

Averaging-based ISS analysis of systems with rapidly varying periodic delays

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Abstract—We study the stability of a linear system with a periodic delay that is rapidly varying around a nominal value. We are inspired by previous results on rapidly varying periodic delays, where the stabilizing effect of the rapidly-varying periodic delay on the system stability was studied via a trajectory-based comparison to a suitable comparison (averaged) system with distributed delays. The stability of the original system was proved to be asymptotic, provided the delay modulation was rapid enough, but quantitative bounds on the delay period which preserve the stability of the original system were not provided. In this work we aim to provide such quantitative bounds for the exponential stability of the original system. We introduce a novel state transformation that explicitly maps the original system into a perturbed version of the comparison system. Then, assuming knowledge of an exponential bound on the norm of the fundamental solution of the comparison system, and augmenting the state of the original system with that of the transformed system, we employ a recently developed trajectory-based inequality to derive constructive conditions on the delay period and on the exponential decay rate of the original system. In the case of perturbed linear systems, we prove that the same delay periods guarantee input-to-state stability (ISS) of the original system. Numerical examples demonstrate the efficacy of the proposed approach.

Index Terms—Time-delay systems, stability, averaging.

I. INTRODUCTION

Systems with periodic or almost periodic components are prevalent in physics and engineering. Among their numerous uses, one can find vibrational control [1], power systems [2] and time-delay systems [3]. An important application stemming from mechanical engineering is the study of the dynamics of rotating cutting machines [4], [5]. Such dynamical models are often characterized by a linear time-delay system with a delay that is rapidly varying around a nominal constant value. In order to avoid undesirable oscillations and chatter, the stability region of the steady-state solution of cutting machines is enlarged by rapidly modulating the rotational speed of the machine around a nominal value, which correspond to rapidly varying time-delay.

The technique of averaging, emanating from perturbation theory, is a widely used analysis tool for studying systems with periodic components [6], which has been successfully employed for systems with oscillatory control inputs [7], [8]

and switched systems [9]–[11]. The central idea in averaging is that stability of the first-order averaged system guarantees stability of the original periodic system for small enough values of the time-scale parameter. However, this often leads to only qualitative existence results, without yielding an efficient and explicit bound on the small parameter for which the stability of the original system is preserved. Recently, the first efficient quantitative methods for stability by averaging were suggested. A constructive time-delay approach to periodic averaging of a system with a single rapid time-scale was suggested in [12]–[14]. A complimentary delay-free technique, which employs simpler analysis tools and is well-suited for handling multi-scale systems was developed in [15], [16].

It is well-known that in some time-delay systems, delay intervals for the constant and time-varying delays may have “opposite” effects on stability. This phenomenon is characterized by stability of the system in the presence of time-varying delays and instability of the same system in the case of constant delays (or vice-versa), and is often referred to as “quenching” [17]. Motivated by this, the stabilizing effect of periodic delay that is rapidly varying around a constant value was studied in [18] for linear systems with constant coefficients, where the system with the constant nominal delay may be unstable. The authors of [18] introduced a linear time-invariant comparison system with a distributed delay, whose analysis is more tractable than the analysis of the original system, and whose trajectories are comparable to the trajectories of the original system in an appropriate sense. The authors then combine asymptotic averaging with estimation of the difference between the trajectories of the original and averaged systems (see Appendix B in [18]) to show that exponential stability of the comparison system implies the asymptotic stability of the original system, provided the delay period is small enough. However, the approach in [18] does not provide upper bounds on the delay period which guarantees exponential stability of the original system. This work builds upon [18] and provides a significant quantitative extension of the results therein by employing a different analysis approach that relies on a novel system transformation and a trajectory-based inequality [20]. Under the assumption that an exponential bound on the norm of the fundamental solution of the comparison system is known, we show how explicit estimates on the delay period which guarantees stability of the system with a rapidly-varying delay can be obtained constructively. We first introduce a novel state transformation that maps the original system into a perturbed version of the comparison system. Then, augmenting the state of the original system with that of the transformed system, we employ a recently developed trajectory-based inequality

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to derive a constructive upper bound on the delay period under which exponential stability of the original system is guaranteed. We further provide explicit estimates on the exponential decay rate of the original system. The availability of explicit and constructive estimates, which are guaranteed by our method, then allows to easily extend the proposed approach to input-to-state stability (ISS) analysis of time-delay systems with rapid time-varying delays, subject to essentially bounded disturbances. Numerical examples validate the presented method and demonstrate its efficacy.

Preliminaries:

The following two lemmas will be used subsequently.

Lemma 1 (Delayed Gronwall-Bellman inequality [19]): Let $t_0 \in \mathbb{R}$, $t_0 < T \leq \infty$ and $c \geq 0$. Let $a : [t_0, T) \rightarrow [0, \infty)$ be locally integrable. Assume $r \geq 0$ and $\tau : [t_0, T) \rightarrow [0, \infty)$ be measurable such that

$$t_0 - r \leq t - \tau(t), \quad t \in [t_0, T).$$

If $x : [t_0 - r, \infty) \rightarrow [0, \infty)$ is Borel measurable and locally bounded such that

$$x(t) \leq c + \int_{t_0}^t a(u)x(u - \tau(u))du, \quad t \in [t_0, T),$$

then

$$x(t) \leq K \exp \left(\int_{t_0}^t a(s)ds \right), \quad t \in [t_0, T)$$

where

$$K := \max \left(c, \|x\|_{L^\infty[t_0-r, t_0]} \right).$$

Lemma 2 (Trajectory-based inequality [20]): Let $T > 0$ and $\psi = (\psi_1, \dots, \psi_n)^\top : [-T, +\infty) \rightarrow [0, \infty)^n$ satisfy

$$\psi(t) \leq S \begin{bmatrix} \sup_{\ell \in [t-T, t]} \psi_1(\ell) \\ \vdots \\ \sup_{\ell \in [t-T, t]} \psi_n(\ell) \end{bmatrix}, \quad t \geq 0,$$

where the inequality is understood componentwise. Assume that $S > 0$ (entrywise) is Schur stable and let $v \in [1, +\infty)^n$ and $\chi \in (0, 1)$ be the Perron-Frobenius eigenvector-eigenvalue pair of S , satisfying $Sv = \chi v$. Then, componentwise, we have

$$\psi(t) \leq \left(\sum_{j=1}^n \sup_{\ell \in [-T, 0]} \psi_j(\ell) \right) e^{\frac{\ln(\chi)t}{T}} v, \quad t \geq 0.$$

II. STABILITY OF A LINEAR SYSTEM WITH RAPIDLY-VARYING PERIODIC DELAYS

A. Problem formulation

We consider the following system with time-varying delay

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bx(t - \tau(t)), \\ x(t) &= \phi(t), \quad t \in [-\tau_M, 0] \end{aligned} \quad (1)$$

where $x(t) \in \mathbb{R}^{n_x}$, $A, B \in \mathbb{R}^{n_x \times n_x}$, $\tau_M > 0$ and $\phi \in C[-\tau_M, 0]$. The delay $\tau(t) \geq 0$ may be rapidly-varying and with a derivative that is not bounded uniformly in $\epsilon > 0$, and has the form

$$\tau(t) = \tau_0 + \delta f\left(\frac{t}{\epsilon}\right) \quad (2)$$

with $\epsilon, \delta, \tau_0 > 0$, $\tau_0 \geq \delta$ and $f : \mathbb{R} \rightarrow [-1, 1]$ a T -periodic function with $\int_0^T f(s)ds = 0$. Our assumption on f guarantees

two things. First, the delay $\tau(t)$ is ϵT -periodic, whence ϵ plays the role of delay frequency. Second, the average value of the delay $\tau(t)$ is exactly the nominal delay τ_0 , since

$$\frac{1}{\epsilon T} \int_0^{\epsilon T} \tau(s)ds - \tau_0 = \frac{\delta}{T} \int_0^T f(v)dv = 0.$$

We further denote $\tau_M := \tau_0 + \delta$. Thus, we consider an interval (or non-small) delay $\tau(t) \in [\tau_0 - \delta, \tau_0 + \delta]$ that can be treated by Lyapunov methods or input-output approach provided (1) with $\delta = 0$ (i.e. with the constant discrete delay τ_0) is exponentially stable [21]. However, it may be that (1) with $\delta = 0$ is unstable. In this case, the so called "quenching" phenomenon may manifest. "Quenching" happens when the addition of (rapidly oscillating) time-varying delays to the nominal constant delay, via $\delta > 0$, leads to stability of (1). In this case, it is challenging to employ Lyapunov methods in order to obtain a bound on the delay period which guarantees stability of (1). Here, we aim to derive stability conditions that may hold when (1) with $\delta = 0$ is unstable and "quenching" occurs. In order to simplify the analysis, this section considers asymptotic stability of the system (1). ISS of (1) subject to locally essentially bounded disturbances will be treated subsequently.

