European Journal of Control xxx (xxxx) xxx

Contents lists available at ScienceDirect

European Journal of Control

journal homepage: www.sciencedirect.com/journal/european-journal-of-control

An extremum seeking algorithm for 1D static maps with delay-based derivative estimation

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ARTICLE INFO

Recommended by T. Parisini

Keywords:
Extremum seeking
Time-delay systems

ABSTRACT

For an even power convex or concave function of a scalar variable having a global and unique extremum, an algorithm of the extremum seeking is proposed, which does not use any dither excitation signal, hence, being asymptotically exact, and it is based on online time derivative estimation of the measured output. Two approaches are discussed, first, with utilization of the super-twisting differentiator, and second, where the derivative is estimated via the time-delay method. For analysis of the latter, an extension of the invariance principle is formulated for functional differential inclusions. The efficiency of the suggested extremum seeking algorithms is illustrated through numeric experiments.

1. Introduction

Extremum seeking control and optimization found applications in many areas of science and technology (Krstić & Wang, 2000; Scheinker & Krstić, 2017; Zhang & Ordóñez, 2012). The basic problem consists in determining an extremum of a convex/concave uncertain map by controlling its argument and by measuring the respective value of this function, with absence of the access to its gradient, which is usually numerically reconstructed. To this end, there are perturbation-based and model-based approaches to extremum seeking (Dochain, Perrier, & Guay, 2011). The former uses dither signals, harmonic or stochastic, in order to evaluate the gradient through averaging approaches, while the latter is oriented on estimation of the map itself with posterior utilization of the gradient directly. After derivation of the estimates of the gradient, different nonlinear system dynamics may be used for extremum seeking, usually together with the dither signals (Angulo, 2015; Labar, Garone, Kinnaert, & Ebenbauer, 2019; Nesić, Tan, Manzie, Mohammadi, & Moase, 2012; Suttner & Krstić, 2024; Zhang & Ordóñez, 2007). Often, only convergence to a vicinity of the optimum is guaranteed even in the noise-free setting (due to introduction of auxiliary perturbations).

Estimation of derivative of a sufficiently smooth signal through its noisy measurements in real time is a well-known issue, which has many popular and well-established solutions (Cruz-Zavala, Moreno, & Fridman, 2011; Holloway & Krstić, 2019; Khalil, 2017; Levant, 1998; Lopez-Ramirez, Polyakov, Efimov, & Perruquetti, 2018; Orlov, 2022; Perruquetti, Floquet, & Moulay, 2008). Nevertheless, it is still an active area of research. The issue of numeric differentiation of a noisy signal can be reformulated as a state estimation problem (Kairuz, Orlov, & Aguilar, 2022; Orlov & Kairuz, 2022; Reichhartinger, Efimov, & Fridman, 2018).

In this note, pursuing an asymptotically exact search for the optimum, we are going to present our results on a new approach for extremum seeking, which is not based on dither signal (similarly to Hunnekens, Haring, van de Wouw, & Nijmeijer, 2014; Utkin, 1992) or model identification, but utilizes time derivative of the output (in order to evaluate the gradient of the map, similarly as it has been suggested in niga, López-Caamal, Hernández-Escoto, & Alcaraz-González, 2021) together with a special nonlinear dynamical optimization algorithm of the second order. In the noise-free setting our algorithm achieves the exact asymptotic convergence to the optimum. Two methods for online derivative estimation are used: the super-twisting differentiator, which provides the value of derivative in a finite time for noise-free setting, and time-delay approach (in Hunnekens et al., 2014 the delayed values of the output have been used to estimate the gradient in the model-identification framework). The implementable version of the proposed algorithm is represented by discontinuous dynamical (time-delay) systems. Utilization of discontinuous dynamics for extremum seeking has been already reported in Korovin and Utkin (1974), Teixeira and Zak (1998), Utkin (1992) (sliding mode control), in this note another algorithm is proposed, which is not based on a line search method (Nusawardhana & Zak, 2003), and whose performances are evaluated using a Lyapunov function (a Lyapunov-Krasovskii functional). The

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https://doi.org/10.1016/j.ejcon.2025.101339
Received 23 May 2025; Received in revised form 10 July 2025; Accepted 10 July 2025
Available online 29 July 2025

algorithm has only three tuning parameters. The convergence results are obtained in weak sense (for some solutions of the discontinuous system), however, due to absence of the dither signal the steady-state error is avoided.

