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Brief paper

Stability of homogeneous systems with distributed delay and time-varying perturbations[☆]

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ABSTRACT

For a class of nonlinear systems with homogeneous right-hand sides of non-zero degree and distributed delays, the problem of stability robustness of the zero solution with respect to time-varying perturbations multiplied by a nonlinear functional gain is studied. It is assumed that the disturbance-free and delay-free system (that results after substitution of non-delayed state for the delayed one) is globally asymptotically stable. First, it is demonstrated that in the disturbance-free case the zero solution is either locally asymptotically stable or practically globally asymptotically stable, depending on the homogeneity degree of the delay-free counterpart. Second, using averaging tools several variants of the time-varying perturbations are considered and the respective conditions are derived evaluating the stability margins in the system. The results are obtained by a careful choice and comparison of Lyapunov–Krasovskii and Lyapunov–Razumikhin approaches. Finally, the obtained theoretical findings are illustrated on two mechanical systems.

1. Introduction

The problem of stability analysis for nonlinear time-delay systems influenced by time-varying perturbations is rather complex (Fridman, 2014; Gu et al., 2003; Hale, 1977; Kolmanovskii & Myshkis, 1999; Richard, 2003). Despite that, nowadays it is gaining a practical importance with raising the internet of things and cyber–physical systems technologies (Hetel et al., 2017; Karafyllis et al., 2016). There are two main methods for stability analysis of time-delay systems: Lyapunov–Krasovskii (LK) and Lyapunov–Razumikhin (LR) approaches (Fridman, 2014). The former uses LK functionals (LKFs) and it has been proven to give the necessary and sufficient conditions of stability (Efimov & Fridman, 2020; Pepe et al., 2017), while the latter is based on usual Lyapunov function analysis (under additional restrictions), and it provides only sufficient conditions of stability, which however may be easier to check in applications. Both methods have their own extensions to the input-to-state stability (ISS) verification (Fridman et al., 2008; Teel, 1998).

Distributed delays can be induced by communication network or realization of the control/estimation algorithms (Anan'evskii & Kolmanovskii, 1989; Feng & Lam, 2012; Feng et al., 2020), and their stability analysis requires special extensions of the previously mentioned methods (Solomon & Fridman, 2013; Xie et al., 2001). The investigation becomes even more complex if there are external perturbations and we would like to evaluate the shape of admissible upper bound of the disturbance in terms of the delayed state (*i.e.*, evaluate the asymptotic gains in terms of ISS). The obtained bounds can be less conservative if the features of the time-varying perturbations are taken into account. For example, if the perturbations are periodic or almost periodic, then the averaging method has been proven to be very efficient (Bogoliubov & Mitropolsky, 1961; Khapaev, 1993).

In this work we will focus on homogeneous models, which in the delay-free case have a lot of useful properties that helped them to gain popularity (Efimov & Polyakov, 2021; Qian et al., 2015), and there are also their extensions to infinite-dimensional systems (Efimov et al., 2014, 2016). It has been shown that for systems with discrete time-delays, if the delay-free counterparts

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(the systems with zero delay) are globally asymptotically stable (AS) at the origin, then for any delay value the original dynamics are locally AS at the origin for positive homogeneity degree (Aleksandrov & Zhabko, 2012, 2014; Efimov et al., 2014) or practically globally AS (we will call later this property uniform ultimate boundedness (UUB) of the solutions) for negative degree (Zimenko et al., 2017). In above works, the LR method is used, and the LK approach has been developed to this case recently in Aleksandrov et al. (2015) and Efimov and Aleksandrov (2021).

In this work, these results will be extended to the case of distributed delays, and next robust stability conditions will be developed evaluating the admissible shapes of perturbations without, and next with, the averaging approach.

The outline of this work is as follows. Preliminaries are given in Section 2. The considered analysis problem is described in Section 3. The general stability and robustness results are formulated in Sections 4 and 5. Using the averaging approach, for particular classes of perturbations and the systems, the respective stability results are presented in Section 6 (for any sign of homogeneity degree) and 7 (for the positive degree of homogeneity). The illustrative examples are shown in Section 8.

2. Preliminaries

The real numbers are denoted by $\mathbb{R}$, and $|s|$ is an absolute value for $s \in \mathbb{R}$. Euclidean norm for a vector $x \in \mathbb{R}^n$ is defined as $\|x\|$. We denote by $C([a,b],\mathbb{R}^n)$, $-\infty < a < b < +\infty$ the Banach space of continuous functions $\phi : [a,b] \to \mathbb{R}^n$ with the uniform norm $\|\phi\|_C = \sup_{a\le\varsigma\le b}\|\phi(\varsigma)\|$.

A sequence of integers, $m, m+1, \ldots, n$ for $m < n$ is further denoted by $\overline{m,n}$.

A continuous function $\sigma : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to class $\mathcal{K}$ if it is strictly increasing and $\sigma(0) = 0$; it belongs to class $\mathcal{K}_\infty$ if it is also radially unbounded. A continuous function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ belongs to class $\mathcal{KL}$ if $\beta(\cdot, r) \in \mathcal{K}$ and $\beta(r,\cdot)$ is a strictly decreasing to zero for any fixed $r \in \mathbb{R}_+$.

The standard definitions of stability and related properties for time-delay systems can be found in Fridman (2014), Hale (1977) and Kolmanovskii and Myshkis (1999), and for delay-free dynamics in Khalil (2002).

2.1. Useful inequalities

The *Young's inequality* claims that for any $\mathfrak{a}, \mathfrak{b} \in \mathbb{R}_+$ and $\gamma > 0$, $\delta > 0$ (Bacciotti & Rosier, 2001):

$$\mathfrak{a}^\gamma \mathfrak{b}^\delta \le \frac{1}{p}\mathfrak{a}^{\gamma p} + \frac{p-1}{p}\mathfrak{b}^{\frac{\delta p}{p-1}}$$

for any $p > 1$, while *Hölder's inequality* for any $f, g : I \to \mathbb{R}$ with $I \subset \mathbb{R}$ ensures that (Bacciotti & Rosier, 2001):

$$\int_I |f(s)g(s)|ds \le \left(\int_I |f(s)|^p ds\right)^{\frac{1}{p}} \left(\int_I |g(s)|^{\frac{p}{p-1}} ds\right)^{\frac{p-1}{p}}$$

for any $p > 1$. Using the properties of homogeneous functions the following result can be obtained (Efimov & Aleksandrov, 2021):

