Brief paper

Detectability and global observer design for 2D Navier–Stokes equations with uncertain inputs

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A B S T R A C T

We present simulation friendly detectability conditions for 2D Navier–Stokes Equation (NSE) with periodic boundary conditions, and describe a generic class of “detectable” observation operators: it includes pointwise evaluation of NSE’s solution at interpolation nodes, and spatial average measurements. For “detectable” observation operators we design a global infinite-dimensional observer for NSE with uncertain possibly destabilizing inputs: in our numerical experiments we illustrate 𝐻¹-sensitivity of NSE to small perturbations of initial conditions, yet the observer converges for known and uncertain inputs.

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1. Introduction

Navier–Stokes Equation (NSE)

\[
\frac{d\vec{u}}{dt} + (\vec{u} \cdot \nabla)\vec{u} - \nu \Delta \vec{u} + \nabla p = \vec{f}
\]  

(1.1)

is a basic mathematical model of fluid dynamics: it describes evolution of fluid’s velocity vector-field \( \vec{u} \) and scalar pressure field \( p \) as a function of initial and boundary conditions, input \( \vec{f} \) and coefficient \( \nu > 0 \). NSE has applications in biology, weather prediction, energy forecasting, etc. It also serves as a mathematical model of turbulence: yet an open problem in 3D, in 2D NSE is used to study turbulence and, in particular, to reveal its connection to deterministic chaos (Foias et al., 2001). Indeed, for certain \( \nu \) and \( \vec{f} \) NSE (1.1) has the unique attractor, which is globally exponentially stable, e.g. \( \vec{u} = 0 \) for \( \vec{f} = 0 \) and any \( \nu > 0 \). However, for small enough \( \nu > 0 \) and a destabilizing input \( \vec{f} \), NSE’s attractor could be a multidimensional manifold (e.g. Foias et al. (2001), Ilyin and Titi (2006)), moreover, the attractor could also be chaotic, i.e. initially close-by trajectories might diverge over time on the attractor. For such destabilizing inputs classical control problems, e.g. observer design, become challenging, as was noted in Vazquez and Krstic (2005), especially when attractor’s structure depends on an uncertain input \( \vec{f} \).

In this work we generalize the classical notion of detectability for LTI evolution equations (Bensoussan et al., 2007) to periodic 2D NSE, and introduce detectability conditions: we describe a generic class of “detectable” observation (output) operators \( \mathcal{C} \) verifying the proposed conditions (our 1st contribution). Intuitively, “detectable” output \( \mathcal{C}\vec{u} \) of (1.1) must provide information about a finite-dimensional subspace where the nonlinear advection term \((\vec{u} \cdot \nabla)\vec{u}\) is not dominated by the diffusion term \(-\nu \Delta \vec{u}\) (see also the mechanism of energy cascades (Foias et al., 2001)).

Given such output \( \mathcal{C}\vec{u}(t) \) one can then reconstruct the entire state \( \vec{u}(t) \) by designing an observer/filter which injects \( \mathcal{C}\vec{u} \) so that the reconstruction error in the “unmeasured” infinite-dimensional orthogonal complement of the range of \( \mathcal{C} \) will decay thanks to the stabilizing effect of \(-\nu \Delta\vec{u}\) akin to the case of detectable LTI systems. More specifically, by employing direct Lyapunov method, we design a Luenberger-type observer for NSE (1.1) which, in fact, is an approximation of minimax filter: we show that \( \hat{\vec{u}}(\vec{t}) \), the solution of (1.1) for an unknown initial condition \( \vec{u}(0) = \vec{u}_0 \), and for \( \vec{f} = \vec{g} + d, \vec{g} \) - a destabilizing known input, \( d \) - an uncertain bounded input, belongs to \( H^1 \)-ellipsoid, centered at the state \( \vec{x}(t) \) of the observer, and its radius decays to 0 as \( t \to \infty \) independently of \( \vec{u}_0 \); provided the uncertain input \( d \) “perturbs” \( \vec{g} \) less and less frequently as \( t \to \infty \), or \( L^2 \)-norm of this perturbation (in space) decays as \( t \to \infty \), a remarkable result for turbulent systems which are highly sensitive to small perturbations (our 2nd contribution).

Infinite-dimensional observers for NSEs, employing backstepping as a design technique, were proposed in Vazquez and Krstic (2005) for Poiseuille flows, a benchmark for turbulence estimation, and in He, Hu, and Zhang (2018) for local observer-based
stabilization of NSEs. Disturbance estimation by means of backstepping for linear 1D heat equation was proposed in Feng and Guo (2017).

The proposed class of “detectable” output operators $C$ generalizes the notions of so-called determining modes, e.g., spatial averages $\int_{\Omega_1} \delta x \Delta u$ of the solution $u$ over squares $\Omega_1$ covering the domain $\Omega$ (see e.g. Fridman and Bar Am (2013), Fridman and Blighovsky (2012), Kang and Fridman (2020)), and determining modes, e.g. pointwise evaluation $u(x, \tau)$ at finitely many interpolation nodes $x_1$ (see e.g. Azouani, Olson, and Titi (2014)), which are well-known in NSE's literature (Foias et al., 2001, p.123,p.131). Our class contains those cases, and more generally it consists of all closed linear operators $C$ verifying certain inequalities; e.g., spatial averages output $\vec{u}C = \sum_{j=1}^J \int_{\Omega_j} \hat{u} \delta x, \hat{u}$ - normalized indicator of $h \times h$-square $\Omega_1$, is detectable if $h^2$, the area of $\Omega_1$ is small enough for $\vec{u} = \hat{u}$ to be “controlled” by $h\vec{u}$ (Poincaré inequality). These inequalities do not cover boundary-type observations (see Vazquez and Krstic (2008), Vazquez, Schuster, and Krstic (2008)).

We stress that spatial averages outputs are important in real-world applications: for example, short-term solar energy forecasting from a sequence of Cloud Optical Depth (COD) images relies upon estimating averaged velocities of clouds from COD images and uses those as measurements in periodic 2D NSE for cloud velocity prediction (Akhirov et al., 2017); similarly 2D NSE with averaged velocity measurements obtained from Sea Surface Temperature (SST) satellite images were used in Herlin et al. (2012) to predict short-term SST dynamics.