Following [18], we employ the existence of an integrable function $\omega : [-1, 1] \rightarrow [0, \infty)$ satisfying

$$\int_{-1}^1 \alpha(s)\omega(s)ds = \frac{1}{T} \int_0^T \alpha(-f(s))ds \quad (3)$$

for every continuous function $\alpha : [-1, 1] \rightarrow \mathbb{R}$. In particular, setting $\alpha \equiv 1$

$$\int_{-1}^1 \omega(s)ds = 1. \quad (4)$$

The existence of such a function ω is guaranteed by application of the Radon-Nikodym theorem (see [18]).

Remark 1: Since $\alpha \circ (-f)$ is T -periodic, we have that (3) continues to hold if the right-hand side is replaced by $\frac{1}{T} \int_t^{t+T} \alpha(-f(s))ds$ with arbitrary $t \in \mathbb{R}$.

Consider the comparison system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + \int_{-\tau_M}^0 A_1(\xi)x(t + \xi)d\xi, \\ A_1(\xi) &:= \frac{1}{\delta} B\omega\left(\frac{\xi + \tau_0}{\delta}\right) \mathbb{1}_{[-\tau_M, -\tau_0 + \delta]}(\xi) \end{aligned} \quad (5)$$

where $\mathbb{1}_{\mathcal{I}}$ is the indicator function of the set $\mathcal{I} \subseteq \mathbb{R}$. Note that the right-hand side of (5) is well-defined since $\xi \in [-\tau_0 - \delta, -\tau_0 + \delta]$ implies that $\frac{\xi + \tau_0}{\delta} \in [-1, 1]$. Throughout this note we denote by $\Phi(t)$ the fundamental solution of (5), namely the solution to the *matrix* system (5) with $x(t)$ replaced by $\Phi(t)$ and initial condition

$$\Phi(t) = \begin{cases} 0, & t < 0 \\ I_n & t = 0 \end{cases}. \quad (6)$$

Here, $I_n \in \mathbb{R}^{n \times n}$ is the identity matrix.

By [18, Theorem 1], exponential stability of the system (5) implies that the system (1) is asymptotically stable for small enough $\epsilon > 0$. However, the results of [18] do not provide an explicit bound on $\epsilon > 0$ which guarantees asymptotic stability of (1). The aim of this note is to provide explicit conditions on $\epsilon > 0$, which guarantee the asymptotic stability

of (1), provided an exponential upper bound on the norm of the fundamental solution of (5) is available. Since we assume the same conditions as [18], we have the following assumption:

Assumption 1: There exist known constants $\sigma > 0$ and $M_\Phi \geq 1$ such that the fundamental solution of the comparison system (5) satisfies

$$\|\Phi(t)\| \leq M_\Phi e^{-\sigma t}, \quad t \geq 0. \quad (7)$$

To apply our method, we only need to have *some* constants on M_Φ and σ for which (7) holds. Such constants can be estimated from a *single simulation* of the comparison system (5), *which does not involve* ϵ . For some cases of the density ω in (5), M_Φ and σ can be upper bounded analytically. However, to the best of our knowledge, there does not exist a theoretical result that specifies an explicit relation between M_Φ , σ and ω , in general.

The next assumption is a technical one and aimed at simplifying the expressions and bounds which will be derived subsequently. One can replace the upper bound therein by any fixed constant, while preserving the validity of the results.

Assumption 2: The parameter $\epsilon > 0$ satisfies $\epsilon T \|B\| < 2$. Note that since T and $\|B\|$ are assumed known, ϵ can always be chosen such that Assumption 2 holds.

Before proceeding, we give an a priori bound on the growth of the solution to (1), which will be used below.

Proposition 1: Consider the system (1) and fix $\beta > 0$. There exists an explicit constant $M_x(\beta) \geq 1$ such that

$$|x(t)| \leq M_x(\beta) \|\phi\|_{C[-\tau_M, 0]}, \quad t \in [0, \beta]. \quad (8)$$

Proof: Employing variation of constants in (1), we have

$$\begin{aligned} |x(t)| &\leq \|e^{At}\| |x(0)| + \int_0^t \|e^{A(t-s)}\| \|B\| |x(s - \tau(s))| ds \\ &\leq e^{\|A\|\beta} |x(0)| + e^{\|A\|\beta} \|B\| \int_0^t |x(s - \tau(s))| ds, \end{aligned}$$

where $t \in [0, \beta]$. We can now use Lemma 1 to obtain (8) with

$$M_x(\beta) = e^{(\|A\| + \|B\| e^{\|A\|\beta})\beta}.$$

This completes the proof of the proposition. $\blacksquare$

B. System transformation

We now introduce a transformation which explicitly relates the systems (1) and (5). Inspired by [15], we define

$$G(t) = \frac{1}{\epsilon T} B \int_t^{t+\epsilon T} (t + \epsilon T - s) x(t - \tau(s)) ds, \quad t \geq 0. \quad (9)$$

Recalling (1), we note that $G(t)$ is differentiable for $t \geq \tau_M$. Differentiating $G(t)$ for $t \geq \tau_M$, we have

$$\begin{aligned} \dot{G}(t) &= -Bx(t - \tau(t)) + \frac{1}{\epsilon T} B \int_t^{t+\epsilon T} x(t - \tau(s)) ds \\ &\quad + \frac{1}{\epsilon T} B \int_t^{t+\epsilon T} (t + \epsilon T - s) \dot{x}(t - \tau(s)) ds. \end{aligned} \quad (10)$$

Substituting (1) into the last term on the right-hand side of (10), we obtain

$$\begin{aligned} &\frac{1}{\epsilon T} B \int_t^{t+\epsilon T} (t + \epsilon T - s) \dot{x}(t - \tau(s)) ds \\ &= \frac{1}{\epsilon T} B A \int_t^{t+\epsilon T} (t + \epsilon T - s) x(t - \tau(s)) ds \\ &\quad + \frac{1}{\epsilon T} B^2 \int_t^{t+\epsilon T} (t + \epsilon T - s) x(t - \rho(t, s)) ds \end{aligned} \quad (11)$$

where $t \geq \tau_M$ and

$$\rho(t, s) := \tau(s) + \tau(t - \tau(s)), \quad \tau_M \leq t \leq s \leq t + \epsilon T \quad (12)$$

satisfies $|\rho(t, s)| \leq 2\tau_M$ for any $\tau_M \leq t \leq s \leq t + \epsilon T$. Denoting $\alpha_t(\mu) := x(t - \tau_0 + \delta\mu)$, where $t \geq \tau_M$ is fixed. Recalling (2), we see that the second term on the right hand-side of (10) can be presented as

$$\begin{aligned} &\frac{1}{\epsilon T} B \int_t^{t+\epsilon T} x(t - \tau(s)) ds = \frac{1}{T} B \int_{\frac{t}{\epsilon}}^{\frac{t}{\epsilon}+T} \alpha_t(-f(\xi)) d\xi \\ &= B \int_{-1}^1 \alpha_t(\xi) \omega(\xi) d\xi = \int_{-\tau_M}^0 A_1(\xi) x(t + \xi) d\xi. \end{aligned} \quad (13)$$

Here the second equality follows by application of the identity (3) and Remark 1.

