix} \alpha_2 G_1 & 0 \\ \tilde{G}_2^{(j)} & \tilde{G}_3^{(j)} \end{bmatrix} \tag{50c}$$

$$\tilde{H}_j = \begin{bmatrix} \alpha_3 G_1^{(j)} & 0 \\ \tilde{H}_2^{(j)} & \tilde{H}_3^{(j)} \end{bmatrix} \tag{50d}$$

$$Q_j = \begin{bmatrix} Q_1^{(j)} & 0 \\ Q_2^{(j)} & Q_3^{(j)} \end{bmatrix} \tag{50e}$$

for some positive scalars $\alpha_i, i = 1, 2, 3$, we apply Lemma 7 to Theorem 2 and obtain the following.

Theorem 3

For given positive scalars γ and \bar{w} , there exists a state-feedback gain K that internally stabilizes (9), over the uncertainty polytope Ω and leads to (8) for all delays τ that satisfy (3), if for some positive tuning parameters $\varepsilon, \bar{\varepsilon}$ and $\alpha_i, i = 1, 2, 3$ there exist $Q_j \in \mathcal{R}^{2n \times 2n}$ of the structure (50e), with $0 < Q_1^{(j)}, \hat{G}_j, \hat{H}_j \in \mathcal{R}^{(2n+p) \times (2n+p)}$ of the structure (50a) and (50b), with $0 < G_1, \tilde{G}_j, \tilde{H}_j \in \mathcal{R}^{2n \times 2n}$ of the structure (50c) and (50d), $Z_j \in \mathcal{R}^{(2n+p) \times (2n+p)}, 0 < Q_{1j}, R_j \in \mathcal{R}^{n \times n}, j = 1, \dots, \bar{N}$,

positive scalars $\lambda, \bar{\lambda}, \beta, \gamma_i, i = 0, \dots, 3, S \in \mathcal{R}^{n \times n}$ and $Y, G \in \mathcal{R}^{m \times n}$ that satisfy the following for $j = 1, \dots, \bar{N}$:

$$\begin{bmatrix} \sum_1^{(j)} & -\hat{Q}_j^T + \hat{G}_j^T - \hat{A}_0^{(j)} \hat{H}_j - \alpha_1 \begin{bmatrix} 0 \\ B_2 \\ D_{12} \end{bmatrix} (D_j Y + D_j^- G) [I_n \ 0] & \mathcal{Q}_j^T \begin{bmatrix} I & 0 \\ 0 & hI \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ B_1^{(j)} & (\bar{\epsilon} - 1) A_h^{(j)} S \end{bmatrix} \\ * & -\hat{H}_j - \hat{H}_j & 0 & 0 \\ * & * & -\begin{bmatrix} S & 0 \\ * & hR_j \end{bmatrix} & 0 \\ * & * & * & -\begin{bmatrix} \gamma^2 I_q & 0 \\ * & (1-d)S \end{bmatrix} \end{bmatrix} \leq 0 \tag{51a}$$

$$\begin{aligned} \sum_1^{(j)} &= \hat{G}_j^T \hat{A}_0^{(j)T} + \hat{A}_0^{(j)} \hat{G}_j + \begin{bmatrix} 0 \\ B_2 \\ D_{12} \end{bmatrix} (D_j Y + D_j^- G) [I_n \ 0] + \begin{bmatrix} I_n \\ 0 \end{bmatrix} (Y^T D_j + G^T D_j^-) \begin{bmatrix} 0 \\ B_2 \\ D_{12} \end{bmatrix}^T \\ &+ \text{diag}\{hZ_j, 0\} \end{aligned} \tag{51b}$$

$$\begin{bmatrix} \sum_2^{(j)} & -\text{diag}\{Q_{1j}, Q_{1j}\} + \tilde{G}_j - \hat{A}_0^{(j)} \tilde{H}_j - \alpha_2 \begin{bmatrix} I_n \\ 0 \end{bmatrix} B_2 (D_j Y + D_j^- G) \begin{bmatrix} I_n \\ I_n \end{bmatrix}^T & \begin{bmatrix} I_n \\ 0 \end{bmatrix} \begin{bmatrix} B_1^{(j)} & (1-\epsilon)A_h & \epsilon h A_h Q_{1j} & \epsilon h B_1^{(j)} \end{bmatrix} \\ * & -\tilde{H}_j - \tilde{H}_j^T & 0 \\ * & * & -\text{diag}\{\lambda I_q, \gamma_0 Q_{1j}, h\gamma_2 Q_{1j}, \gamma_3 I_q\} \end{bmatrix} \leq 0 \tag{52a}$$