The proof is grounded on a reformulation of the integral invariance principle presented in Byrnes and Martin (1995), which is a weaker form of the LaSalle invariance principle (LaSalle, 1960), extended in Ryan (1998) to ordinary differential inclusions, with further generalization in Desch, Logemann, Ryan, and Sontag (2001). The LaSalle invariance principle for time-delay systems has been obtained in Hale (1965).

The paper is organized as follows. The brief preliminaries are given in Section 2, where also an extension of LaSalle invariance principle is formulated for functional differential inclusions (our technical contribution). The problem statement is introduced in Section 3. A preliminary nonlinear extremum-seeking algorithm of second order based on the use of the exact derivative is presented in Section 4, clarifying the underlying idea of the proposed approach. Its extension (our main contribution), where the derivative is replaced with its estimation through the time-delay approach, is developed in Section 5. The results of numeric simulation of the designed algorithms with inclusion of a measurement noise are shown in Section 6.

Notation

- $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$, where $\mathbb{R}$ is the set of real numbers, $\mathbb{Z}$ is the set of integer numbers, $\mathbb{Z}_+ = \mathbb{Z} \cap \mathbb{R}_+$.
- $|\cdot|$ denotes the absolute value in $\mathbb{R}$ or the Euclidean norm on $\mathbb{R}^n$.
- $C^n_{[a,b]}$, $-\infty < a < b < +\infty$ denotes the Banach space of continuous functions $\phi : [a,b] \to \mathbb{R}^n$with the uniform norm $\|\phi\| = \sup_{a\leq\varsigma\leq b} |\phi(\varsigma)|$. For a set $\mathcal{M} \subset C^n_{[a,b]}$ and $\phi \in C^n_{[a,b]}$ let $\mathrm{dist}(\phi, \mathcal{M})$ denote the distance between ϕ and the set $\mathcal{M}$.
- $\mathrm{cl}(\cdot)$ denotes the closure of the argument set.
- Denote $\mathbf{e} = \exp(1)$.

2. Preliminaries

The definitions of standard stability notions used below can be found in Khalil (2002).

2.1. Exact super-twisting differentiator

The following result has been proven in Levant (1998) (with numerous extensions obtained later, as for example in Cruz-Zavala et al., 2011):

Lemma 1. *Let $y : \mathbb{R}_+ \to \mathbb{R}$ be a twice continuously differentiable signal with $\sup_{t\geq 0} |\ddot{y}(t)| < +\infty$, then there exist $0 < \lambda_1 < \lambda_2 < +\infty$ and $T > 0$ such that*

$$z_2(t) = \dot{y}(t),\ \forall t \geq T,$$

where $z_2(t)$ is the output of the following system:

$$\begin{aligned} \dot{z}_1(t) &= -\lambda_1\sqrt{|z_1(t) - y(t)|}\mathrm{sign}(z_1(t) - y(t)) + z_2(t), \\ \dot{z}_2(t) &= -\lambda_2\mathrm{sign}(z_1(t) - y(t)), \\ z_1(0) &= z_2(0) = 0. \end{aligned} \quad (1)$$

In the same work it has been also shown that if noisy measurements $y(t) + v(t)$ are available, where $v : \mathbb{R}_+ \to \mathbb{R}$ is a bounded Lebesgue measurable perturbation, then the estimation error $z_2(t) - \dot{y}(t)$ stays bounded. There are also many works providing tuning of the gains λ_1 and λ_2, with estimation of the settling time T, see for example (Cruz-Zavala & Moreno, 2016; Mojallizadeh, Brogliato, & Acary, 2021; Seeber, 2023).