Lemma 1. *Let $\mathfrak{a}, \mathfrak{b} \in \mathbb{R}_+$ and $\ell > 0$, $\alpha > 0$, $\beta > 0$, $\gamma > 0$, $\delta > 0$ be given, then $\mathfrak{a}^\alpha + \mathfrak{b}^\beta - \ell\mathfrak{a}^\gamma\mathfrak{b}^\delta \ge 0$ provided that*

(1) $\max\{\mathfrak{a}^\alpha, \mathfrak{b}^\beta\} \ge \ell^{\frac{1}{1-\frac{\gamma}{\alpha}-\frac{\delta}{\beta}}}$ *and* $\frac{\gamma}{\alpha} + \frac{\delta}{\beta} < 1$,

(2) $\max\{\mathfrak{a}^\alpha, \mathfrak{b}^\beta\} \le \ell^{\frac{1}{1-\frac{\gamma}{\alpha}-\frac{\delta}{\beta}}}$ *and* $\frac{\gamma}{\alpha} + \frac{\delta}{\beta} > 1$.

2.2. Homogeneity

For any $r_i > 0$, $i = \overline{1,n}$ and $\lambda > 0$, define the *vector of weights* $\mathbf{r} = [r_1, \ldots, r_n]$ and the *dilation matrix* $\Lambda_r(\lambda) = \mathrm{diag}\{\lambda^{r_i}\}_{i=1}^n$; $r_{\min} = \min_{i=\overline{1,n}} r_i$ and $r_{\max} = \max_{i=\overline{1,n}} r_i$.

Definition 1 (*Efimov & Polyakov, 2021; Zubov, 1958*)**.** The function $h : \mathbb{R}^n \to \mathbb{R}$ is called $\mathbf{r}$-*homogeneous*, if for any $x \in \mathbb{R}^n$ the relation

$$h(\Lambda_r(\lambda)x) = \lambda^\nu h(x)$$

holds for some $\nu \in \mathbb{R}$ and all $\lambda > 0$.

The vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ is called $\mathbf{r}$-*homogeneous*, if for any $x \in \mathbb{R}^n$ the relation

$$f(\Lambda_r(\lambda)x) = \lambda^\nu \Lambda_r(\lambda) f(x)$$

holds for some $\nu \ge -r_{\min}$ and all $\lambda > 0$.

In both cases, the constant ν is called the *degree* of homogeneity.

For any $x \in \mathbb{R}^n$ and $\varpi \ge r_{\max}$, a *homogeneous norm* can be defined as follows

$$\|x\|_r = \left(\sum_{i=1}^n |x_i|^{\varpi/r_i}\right)^{1/\varpi}.$$

For all $x \in \mathbb{R}^n$, its Euclidean norm $\|x\|$ is related with the homogeneous one:

$$\underline{\sigma}_r(\|x\|_r) \le \|x\| \le \bar{\sigma}_r(\|x\|_r)$$

for some $\underline{\sigma}_r, \bar{\sigma}_r \in \mathcal{K}_\infty$ (Efimov et al., 2018). In the following, due to this "equivalence", stability analysis with respect to the norm $\|x\|$ can be substituted with analysis for the norm $\|x\|_r$. The homogeneous norm has an important property: it is $\mathbf{r}$-homogeneous of degree 1, that is $\|\Lambda_r(\lambda)x\|_r = \lambda\|x\|_r$ for all $x \in \mathbb{R}^n$ and $\lambda > 0$.

3. Statement of the problem

Consider a nonlinear time-delay system without disturbances

$$\dot{x}(t) = F(x(t)) + \int_{t-\tau}^{t} G(x(s))ds, \tag{1}$$

where $x(t) \in \mathbb{R}^n$, vector fields $F(x)$ and $G(x)$ are continuous for $x \in \mathbb{R}^n$ and $\mathbf{r}$-homogeneous of the degree ν with respect to weights $\mathbf{r} = (r_1, \ldots, r_n)$, $\nu + r_{\min} > 0$; $\tau = \text{const} > 0$ is the delay; the initial functions for (1) belong to the space $C([-\tau,0],\mathbb{R}^n)$. Denote by x_t the restriction of a solution $x(t)$ to the segment $[t-\tau, t]$, *i.e.*, $x_t : \xi \mapsto x(t+\xi)$, $\xi \in [-\tau, 0]$. The system (1) admits the zero solution.

Our objective is to derive the conditions of *asymptotic stability* for the zero solution (there exists $\beta \in \mathcal{KL}$ such that $\|x(t)\| \le \beta(\|x_0\|_C, t)$ for all $t \ge 0$ and $\|x_0\|_C < \bar{\rho}$ for some $\bar{\rho} > 0$), or we will look for *UUB* conditions in (1) (there exist $\beta \in \mathcal{KL}$ and $\tilde{\rho} > 0$ such that $\|x(t)\| \le \beta(\|x_0\|_C, t) + \tilde{\rho}$ for all $t \ge 0$ and $x_0 \in C([-\tau,0],\mathbb{R}^n)$), depending on the sign ν of the homogeneity degree of F and G (see Section 4). Further, we will study, first, the impact on the dynamics of (1) of general non-stationary perturbations (in Section 5). Next, disturbances admitting averaging (*e.g.*, periodic or almost periodic) with a nonlinear gain containing distributed delay (for all signs of homogeneity degree in Section 6 and an improvement for positive degree in Section 7).

The results of Sections 6 and 7 in this paper are obtained with the aid of a special approach extending the averaging method (see Fridman and Zhang (2020) and Zhang and Fridman (2022)). Using this tool the conditions are derived under which the perturbations do not destroy the asymptotic stability or ultimate boundedness in (1).

4. Basic stability and UUB conditions

The main hypothesis for us is the asymptotic stability of the origin for the delay-free counterpart of (1), which is obtained after substitution of $x(t)$ in place of $x(s)$:

Assumption 1. The zero solution of the auxiliary delay-free system

$$\dot{x}(t) = F(x(t)) + \tau G(x(t)), \quad x(t) \in \mathbb{R}^n, \ t \geq 0,$$

is AS.

Remark 1. In the case of linear system (1) with $F(x) = Ax$ and $G(x) = A_d x$, where A and $A_d \in \mathbb{R}^{n\times n}$, Assumption 1 means that $A + \tau A_d$ is Hurwitz, which guarantees the stability of the system provided that an additional condition holds (see Section 3.8 in Fridman (2014)). Surprisingly, for homogeneous systems with $\nu \neq 0$, Assumption 1 appears to be sufficient for stability of (1) (this will be demonstrated in Theorem 1). Avoiding additional hypotheses, for brevity of exposition in the present work only the case $\nu \neq 0$ is studied.