$$\sum_2^{(j)} = \hat{A}_0^{(j)} \tilde{G}_j + \tilde{G}_j \hat{A}_0^{(j)T} + \begin{bmatrix} I_n \\ 0 \end{bmatrix} B_2 (D_j Y + D_j^- G) [I_n \ I_n] + \begin{bmatrix} I_n \\ I_n \end{bmatrix} (Y^T D_j + G^T D_j^-) B_2^T [I_n \ 0] \tag{52b}$$

$$\begin{bmatrix} \beta & g_i \\ * & \bar{u}_i^2 Q_1^{(j)} \end{bmatrix} \geq 0 \tag{53a}$$

and

$$\begin{bmatrix} Q_1^{(j)} & I \\ * & \bar{\lambda} I \end{bmatrix} \geq 0 \tag{53b}$$

where $\bar{\lambda}$ is minimized and where

$$\hat{A}_0^{(j)} = \begin{bmatrix} 0 & I_n & 0 \\ A^{(j)} + \varepsilon A_h^{(j)} & -I_n & 0 \\ C^{(j)} & 0 & -I_p \end{bmatrix} \quad \text{and} \quad \hat{\lambda}_0^{(j)} = \begin{bmatrix} A^{(j)} + \varepsilon A_h^{(j)} + \phi_1 I_n & \varepsilon h A^{(j)} \\ 0 & -\frac{h}{2} \gamma_1 I_n \end{bmatrix}$$

The feedback gain matrix which achieves the required performance is given by:

$$K = Y(G_1)^{-1} \quad (54)$$

4. EXAMPLES

4.1. Stabilization

We consider the stabilization example of Reference [8]. The system of (9) was considered there where

$$A = \begin{bmatrix} 0.5 & -1 \\ 0.5 & -0.5 \end{bmatrix}, \quad A_d = \begin{bmatrix} 0.6 & 0.4 \\ 0 & -0.5 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and where $\bar{u} = 5$.

In Reference [8] stabilization via state-feedback was obtained for delays that are less than or equal to $h = 0.35$. For $\tau = 0.35$ the maximum radius of the stability ball achieved was 0.968. Applying the theory of Theorem 1 a stabilizing state-feedback controller has been obtained for all delays that are less than or equal to $h = 1.854$. For the latter delay, with $d = 0$, namely constant delay of $\tau = 1.854$ the stabilizing gain was $K = -[25.8809 \ 4.9315]$ with a stability ball radius of $\delta_1 = \delta_2 = 0.091$. This result was obtained for $\varepsilon = 0.89$ and $\beta = 1$. The latter radius increases significantly when h decreases. For, say, $h = 0.35, 1, 1.8$ the corresponding radii were (again $\delta_1 = \delta_2$) 2.852, 1.7442, 0.8032, respectively. The stabilization theory of Theorem 1 results in state trajectories, for $h = 0.35$, which begin on the periphery of the inner circle, never leave the outer ellipse and end up at the origin (see Figure 1).

4.2. H_∞ control

We consider the example that appeared in References [11] and [8]. The system (1) is described by:

$$A = \begin{bmatrix} 1 & 1.5 \\ 0.3 & -2 \end{bmatrix}, \quad A_h = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 10 \\ 1 \end{bmatrix} \quad (55)$$

We assume that $h = 1$ s, $d = 0$ and $u_0 = 15$. In Reference [11], quadratically local stabilization was achieved for all initial conditions in χ_δ (and $\beta = 0$ in their paper) with a $\delta_{\max} < 58.395$.

Application of Theorem 1 (in the present paper), resulted in a value of $\delta_1 = \delta_2 = \delta_{\max} = 79.43$, which was obtained for $\varepsilon = 1$ and $\beta = 1$. The corresponding state-feedback gain and H were:

$$K = [-7.913 \ 0.7323] \quad \text{and} \quad H = -[0.1534 \ 0.0164]$$

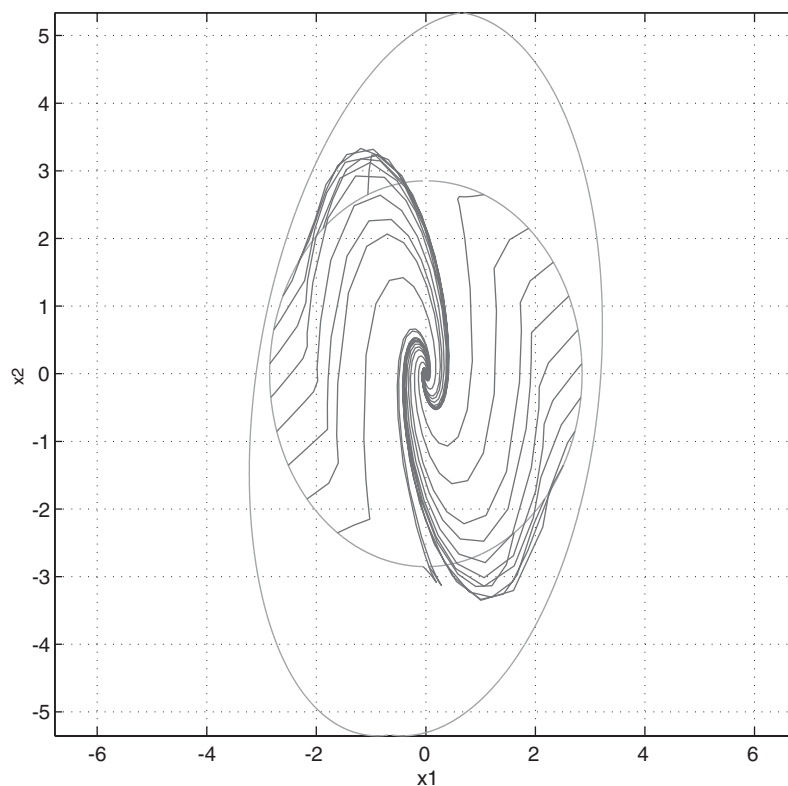


Figure 1. Stabilization result: delay = 0.35 s, inner circle radius = $\delta_1 = \delta_2 = 2.852$, outer ellipse $x^T P_1 x \leq \beta^{-1} = 1$.

At this juncture we note the following points:

- The above value of H implies that the input to the actuators may exceed \bar{u} during operation.
- As should be expected, the result is insensitive to the value of β . The task of the latter parameter is to scale the elements in P .
- The result we obtained for δ_{\max} is better than the one obtained by Tarbouriech *et al.* [11] and Cao *et al.* [8] for initial functions with small enough derivatives. Taking for example $\delta_2 = 0$, a maximum value of $\delta_1 = 97.19$ is achieved. The improvement in δ_{\max} is partially due to the delay-dependent criterion used.
- The strength of the descriptor approach lies in its delay-dependent conditions. The theory developed in Reference [11] is, however, delay-independent. Applying our delay-independent version of Theorem 1 (letting $\varepsilon = 0$ in (21)), values of $\delta_1 = \delta_2 = 63.79$ were achieved along with a gain vector of

$$K = -10^9 [5.223 \ 2.596]$$

Even this delay-independent result compares favourably to the one which appeared in Reference [11].

- The above numerical results were obtained for $d = 0$, namely for the situation where the delay τ can reside anywhere in $[0 \ h]$ but is time invariant. The fact that our solution was obtained for $\varepsilon = 1$ implies, however, that it holds true for any time varying delay which satisfies (3a) [21].

The issue of H_∞ performance was addressed for a $\bar{w} = 0.1$, and respective B_1 C and D matrices of the form:

$$B_1 = [0 \ 3]^T, \quad C = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad D_{12} = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}$$

With the use of Theorem 2 we obtained a minimum value of $\gamma = 0.0714$ for $\beta = 0.5$, $\varepsilon = 0$, $\bar{\varepsilon} = 0.2$, $\gamma_0 = \gamma_1 = 0.84$, $\gamma_2 = 0.28$, $\gamma_3 = 0.0014$, and $\lambda = 0.1$. The resulting state-feedback gain was:

$$K = -[2.0743 \ 0.1266] \quad (56)$$

and the corresponding eigenvalues of βP_1 were 0.0006 and 0.0209.

The following points should be noted.