In this work we build upon preliminary results presented in our conference papers (Kang, Fridman, & Zhuk, 2019; Zayats, Fridman, & Zhuk, 2021): key differences include detectability conditions and more general case of uncertain inputs. Qualitatively similar results were obtained in Azouani et al. (2014) where sufficient conditions for convergence of Luenberger observer were proposed for periodic boundary conditions and known inputs. The authors make use of Brezis inequality to bound the nonlinear advection term in the error equation, and derive conditions for observer's gain and output (e.g. number and size of squares $\Omega_1$ for the case of spatial averages outputs), sufficient for convergence. An experimental assessment of these conditions was recently provided in (Franz et al. 2022). In contrast, our convergence analysis relies upon a novel one-parametric inequality relating $L^\infty$ and $H^2$-norms of periodic vector-functions, which for certain values of the parameter reduces to Agmon and Brezis inequalities, and S-procedure widely used in Lyapunov stability analysis. As a result, we get simulation friendly and less conservative detectability conditions: e.g. for the case of spatial averages outputs our estimate of $h^2$, the area of squares $\Omega_1$, is at least one order of magnitude better for small $\nu > 0$, which is of high interest in the case of turbulent flows, and hence for convergence, observer requires averaging the velocity $\vec{u}$ over larger $\Omega_1$ (see Remark 3.3) so less of squares (“sensors”) is needed, and more importantly, the averaging over larger $\Omega_1$ helps reducing the noise hence improving convergence in practice.

1.1. Mathematical preliminaries

Notation. Let $\mathbb{R}^n$ denote Euclidean space of dimension $n$ with inner product $\langle \vec{u}, \vec{v} \rangle = \sum_{i=1}^n u_i v_i$, $\mathbb{R}^n_+$ – non-negative orthant of $\mathbb{R}^n$, $\mathbb{R}^n_{++}$ = $\mathbb{R}^n_+$. and for $k, \ell \in \mathbb{R}^n$ set $\mathbb{R}^n_{++} = \{ x = \ell / \ell \pm \mathbb{R}^n, \ell \neq 0 \}$. Let $\mathcal{X}(H)$ denote the space of all closed linear operators $C$ acting in a Hilbert space $H$ with domain $\mathcal{D}(C) \subset H$. The following functional spaces are standard in NSE’s theory (Foias et al., 2001, p.45-p.48):

- $L^2(\Omega) – \text{space of } \Omega \text{-periodic functions } u : \Omega \subset \mathbb{R} \to \mathbb{R}$ with inner product $\langle u, v \rangle = \int_{\Omega} u v \, dx$ and norm $\| u \|_{L^2} = \| u \|_2$.
- $H^1(\Omega) = \{ u \in L^2(\Omega) : \| \nabla u \|_{L^2} \}$. $H^1(\Omega) \equiv \{ u \in L^2(\Omega) : u|_{\Sigma_1} \in H^1(\Omega) \}$ with norm $\| u \|_{H^1} = \| u \|_{L^2} + \| \nabla u \|_{L^2}$.
- $\mathcal{H} = \{ \vec{v} \in L^2(\Omega) : \vec{v} = \vec{v} \}$ on $\mathcal{H} = \mathcal{H}$ – subspace of $\mathcal{H}$ with zero mean components.
- $\mathcal{V} = \{ \vec{v} \in H^1(\Omega) : \vec{v} = \vec{v} \} = \mathcal{V}$ on $\mathcal{V} = \mathcal{V}$
- $L^2(0,T,H)$ – space of $H$-valued functions $t \mapsto u(t)$ in $H$ with finite norm $\| u \|_{L^2} = \int_0^T \| u(t) \|_{L^2}^2 dt \mathrm{d}t$ for $T \in (0, +\infty), e.g. L^2(0,T,H)$ – space of $\vec{v}(t, x, \tau)$ such that $\int_0^T \int_{\Omega} \vec{v}(t, x, \tau) \| dx dt < +\infty$ and $\vec{u}(t, \cdot)$ has zero divergence and zero mean for almost all $t \in (0, T)$.
- $L^\infty(0,T,H)$ – space of $H$-valued functions $t \mapsto u(t)$ in $H$ such that $\| u(t) \|_H \leq C < +\infty$ for $C > 0$ and almost all $t \in (0, T), T \in (0, +\infty)$ with finite norm $\| u \|_H = \max \{ C > 0 : \| u(t) \|_H \leq C \}$. The proof is given in the Appendix.

1.1.2. Navier–Stokes equation: weak formulation and well-posedness in 2D

The classical NSE in 2D is a system of two PDEs defining dynamics of the scalar pressure field $p(x, y)$ and the viscous fluid velocity field $\vec{u}(x, y)$ which depends on the initial condition $\vec{u}(0) = \vec{u}_0 \in \mathcal{H}$, input (e.g. forcing) $f = [f_1, f_2]$ and Boundary Conditions (BC), e.g. periodic BC $u_{1,2}(x + \ell_1, y, + \ell_2) = u_{1,2}(x, y)$, $u_{1,2}(x + \ell_1, y, + \ell_2) = u_{1,2}(x, y)$, $u_{1,2}(x + \ell_1, y, + \ell_2) = u_{1,2}(x, y)$, $u_{1,2}(x + \ell_1, y, + \ell_2) = u_{1,2}(x, y)$. In the vector form it reads as follows:

$$\frac{d\vec{u}}{dt} + (\vec{u} \cdot \nabla)\vec{u} - \nu \Delta \vec{u} + \nabla p = \vec{f}, \ \nabla \cdot \vec{u} = 0 \tag{1.5}$$

To eliminate pressure $p$ and obtain an evolution equation just for $\vec{u}$ it is common to use Leray projection (Foias et al., 2001, p.38):

$$\frac{d\vec{u}}{dt} + (\vec{u} \cdot \nabla)\vec{u} - \nu \Delta \vec{u} = \vec{f}, \ \nabla \cdot \vec{u} = 0$$