We are now ready to finally introduce the proposed system transformation. Recalling (1) and (9), we employ the following transformation

$$z(t) = x(t) + G(t), \quad t \geq 0. \quad (14)$$

Taking into account (1), (10), (11) and (13), and differentiating $z(t)$ for $t \geq \tau_M$, we obtain

$$\begin{aligned} \dot{z}(t) &= Az(t) + \int_{-\tau_M}^0 A_1(\mu) z(t + \mu) d\mu - AG(t) \\ &\quad + H_0(t) + H_1(t) + H_2(t), \quad t \geq \tau_M, \end{aligned} \quad (15)$$

where for $t \geq \tau_M$

$$\begin{aligned} H_0(t) &:= -B \int_{t-\tau_0-\delta}^{t-\tau_0+\delta} \frac{1}{\delta} \omega\left(\frac{\mu-t+\tau_0}{\delta}\right) G(\mu) d\mu, \\ H_1(t) &:= \frac{1}{\epsilon T} B A \int_t^{t+\epsilon T} (t + \epsilon T - s) x(t - \tau(s)) ds, \\ H_2(t) &:= \frac{1}{\epsilon T} B^2 \int_t^{t+\epsilon T} (t + \epsilon T - s) x(t - \rho(t, s)) ds. \end{aligned} \quad (16)$$

Note that for small $\epsilon > 0$ and $x(t)$ of the order of $O(1)$, we have $G(t) = O(\epsilon)$ in (9), as well as $H_i(t) = O(\epsilon)$, $i = 0, 1, 2$ in (16). Thus, (15) forms an $O(\epsilon)$ perturbation of the comparison (i.e. averaged) system (5).

Differently from the analysis in [18], we will analyze the *coupled* systems (1) and (15) (note that the coupling is manifested through both the transformation (14), and through the last four terms on the right-hand side of (15)). Subject to Assumption 1, we employ a recently established trajectory-based inequality [20], [22], to derive estimates on $\epsilon > 0$ in (2) for which exponential stability of (1) holds. We conclude this section with two a priori estimates on $G(t)$ and $z(t)$.

Proposition 2: Let $\beta > 0$ and recall the notation $M_x(\beta)$ from Proposition 1. Then, the following inequalities hold

$$\begin{aligned} |G(t)| &\leq M_x(\beta) \|\phi\|_{C[-\tau_M, 0]}, \quad t \in [0, \beta], \\ |z(t)| &\leq 2M_x(\beta) \|\phi\|_{C[-\tau_M, 0]}, \quad t \in [0, \beta]. \end{aligned} \quad (17)$$

Proof: The first inequality follows from Proposition 1, Assumption 2 and the following upper bound, as follows

$$\begin{aligned} |G(t)| &\leq \frac{\epsilon T \|B\|}{2} \sup_{\mu \in [t-\tau_M, t]} |x(\mu)| \\ &\leq \frac{\epsilon T \|B\| M_x(\beta)}{2} \|\phi\|_{C[-\tau_M, 0]}, \quad t \in [0, \beta]. \end{aligned}$$

As for the second inequality, by (14) and the first line of (17)

$$|z(t)| \leq |x(t)| + |G(t)| \leq 2M_x(\beta) \|\phi\|_{C[-\tau_M, 0]}, \quad t \in [0, \beta]. \quad \blacksquare$$

C. Trajectory-based analysis of the (x, z) system

Here we analyze the trajectories of the (x, z) system directly and employ a recently developed trajectory-based inequality to infer stability of the system. The next proposition is used in the application of the trajectory-based inequality of Lemma 2.

Proposition 3: The following upper bounds are satisfied:

$$\begin{aligned} |G(t)| &\leq \frac{\epsilon T \|B\|}{2} \sup_{s \in [t-\tau_M, t]} |x(s)|, \quad t \geq 0, \\ |H_0(t)| &\leq \frac{\epsilon T \|B\|^2}{2} \sup_{s \in [t-2\tau_M, t]} |x(s)|, \quad t \geq \tau_M, \\ |H_1(t)| &\leq \frac{\epsilon T \|A\| \|B\|}{2} \sup_{s \in [t-\tau_M, t]} |x(s)|, \quad t \geq \tau_M, \\ |H_2(t)| &\leq \frac{\epsilon T \|B\|^2}{2} \sup_{s \in [t-2\tau_M, t]} |x(s)|, \quad t \geq \tau_M. \end{aligned} \quad (18)$$

Proof: The estimate on $|G(t)|$, $t \geq 0$ follows immediately from (9) and the fact that

$$\frac{1}{\epsilon T} \|B\| \int_t^{t+\epsilon T} (t + \epsilon T - s) ds = \frac{\epsilon T \|B\|}{2}.$$

For $H_0(t)$, $t \geq \tau_M$, we recall (4) and employ (16) to obtain

$$\begin{aligned} |H_0(t)| &\leq \|B\| \sup_{\mu \in [t-\tau_M, t]} |G(\mu)| \\ &\leq \frac{\epsilon T \|B\|^2}{2} \sup_{s \in [t-2\tau_M, t]} |x(s)|, \quad t \geq \tau_M \end{aligned}$$

where we used the upper bound on $G(t)$, derived earlier. The remaining upper bounds on $|H_1(t)|$, $t \geq \tau_M$ and $|H_2(t)|$, $t \geq \tau_M$ are proved in a similar manner. ■

We now recall the transformation (14) and note that it implies

$$\begin{aligned} |x(t)| &\leq |z(t)| + \frac{\epsilon T \|B\|}{2} \sup_{s \in [t-\tau_M, t]} |x(s)| \\ &\leq \left[\frac{\epsilon T \|B\|}{2} \mathbf{1} \right] \left[\sup_{s \in [t-2\tau_M-\tau_*, t]} |x(s)| \right] \left[\sup_{s \in [t-2\tau_M-\tau_*, t]} |z(s)| \right], \quad t \geq 2\tau_M + \tau_* \end{aligned} \quad (19)$$

where $\tau_* > 0$ will be specified later. In order to obtain a similar estimate on $z(t)$, we first employ (15) and the variation of constants formula [21, Equation 2.12] to present the solution of (15) as

$$\begin{aligned} z(t+s) &= \Phi(t)z(s) + \int_0^t \Phi(t-\mu)F(\mu)d\mu \\ &\quad + \int_{-\tau_M}^0 \left(\int_0^{\theta+\tau_M} \Phi(t-\rho)A_1(\theta-\rho)d\rho \right) z(\theta+s)d\theta, \end{aligned} \quad (20)$$

where $t \geq 0$, $s \geq \tau_M$ and

$$F(q) := -AG(q+s) + \sum_{i=0}^2 H_i(q+s), \quad q \geq 0. \quad (21)$$

Note that here [21, Equation 2.12] was employed as follows. First, recalling (15), we choose $A_0 = A$, $A_i = 0$ for all $i = 1, \dots, K$ and $A_d = A_1$. Second, consider the remaining term $\Sigma(t) = -AG(t) + H_0(t) + H_1(t) + H_2(t)$ in (15). Noting that $\Sigma(t)$ depends only on $x(t)$, which is continuous, and that the (x, z) system has a cascade structure in which $x(t)$ flows into the z -system, we see that $\Sigma(t)$ can be substituted for $f(t)$ in [21, Equation 2.12],