- With β as a tuning parameter, the minimum achievable γ became a monotonously increasing function of β . For instance, with $\beta = 0.01$, a minimum value of $\gamma = 0.2687$ resulted. A clear trade-off between the minimum value of γ and the size of $\mathcal{X}_{P_1, \beta}$ is observed. In the latter case of $\beta = 0.01$ the eigenvalues of βP_1 are 0.0002 and 0.0021.
- The state-feedback gain of (56) was applied to the system (1) with the parameters given in (55). A frequency sweep of bounded sine waves was used for $w(t)$ in the Simulink software package [24]. The values of the ratio between the resulting $\|z\|_2^2$ and $\|w\|_2^2$ were recorded and the maximum ratio was 0.058^2 . The frequency sweep calculations of the ratio $\|z\|_2^2/\|w\|_2^2$ (H_∞ results) are depicted in Figure 2. The latter ratio is clearly less than the γ^2 achieved (0.0714^2). The difference between the two numbers reflects the overdesign that is entailed in applying the descriptor model transformation.
- The inequalities of Theorems 2 and 3 are multilinear in the decision variables. In Theorem 2 bilinearity is achieved if β is prescribed. In order to apply LMI based solution methods, the scalar decision variables should be considered as tuning variables. The tuning of a variable may be performed manually, especially when the cost function is a convex function of this variable. When the number of tuning variables is greater than one, the required inequalities may be solved iteratively or by a relaxation method, that is, for some initially prescribed values of the tuning variables, the remaining decision variables are obtained by solving the given set of LMIs. These latter decision variables are then inserted into the set of LMIs which are once again solved for the tuning variables. This procedure is repeated such that, at each iteration step, new values of the tuning parameters are found which minimize a cost function, thus guaranteeing convergence to a local minimum, for instance, minimizing γ in Theorem 2. In Theorem 2, the scalar β was used as a single tuning parameter. The results for the above example were achieved by applying this latter relaxation method.

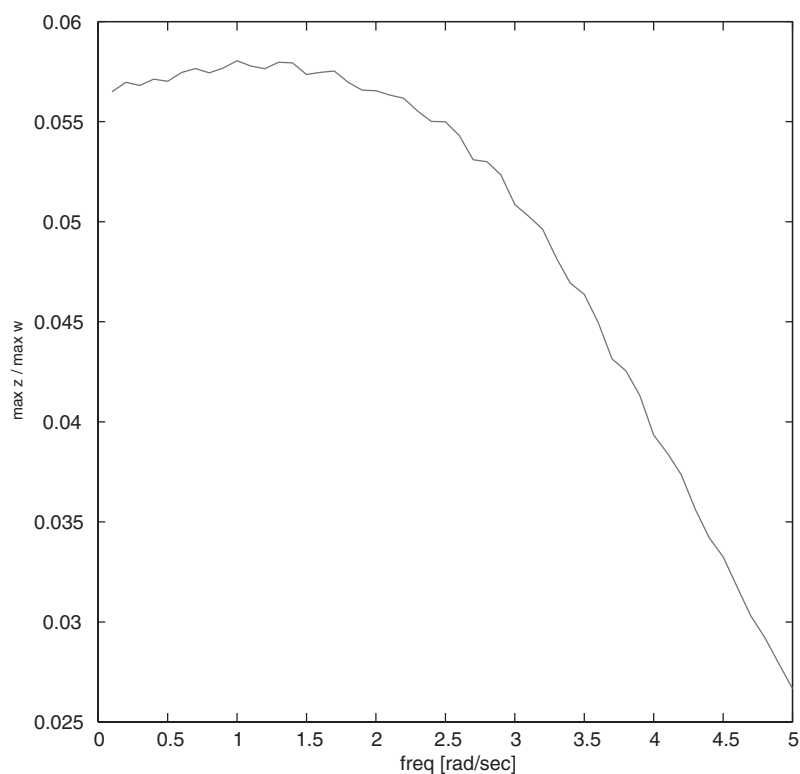


Figure 2. Deterministic frequency sweep from 0 to 5 rad/s with 20 random phase values at each frequency.

5. CONCLUSIONS

In the present paper, a delay-dependent LMI based sufficient condition has been proposed for stabilization and H_∞ control of linear systems with time varying delay. This feat was accomplished by combining the transformation of a single linear system with m possibly saturated actuator channels into a set of 2^m linear systems embedded within a convex polytope with the Lyapunov-Krasovskii technique via descriptor model transformation. The merits of the descriptor model approach lie in the fact that a smaller number of cross terms require bounding, thus resulting in a reduction of the over-design. The boundedness of the trajectories for systems with bounded peak inputs has been treated by Razumikhin approach via first order model transformation.

In both the designs for stabilization and performance satisfaction, a serious source of over-design arises from the quadratic stabilizability inherent in the design procedure for polytopic systems. In order to alleviate this problem, a method wherein a different Lyapunov candidate function is assigned to each vertex of the polytope, was used, thus resulting in a further reduction of the conservativeness of the design. For the express purpose of comparing delay-dependent and delay-independent designs, a tuning parameter ε was

introduced which facilitates the switch between delay-dependent ($\varepsilon \neq 0$) and delay-independent designs ($\varepsilon = 0$).

One of the drawbacks of the proposed method is that the domain of attraction depends on the derivative of the initial function. The proposed method for regional stabilization may thus be useful in the case of neutral type systems where it is natural to consider initial functions from the space C^1 .

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