2.2. Stability of functional differential inclusions

Consider a functional differential inclusion of retarded type:

$$\dot{x}(t) \in F(x_t),\ t \geq 0, \quad (2)$$

where $x(t) \in \mathcal{X} \subseteq \mathbb{R}^n$ and $x_t \in \mathbb{X} = \{\phi \in C^n_{[-\tau,0]} : \phi(s) \in \mathcal{X}, \forall s \in [-\tau, 0]\}$ is the state function, $x_t(s) = x(t+s)$, $-\tau \leq s \leq 0$ and $\tau > 0$ is a finite delay, $\mathcal{X}$ is open and contains the origin; the map $F(\phi) \subset \mathbb{R}^n$, $\phi \in \mathbb{X}$ is upper semicontinuous taking nonempty, convex and compact values on each bounded subset of $\mathbb{X}$, which guarantees that for each $x_0 \in \mathbb{X}$ there exists a non-empty set $S(x_0)$ of solutions in forward time for the system (2) (Kolmanovskii & Myshkis, 1999). Denote such a solution by $x(\cdot, x_0) \in S(x_0)$, which is absolutely continuous and defined on some time interval $[-\tau, T_{x_0})$ for $T_{x_0} \in \mathbb{R}_+ \cup \{+\infty\}$ (then $x_t(x_0) \in \mathbb{X}$ or $x(t, x_0) \in \mathcal{X}$ denote realizations of the solution for given t), and it satisfies (2) for almost all instant of time in $[-\tau, T_{x_0})$ (we assume that the solutions in $S(x_0)$ are all maximal, i.e., they do not have a proper right extension that is also a solution of (2)). Let $\{0\} \subseteq F(0)$, then (2) admits the zero solution, and denote $B_\delta = \{\phi \in C^n_{[-\tau,0]} : \|\phi\| < \delta\}$.

Definition 1. *The zero solution of (2) is said to be strongly (weakly) stable if for all $\epsilon > 0$ there exits $\delta > 0$ such that $x_0 \in B_\delta \cap \mathbb{X}$ implies that $x(t, x_0)$ is defined for $t \geq 0$ and $|x(t, x_0)| < \epsilon$ for $t \geq 0$ for all (for at least one) $x(\cdot, x_0) \in S(x_0)$. If, in addition, there is $\rho > 0$ such that $\lim_{t\to+\infty} |x(t, x_0)| = 0$ for all (for at least one) $x(\cdot, x_0) \in S(x_0)$ for any $x_0 \in B_\rho \cap \mathbb{X}$, then the zero solution is said to be strongly (weakly) asymptotically stable.*

Definition 2. *For all $x_0 \in \mathbb{X}$, $\Omega(x_0) \subset C^n_{[-\tau,0]}$ is the strong (weak) ω-limit set of x_0, if any (at least one) $x(\cdot, x_0) \in S(x_0)$ is defined for all $t \geq 0$ and there is a sequence $t_n \in \mathbb{R}_+$, $n \in \mathbb{Z}_+$ with $\lim_{n\to+\infty} t_n = +\infty$ such that $\|x_{t_n}(x_0) - \phi\| \to 0$ as $n \to +\infty$ for some $\phi \in \Omega(x_0)$ (and $\Omega(x_0)$ is minimal such a set).*

Definition 3. *A set $\mathcal{M} \subset \mathbb{X}$ is called forward strongly (weakly) invariant if for any $x_0 \in \mathcal{M}$, $x(t, x_0)$ is defined for $t \geq 0$ and $x_t(x_0) \in \mathcal{M}$ for $t \geq 0$ for all (for at least one) $x(\cdot, x_0) \in S(x_0)$.*