This condition is necessary: lemma does not hold for $u = \text{const}$ and small enough $\ell_{1,2}$.
every vector-field \( \ddot{u} \) in \( \mathbb{R}^2 \) admits Helmholtz–Leray decomposition, \( \ddot{u} = \nabla p + \ddot{v} \) with \( \nabla \cdot \ddot{u} = 0 \) which in turn defines Leray projector, \( P(\ddot{u}) = \ddot{v} \) – an orthogonal projector onto \( \mathcal{H} \) (e.g. Foias et al. (2001, p.36)). Multiplying \((1.5)\) by a test function \( \phi \in \mathcal{Y} \) (the projection step), and integrating by parts in \( \Omega \) allows one to obtain Leray’s weak formulation of NSE in 2D:

\[
\frac{d}{dt}(\ddot{u}, \phi) + b(\ddot{u}, \ddot{v}, \phi) + \nu((\ddot{u}, \ddot{v})) = (\ddot{f}, \phi), \quad \forall \phi \in \mathcal{Y} \tag{1.6}
\]

with initial condition \((\ddot{u}(0), \ddot{\phi}(0)) = (\bar{u}_0, \ddot{\phi}_0).\) Here

\[
b(\ddot{u}, \ddot{v}, \ddot{\phi}) = (\ddot{u} \cdot \nabla w_1(\phi_1) + (\ddot{u} \cdot \nabla w_2(\phi_2), \phi_3),
\]

\[
((\ddot{u}, \ddot{\phi})) = (\nabla u_1, \nabla \phi_1) + (\nabla u_2, \nabla \phi_2).
\]

In what follows we will be using some properties of the trilinear form \( b \) and Stokes operator \( \ddot{u} \mapsto \ddot{A}\ddot{u} = -P_{\lambda} \ddot{u}, \) a self-adjoint positive operator with compact inverse, which coincides with \( \Delta \ddot{u} \) for periodic BC (see Foias et al. (2001, p.52)): for \( \ddot{u} \in \mathcal{D}(A) \) and \( \dddot{\phi} \in \mathcal{Y} \)

\[
(A\ddot{u}, \dddot{\phi}) = ((\ddot{u}, \dddot{\phi}), (\ddot{u}, \dddot{\phi})) \geq \lambda_1(\ddot{u}, \dddot{\phi}) \tag{1.7}
\]

\[
(A\ddot{u}, \dddot{\phi}) = (A(\dddot{\phi})^2, \dddot{\phi}) \geq \lambda_1(\ddot{u}, \dddot{\phi}) \tag{1.8}
\]

\[
\lambda_1 = 4\pi^2/\max(\ell_1, \ell_2)^2 \tag{1.9}
\]

\[
b(\ddot{u}, \dddot{\phi}, \dddot{\phi}) = -b(\dddot{u}, \dddot{\phi}, \dddot{\phi}) = 0 \tag{1.10}
\]

\[
b(\ddot{u}, \dddot{\phi}, \dddot{\phi}) = 0, \quad b(\dddot{u}, \dddot{\phi}, \dddot{\phi}) = 0 \tag{1.11}
\]

Next lemma collects results from (Foias et al., 2001, p.58, Th.7.4, p.99, Th.1.3, Th.4.42, p.102, Th.6.65–67) on existence, uniqueness, regularity and input-to-state stability of NSE’s weak (strong) solution \( \ddot{u} \), and bounds for \( A\ddot{u}. \) Classical smoothness of \( \ddot{u} \) requires further constraining of \( f, \) \( \bar{u}_0 \) (Foias et al., 2001, p.59):

**Lemma 1.2.** Let \( \bar{u}_0 \in \mathcal{H} \) and \( \ddot{f} \in L^2(0, \dddot{\phi}, \mathcal{H}). \) Then, on \([0, \dddot{\phi}, \mathcal{H}]) \) there exist the unique weak solution \( \ddot{u} \in C(0, \dddot{\phi}, \mathcal{H}) \) of \( \text{(1.6)}, \) and the components of \( \ddot{u} = (u_1, u_2) \) verify: \( u_1, (u_1)_{x,y} \in L^2(\Omega \times (0, \dddot{\phi})). \) If \( \bar{u}_0 \in \mathcal{H} \) then the weak solution coincides with the strong solution of \( \text{(1.6)}, \)

\[
du dt ([u_1, \quad u_2] \in L^2((0, \dddot{\phi})) \quad \forall \dddot{\phi} \geq 0 \quad \forall \dddot{\phi}, \mathcal{H}). \]

\[
\bar{u}_0 \in \mathcal{H} \quad \limsup_{t \rightarrow \infty} ||\dddot{u}(t)||_2 \leq \lambda^* \quad \forall \dddot{\phi} \in \mathcal{H} \quad \forall \dddot{\phi}, \mathcal{H} \]
3. Main results

3.1. Sufficient conditions for detectability

Recall the classes \(\varphi_1(h, \Omega)\) and \(\varphi_2(h, \Omega)\) of observation operators \(c\) introduced above in Definition 2.2. The following proposition demonstrates existence of a constant \(h > 0\) for each of \(\varphi_1(h, \Omega)\) and \(\varphi_2(h, \Omega)\) such that NSE is detectable (as per Definition 2.1). However, as this is, this proposition is of rather theoretical value as to compute \(h > 0\) for numerical simulations one needs to know certain interpolation constants which are either unknown or very conservative. The question of how to use this result for estimating \(h\) in numerical simulations will be addressed below.

Proposition 1. Recall Definition 2.1: let \(\tilde{z}, \tilde{u}\) solve (2.1)-(2.2) for \(\tilde{f}, \tilde{F} \in L^\infty(\mathbb{R}_+, \mathbb{R})\). Recall from (Foias et al., 2001, p.100,(A.47)) an interpolation inequality for \(v\tilde{u}\), and define constants \(c_1, c_2, c_3\):

\[
\forall \tilde{u} \in D(A): \|\tilde{v}\|_{L^2} \leq c_1 \|\tilde{u}\|_{L^2} \|\tilde{u}\|_{L^2}
\]

\[
C_2^2 = \frac{\|\tilde{u}\|_{L^2}^2}{(2\pi)^3}
\]

Take \(c \in \varphi_1(h, \Omega)\). Then NSE is detectable in \(\hat{V}\) if for some \(1 < \kappa c 2\) and \(\Gamma > 0\)

\[
h \leq \frac{\lambda^2 (\tilde{v}^2 + c_3)}{c_1 c_2 c_3^2} \|\tilde{v}\|_{L^2} \|\tilde{u}\|_{L^2} \log^2 (1 + 4\pi^2 \|\tilde{u}\|_{L^2}^2 \tilde{c}^2 c_3^2)
\]

moreover, if \(c \in \varphi_2(h, \Omega)\) then NSE is detectable in \(\hat{V}\) if \(h\) verifies (3.3) without \(\lambda^2\) in the numerator.

The proof is provided in the Appendix.

3.2. Observer design

As noted above, condition (3.3) is hard to use in simulations. Below we build on (3.3) and propose “simulation friendly” conditions for \(h\) (see C1, C2 of Theorem 3.1): it is demonstrated that plugging Luhenberg-type output feedback \(\tilde{F} = \tilde{g} + Lc(\tilde{u} - \tilde{z})\) into (2.2) ensures output and input convergence (conditions (A) and (B) of Definition 2.1), and as a result implies state convergence in \(\hat{V}: \|\tilde{v}(t) - \tilde{z}(t)\|_{L^2} \to 0\).