Recalling (7), we proceed with upper bounding the terms on the right-hand side of (20). First, we have

$$|\Phi(t)z(s)| \stackrel{(7)}{\leq} M_\Phi e^{-\sigma t} |z(s)|. \quad (22)$$

Second, employing Fubini's theorem, we obtain

$$\begin{aligned} &\left| \int_{-\tau_M}^0 \left(\int_0^{\theta+\tau_M} \Phi(t-\rho)A_1(\theta-\rho)d\rho \right) z(\theta+s)d\theta \right| \\ &\leq M_\Phi \int_{-\tau_M}^0 \left(\int_0^{\theta+\tau_M} e^{-\sigma(t-\rho)} \|A_1(\theta-\rho)\| d\rho \right) d\theta \\ &\quad \times \sup_{q \in [s-\tau_M, s]} |z(q)| \\ &\leq M_\Phi \int_0^{\tau_M} \left(\int_{\rho-\tau_M}^0 e^{-\sigma(t-\rho)} \|A_1(\theta-\rho)\| d\theta \right) d\rho \\ &\quad \times \sup_{q \in [s-\tau_M, s]} |z(q)| \\ &= M_\Phi \int_0^{\tau_M} e^{-\sigma(t-\rho)} \left(\int_{-\tau_M}^{-\rho} \|A_1(\theta)\| d\theta \right) d\rho \\ &\quad \times \sup_{q \in [s-\tau_M, s]} |z(q)| \\ &\leq \frac{M_\Phi \|B\|}{\sigma} (e^{\sigma\tau_M} - 1) e^{-\sigma t} \sup_{q \in [s-\tau_M, s]} |z(q)|. \end{aligned} \quad (23)$$

Finally, we proceed to upper bound the term in (20), which involves F , defined in (21). We do this by employing Proposition 3 and obtain

$$\begin{aligned} &\left| \int_0^t \Phi(t-\mu)AG(\mu+s)d\mu \right| \\ &\leq M_\Phi \|A\| \int_0^t e^{-\sigma(t-\mu)} |G(\mu+s)| d\mu \\ &\stackrel{(18)}{\leq} \frac{M_\Phi \epsilon T \|A\| \|B\|}{2} \left(\int_0^t e^{-\sigma(t-\mu)} d\mu \right) \sup_{q \in [s-\tau_M, s+t]} |x(q)| \\ &= \frac{M_\Phi \epsilon T \|A\| \|B\|}{2\sigma} (1 - e^{-\sigma t}) \sup_{q \in [s-\tau_M, s+t]} |x(q)|. \end{aligned} \quad (24)$$

By similar arguments, the following bounds are also satisfied

$$\begin{aligned} &\left| \int_0^t \Phi(t-\mu)H_0(\mu+s)d\mu \right| \\ &\leq \frac{M_\Phi \epsilon T \|B\|^2}{2\sigma} (1 - e^{-\sigma t}) \sup_{q \in [s-2\tau_M, s+t]} |x(q)|, \\ &\left| \int_0^t \Phi(t-\mu)H_1(\mu+s)d\mu \right| \\ &\leq \frac{M_\Phi \epsilon T \|A\| \|B\|}{2\sigma} (1 - e^{-\sigma t}) \sup_{q \in [s-\tau_M, s+t]} |x(q)|, \\ &\left| \int_0^t \Phi(t-\mu)H_2(\mu+s)d\mu \right| \\ &\leq \frac{M_\Phi \epsilon T \|B\|^2}{2\sigma} (1 - e^{-\sigma t}) \sup_{q \in [s-2\tau_M, s+t]} |x(q)|. \end{aligned} \quad (25)$$

By combining (20)-(25) and replacing the variables as $t \rightarrow t+s$, $\tau_* \rightarrow t$ and $t-\tau_* \rightarrow s$, we have for $t \geq \tau_M + \tau_*$

$$\begin{aligned} |z(t)| &\leq M_\Phi e^{-\sigma\tau_*} |z(t-\tau_*)| \\ &\quad + \frac{M_\Phi \|B\|}{\sigma} (e^{\sigma\tau_M} - 1) e^{-\sigma\tau_*} \sup_{q \in [t-\tau_*, t-\tau_M, t]} |z(q)| \\ &\quad + \frac{M_\Phi \epsilon T \|A\| \|B\|}{\sigma} (1 - e^{-\sigma\tau_*}) \sup_{q \in [t-\tau_*, t-\tau_M, t]} |x(q)| \\ &\quad + \frac{M_\Phi \epsilon T \|B\|^2}{\sigma} (1 - e^{-\sigma\tau_*}) \sup_{q \in [t-\tau_*, t-2\tau_M, t]} |x(q)|. \end{aligned} \quad (26)$$

Denoting

$$\begin{aligned} L_1(\sigma, \tau_*, \tau_M) &:= M_\Phi e^{-\sigma\tau_*} [1 + \sigma^{-1} \|B\| (e^{\sigma\tau_M} - 1)], \\ L_2(\sigma, \tau_*) &:= \sigma^{-1} M_\Phi T \|B\| (\|A\| + \|B\|) (1 - e^{-\sigma\tau_*}) \end{aligned} \quad (27)$$

we conclude that

$$\begin{aligned} |z(t)| &\leq [\epsilon L_2(\sigma, \tau_*) L_1(\sigma, \tau_*, \tau_M)] \\ &\quad \times \left[\sup_{s \in [t-2\tau_M-\tau_*, t]} |x(s)| \right] \left[\sup_{s \in [t-2\tau_M-\tau_*, t]} |z(s)| \right], \quad t \geq 2\tau_M + \tau_*. \end{aligned} \quad (28)$$

Overall, from (19) and (28) we have for $t \geq 2\tau_M + \tau_*$

$$\begin{aligned} \begin{bmatrix} |x(t)| \\ |z(t)| \end{bmatrix} &\leq \Upsilon(\epsilon, \sigma, \tau_*, \tau_M) \begin{bmatrix} \sup_{s \in [t-2\tau_M-\tau_*, t]} |x(s)| \\ \sup_{s \in [t-2\tau_M-\tau_*, t]} |z(s)| \end{bmatrix}, \\ \Upsilon(\epsilon, \sigma, \tau_*, \tau_M) &:= \begin{bmatrix} \frac{\epsilon T \|B\|}{2} & 1 \\ \epsilon L_2(\sigma, \tau_*) L_1(\sigma, \tau_*, \tau_M) \end{bmatrix}, \end{aligned} \quad (29)$$

where the inequality is interpreted coordinatewise. Note that

aside from ϵ , and τ_* , all parameter in $\Upsilon(\epsilon, \sigma, \tau_*, \tau_M)$ are assumed to be known. The parameters ϵ, τ_* in (29) can then be tuned to guarantee Schur stability of $\Upsilon(\epsilon, \sigma, \tau_*, \tau_M)$. By [20, Lemma 5], Schur stability of $\Upsilon(\epsilon, \sigma, \tau_*, \tau_M)$ implies that for $t \geq 2\tau_M + \tau_*$

$$\begin{aligned} \begin{bmatrix} |x(t)| \\ |z(t)| \end{bmatrix} &\leq v e^{\frac{\ln(\chi)}{2\tau_M + \tau_*}(t - 2\tau_M - \tau_*)} \\ &\times \left[\sup_{s \in [0, 2\tau_M + \tau_*]} |x(s)| + \sup_{s \in [0, 2\tau_M + \tau_*]} |z(s)| \right] \end{aligned} \quad (30)$$

where $\chi \in (0, 1)$ is the Perron eigenvalue (the real eigenvalue of largest magnitude) and $v \in [1, \infty)^2$ is the corresponding right Perron eigenvector of the matrix $\Upsilon(\epsilon, \sigma, \tau_*, \tau_M)$ with positive entries. The following proposition provides a sufficient condition for Schur stability of $\Upsilon(\epsilon, \sigma, \tau_*, \tau_M)$. For brevity, we suppress the dependencies of the expressions on their respective arguments.