Remark 2. It is well known (see Rosier (1992) and Zubov (1958)) that, under Assumption 1, the delay-free system admits a Lyapunov function $V(x)$ with the following properties:
(i) $V(x)$ is twice continuously differentiable for $x \in \mathbb{R}^n$;
(ii) $V(x)$ is positive definite;
(iii) $V(x)$ is $\mathbf{r}$-homogeneous of the degree $\mu > 2r_{\max}$;
(iv) the function $(\partial V(x)/\partial x)^\top (F(x) + \tau G(x))$ is negative definite.

In Aleksandrov et al. (2015) and Efimov and Aleksandrov (2021), an approach to the so-called complete-type LKFs construction for homogeneous systems with discrete delays was developed. In the present contribution, we will extend this approach to homogeneous systems with distributed delays.

Theorem 1. *Let Assumption 1 be fulfilled,*
(i) if $\nu > 0$, then the zero solution of (1) is AS;
(ii) if $\nu < 0$, then solutions of (1) are UUB.

Proof. Choose a LKF for (1) as follows:

$$\widetilde{V}(x_t) = V(x) + \left(\frac{\partial V(x)}{\partial x}\right)^T \int_{t-\tau}^{t} (s + \tau - t) G(x(s)) ds$$
$$+ \int_{t-\tau}^{t} (\alpha + \beta(s + \tau - t)) |x(s)|_r^{\mu+\nu} ds, \tag{2}$$

where α, β are positive parameters and $V(x)$ is a Lyapunov function satisfying the conditions specified in Remark 2.

Using properties of homogeneous functions (see Efimov and Polyakov (2021)), we arrive at the estimates

$$c_1|x(t)|_r^\mu - c_3 \sum_{i=1}^{n} |x(t)|_r^{\mu-r_i} \int_{t-\tau}^{t} |x(s)|_r^{\nu+r_i} ds$$
$$+\alpha \int_{t-\tau}^{t} |x(s)|_r^{\mu+\nu} ds \leq \widetilde{V}(x_t) \leq c_2|x(t)|_r^\mu$$
$$+c_3 \sum_{i=1}^{n} |x(t)|_r^{\mu-r_i} \int_{t-\tau}^{t} |x(s)|_r^{\nu+r_i} ds + (\alpha + \beta\tau) \int_{t-\tau}^{t} |x(s)|_r^{\mu+\nu} ds,$$

where c_1, c_2, c_3 are positive constants.

Considering the derivative of the functional (2) along the solutions of (1) we obtain:

$$\dot{\widetilde{V}} \leq -(c_4\alpha - \beta\tau)|x(t)|_r^{\mu+\nu} - \alpha|x(t-\tau)|_r^{\mu+\nu}$$
$$+ c_5 \sum_{i,j=1}^{n} |x(t)|_r^{\mu-r_i-r_j} \left(|x(t)|_r^{\nu+r_j} + \int_{t-\tau}^{t} |x(s)|_r^{\nu+r_j} ds\right)$$
$$\times \int_{t-\tau}^{t} |x(s)|_r^{\nu+r_i} ds - \beta \int_{t-\tau}^{t} |x(s)|_r^{\mu+\nu} ds,$$

where $c_4 > 0$, $c_5 > 0$.

Take $\alpha + \beta\tau < \frac{c_4}{4}$, then with the aid of Young's and Hölder's inequalities and Lemma 1, it is straightforward to prove the existence of positive numbers Δ_1, Δ_2 such that
(i) if $\nu > 0$, then

$$\frac{1}{2}c_1|x(t)|_r^\mu + \frac{1}{2}\alpha \int_{t-\tau}^{t} |x(s)|_r^{\mu+\nu} ds \leq \widetilde{V}(x_t)$$
$$\leq 2c_2|x(t)|_r^\mu + 2(\alpha + \beta\tau) \int_{t-\tau}^{t} |x(s)|_r^{\mu+\nu} ds,$$
$$\dot{\widetilde{V}} \leq -\frac{1}{2}c_4|x(t)|_r^{\mu+\nu} - \frac{1}{2}\beta \int_{t-\tau}^{t} |x(s)|_r^{\mu+\nu} ds$$

for $|x(t)|_r^\mu + \int_{t-\tau}^{t} |x(s)|_r^{\mu+\nu} ds < \Delta_1$;
(ii) if $\nu < 0$, then these inequalities hold for $|x(t)|_r^\mu + \int_{t-\tau}^{t} |x(s)|_r^{\mu+\nu} ds > \Delta_2$.

Hence (see Efimov and Aleksandrov (2021)), one can choose positive constants $h_1, h_2, \widetilde{\Delta}_1, \widetilde{\Delta}_2$ such that

$$\dot{\widetilde{V}} \leq -h_1 \widetilde{V}^{1+\frac{\nu}{\mu}}$$

for $\nu > 0$ in the domain where $\widetilde{V} < \widetilde{\Delta}_1$, and

$$\dot{\widetilde{V}} \leq -h_2 \widetilde{V}^{1+\frac{\nu}{\mu}}$$

for $\nu < 0$ in the domain $\widetilde{V} > \widetilde{\Delta}_2$. This completes the proof.

Note that the results of this work deal with non-zero homogeneity degree ν, which means that the linear systems (that are homogeneous with zero degree $\nu = 0$ (Efimov & Polyakov, 2021) and mostly studied in the literature (Fridman, 2014)) or nonlinear homogeneous systems of zero degree (Yu & Lin, 2022; Zhao & Lin, 2022) are not considered.

5. Stability of the perturbed system with distributed delay

Next, consider a perturbed system

$$\dot{x}(t) = F(x(t)) + \int_{t-\tau}^{t} G(x(s)) ds + \int_{t-\tau}^{t} R(s, x(s)) ds, \tag{3}$$

where vector function $R(t, x)$ is continuous for $t + \tau \geq 0$ and $x \in \mathbb{R}^n$. Here we do not introduce any additional constraint on the form of dependence of the perturbation R in the time argument t (this will be done in the next sections).

Assumption 2. The inequalities

$$|R_i(t, x)| \leq a|x|_r^{r_i+\varrho}, \quad i = \overline{1, n},$$

are valid for $t + \tau \geq 0$, $x \in \mathbb{R}^n$, where $a > 0$, $\varrho + r_{\min} > 0$ and $R_i(t, x)$ are components of the vector function $R(t, x)$.