Lemma 3.1 (Wellposedness). Let \(\tilde{u}\) solve (2.1). For any \(\tilde{g} \in L^\infty(\mathbb{R}_+, \mathbb{R})\) there is the unique \(\tilde{z} \in L^\infty(\mathbb{R}_+, \mathbb{R}) \bigcap L^2(\mathbb{R}_0, t_1, D(A))\), \(0 < t_0 < t_1 < +\infty\) such that

\[
\frac{d}{dt}(\tilde{z}, \phi) + b(\tilde{z}, \phi) + n(\tilde{z}, \phi) = (\tilde{F}, \phi)
\]

\[
\tilde{F} = \tilde{g} + Lc(\tilde{u} - \tilde{z})
\]

provided \(c \in \varphi_1(h, \Omega)\) and \(L c^2 \Omega = 2\nu\), or \(c \in \varphi_2(h, \Omega)\) and \(L c^2 \Omega \leq \nu\).

The proof is given in the Appendix.

Theorem 3.1. Assume that (i) \(\tilde{u}\) solves (2.1) for unknown \(\tilde{u}(0) \in \tilde{V}\) and \(f\), and (ii) \(\tilde{g} = \tilde{g} + \tilde{d}\) for a known \(\tilde{g} \in L^\infty(\mathbb{R}_+, \mathbb{R})\), \(\tilde{g}_k = [\tilde{g} \big|_{\Omega \times (t_0, t_1)}] \bigcap R\) of radius \(R > 0\). Let \(\tilde{z}\) solve (3.4) for \(\tilde{z}(0) = 0\), \(L > 0\) and \(\tilde{g}\) as in (ii) above. For \(\epsilon > 0\) define \(\kappa = 1 + \epsilon\) and functions of a parameter \(\Gamma\):

\[
\hat{L}(\Gamma) = \frac{2(\hat{v}^2 - \sqrt{32\pi^3 \lambda_1 \Gamma})}{\sqrt{32\pi^3 \lambda_1 \Gamma}} / C_2 \Omega.
\]

\[
\hat{L}_\Lambda(\Gamma) = \frac{(\kappa + \lambda_2^2) C_2^2}{\nu} \log^2 (1 + \frac{4\pi^2 \kappa C_2^2 \Gamma^2}{\nu^2 \epsilon_1 \epsilon_2})
\]

\[
\Theta(\Gamma) = \frac{\hat{v}(\Gamma)}{(2\hat{L}_\Lambda(\Gamma))}
\]

and let \(\Gamma_{\max} > 0\) maximize \(\Theta(\Gamma)\). Finally assume that

\[
\exists T > 0: \Sigma_\epsilon(t) = \sup_{t \geq T} \frac{1}{T} \int_t^{t+T} \|\hat{d}(\tau)\|_{L^2} d\tau \xrightarrow{s \to \infty} 0
\]

and take minimal \(T_* > 0\) and \(T_1 \geq T_*\) verifying

\[
\hat{d}(\tau) + \frac{2(\hat{z}^2 + R_{\gamma}^2)}{C_2^2 \hat{v}^2 \tau_1 \lambda_1} + \frac{2v_2 \gamma_1 \epsilon_2 \gamma_1 \epsilon_2}{\nu^2 \epsilon_1 \epsilon_2} \leq \epsilon
\]

Then, for \(s \geq T_*\), \(V(t) = \|\tilde{v}(\tilde{u}(t) - \tilde{z}(t))\|_{L^2}^2\) verifies

\[
V(t) \leq \epsilon e^\epsilon T_1 V(s) e^{\left(\epsilon^2 - \epsilon\right) \epsilon T_1} + \frac{T_1 e^\epsilon (t_*)}{C_2 (1 - \beta) \hat{L}_V(\Gamma_{\max})} \Sigma_\epsilon(t)
\]

provided \(\tilde{r}(t)\) is large enough for the impact of the 1st term in r.h.s. of (3.11) be negligible compared to the 2nd term. If \(\Sigma_\epsilon(t) = 0\) for \(s > t^*\) then \(V(t) \to 0\) exponentially after \(t = \max(s, t^*)\). In fact, \(V \leq V_1 = \|\tilde{z} - \tilde{u}\|_{L^2(\tilde{u})}\) and by (1.7) \(V_1 \leq (1 + \gamma_2^\epsilon)^{\epsilon/\gamma_2^\epsilon}\) hence theorem proposes an approximation of minimax filter in the following sense: by (3.11) \(\tilde{v}(t)\) belongs to \(H^1\)-ellipsoid, centered at \(\tilde{z}(t)\), and its radius is given by r.h.s. of (3.11).

Remark 3.1. Condition (3.9) is verified if either (i) the average of \(L^2\)-energy of the uncertain input \(d\) over time window \((t, t + T)\) decays to 0, or (ii) the measure of time instants \(s\) within a “window” \((t, t + T)\), where \(d(s, \cdot)\) is “active”, decays to 0 as the window \((t, t + T)\) slides to infinity \((t \to \infty)\). In other words, the case (ii) does not require \(\|d(s, \cdot)\|_{L^2} \to 0\) but requires that the uncertain input “perturbs” \(\tilde{g}\) less and less frequently asymptotically.

The rate of decay of \(\Sigma_\epsilon(s)\) to 0 determines the rate of decay of \(V\): indeed, (3.11) implies that

\[
\sup_{t \geq \tilde{s}(t)} V(t) \leq \epsilon e^\epsilon T_1 V(s) e^{\left(\epsilon^2 - \epsilon\right) \epsilon T_1} + \frac{T_1 e^\epsilon (t_*)}{C_2 (1 - \beta) \hat{L}_V(\Gamma_{\max})} \Sigma_\epsilon(t)
\]
Remark 3.3. Let us compare our upper bound for $h^2$ obtained above for $c' \in C'(\nu, \Omega)$, namely $h^2 < \beta \Theta(T_{\max})$ to the state-of-the-art result of Azouani et al. (2014, Prop.2):

$$h^2 \leq \nu T(v) = C_0 \left(3|\lambda_1| (2c \log(2c)|\nu| + 8c \log(1 + G))G^{-1}\right), \quad (3.12)$$