Proposition 4: Υ is Schur stable if

$$\begin{aligned} \varphi_1 + \sqrt{(1 + \varphi_1)^2 - 4\varphi_2^2} &< 1, \\ \varphi_1 &:= \frac{1}{4}(\epsilon T \|B\|)^2 + \epsilon^2 L_2^2 + L_1^2, \quad \varphi_2 := \frac{1}{2}\epsilon T \|B\| L_1 - \epsilon L_2. \end{aligned} \quad (31)$$

Proof: Recall that $\|\Upsilon\|^2 = \lambda_{\max}(\Upsilon^\top \Upsilon)$. The characteristic polynomial of $\Upsilon^\top \Upsilon$ is given by

$$p(s) = s^2 - \|\Upsilon\|_F^2 s + (\det(\Upsilon))^2 = s^2 - (1 + \varphi_1)s + \varphi_2^2,$$

where $\|\cdot\|_F$ is the Frobenius norm. It can be verified that the discriminant of $p(s)$ is positive. Hence,

$$\|\Upsilon\|^2 < 1 \Leftrightarrow \frac{1 + \varphi_1 + \sqrt{(1 + \varphi_1)^2 - 4\varphi_2^2}}{2} < 1,$$

which is equivalent to (31). Since $\|\Upsilon\|$ is an upper bound on the spectral radius of Υ , the claim of the proposition follows. ■

Summarizing, we arrive at

Theorem 1: Consider the system (1) with a delay $\tau(t)$ of the form (2) with $\epsilon, \delta > 0$, $\tau_0 \geq \delta$, $\tau_0 + \delta = \tau_M$ and $f: \mathbb{R} \rightarrow [-1, 1]$ a T -periodic function with $\int_0^T f(s)ds = 0$. Consider the comparison (averaged) system (5) with the integrable kernel $\omega: [-1, 1] \rightarrow [0, \infty)$ satisfying (3) for every continuous function $\alpha: [-1, 1] \rightarrow \mathbb{R}$. Let Assumption 1 hold with known constants $\sigma > 0$ and $M_\Phi \geq 1$. Let $\tau_* > 0$ and $\epsilon_* > 0$ be tuning parameters such that $\epsilon T \|B\| < 2$ and $\Upsilon(\epsilon_*, \sigma, \tau_*, \tau_M)$, given in (29), is Schur stable. Then, for all $\epsilon \in (0, \epsilon_*]$, the system (1) is uniformly exponentially stable, meaning that there exist constants $M_* \geq 1$ and $\kappa > 0$, independent of $\epsilon \in (0, \epsilon_*]$, satisfying

$$|x(t)| \leq M_* e^{-\kappa t} \|\phi\|_{C[-\tau_M, 0]}, \quad t \geq 0, \quad \forall \epsilon \in (0, \epsilon_*]. \quad (32)$$

Moreover, there always exist $\tau_* > 0$ large enough and $\epsilon_* > 0$ small enough such that $\Upsilon(\epsilon_*, \sigma, \tau_*, \tau_M)$ is Schur stable.

Proof: For any $\epsilon \in (0, \epsilon_*]$, let $\chi(\epsilon)$ and $v(\epsilon) \in [1, \infty)^2$ be the Perron eigenvalue and right eigenvector of $\Upsilon(\epsilon, \sigma, \tau_*, \tau_M)$, respectively. Consider first $\Upsilon(0, \sigma, \tau_*, \tau_M)$. A direct computation of its characteristic polynomial yields

$$p_0(z) = z(z - L_1(\sigma, \tau_*, \tau_M)). \quad (33)$$

Note that both eigenvalues of $\Upsilon(0, \sigma, \tau_*, \tau_M)$ are simple.

Denote by $v(0)$ the eigenvector of $\Upsilon(0, \sigma, \tau_*, \tau_M)$ corresponding to the eigenvalue $L_1(\sigma, \tau_*, \tau_M)$, where $L_1(\sigma, \tau_*, \tau_M) \in (0, 1)$ provided τ_* is large enough.

Let $\tau_*, \epsilon_* > 0$ be such that $\Upsilon(\epsilon_*, \sigma, \tau_*, \tau_M)$ is Schur stable. Then, for any $0 < \epsilon_1 \leq \epsilon_2 \leq \epsilon_*$, we have $0 < \Upsilon(\epsilon_1, \sigma, \tau_*, \tau_M) \leq \Upsilon(\epsilon_2, \sigma, \tau_*, \tau_M)$, where the inequality is understood coordinatewise. By [23, Corollary 8.1.19], the spectral radius of $\Upsilon(\epsilon, \sigma, \tau_*, \tau_M)$ is monotonically decreasing as $\epsilon \searrow 0$. Hence, $\Upsilon(\epsilon, \sigma, \tau_*, \tau_M)$ is Schur for all $0 < \epsilon \leq \epsilon_*$. Furthermore, $0 < \chi(\epsilon) \leq \chi(\epsilon_*) =: q < 1$ for all $\epsilon \in (0, \epsilon_*]$.

As the eigenvalue $L_1(\sigma, \tau_*, \tau_M)$ of $\Upsilon(0, \sigma, \tau_*, \tau_M)$ is simple, we have [24, Chapter 2]

$$\lim_{\epsilon \rightarrow 0^+} v(\epsilon) = v(0), \quad \lim_{\epsilon \rightarrow 0^+} \chi(\epsilon) = L_1(\sigma, \tau_*, \tau_M) \in (0, q]. \quad (34)$$

Hence, the function $\epsilon \mapsto \|v(\epsilon)\|_\infty$ is continuous on $[0, \epsilon_*]$, due to $\chi(\epsilon)$ being a simple eigenvalue for all $\epsilon \in (0, \epsilon_*]$ [24, Chapter 2]. Thus, there exists some $N \geq 1$ such that $\|v(\epsilon)\|_\infty \leq N$ for all $\epsilon \in [0, \epsilon_*]$.

Now, let $\epsilon \in (0, \epsilon_*]$. By (30), we have that for $t \geq 2\tau_M + \tau_*$

$$\begin{aligned} |x(t)| &\leq v_1(\epsilon) e^{\frac{\ln(\chi(\epsilon))}{2\tau_M + \tau_*}(t - 2\tau_M - \tau_*)} \\ &\times \left[\sup_{s \in [0, 2\tau_M + \tau_*]} |x(s)| + \sup_{s \in [0, 2\tau_M + \tau_*]} |z(s)| \right] \end{aligned}$$

Employing Propositions 1 and 2 with $\beta_* = 2\tau_M + \tau_*$, we have

$$\begin{aligned} \sup_{s \in [0, \beta_*]} |x(s)| &\leq M_x(\beta_*) \|\phi\|_{C[-\tau_m, 0]}, \\ \sup_{s \in [0, \beta_*]} |x(s)| &\leq 2M_x(\beta_*) \|\phi\|_{C[-\tau_m, 0]}. \end{aligned}$$

Therefore, for $t \geq \beta_*$

$$\begin{aligned} |x(t)| &\leq 3M_x(\beta_*) \|v(\epsilon)\|_\infty e^{\frac{\ln(\chi(\epsilon))}{\beta_*}(t - \beta_*)} \|\phi\|_{C[-\tau_m, 0]} \\ &\leq M_1(\epsilon) e^{-\kappa_1(\epsilon)t} \|\phi\|_{C[-\tau_m, 0]} \end{aligned} \quad (35)$$

with

$$\kappa_1(\epsilon) = -\frac{\ln(\chi(\epsilon))}{\beta_*} > 0, \quad M_1(\epsilon) = 3M_x(\beta_*) N e^{-\ln(\chi(\epsilon))}. \quad (36)$$

Furthermore, Proposition 1 implies that (35) also holds on $t \in [0, \beta_*]$, which yields (32) with M_* replaced by $M_1(\epsilon)$ and κ replaced by $\kappa_1(\epsilon)$. To see that, given $\tau_* > 0$, the constants M_* and $\kappa > 0$ in (32) can be chosen independently of $\epsilon \in (0, \epsilon_*]$, note that by (34), $\chi(\epsilon)$ is bounded away from zero for $\epsilon \in (0, \epsilon_*]$. Hence, by continuity of the logarithm and (36), there exist $\kappa > 0$ and $M_* \geq 1$ such that $\kappa_1(\epsilon) \geq \kappa > 0$ and $M_1(\epsilon) \leq M_*$ for all $\epsilon \in (0, \epsilon_*]$.