This kind of upper estimates for R can be satisfied if, for example, for each fixed value of t the vector field R is $\mathbf{r}$-homogeneous of the degree ϱ.

Theorem 2. *Let Assumptions 1 and 2 be fulfilled, then*
(i) for $\nu > 0$, $\varrho > \nu$ the zero solution of (3) is AS;
(ii) for $\nu < 0$, $\varrho < \nu$ solutions of (3) are UUB.

Proof. Constructing a LKF candidate in the form (2) and considering its derivative along the solutions of the perturbed system, we obtain:

$$\begin{aligned}\dot{\tilde V} &= \Xi(x_t) + \left(\frac{\partial V(x(t))}{\partial x}\right)^{\top}\int_{t-\tau}^{t} R(s,x(s))ds\\
&+ \sum_{i,j=1}^{n}\frac{\partial^2 V(x(t))}{\partial x_i\partial x_j}\int_{t-\tau}^{t} R_j(s,x(s))ds\int_{t-\tau}^{t}(s+\tau-t)G_i(x(s))ds\\
&\le -p_1|x(t)|_r^{\mu+\nu} + p_2\sum_{i=1}^{n}|x(t)|_r^{\mu-r_i}\int_{t-\tau}^{t}|x(s)|_r^{\varrho+r_i}ds\\
&+ p_3\sum_{i,j=1}^{n}|x(t)|_r^{\mu-r_i-r_j}\left(|x(t)|_r^{\nu+r_j}+\int_{t-\tau}^{t}|x(s)|_r^{\nu+r_j}ds\right.\\
&\left.+\int_{t-\tau}^{t}|x(s)|_r^{\varrho+r_j}ds\right)\int_{t-\tau}^{t}|x(s)|_r^{\nu+r_i}ds\\
&+(\alpha+\beta\tau)|x(t)|_r^{\mu+\nu} - \alpha|x(t-\tau)|_r^{\mu+\nu} - \beta\int_{t-\tau}^{t}|x(s)|_r^{\mu+\nu}ds,\end{aligned}$$

where $\Xi(x_t)$ is the derivative of (2) along the solutions of (1) and p_1, p_2, p_3 are positive constants. The subsequent proof is similar to that of Theorem 1.

Remark 3. Theorems 1–2 can be proven by applying the LR approach, as well.

In the next section we will consider systems with non-stationary perturbations of a special form.

6. Stability and UUB analysis via averaging

In many cases the time dependence of perturbation has periodic or almost periodic nature, which can be effectively used in the analysis improving the results of Theorem 2.

6.1. Application of the Lyapunov–Krasovskii approach

Consider a perturbed system of the form

$$\dot x(t) = F(x(t)) + \int_{t-\tau}^{t} G(x(s))ds + B\left(\frac{t}{\varepsilon}\right)\int_{t-\tau}^{t} Q(x(s))ds, \qquad (4)$$

where ε is a positive parameter, the matrix $B(t)$ is continuous for $t\in(-\infty,+\infty)$, vector function $Q(x)$ is continuous for $x\in\mathbb{R}^n$.

Assumption 3. Let $\|B(t)\|\le M$ for $t\in(-\infty,+\infty)$, $M=\mathrm{const}>0$.

Remark 4. Under Assumption 3, there exists a number $\delta\in[0,M]$ such that

$$\frac{1}{\varepsilon}\left\|\int_{t-\varepsilon}^{t} B\left(\frac{\theta}{\varepsilon}\right)d\theta\right\| \le \delta \qquad (5)$$

for $t\ge 0$, $\varepsilon>0$. For instance, $\delta=0$ if all elements of $B(\cdot)$ are zero-mean 1-periodic signals.

Remark 5. Under Assumption 3, the estimate

$$\frac{1}{\varepsilon}\left\|\int_{t-\varepsilon}^{t}(\theta+\varepsilon-t)B\left(\frac{\theta}{\varepsilon}\right)d\theta\right\| \le \frac{1}{2}\varepsilon M$$

holds for $t\ge 0$, $\varepsilon>0$.

Assumption 4. For any fixed $t\in(-\infty,+\infty)$, $B(t)Q(x)$ is $\mathbf{r}$-homogeneous function of x of the degree $\sigma > -r_{\min}$.

Under this hypothesis, Assumption 2 is satisfied for $R(t,x)=B(t)Q(x)$ with $\varrho=\sigma$.

Theorem 3. *Let Assumptions 1, 3, 4 be fulfilled. Then one can choose positive numbers ε_0 and δ_0 such that if $0<\varepsilon<\varepsilon_0$ and constant δ in the estimate (5) satisfies the condition $0\le\delta<\delta_0$, then*
(i) for $\nu>0$, $\sigma\ge\nu$ the zero solution of (4) is AS;
(ii) for $\nu<0$, $\sigma\le\nu$ solutions of (4) are UUB.

Proof. Applying the approach proposed in Fridman and Zhang (2020), choose a LKF for (4) in the form

$$\begin{aligned}\widetilde W(t,x_t) &= \widetilde V(x_t) - \frac{1}{\varepsilon}\left(\frac{\partial V(x(t))}{\partial x}\right)^{\top}\\
&\times\int_{t-\varepsilon}^{t}(\theta+\varepsilon-t)B\left(\frac{\theta}{\varepsilon}\right)d\theta\int_{t-\tau}^{t} Q(x(s))ds,\end{aligned}$$

where $V(x)$ is a Lyapunov function satisfying the conditions specified in Remark 2 and $\widetilde V(x_t)$ is the functional (2) constructed in the proof of Theorem 1. Then

$$\begin{aligned}&c_1|x(t)|_r^{\mu} - c_3\sum_{i=1}^{n}|x(t)|_r^{\mu-r_i}\int_{t-\tau}^{t}|x(s)|_r^{\nu+r_i}ds\\
&+\alpha\int_{t-\tau}^{t}|x(s)|_r^{\mu+\nu}ds - \varepsilon\hat c\sum_{i=1}^{n}|x(t)|_r^{\mu-r_i}\int_{t-\tau}^{t}|x(s)|_r^{\sigma+r_i}ds\\
&\le \widetilde W(t,x_t) \le c_2|x(t)|_r^{\mu} + c_3\sum_{i=1}^{n}|x(t)|_r^{\mu-r_i}\int_{t-\tau}^{t}|x(s)|_r^{\nu+r_i}ds\\
&+(\alpha+\beta\tau)\int_{t-\tau}^{t}|x(s)|_r^{\mu+\nu}ds\\
&+\varepsilon\hat c\sum_{i=1}^{n}|x(t)|_r^{\mu-r_i}\int_{t-\tau}^{t}|x(s)|_r^{\sigma+r_i}ds,\end{aligned}$$

where $c_1, c_2, c_3, \hat c$ are positive constants.