where $G = \frac{||f||_{\infty} \|\nabla u\|_{L^2}}{\lambda_1^{1/2}}$ and the constant $c$ comes from Brezis inequality $\|u\|_{L^\infty} \leq c \|\nabla u\|_{L^2}(1 + \log \frac{\|\nabla u\|_{L^2}}{\lambda_1^{1/2}})$ for $u \in \mathcal{D}(A)$ (Foias et al., 2001, p.100, (A.50)). Clearly,

$$c = \sup_{\|u\|_{L^\infty}} \frac{\|u\|_{L^\infty}}{\|\nabla u\|_{L^2}}^{1/2} \left(1 + \log \frac{\|\nabla u\|_{L^2}}{\lambda_1^{1/2}}\right)^{-1/2},$$

and so $c \geq c_1$ for $c_1 = \|u_1\|_{L^\infty} \|\nabla u_1\|_{L^2}^{-1}(1 + \log \frac{\|\nabla u_1\|_{L^2}}{\lambda_1^{1/2}})^{-1/2}$ and $u_1 \in \mathcal{D}(A)$ with components $u_1 = -\cos(2\pi x) \sin(2\pi y)$ and $u_2 = \cos(2\pi y) \sin(2\pi x)$. It is easy to compute that $c_1 = (2\pi)^{-1}$. Since the r.h.s. of (3.12), $\tau(v)$ increases if we substitute $c$ with $c_1$, and so the upper bound on $h^2$ improves, below we compare $\Theta(T_{\max})$ vs. $\tau(v)$ with $c = c_1 = (2\pi)^{-1}$ over the interval $v \in [10^{-6}, 10^{-1}]$. To match the setting of Azouani et al. (2014) we assume that $\xi_{1,2} = 1$, $f(s) = g(s)$ for $s > s^*$ so we can use $\beta = 1$, we also take $\|\nabla f\|_{L^\infty} = C_f = 1$ and $k = 2$. We get that $\log_{10} \left(\Theta(T_{\max})\right) = 1.33$ for $v = 10^{-6}$ and $0.58$ for $v = 10^{-1}$ and LogLog-plot of $\Theta$ and $\tau$ over $v \in [10^{-6}, 10^{-1}]$ is given in Fig. 1. Obviously, for small $v$ our upper bound is at least one order of magnitude better.

4. Experiments

In what follows we first illustrate sensitivity of NSE with destabilizing Kolmogorov input to small perturbations of initial conditions. Then we illustrate convergence of the observer for the case of known inputs. And finally, we perform a crash-test: we take observations generated by a numerical method, and use them in the observer discretized by a different method.

For the crash-test we generate spatial averages outputs by a numerical solver, referred to as FFT-solver. FFT-solver is a pseudo-spectral numerical method, which relies upon vorticity-streamfunction formulation of NSE. It is exactly divergence free (as required by continuous formulation) and has spectral convergence property in space: its convergence rate in $H^1$ automatically increases with the degree of smoothness of initial conditions and inputs. For time discretization we used 2nd order implicit midpoint with 5 iterations; an open-source implementation of FFT-solver with different time-stepping is available in jax-cfd package. Then, we discretize the observer by a less accurate solver, referred to as FEM-solver. FEM-solver is implemented using Finite Element Method (based on Oasis Python package (Mortensen & Valen-Sendstad, 2015)) with 2nd order triangular elements providing global 1st order convergence rate in space. It also employs 2nd order Backward Differentencing scheme for time discretization. FEM-solver is not divergence free as it relies upon iterative minimization of velocity divergence at every time step. The immense differences between those solvers are pronounced on finite grids used below and their impact on observer, discretized by FEM-solver, is described as an unknown bounded disturbance $d$.

**Experiment setup.** We take a shifted domain $\Omega = [0, \ell_1] \times [0, \ell_2]$ with $\ell_1 = \ell_2 = 2\pi$, and a destabilizing input is taken to be $g(x, y) = [-5 \sin(10y), 0]^T$. The initial velocity $u(0)$ is generated randomly and taken such that $\|u(0)\|_{H^s} \approx 9.6$. Both solvers do 1000 steps forward in time with timestep $\Delta t = 0.01$. Spatial resolution varies as detailed below.

**Turbulent behavior.** Top panel of Fig. 2 illustrates the sensitivity of NSE to small perturbations of the Initial Condition (IC) measured in $H^1$-norm: red curve shows dynamics of $H^1$-distance between two trajectories, $\tilde{u}$ and $u$ obtained by high-precision FFT-solver on $256 \times 256$-grid with $\Delta t = 0.01$ and the same input $g$. $\tilde{u}$ and $\tilde{v}$ are close by initially, $\|\tilde{u}(0) - u(0)\|_{H^1} \approx 10^{-5}$. If we repeat the same simulation but for $v = 0.1$ NSE becomes stable: $H^1$-distance between two trajectories decays (blue curve). Bottom panel of Fig. 2 shows dynamics of $H^1$-norm of two trajectories with same ICs and input but different $v$: for $v = 0.1$ $H^1$-norm levels off, and the flow is laminar (stable) as shown in Fig. 3(a), in contrast, for $v = 0.01$ $H^1$-norm is changing and the flow is turbulent as shown in Fig. 3(b).

**Known input.** In this test we show that detectability conditions of Theorem 3.1 are indeed simulation friendly: $c$ is taken to be spatial averages over squares $\Omega_i$ covering $\Omega$, and $h^2$ in (2.3) is the area of the largest $\Omega_i$. Thus $h^2$ in fact determines the number of squares (sensors) required for convergence. $h^2$ is found from C1 (Theorem 3.1): since $\delta = 0$ it follows that we can set $R = 0$, $\beta = 1$ (as per Remark 3.2). Also $\Sigma_T = 0$ for any $T > 0$.2

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2 https://github.com/google/jax-cfd
3 https://github.com/mikaem/Oasis
Fig. 3. Turbulence: velocity snapshots.