Finally, note that there always exist $\tau_* > 0$ large enough and $\epsilon_* > 0$ small enough such that $\Upsilon(\epsilon_*, \sigma, \tau_*, \tau_M)$ is Schur stable. Indeed, setting $\epsilon_* = 0$ and considering (27) and (33), we see that $\lim_{\tau_* \rightarrow \infty} L_1(\sigma, \tau_*, \tau_M) = 0$, whence for $\tau_* > 0$ large enough, $\Upsilon(0, \sigma, \tau_*, \tau_M)$ is Schur stable. By continuity of eigenvalues, $\Upsilon(\epsilon_*, \sigma, \tau_*, \tau_M)$ is also Schur stable for small enough $\epsilon_* > 0$. ■

Remark 2: By the proof of Theorem 1, finding $\tau_* > 0$ which guarantees Schur stability of $\Upsilon(0, \sigma, \tau_*, \tau_M)$ reduces to solving $M_\Phi [1 + \sigma^{-1} \|B\| (e^{\sigma \tau_M} - 1)] < e^{\sigma \tau_*}$ for $\tau_* > 0$. Note that this condition only involves the delay bound $\tau_M > 0$ and the system matrices in (1) and σ, M_Φ in Assumption 1.

D. Implications for input-to-state stability

The results of Theorem 1 can be immediately applied to ISS analysis of the following system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bx(t - \tau(t)) + d(t), \\ x(t) &= \phi(t), \quad t \in [-\tau_M, 0] \end{aligned} \quad (37)$$

where $d : [0, \infty) \rightarrow \mathbb{R}^n$ is a locally essentially bounded disturbance.

The study of ISS of systems of the form (37) was not carried out in [18], where asymptotic stability was achieved. Indeed, the trajectory-based approach therein only establish that exponential stability of the comparison system (5) implies the stability of (1) for small enough $\epsilon > 0$, without estimates on how small ϵ must be. The lack of explicit estimates on the solutions of (1) in [18] prohibits one from extending the stability analysis to an ISS analysis. As the previous section show, our approach yields explicit estimates on $\epsilon_* > 0$ such that for $\epsilon \in (0, \epsilon_*]$, the stability of (1) is guaranteed, as well as on the corresponding exponential decay rate. Summarizing,

Theorem 2: Consider the system (37) with delay $\tau(t)$ of the form (2) with $\epsilon, \delta > 0$, $\tau_0 \geq \delta$, $\tau_0 + \delta = \tau_M$ $f : \mathbb{R} \rightarrow [-1, 1]$ a T -periodic function with $\int_0^T f(s)ds = 0$ and a piecewise continuous disturbance $d : [0, \infty) \rightarrow \mathbb{R}^n$. Consider the comparison (averaged) disturbance-free system (5) with the integrable kernel $\omega : [-1, 1] \rightarrow [0, \infty)$ satisfying (3) for every continuous function $\alpha : [-1, 1] \rightarrow \mathbb{R}$. Let the assumptions of Theorem 1 hold. Then, for all $\epsilon \leq \epsilon_*$, the system (37) is ISS. More precisely, there exist constants $M_* \geq 1$, $\gamma_D > 0$ and $\kappa > 0$, independent of $\epsilon \in (0, \epsilon_*]$, satisfying

$$\begin{aligned} |x(t)| &\leq M_* e^{-\kappa t} \|\phi\|_{C[-\tau_M, 0]} \\ &\quad + \gamma_D \mathcal{D}(t), \quad t \geq 0, \quad \forall \epsilon \in (0, \epsilon_*], \end{aligned} \quad (38)$$

where $\mathcal{D}(t) := \sup_{s \in [0, t]} |d(s)|$, $t \geq 0$.

Proof: Denote by $\Psi(t, \epsilon)$ the fundamental solution of (1). By Theorem 1, there exist constants $M_* \geq 1$ and $\kappa > 0$ such that for $\epsilon \in (0, \epsilon_*]$, the estimate (32) holds for any solution of (1) (note that M_* and κ can be estimated explicitly, as follows from the proof of Theorem 1. The latter implies

$$\|\Psi(t, \epsilon)\| \leq M_* e^{-\kappa t}, \quad t \geq 0, \quad \epsilon \in (0, \epsilon_*]. \quad (39)$$

By the variations of constants formula [21, Equation 2.12], the solution to (37) is given by

$$x(t) = x_h(t) + \int_0^t \Psi(t - s, \epsilon) d(s) ds,$$

where $x_h(t)$ is the solution to the homogeneous system corresponding to (37). Therefore, by Theorem 1 we have

$$\begin{aligned} |x(t)| &\leq |x_h(t)| + \int_0^t \|\Psi(t - s, \epsilon)\| |d(s)| ds \\ &\leq M_* e^{-\kappa t} \|\phi\|_{C[-\tau_M, 0]} + \frac{M_*}{\kappa} \mathcal{D}(t), \quad t \geq 0 \end{aligned}$$

which is the desired result with $\gamma_D = \frac{M_*}{\kappa}$. ■

Remark 3: For a fixed $\epsilon \in (0, \epsilon_*]$, where $\epsilon_* > 0$ given in Theorem 1, one can obtain tighter (non-uniform in ϵ) ISS estimates than (38), by replacing M_* and κ therein with $M_1(\epsilon)$ and $\kappa_1(\epsilon)$, given in (36).

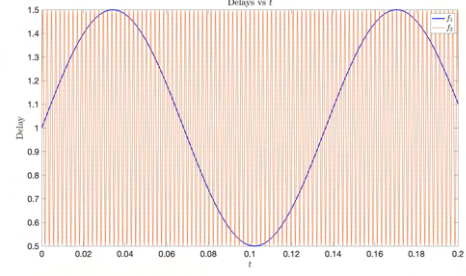


Fig. 1. Rapidly-varying delays $1 + \frac{1}{2}f_i\left(\frac{t}{\epsilon}\right)$, $i = 1, 2$. Here $\epsilon = 0.0217$ for f_1 , whereas $\epsilon = 0.00033$ for f_2 .

III. NUMERICAL EXAMPLES

In this section, we test our approach on the examples presented in [18], and demonstrate the efficacy of our method in providing explicit estimates on $\epsilon_* > 0$ for which the exponential stability of (1) with $\epsilon \in (0, \epsilon_*]$ is guaranteed. Recall that this is done by tuning ϵ and τ_* to obtain Schur stability of $\Upsilon(\epsilon, \sigma, \tau_*, \tau_M)$ in (29) (see Theorems 1 and (2)).

A. Scalar system

We consider the following scalar equation from [18],

$$\dot{x}(t) = -x(t) + bx \left(t - 1 - \delta f \left(\frac{t}{\epsilon} \right) \right) \quad (40)$$

where $\epsilon > 0$ and $\delta \in (0, 1]$. The function f will have one of the following forms

$$\begin{aligned} f_1(t) &= \begin{cases} \frac{2}{\pi} \left(t - \frac{\pi}{2} \right), & t \in [0, \pi) \\ \frac{2}{\pi} \left(\frac{3\pi}{2} - t \right), & t \in [\pi, 2\pi) \end{cases} \Rightarrow \omega_1(t) \equiv \frac{1}{2}, \\ f_2(t) &= \sin(t) \Rightarrow \omega_2(t) = \frac{1}{\pi\sqrt{1-t^2}}. \end{aligned} \quad (41)$$

The parameters b and δ are free, whereas f_1 and f_2 are considered as $T = 2\pi$ periodic functions on $\mathbb{R}$. The delays $1 + \frac{1}{2}f_i\left(\frac{t}{\epsilon}\right)$, $i = 1, 2$ are given in Figure 1.