Differentiating $\widetilde W(t,x_t)$ along the solutions of (4), we obtain

$$\begin{aligned}\dot{\widetilde W} &\le -(c_4-\alpha-\beta\tau)|x(t)|_r^{\mu+\nu} - \alpha|x(t-\tau)|_r^{\mu+\nu}\\
&-\beta\int_{t-\tau}^{t}|x(s)|_r^{\mu+\nu}ds + c_5\sum_{i,j=1}^{n}|x(t)|_r^{\mu-r_i-r_j}[|x(t)|_r^{\nu+r_j}\\
&+\int_{t-\tau}^{t}|x(s)|_r^{\nu+r_j}ds + \int_{t-\tau}^{t}|x(s)|_r^{\sigma+r_j}ds]\\
&\times\left(\int_{t-\tau}^{t}|x(s)|_r^{\nu+r_i}ds + \varepsilon\int_{t-\tau}^{t}|x(s)|_r^{\sigma+r_i}ds\right)\\
&+\delta c_6\sum_{i=1}^{n}|x(t)|_r^{\mu-r_i}\int_{t-\tau}^{t}|x(s)|_r^{\sigma+r_i}ds\\
&+\varepsilon c_7\sum_{i=1}^{n}|x(t)|_r^{\mu-r_i}\left(|x(t)|_r^{\sigma+r_i}+|x(t-\tau)|_r^{\sigma+r_i}\right),\end{aligned}$$

where $c_i>0$, $i=\overline{4,7}$.

Similarly to the proof of Theorem 1, with the aid of Young's and Hölder's inequalities and Lemma 1, it can be shown that if $\alpha+\beta\tau<c_4/4$ and ε,δ are sufficiently small, then
(*i*) in the case where $\nu>0$, $\sigma\ge\nu$ there exists $\Delta_1>0$ such that

$$\begin{aligned}&\frac{1}{2}c_1|x(t)|_r^{\mu} + \frac{1}{2}\alpha\int_{t-\tau}^{t}|x(s)|_r^{\mu+\nu}ds \le \widetilde W(t,x_t)\\
&\le 2c_2|x(t)|_r^{\mu} + 2(\alpha+\beta\tau)\int_{t-\tau}^{t}|x(s)|_r^{\mu+\nu}ds,\\
&\dot{\widetilde W} \le -\frac{1}{2}c_4|x(t)|_r^{\mu+\nu} - \frac{1}{2}\beta\int_{t-\tau}^{t}|x(s)|_r^{\mu+\nu}ds\end{aligned}$$

for $|x(t)|_r^{\mu} + \int_{t-\tau}^{t}|x(s)|_r^{\mu+\nu}ds < \Delta_1$;

(ii) in the case where $\nu < 0$, $\sigma \le \nu$ there exists $\Delta_2 > 0$ such that these inequalities hold for $|x(t)|_r^\mu + \int_{t-\tau}^t |x(s)|_r^{\mu+\nu} ds > \Delta_2$.

This completes the proof.

Remark 6. It is worth noticing that we failed to prove Theorem 3 on the basis of the LR approach.

6.2. Application of the LR approach

Next, consider a perturbed system of the form

$$\dot{x}(t) = F(x(t)) + \int_{t-\tau}^t G(x(s))ds + \int_{t-\tau}^t B\left(\frac{s}{\varepsilon}\right) Q(x(s))ds, \tag{6}$$

where again ε is a positive parameter, the matrix $B(t)$ is continuous for $t \in (-\infty, +\infty)$, and vector function $Q(x)$ is continuous for $x \in \mathbb{R}^n$. The difference of this model with respect to (4) is that in the former the time-varying averaging perturbation is distributed: for example, if the integral term with $Q(x(s))$ is originated by implementation of a control, then in (4) the time-varying perturbations act in the control channel, while in (6) they come from the sensor.

Theorem 4. *Let Assumptions 1, 3, 4 be fulfilled. Then there exist positive numbers ε_0 and δ_0 such that if $0 < \varepsilon < \varepsilon_0$ and constant δ in the estimate (5) satisfies the condition $0 \le \delta < \delta_0$, then*
(i) for $\nu > 0$, $\sigma \ge \nu$ the zero solution of (6) is AS;
(ii) for $\nu < 0$, $\sigma \le \nu$ solutions of (6) are UUB.

Proof. The system (6) can be rewritten as follows:

$$\dot{x}(t) = F(x(t)) + \tau G(x(t)) + \int_{t-\tau}^t B\left(\frac{s}{\varepsilon}\right) ds\, Q(x(t)) \tag{7}$$
$$+ \int_{t-\tau}^t (G(x(s)) - G(x(t)))ds$$
$$+ \int_{t-\tau}^t B\left(\frac{s}{\varepsilon}\right)(Q(x(s)) - Q(x(t)))ds.$$

It should be noted that the condition (5) is equivalent to the following one:

$$\left\| \int_{t-1}^t B(u)du \right\| \le \delta$$

for $t \ge 0$. Therefore,

$$\left\| \int_{t-\tau}^t B\left(\frac{s}{\varepsilon}\right) ds \right\| = \varepsilon \left\| \int_{t/\varepsilon-\tau/\varepsilon}^{t/\varepsilon} B(u)\, du \right\| \le \tau\delta + \varepsilon M.$$

Let $V(x)$ be a Lyapunov function possessing the properties specified in Remark 2. Differentiating this function along the solutions of (7), we obtain

$$\dot{V} \le -c_1|x(t)|_r^{\mu+\nu} + c_2(\tau\delta + \varepsilon M)|x(t)|_r^{\mu+\sigma}$$
$$+ c_3 \sum_{i=1}^n |x(t)|_r^{\mu-r_i} \int_{t-\tau}^t |G_i(x(s)) - G_i(x(t))|ds$$
$$+ c_4 \sum_{i=1}^n |x(t)|_r^{\mu-r_i} \int_{t-\tau}^t |H_i(s, x(s)) - H_i(s, x(t))|ds$$

where $H_i(s, x)$ are components of the vector $B(s/\varepsilon)Q(x)$ and c_1, c_2, c_3, c_4 are positive constants.