Hence (3.10) holds for every $T_1 = T > 0$ and $\varepsilon > 0$ such that $\frac{2}{(\nu \lambda_1)} + 2 \varepsilon e^{-\lambda_1 T^*} \| \nabla u_0 \|_{L^2}^2 C_{\delta}^2 \leq \varepsilon T_1$ as $\delta(t^*) = 0$. We set $\varepsilon = 0.001$ and $T_1 = 1.01 \times 2/(\nu \lambda_1)$ so (3.10) holds for $t^*$ such that $2$nd term of the last inequality is less than $0.01 \times 2/(\nu \lambda_1)$. We plug $\kappa = 1 + \varepsilon$ into (3.8) and maximize $\Theta$ by using grid search: we find $T_{\text{max}} = 611$. Hence $h^2 = 0.0029$ as per C1, and $L = 262$. To get outputs we discretize NSE by FEM-solver with quadratic triangular elements constructed on a uniform grid of $256 \times 256$ nodes. Then the outputs are plugged into the observer discretized by the same FEM-solver. The $H^1$-estimation error is given in Fig. 4 for 1, 10 and 50 pressure corrections, which are used in FEM-solver to minimize the numerical divergence: 50 corrections have smallest divergence (red curve). Clearly, reduction of the numerical divergence implies reduction of $H^1$-estimation error.

Uncertain input. In this test we pick $\varepsilon$, $T_1$ and $h$, $L$ as above but use FFT-solver to generate the outputs. The differences between discrete NSE obtained by FEM-solver and FFT-solver on finite grids are significant, and in fact the former can be seen as the latter but with an additive uncertain input $d$ which is expected to get “smaller” for finer grids. And this is exactly what we see in Fig. 5: due to the presence of the disturbance the observer converges into a “zone” which shrinks (in $H^1$-norm) when spatial resolution increases from $256 \times 256$ to $512 \times 512$.

5. Conclusions

We proposed simulation friendly detectability conditions for 2D Navier–Stokes Equation, and designed infinite-dimensional globally converging Luenberger observer for continuous in time measurements. Promising research directions include extending our approach to sampled measurements, and to pointlike measurement where $\Omega_j$ do not necessarily cover the entire domain $\Omega$ as studied in Selivanov and Fridman (2019) for the case of 2D heat equation.

Appendix. Proofs

Proof of Lemma 1.1. Take $\bar{u} = \begin{bmatrix} u & v \end{bmatrix} \in [H^2_\mu(\Omega)]^2$. By Sobolev embedding theorems (see Hutson and Pym (1980, Th.11.3.14)) $H^2(\Omega)$ is continuously embedded into the space of continuous functions $C(\Omega)$ hence the vector-function $\bar{u}$ is continuous. Since $\bar{u}$ is also periodic by Weierstrass approximation theorem (Grafakos, 2008, Corollary 3.1.11.) $\bar{u}$ can be represented as a uniform limit of its Fourier series:

$$\lim_{N \to \infty} \sup_{x \in \Omega} \| \bar{u}(x) - \sum_{|k| \leq N} [u_k \bar{v}_k] e^{2\pi i \frac{k}{L} x} \|_{L^2} = 0 \quad (A.1)$$

with $u_k$ and $v_k$ being complex conjugates of $u_{-k}$ and $v_{-k}$ (since $\bar{u}$ is real-valued), and $u_k = (\epsilon_1 \epsilon_2)^{-1} \int_{\Omega} \bar{u}(x) e^{2\pi i \frac{k}{L} x} dx$, and $v_k$ defined as $u_k$ but with $\epsilon_2$, provided $e_{1,2}$ – canonical basis of $\mathbb{R}^2$. 

Fig. 4. $H^1$ rel. est. error (log-scale) over time (1000 timesteps, $\Delta t = 0.01$): output and observer generated by FEM-solver.

Fig. 5. $H^1$ rel. est. error (log-scale) over time (1000 timesteps, $\Delta t = 0.01$): output generated by FFT-solver, observer – by FEM-solver.
We have:
\[
\|\hat{u}(x)\|_{L^{2}} \leq \|\hat{u}\|_{L^{2}} - \sum_{k \in \mathbb{Z}^{2}} \left( u_{k}^{2} + |v_{k}|^{2} \right)^{2} \|\hat{u}\|_{L^{2}} \tag{A.2}
\]
\[
+ \sum_{k \in \mathbb{Z}^{2}} (|u_{k}|^{2} + |v_{k}|^{2})^{2} \|\hat{u}\|_{L^{2}} \tag{A.3}
\]
\[
\leq \sum_{k \in \mathbb{Z}^{2}} (|u_{k}|^{2} + |v_{k}|^{2})^{2} \tag{A.4}
\]
To get (A.4) one sends $N \to \infty$, invokes (A.1) and recalls that $\|\hat{u}\|_{L^{2}} = 1$ and that $\hat{u} \in L^{2}(\mathbb{R}^{2})$, as if it is continuous, hence the series in (A.4) is converging.

Let us compute $\|\hat{u}\|_{L^{2}}^{2} = \|u\|_{L^{2}}^{2} + \|v\|_{L^{2}}^{2}$, $\|\hat{v}\|_{L^{2}}^{2} = \|u\|_{L^{2}}^{2} + \|v\|_{L^{2}}^{2}$, $\|\hat{\Delta u}\|_{L^{2}}^{2} = \|\Delta u\|_{L^{2}}^{2} + \|\Delta v\|_{L^{2}}^{2}$. Recall Parseval’s identity
\[
\|\hat{u}\|_{L^{2}}^{2} = \ell_{1}\ell_{2} \sum_{k \in \mathbb{Z}^{2}} (|u_{k}|^{2} + |v_{k}|^{2}) \tag{A.5}
\]
and classical relations between the smoothness of a function and decay of its Fourier coefficients (see Grafakos (2008, Theorem 3.2.9)), and differentiate Fourier series of a function of $H^{2}$-class (e.g. Grafakos (2008, p.182)) to compute norms of $\hat{u}$ and $\hat{\Delta u}$:
\[
\|
\hat{v}\|_{L^{2}}^{2} = 4\pi^{2}\ell_{1}\ell_{2} \sum_{k \in \mathbb{Z}^{2}} \left( \frac{\ell}{\ell} \right) (|u_{k}|^{2} + |v_{k}|^{2})^{2} \tag{A.6}
\]
Now we split (A.4) into a finite sum and the remainder:
\[
\|\hat{u}\|_{L^{2}}^{2} \leq \sum_{k \in \mathbb{Z}^{2}} \left( \ell_{1}\ell_{2} \left( 1 + 4\pi^{2}\ell_{1}\ell_{2} \|\hat{u}\|_{L^{2}}^{2} \right) \right) \left( \|\hat{u}\|_{L^{2}}^{2} \right)^{2} \frac{1}{2} \tag{A.7}
\]
and make use of (A.5)-(A.6) to derive (1.2):
\[
\|u\|_{L^{2}} \leq \|\hat{u}\|_{L^{2}} \left( \sum_{k \in \mathbb{Z}^{2}} \left( \ell_{1}\ell_{2} + 4\pi^{2}\ell_{1}\ell_{2} \|\hat{u}\|_{L^{2}}^{2} \right) \right)^{-1} \tag{A.8}
\]
To compute the latter line integral along the circle set $x_{1} = r\sin(t)$ and $x_{2} = r\cos(t)$:
\[
I_{1} = \int_{0}^{2\pi} \int_{0}^{\ell_{1}\ell_{2} + 4\pi^{2}\ell_{1}\ell_{2} \|\hat{u}\|_{L^{2}}^{2}} \frac{drdt}{\pi}\tag{A.9}
\]
\[
\int_{0}^{2\pi} \left( \ell_{1}^{2} + 4\pi^{2}\ell_{2}^{2} \right) \left( \ell_{2}^{2} + 4\pi^{2}\ell_{1}^{2} \right) \frac{drdt}{\pi}\tag{A.10}
\]
\[
\leq \int_{0}^{2\pi} \left( \ell_{1}^{2} + 4\pi^{2}\ell_{2}^{2} \right) \left( \ell_{2}^{2} + 4\pi^{2}\ell_{1}^{2} \right) \frac{drdt}{\pi}\tag{A.11}
\]
Analogously we compute $I_{2} = \frac{\pi^{2}}{2}\|\hat{u}\|_{L^{2}}^{2}$.