Consider first the case of f_1 in (41). For $\delta = 0$, we employ [21, Example 2.4] to infer the necessary condition $b < 1$ for stability of the system (40). Furthermore, the analytic calculations in [21, Example 2.4] show that for a unit delay, stability of (40) cannot be achieved for $b \leq -1.78$. Next, consider the case $\delta = \frac{1}{2}$ and the corresponding comparison system (5) with f_1 in (41). As implied by the results of [18], the stability region of the comparison system (5) is enlarged when δ is increased from zero. In particular, the comparison system (5) is stable for values of $b < -1.78$. Choosing $b \in \{-2, -2.5\}$ and $\delta = \frac{1}{2}$, we perform simulations of the system (5) to estimate the norm of the fundamental solutions. The results are shown in Figure 2, where we obtain $\|\Phi(t)\| \leq 2e^{-2t}$, $t \geq 0$ for $b = -2$ and $\|\Phi(t)\| \leq 2e^{-0.05t}$, $t \geq 0$ for $b = -2.5$. For both $b = -2$ and $b = -2.5$, the result of [18] guarantees stability of (40) with f_1 of (41) and $\delta = \frac{1}{2}$ for sufficiently small $\epsilon > 0$, without providing an explicit estimate on $\epsilon > 0$ for which stability is preserved. Employing the bounds on the fundamental solutions in Figure 2 and Theorem 1, we look for $\epsilon_* > 0$ and $\tau_* > 0$ for which $\Upsilon(\epsilon_*, \sigma, \tau_*, \tau_M)$, given in (29), is Schur stable. Note that here $a = -1$ and

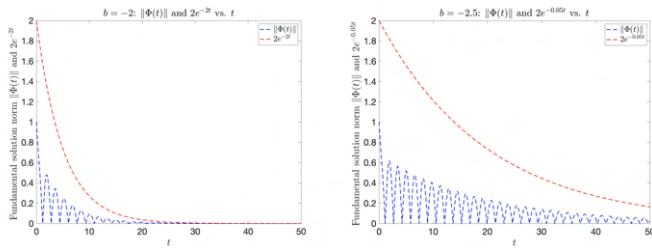


Fig. 2. Scalar system - Fundamental solutions of (5) with f_1 in (41): $b = -2$ (left) and $b = -2.5$ (right).

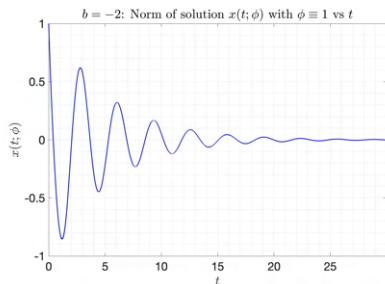


Fig. 3. Scalar system - Stability of (40) with f_1 in (41), $b = -2$ and $\epsilon = 0.0217$.

$\tau_M = \frac{3}{2}$. Recall that the conditions of Theorem 1 guarantee stability of the system (40) with f_1 of (41). For $b = -2$, the conditions of Theorem 1 are satisfied for $\epsilon_* = 0.0227$, which is obtained for $\tau_* = 5$. For $b = -2.5$, the conditions of Theorem 1 are satisfied for $\epsilon_* = 0.0011$, which is obtained for $\tau_* = 33.5$. We fix $b = -2$ and simulate the system (40) with $\epsilon = 0.0227$ and initial condition $\phi(t) \equiv 1$. The Simulation is given in Figure 3 and validates our results. Simulation of (40) with larger values of ϵ show that stability is preserved up to $\epsilon_* = 0.031$, illustrating some conservatism of our results. Derivation of less conservative estimates on ϵ is left for future research. Next, following [18], we consider (40) with $b = -4$, $\delta = 1$ and f_2 in (41). For the comparison system (5) with $b = -4$, $\delta = 1$ and f_2 of (41), we begin by simulating the fundamental solution. The results are shown in Figure 4, where we have $\|\Phi(t)\| \leq 2e^{-0.05t}$, $t \geq 0$. We employ Theorem 1 to obtain $\epsilon_* > 0$ and $\tau_* > 0$ for which $\Upsilon(\epsilon_*, \sigma, \tau_*, \tau_M)$, given in (29), is Schur stable. The conditions of Theorem 1 are satisfied for $\epsilon_* = 0.00033$, obtained for $\tau_* = 180.8$. We then perform simulations of the system system (40) with $\epsilon = 0.00033$ and

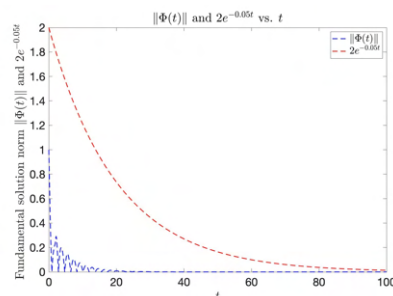


Fig. 4. Scalar system - Fundamental solution of (5) with f_2 in (41).

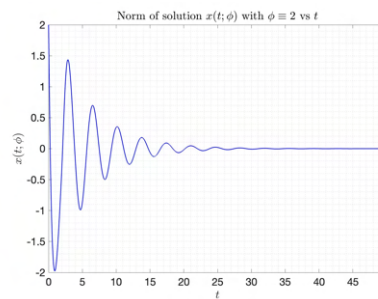


Fig. 5. Scalar system - Stability of (40) with f_2 of (41), $b = -4$ and $\epsilon = 0.00033$.

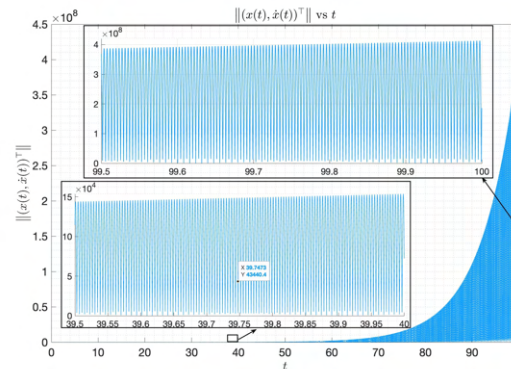


Fig. 6. Instability of the second order system with $\tau_0 = 0.06$ and $\delta = 0$. The insets show the oscillations of the solution.

initial condition $\phi(t) \equiv 2$. The simulation is given in Figure 5 and validates our results.

B. Second order system

We demonstrate our results on a model stemming from variable spindle speed cutting machines, which describes one mode of a mechanical rotational cutting process (see [4], [18] for further details). Consider the system

$$\ddot{x}(t) + 2\xi\omega_n\dot{x}(t) + \omega_n^2x(t) = \frac{k}{m}[x(t - \tau(t)) - x(t)], \quad (42)$$

where $\tau(t)$ is of the form (2) with $\tau_0 = 0.022$, $\delta = 0.05\tau_0$ and $f(t) = f_1(t)$ given in (41). The system (42) can be presented as (1) with

$$A = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 - \frac{k}{m} & -2\xi\omega_n \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \frac{k}{m} \end{bmatrix}.$$

Choosing the parameters $\omega_n = 795$, $k = m = 1$ and $\xi = 0.39585$, we have $\|B\| = 1$ and $\|A\| = 6.3203 \cdot 10^5$. We begin by considering the system (1) with $\tau_0 = 0.06$ and $\delta = 0$, i.e. with a constant delay, and simulate the system with the initial condition $\phi(t) = 1$, $t \in [-\tau_0, 0]$. The simulation results are shown in Figure 6 and show the instability of the system with the constant discrete delay. Next, we choose $\delta = 0.01\tau_0$ and $f = f_1$ of (41) and simulate the fundamental solution of the comparison system (5). The results are given in Figure 7. In particular, we see that the comparison system is stable and the fundamental solution satisfies $\|\Phi(t)\| \leq 800e^{-4t}$, $t \geq 0$ (i.e., $M_\phi = 800$ and $\sigma = 4$). This fact implies the asymptotic

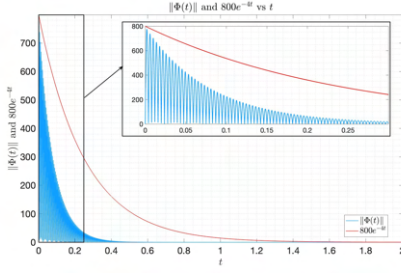


Fig. 7. Second order system - Fundamental solution of (5) with $\delta = 0.01\tau_0$ and f_1 of (41). Exponential upper bound on norm of the fundamental solution.