Assume that $x(t) \ne 0$ and the function $V(x)$ satisfies the LR condition $V(x(\xi)) \le 2V(x(t))$ for $\xi \in [t - 2\tau, t]$. Then $|x(\xi)|_r \le m|x(t)|_r$ for $\xi \in [t - 2\tau, t]$, $m = \text{const} > 1$, and

$$\int_{t-\tau}^t |G_i(x(s)) - G_i(x(t))|ds$$
$$= |x(t)|_r^{\nu+r_i} \int_{t-\tau}^t |G_i(z(t) + \Delta z(t, \xi)) - G_i(z(t))|ds,$$
$$\int_{t-\tau}^t |H_i(s, x(s)) - H_i(s, x(t))|ds$$
$$= |x(t)|_r^{\sigma+r_i} \int_{t-\tau}^t |H_i(s, z(t) + \Delta z(t, \xi)) - H_i(s, z(t))|ds,$$

where $z(t) = \Lambda_r^{-1}(|x(t)|_r)x(t)$, $\Delta z(t, \xi) = \Lambda_r^{-1}(|x(t)|_r)(x(\xi) - x(t))$, $i = \overline{1, n}$.

Using the Mean value theorem, it is direct to verify that there exists a number $\tilde{m} > 0$ such that

$$\|\Delta z(t, \xi)\| \le \tilde{m}(|x(t)|_r^\nu + |x(t)|_r^\sigma).$$

Hence, if $\nu > 0$ and $\sigma > 0$ (if $\nu < 0$ and $\sigma < 0$), then

$$\int_{t-\tau}^t |H_i(s, z(t) + \Delta z(t, \xi)) - H_i(s, z(t))|ds \to 0,$$
$$\int_{t-\tau}^t |G_i(z(t) + \Delta z(t, \xi)) - G_i(z(t))|ds \to 0,$$

as $|x(t)|_r \to 0$ (as $|x(t)|_r \to \infty$), $i = \overline{1, n}$.

Let $c_2(\tau\delta + \varepsilon M) < c_1/2$, then
(i) in the case where $\nu > 0$, $\sigma \ge \nu$ there exists $\Delta_1 > 0$ such that

$$\dot{V} \le -\frac{1}{3}c_1|x(t)|_r^{\mu+\nu}$$

for $|x(t)|_r < \Delta_1$;
(ii) in the case where $\nu < 0$, $\sigma \le \nu$ there exists $\Delta_2 > 0$ such that this estimate holds for $|x(t)|_r > \Delta_2$.

This completes the proof.

Remark 7. It is worth noticing that we failed to prove Theorem 4 on the basis of the LK approach.

In this section, under additional hypotheses imposed on time dependence of the perturbations, in Theorems 3 and 4, the degree of perturbation σ can be bigger/smaller or equal to the degree of the delay-free system ν, while without these restrictions, in Theorem 2, only strict inequalities are allowed between ϱ (an analogue of σ) and ν.

7. A modification of the averaging technique for $\nu > 0$

In this section, a modified approach to a Lyapunov function and a LKF construction for the systems (4) and (6) will be proposed for the case $\nu > 0$. Using this approach, we will show that, under some additional constraints, less conservative asymptotic stability conditions than those in Theorems 3 and 4 can be derived. Unlike results of the previous section, we do not assume that ε is sufficiently small. Moreover, the conditions will be obtained under which the asymptotic stability can be guaranteed even in the case where the degree of perturbations σ is less than one of the unperturbed system ν.

7.1. Application of the LR approach

Consider the system (4) under the following additional constraints.

Assumption 5. Let $\nu > 0$, $\sigma > 0$ and the function $Q(x)$ be continuously differentiable for $x \in \mathbb{R}^n$.

Theorem 5. *Let Assumptions 1, 3, 4, 5 be fulfilled. Then one can choose positive number δ_0 such that if the constant δ in the estimate*

(5) *satisfies the condition* $0 < \delta < \delta_0$, *then the zero solution of* (4) *is AS for* $\sigma \geq \nu$ *and any* $\varepsilon > 0$. *In the case where*

$$\int_{t-1}^{t} B(s)ds = 0 \quad \text{for} \quad t \geq 0 \tag{8}$$

the zero solution of (4) *is AS for* $\sigma > \nu/2$ *and any* $\varepsilon > 0$.

Proof. Choose a Lyapunov function candidate for (4) in the form

$$V_1(t,x) = V(x) - \frac{\tau}{\varepsilon}\left(\frac{\partial V(x)}{\partial x}\right)^{\top} \int_{t-\varepsilon}^{t} (s+\varepsilon-t)B\left(\frac{s}{\varepsilon}\right) ds\, Q(x),$$

where $V(x)$ is a Lyapunov function possessing the properties specified in Remark 2. Then

$$c_1|x|_r^{\mu} - c_2|x|_r^{\mu+\sigma} \leq V_1(t,x) \leq c_3|x|_r^{\mu} + c_2|x|_r^{\mu+\sigma},$$

$$\begin{aligned}\dot{V}_1 &\leq -c_4|x(t)|_r^{\mu+\nu} + c_5\delta|x(t)|_r^{\mu+\sigma}\\ &+ c_6\sum_{i=1}^{n}|x(t)|_r^{\mu-r_i}\int_{t-\tau}^{t}|G_i(x(s)) - G_i(x(t))|ds\\ &+ c_7\sum_{i=1}^{n}|x(t)|_r^{\mu-r_i}\int_{t-\tau}^{t}|H_i(t,x(s)) - H_i(t,x(t))|ds\\ &+ c_8\sum_{i=1}^{n}|x(t)|_r^{\mu+\sigma-r_i}\left(|x(t)|_r^{\nu+r_i} + \int_{t-\tau}^{t}|x(s)|_r^{\mu+r_i}ds\right.\\ &\left.+ \int_{t-\tau}^{t}|x(s)|_r^{\sigma+r_i}ds\right)\end{aligned}$$

where $H_i(t,x)$ are components of the vector $B(t/\varepsilon)\,Q(x)$ and $c_k > 0$, $k = \overline{1,8}$. It should be noted that if the condition (8) is valid, then $\delta = 0$. The subsequent proof is similar to that of Theorem 4.

7.2. Application of the LK approach

Let us consider the system (6) once again.