**Proof of Proposition 1.** Take $C \in \mathcal{A}_{1}(\Omega, \mathbb{R})$ and set $\tilde{z} = \hat{u} - \tilde{z}$. Subtracting (2.2) from (2.1) we get “the error equation”:
\[
\frac{d}{dt}(\hat{e}, \tilde{z}) + \hat{b}(\hat{e}, \tilde{u}, \tilde{z}) + \hat{b}(\tilde{e}, \hat{u}, \tilde{z}) + \nu(\tilde{e}, \hat{\tilde{z}}) = (\hat{f} - F, \tilde{z}) \tag{A.12}
\]
Note that by (1.7) $\|\hat{v}\|_{L^{2}}^{2} = (\hat{e}, \hat{e}) \to 0$ implies that $\|\hat{e}\|_{L^{2}}^{2} = (\hat{e}, \hat{e}) \to 0$ hence, it is sufficient to demonstrate that $\frac{1}{T} \int_{0}^{T} (\tilde{e}, \hat{\tilde{z}})ds \to 0$ and $\frac{1}{T} - \int_{0}^{T} F(\hat{e}, s) - F(\tilde{e}, s)ds \to 0$ (conditions (A) and (B) of Definition 2.1) imply $V = ((\hat{e}, \hat{e})) \to 0$ if (3.3) holds. To this end we plug $\hat{e} = \tilde{z}$ into the “error equation” (*), which apply simple transformations: (i) recalling from (1.7) and (1.8) that $(\hat{e}, \hat{A}) = ((\hat{e}, \hat{e}))$, $(\hat{e}, \hat{A}) = (\hat{e}, \hat{A})$, and (ii) recalling from (1.11) that $b(\hat{e}, \hat{e}, \hat{A}) = 0$ which implies $b(\hat{e}, \hat{A}) = 0$ $b(\hat{e}, \hat{A}) = 0$ so that
\[
b(\hat{e}, \tilde{z}, \hat{A}) = b(\hat{e}, \hat{u}, \hat{A}) + b(\tilde{z}, \hat{e}, \hat{A}) = b(\hat{e}, \hat{u}, \hat{A}) + b(\tilde{z}, \hat{e}, \hat{A}) \tag{A.13}
\]
we get:
\[
\|\hat{v}\|_{L^{2}}^{2} \leq C_{1} \log 2 \left( 1 + \frac{4\pi^{2}\|\hat{u}\|_{L^{2}}^{2}}{\ell_{1}\ell_{2}} \right) \|\hat{v}\|_{L^{2}}^{2} \tag{A.14}
\]
\[
\|\hat{v}\|_{L^{2}}^{2} \leq C_{1} \log 2 \left( 1 + \frac{4\pi^{2}\|\hat{u}\|_{L^{2}}^{2}}{\ell_{1}\ell_{2}} \right) \|\hat{v}\|_{L^{2}}^{2} \tag{A.15}
\]
Now, by (3.1), (1.8) and (2.3) we get:
\[
\|\hat{e}\|_{L^{2}}^{2} \leq C_{1} \log 2 \left( 1 + \frac{4\pi^{2}\|\hat{u}\|_{L^{2}}^{2}}{\ell_{1}\ell_{2}} \right) \|\hat{v}\|_{L^{2}}^{2} \tag{A.16}
\]
Set $\gamma = \|\hat{u}\|_{L^{2}}^{2}$ for some $\gamma > 0$ and define
\[
\hat{C}_{h, r}(t) = C_{1} C_{2} \log 2 \left( 1 + \frac{4\pi^{2}\|\hat{u}\|_{L^{2}}^{2}}{\ell_{1}\ell_{2}} \right) \|\hat{v}\|_{L^{2}}^{2} \tag{A.17}
\]
\[
\hat{C}_{r}(t) = C_{1} C_{2} \log 2 \left( 1 + \frac{4\pi^{2}\|\hat{u}\|_{L^{2}}^{2}}{\ell_{1}\ell_{2}} \right) \|\hat{v}\|_{L^{2}}^{2} \tag{A.18}
\]
Using $\hat{C}_{h, r}$ and $\hat{C}_{r}$ and noting that $2\|\hat{A}\|_{L^{2}} \|\hat{u}\|_{L^{2}} \leq \|\hat{A}\|_{L^{2}} + \|\hat{u}\|_{L^{2}}$ we transform the upper bound for $b$:
\[
b(\hat{e}, \tilde{z}, \hat{A}) \leq \hat{C}_{h, r}(t) \|\hat{A}\|_{L^{2}} \tag{A.19}
\]
By Schwarz inequality:
\[
\|\hat{f} - F, \tilde{z}\| \leq 1/\|\hat{f} - F\|_{L^{2}} + \nu(4\|\hat{A}\|_{L^{2}}) \tag{A.20}
\]
We plug the latter inequality and (A.9) into (A.7) (recall that \( V = (v, e) \)): 
\[
\dot{V} + (3v^4 - \tilde{C}\Gamma(t))(AE, A\tilde{e}) \leq \beta(t) \\
\beta := \left| f - f_0 \left( \frac{\nu}{\tau} + 0.5C_{\Gamma}(\nu) \right) \right| \lesssim \left\| \tilde{e} \right\|^2_{L^2} + \left\| \tilde{A} \right\|^2_{L^2} \] \]
If we set \( \alpha(t) = (3v^4 - \tilde{C}\Gamma(t)) \) then \( V \) verifies the inequality \( \dot{V}(t) + \alpha(t)V(t) \leq \beta(t) \). To show that \( V \to 0 \) we employ Lemma 1.1 from Foias et al. (2001, p.125), a generalization of the classical Gronwall lemma which in our case reads as follows: if \( V \) verifies \( V(t) + \alpha(t)V(t) \leq \beta(t) \) with the just defined \( \alpha, \beta \) then \( \lim_{t \to \infty} V(t) = 0 \). This is true if \( \limsup_{t \to \infty} \frac{1}{T} \int_t^{t+T} \alpha(t)ds = 0 \) and \( \liminf_{t \to \infty} \frac{1}{T} \int_t^{t+T} \alpha(t)ds > 0 \). We claim that conditions (A) and (B) of Definition 2.1 imply the aforementioned condition for \( \alpha \), and (3.3) implies the required condition on \( \alpha \). To show the former recall from Foias et al. (2001, p.101, A.60) that \( \sup_{T \to \infty} \left\| \tilde{A} \right\|_{L^2} \leq C \), for some \( C > 0 \) which depends on \( \left\| \tilde{f} \right\|_{L^2(R^d, \nu)} \). Similar bound holds for \( A \tilde{e} \) but depends on \( \left\| F \right\|_{L^2(R^d, \nu)} \). Hence \( A \tilde{e} = A \tilde{e} - A \tilde{e} \) is bounded for \( t > T_1 \) and so is \( \tilde{C}(t) \). If conditions (A) and (B) of Definition 2.1 hold then \( \frac{1}{T} \int_t^{t+T} \beta(t)ds \to 0 \) as \( t \to \infty \). Now, to demonstrate condition on \( \alpha \) we first show that (3.3) implies
\[
\frac{1}{T} \int_t^{t+T} \tilde{C}(t)\beta(t)(ds) < \frac{3v^4}{4} \tag{A.10} \]
Indeed, as it follows from (1.14) and (1.15) for any \( 1 < \kappa \epsilon 2 \) there exist \( T^* > 0 \) and \( T > 0 \) such that
\[
\frac{1}{T} \int_t^{t+T} ||A\tilde{e}||_{L^2}^2ds \leq \theta_{T, \epsilon} \leq \frac{\kappa}{\epsilon} \left\| \sum_{k=1}^{(2N+1)