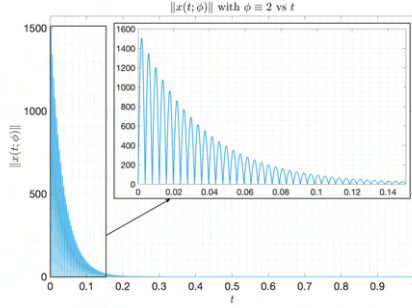


Fig. 8. Second order system - Stability of (42) with $\tau_0 = 0.06$, $\delta = 0.01\tau_0$ and f_1 of (41) and $\epsilon = 0.00123$.

stability of (1) for small enough $\epsilon > 0$, and demonstrates the "quenching phenomenon" in which the addition of the rapidly time-varying delay to the constant delay τ_0 in (1) improves the stability of the system. Next, we employ the bounds on the fundamental solutions in Figure 7 and Theorem 1 to derive $\epsilon_* > 0$ and $\tau_* > 0$ which guarantees stability of (42). The conditions of Theorem 1 are satisfied for $\epsilon_* = 0.00123$, which is obtained for $\tau_* = 122.3$. Finally, we perform simulation of (42) with $\epsilon = 0.00123$ and initial condition $\phi(t) \equiv 2$, $t \in [-\tau_M, 0]$. The simulation is presented in Figure 8 and validates our theoretical results.

IV. CONCLUSION

We studied the stability properties of linear systems with rapidly-varying and periodic delays, which depend on a small positive parameter ϵ . Building upon [18], we employed a comparison (averaged) system whose exponential stability guarantees the asymptotic stability of the original system, provided the time-scale parameter ϵ is small enough. Differently from [18], we introduced a novel transformation which explicitly maps the original system into a perturbed version of the comparison system. Assuming known bounds on the fundamental solution of the comparison system, the introduced transformation allowed to employ a recent trajectory-based inequality to obtain explicit estimates on the small parameter which guarantees exponential stability, as well as on the exponential decay rate, of the original system. We extended our results to ISS analysis of linear systems with rapidly-varying and periodic delays, subject to locally essentially bounded disturbances. Numerical simulations which demonstrate the efficacy of the proposed method have

been presented. Future research may include the improvement of the proposed method in order to obtain better estimates on the small parameter and ISS gains and its extension to nonlinear systems, as well as its application to constructive methods for control problems that involve averaging, such as vibrational control and extremum seeking.

REFERENCES

- [1] X. Cheng, Y. Tan, and I. Mareels, "On robustness analysis of linear vibrational control systems," *Automatica*, vol. 87, pp. 202–209, 2018.
- [2] H. Sandberg and E. Möllerstedt, "Periodic modelling of power systems," *IFAC Proceedings Volumes*, vol. 34, no. 12, pp. 89–94, 2001.
- [3] X. Xie and J. Lam, "Guaranteed cost control of periodic piecewise linear time-delay systems," *Automatica*, vol. 94, pp. 274–282, 2018.
- [4] S. Jayaram, S. Kapoor, and R. DeVor, "Analytical stability analysis of variable spindle speed machining," *J. Manuf. Sci. Eng.*, vol. 122, no. 3, pp. 391–397, 2000.
- [5] J. Sexton, B. Stone *et al.*, "The stability of machining with continuously varying spindle speed," *Annals of the CIRP*, vol. 27, no. 1, pp. 321–326, 1978.
- [6] N. N. Bogoliubov and I. A. Mitropolskij, *Asymptotic methods in the theory of non-linear oscillations*. CRC Press, 1961, vol. 10.
- [7] F. Bullo, "Averaging and vibrational control of mechanical systems," *SIAM Journal on Control and Optimization*, vol. 41, no. 2, pp. 542–562, 2002.
- [8] M. Krstić and H.-H. Wang, "Stability of extremum seeking feedback for general nonlinear dynamic systems," *Automatica*, vol. 36, no. 4, pp. 595–601, 2000.
- [9] D. Liberzon, *Switching in systems and control*. Springer, 2003, vol. 190.
- [10] B. Caiazzo, E. Fridman, and X. Yang, "Averaging of systems with fast-varying coefficients and non-small delays with application to stabilization of affine systems via time-dependent switching," *Nonlinear Analysis: Hybrid Systems*, vol. 48, p. 101307, 2023.
- [11] E. Mostacciolo, S. Trenn, and F. Vasca, "A smooth model for periodically switched descriptor systems," *Automatica*, vol. 136, p. 110082, 2022.
- [12] E. Fridman and J. Zhang, "Averaging of linear systems with almost periodic coefficients: A time-delay approach," *Automatica*, vol. 122, p. 109287, 2020.
- [13] J. Zhang and E. Fridman, "L2-gain analysis via time-delay approach to periodic averaging with stochastic extension," *Automatica*, vol. 137, p. 110126, 2022.
- [14] X. Yang, J. Zhang, and E. Fridman, "Periodic averaging of discrete-time systems: A time-delay approach," *IEEE Transactions on Automatic Control*, vol. 68, no. 7, pp. 4482–4489, 2022.
- [15] R. Katz, E. Fridman, and F. Mazenc, "Constructive method for averaging-based stability via a delay free transformation," *Automatica*, vol. 163, p. 111568, 2024.
- [16] A. Jbara, R. Katz, and E. Fridman, "Averaging-based stability of discrete-time delayed systems via a novel delay-free transformation," *IEEE Transactions on Automatic Control*, 2024.
- [17] J. Louisell, "New examples of quenching in delay differential equations having time-varying delay," in *4th European Control Conference*, 1999.
- [18] W. Michiels, V. Van Assche, and S.-I. Niculescu, "Stabilization of time-delay systems with a controlled time-varying delay and applications," *IEEE Transactions on Automatic Control*, vol. 50, no. 4, pp. 493–504, 2005.
- [19] I. Györi and L. Horváth, "Sharp Gronwall-Bellman type integral inequalities with delay," *Electronic Journal of Qualitative Theory of Differential Equations*, vol. 2016, no. 111, pp. 1–25, 2016.
- [20] F. Mazenc and M. Malisoff, "ISS inequalities for vector versions of Halanay's inequality and of the trajectory-based approach," *European Journal of Control*, vol. 68, p. 100665, 2022.
- [21] E. Fridman, *Introduction to time-delay systems: analysis and control*. Birkhäuser, Systems and Control: Foundations and Applications, 2014.
- [22] F. Mazenc, M. Malisoff, and S.-I. Niculescu, "Stability and control design for time-varying systems with time-varying delays using a trajectory-based approach," *SIAM Journal on Control and Optimization*, vol. 55, no. 1, pp. 533–556, 2017.
- [23] R. A. Horn and C. R. Johnson, *Matrix analysis*. Cambridge university press, 2012.
- [24] J. H. Wilkinson, *The algebraic eigenvalue problem*. Oxford University Press, Inc., 1988.