Theorem 6. *Let Assumptions 1, 3, 4, 5 be fulfilled. Then one can choose positive number* δ_0 *such that if the constant* δ *in the estimate* (5) *satisfies the condition* $0 < \delta < \delta_0$, *then the zero solution of* (6) *is AS for* $\sigma \geq \nu$ *and any* $\varepsilon > 0$. *In the case where the condition* (8) *is valid the zero solution of* (6) *is asymptotically stable for* $\sigma > \nu/2$ *and any* $\varepsilon > 0$.

Proof. Consider an LKF

$$\begin{aligned}\widetilde{V}_1(t,x_t) &= \widetilde{V}(x_t)\\ &+ \left(\frac{\partial V(x(t))}{\partial x}\right)^{\top}\int_{t-\tau}^{t}(s+\tau-t)B\left(\frac{s}{\varepsilon}\right)Q(x(s))ds\\ &- \frac{\tau}{\varepsilon}\left(\frac{\partial V(x(t))}{\partial x}\right)^{\top}\int_{t-\varepsilon}^{t}(s+\varepsilon-t)B\left(\frac{s}{\varepsilon}\right)ds\,Q(x),\end{aligned}$$

where $V(x)$ is a Lyapunov function satisfying the conditions specified in Remark 2 and $\widetilde{V}(x_t)$ is the functional (2) constructed in the proof of Theorem 1. We obtain

$$\begin{aligned}&c_1|x(t)|_r^{\mu} - c_3\sum_{i=1}^{n}|x(t)|_r^{\mu-r_i}\int_{t-\tau}^{t}\left(|x(s)|_r^{\nu+r_i} + |x(s)|_r^{\sigma+r_i}\right)ds\\ &+ \alpha\int_{t-\tau}^{t}|x(s)|_r^{\mu+\nu}ds - c_4|x(t)|_r^{\mu+\sigma} \leq \widetilde{V}_1(t,x_t) \leq\\ &c_2|x(t)|_r^{\mu} + c_3\sum_{i=1}^{n}|x(t)|_r^{\mu-r_i}\int_{t-\tau}^{t}\left(|x(s)|_r^{\nu+r_i} + |x(s)|_r^{\sigma+r_i}\right)ds\\ &+ (\alpha+\beta\tau)\int_{t-\tau}^{t}|x(s)|_r^{\mu+\nu}ds + c_4|x(t)|_r^{\mu+\sigma},\end{aligned}$$

$$\begin{aligned}\dot{\widetilde{V}}_1 &\leq (\alpha+\beta\tau-c_5)|x(t)|_r^{\mu+\nu} + c_6\delta|x(t)|_r^{\mu+\sigma}\\ &- \beta\int_{t-\tau}^{t}|x(s)|_r^{\mu+\nu}ds + c_7\sum_{i,j=1}^{n}|x(t)|_r^{\mu-r_i-r_j}[|x(t)|_r^{\nu+r_j}\\ &+ \int_{t-\tau}^{t}\left(|x(s)|_r^{\nu+r_j} + |x(s)|_r^{\sigma+r_j}\right)ds]\\ &\times\int_{t-\tau}^{t}\left(|x(s)|_r^{\nu+r_i} + |x(s)|_r^{\sigma+r_i}\right)ds\\ &+ c_8\sum_{j=1}^{n}|x(t)|_r^{\mu+\sigma-r_j}[|x(t)|_r^{\nu+r_j}\\ &+ \int_{t-\tau}^{t}\left(|x(s)|_r^{\nu+r_j} + |x(s)|_r^{\sigma+r_j}\right)ds],\end{aligned}$$

where $c_k > 0$, $k = \overline{1,8}$.

Under constraint $\alpha + \beta\tau < c_5/4$, with the aid of Young's and Hölder's inequalities and Lemma 1, it is easy to prove that if one of the following conditions is valid:
(*i*) $\delta > 0$ and $\sigma \geq \nu$;
(*ii*) $\delta = 0$ and $\sigma > \nu/2$,
then there exists a positive number Δ such that

$$\begin{aligned}&\frac{1}{2}c_1|x(t)|_r^{\mu} + \frac{1}{2}\alpha\int_{t-\tau}^{t}|x(s)|_r^{\mu+\nu}ds \leq \widetilde{V}_1(t,x_t)\\ &\leq 2c_2|x(t)|_r^{\mu} + 2(\alpha+\beta\tau)\int_{t-\tau}^{t}|x(s)|_r^{\mu+\nu}ds,\\ &\dot{\widetilde{V}}_1 \leq -\frac{1}{2}c_5|x(t)|_r^{\mu+\nu} - \frac{1}{2}\beta\int_{t-\tau}^{t}|x(s)|_r^{\mu+\nu}ds\end{aligned}$$

for $|x(t)|_r^{\mu} + \int_{t-\tau}^{t}|x(s)|_r^{\mu+\nu}ds < \Delta$. This completes the proof.

Remark 8. Note that the LR approach is used in this section for the system (4) (it was the LK method in Section 7), while for the system (6) the situation is opposite.

8. Examples

8.1. Vector Rayleigh equation

Assume that behavior of a mechanical system is modeled by the equation

$$\ddot{x}(t) + \frac{\partial W(\dot{x}(t))}{\partial \dot{x}} + \frac{\partial \Pi(x(t))}{\partial x} + \int_{t-\tau}^{t}\frac{\partial \widetilde{\Pi}(x(s))}{\partial x}ds = 0, \tag{9}$$

where $x(t) \in \mathbb{R}^n$, $W(\dot{x})$ is a continuously differentiable for $\dot{x} \in \mathbb{R}^n$ positive definite homogeneous of the degree $\varrho + 1 > 1$ function with respect to the standard dilation (*i.e.*, $\mathbf{r} = (1,\ldots,1)$), $\Pi(x)$ and $\widetilde{\Pi}(x)$ are continuously differentiable for $x \in \mathbb{R}^n$ homogeneous of the degree $\rho + 1 > 1$ functions with respect to the standard dilation.

The system (9) is a vector type Rayleigh equation (Rayleigh, 1945) describing the dynamics of mechanical systems with dissipative and potential forces, whereas the term $\int_{t-\tau}^{t}\partial\widetilde{\Pi}(x(s))/\partial x ds$ can be interpreted as an integral part of a PID-like regulator (see Formal'sky (1997) and Radaideh and Hayajneh (2002)).