Proof of Theorem 3.1.} We employ the standard argument (Temam, 1995, p.23,S.33) with a difference in energy bounds due to the term \( Lc\tilde{e} \) which we outline below. Consider \( C \in \mathcal{V}(h, \Omega) \). Galerkin projection of (3.4) onto \( W_m \), the span of \( m \) eigen-functions of the Stokes operator \( A \), is obtained by substituting \( \tilde{Z} \) with \( z_m \), the projection of \( \tilde{Z} \) onto \( W_m \), and restricting test-functions to \( \phi \in W_m \), specifically for \( \phi = Az_m \) one gets: 
\[
\frac{d}{dt}(\langle z_m, z_m \rangle) + v\langle Az_m, Az_m \rangle = (\tilde{G} + Lc\tilde{e}, Az_m) \] 
Adding and subtracting \( L(z_m, Az_m) \) and invoking (2.3) after simple manipulations one finds:
\[
\frac{d}{dt}(\langle z_m, z_m \rangle) + v/4(Az_m, Az_m) \leq 1/v\left\| \tilde{G} + Lc\tilde{e} \right\|_{L^2}^2 \tag{A.12} \]
provided \( \nu h^2C_{\mathcal{D}}/2 \leq \nu \). Note that (A.12) is similar to the classical a-priori energy bound for 2D NSE with periodic BC, e.g. Foias et al. (2001, p.102,(A.65)). Then taking \( m \to \infty \) and using the compactness argument (Temam, 1995, p.23,S.33) one deduces lemma’s statement. The case of \( C \in \mathcal{V}(h, \Omega) \) follows by the same logic. \]
\]
with \( \theta \), \( \tau \), defined in (1.15). Now, we bound \( \theta \tau \); recall definitions of \( \kappa \) and \( \delta \), and plug \( \tilde{f} = \tilde{g} + \tilde{d} \) into (1.15). Noting that \( ||f||_{L(\mathcal{X},\mathcal{Y})} \leq R + C_2 \) as \( \tilde{d} \in \delta \), we find that first term in (1.15) is bounded by the 2nd term of (3.10), and, by Cauchy–Schwarz inequality, the 2nd term in (1.15) is bounded by \( \frac{C^2}{\nu^2} + \delta(t^*) \) hence by (3.10):

\[
\theta \tau \leq \frac{2(C^2 + R^2)}{C^2
\tau \nu \lambda_1} + \frac{2\nu^{-1} \nu \| \nabla \tilde{u} \|_{L^2}}{T
\tau \nu} + \delta(t^*) \times \frac{C^2}{\nu^2} + \frac{C^2}{\nu^2} \leq \frac{C^2}{\nu^2}, \quad t > t^*, \quad \tau \geq T
\]

(A.23)

Substituting (A.18), (A.22) and (A.23) into l.h.s. of (A.21), noting that \( \Lambda^\ast \gamma \leq \frac{L^2}{4\beta \psi \Gamma \nu} \) and recalling definition of \( L \) we obtain (A.21).

Let us show that \( W(L, \Gamma) < 0 \) provided \( h \) and \( L \) are chosen as in (C1). Indeed, \( W \) is a quadratic polynomial (in \( L \)) with two discrete real roots \( \tilde{L}_0 = (1 \pm \sqrt{1 - 4a \tilde{L}_0})/2a \), \( a = \tilde{h}^2/(2\beta \tilde{L} \gamma \Gamma) \) provided the discriminant of \( W \) is positive: \( 1 - 4a \tilde{L}_0 > 0 \). The latter implies \( 0 < \tilde{L}_0 < L \), and since \( W(0, \Gamma) = \tilde{L}_0 > 0 \) it follows that \( W(L, \Gamma) > 0 \) for any \( L \in (L_0, L_0) \). Hence, \( W(L, \Gamma) < 0 \) for \( \tilde{L}_0 < L \), \( L = \beta \tilde{h}^2 \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gamma \gy
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