Let $\varrho = 2\rho/(\rho+1)$, then the corresponding first-order system

$$\begin{aligned}&\dot{x}(t) = y(t), \quad \dot{y}(t) = -\frac{\partial W(y(t))}{\partial \dot{x}} - \frac{\partial \Pi(x(t))}{\partial x}\\ &- \int_{t-\tau}^{t}\frac{\partial \widetilde{\Pi}(x(s))}{\partial x}ds\end{aligned}$$

is $\mathbf{r}$-homogeneous of the degree $(\rho-1)/2$ with respect to the dilation $\mathbf{r} = (r_1,\ldots,r_{2n})$, where $r_i = 1$ for $i = \overline{1,n}$, and $r_i = (\rho+1)/2$ for $i = \overline{n+1,2n}$.

In addition, assume that the function $\Pi(x)+\tau\widetilde{\Pi}(x)$ is positive definite. It is known (see Rouche et al. (1977)) that, under this assumption, the zero solution of the auxiliary system

$$\dot{x}(t)=y(t), \quad \dot{y}(t)=-\frac{\partial W(y(t))}{\partial \dot{x}}-\frac{\partial \Pi(x(t))}{\partial x}-\tau\frac{\partial \widetilde{\Pi}(x(t))}{\partial x}$$

is AS.

Applying Theorem 1, we obtain that
(i) for $\rho > 1$ the equilibrium position $x=\dot{x}=0$ of (9) is AS;
(ii) for $\rho < 1$ solutions of (9) are UUB.

Next, consider the associated perturbed system:

$$\ddot{x}(t)+\frac{\partial W(\dot{x}(t))}{\partial \dot{x}}+\frac{\partial \Pi(x(t))}{\partial x}+\int_{t-\tau}^{t}\frac{\partial \widetilde{\Pi}(x(s))}{\partial x}ds$$
$$+B\left(\frac{t}{\varepsilon}\right)\int_{t-\tau}^{t}Q(x(s))ds=0, \tag{10}$$

where as before we assume that ε is a positive parameter, the matrix $B(t)$ is continuous and bounded for $t\in(-\infty,+\infty)$, vector function $Q(x)$ is continuous for $x\in\mathbb{R}^n$ and homogeneous of the order $\eta > 0$ with respect to the standard dilation.

Using Theorem 3, we obtain that there exist positive numbers ε_0 and δ_0 such that if $0<\varepsilon<\varepsilon_0$ and constant δ in the estimate (5) satisfies the condition $0\le\delta<\delta_0$, then
(i) for $\eta\ge\rho>1$ the zero solution of (10) is AS;
(ii) for $\eta\le\rho<1$ solutions of (10) are UUB.

8.2. Damping angular motions of a rigid body

It is worth noting that the approaches developed in this paper can be applied to strongly nonlinear systems that are not homogeneous. To illustrate this case, consider the problem of damping the angular motions of a rigid body.

Let a rigid body be rotating in an inertial space with angular velocity $\omega(t)\in\mathbb{R}^3$ around its center of inertia C. Denote by $Cxyz$ the principal central axes of inertia of the body. The attitude motion of the body under the action of a torque M is modeled by the Euler equations

$$J\dot{\omega}(t)+\omega(t)\times J\omega(t)=M(t),$$

where $J=\mathrm{diag}\{A_1,A_2,A_3\}$ is the body inertia tensor in the axes $Cxyz$ (Merkin, 1997).

Let the torque M be of the form

$$M(t)=F(\omega(t))+\int_{t-\tau}^{t}(D+B(s))U(\omega(s))ds,$$

where $F(\omega)\in\mathbb{R}^3$ is a continuous for $\omega\in\mathbb{R}^3$ homogeneous of the degree $\rho>1$ with respect to the standard dilation function, $U(\omega)\in\mathbb{R}^3$ is a control vector, $D\in\mathbb{R}^{3\times3}$ is a constant matrix, $B(t)\in\mathbb{R}^{3\times3}$ is a continuous for $t\in(-\infty,+\infty)$ matrix that characterizes control deviations from a prescribed values caused by non-stationary disturbances, τ is a positive constant delay, whose appearance may be related with network communication of the control signal.

We are going to design a control law $U(\omega)$ providing the asymptotic stability of the equilibrium position $\omega=0$ of the body. With the aid of arguments similar to those used in the proof of Theorem 6, it is easy to verify that, under the following conditions:
(i) the matrix $B(t)$ is bounded for $t\in(-\infty,+\infty)$;
(ii) $\left\|\int_{t-1}^{t}B(s)ds\right\|\le\delta$ for $t\ge0$, where δ is a sufficiently small positive constant;
(iii) $U(\omega)$ is continuously differentiable for $\omega\in\mathbb{R}^3$;
(iv) $U(\omega)$ is homogeneous function of the degree ρ with respect to the standard dilation;
(v) the function $\omega^\top(F(\omega)+\tau DU(\omega))$ is negative definite,
the equilibrium position $\omega=0$ of the corresponding closed-loop system is AS.

9. Conclusions

For a nonlinear system with distributed delay and homogeneous right-hand side (in the delay-free setting), several conditions of robust stability in the presence of time-varying perturbations weighted by a nonlinear functional gain as in (4) or (6) have been obtained. In Theorem 1 it has been demonstrated that in the disturbance-free case the zero solution is either locally asymptotically stable (for homogeneity degree of the system $\nu>0$) or practically globally asymptotically stable (for $\nu<0$). Theorem 2 discovers next generic robustness properties (in ISS sense) of the system without using the averaging approach (without assuming certain periodicity of the perturbations), where it has been shown that ν has to be strictly bigger/smaller than the order of the disturbance ϱ for positive/negative scenarios. Theorems 3–6 formulate several sets of restrictions on the time-varying perturbations, based on the averaging techniques, improving the result of Theorem 2 (the homogeneity degree of the perturbation σ (an analogue of ϱ) can be equal to ν (Theorems 3–4), or in the case of positive degree (Theorems 5–6), even smaller). The results are based on utilization of LK and LR approaches, and interestingly to note, in many cases only one of these tools can be applied. To show efficiency of the approach, the models of mechanical systems have been explored, described by a vector Rayleigh equation and Euler equation of angular motions for a rigid body. Future research may include analysis of more general classes of nonlinear systems with delays.

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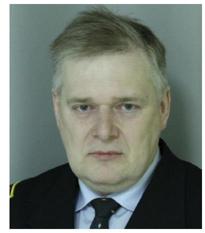

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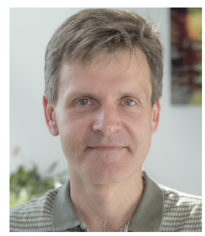

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