The following lemma can be proven repeating the argumentation of Hale (1965, Lemma 2):

Lemma 2. *For all $x_0 \in \mathbb{X}$, if any (at least one) $x(\cdot, x_0) \in S(x_0)$ is defined for all $t \geq 0$ and $|x(t, x_0)| < +\infty$ for $t \geq 0$, then $\Omega(x_0)$ is a nonempty, compact and forward strongly (weakly) invariant set with $\mathrm{dist}(x_t(x_0), \Omega(x_0)) \to 0$ as $t \to +\infty$.*

If $V : \mathbb{X} \to \mathbb{R}_+$ is a continuous functional, for any $\phi \in \mathbb{X}$ define

$$DV(\phi)F(\phi) = \sup_{x(\cdot,\phi)\in S(\phi)} \limsup_{h\to 0^+} \frac{V(x_h(\phi)) - V(\phi)}{h},$$

which is a directional derivative of V on solutions of (2) (it allows the decay of just continuous V on the trajectories of (2) to be evaluated). Combining the results of Desch et al. (2001), Ryan (1998) and Theorem 1 of Hale (1965) we get:

Theorem 1. *Let $g : \mathbb{X} \to \mathbb{R}_+$ be a lower semicontinuous functional. If for some $x_0 \in \mathbb{X}$ any (at least one) $x(\cdot, x_0) \in S(x_0)$ is bounded with $\mathrm{cl}(x(\mathbb{R}_+, x_0)) \subset \mathcal{X}$ and $\int_0^{+\infty} g(x_t(x_0))dt < +\infty$, then such $x_t(x_0)$ approaches for $t \to +\infty$ the largest forward strongly (weakly) invariant set in $\Sigma = \{\phi \in \mathbb{X} : g(\phi) = 0\}$.*

Proof. Boundedness of $x(\cdot, x_0)$ guarantees that this solution is defined for all $t \geq 0$, and by Lemma 2 it implies that $\Omega(x_0)$ is a forward strongly (weakly) invariant set, which is nonempty and compact, moreover $\Omega(x_0) \subset \mathbb{X}$. By the properties of F, in such a case $|\dot{x}(t, x_0)| < +\infty$ for all $t \geq 0$, then $x(\cdot, x_0)$ is uniformly continuous, and $g(t) = g(x_t(x_0))$ is meagre (Desch et al., 2001), then $\Omega(x_0) \subset \Sigma$ by Lemma 9 in Desch et al. (2001). Consequently, $\mathrm{dist}(x_t(x_0), \Omega(x_0) \cap \Sigma) \to 0$ as $t \to +\infty$, and it reaches the largest forward strongly (weakly) invariant subset there. □

Theorem 2. *Let $V : \mathbb{X} \to \mathbb{R}_+$ be a continuous functional and $DV(\phi)F(\phi) \le 0$ for all $\phi \in \mathbb{X}$, and denote $\mathcal{R} = \{\phi \in \mathbb{X} : DV(\phi)F(\phi) = 0\}$. If for some $x_0 \in \mathbb{X}$ any (at least one) $x(\cdot, x_0) \in S(x_0)$ is bounded with $\mathrm{cl}(x(\mathbb{R}_+, x_0)) \subset \mathcal{X}$, then such $x_t(x_0)$ approaches for $t \to +\infty$ the largest forward strongly (weakly) invariant set in $\mathcal{R}$.*

Proof. Again, boundedness of $x(\cdot, x_0)$ by Lemma 2 implies that $\Omega(x_0)$ is a forward strongly (weakly) invariant set, which is nonempty and compact, moreover $\Omega(x_0) \subset \mathbb{X}$. Then a continuous functional V has a limit on this trajectory: $V(x_t(x_0)) \to \ell$ as $t \to +\infty$. Moreover, there exist $\phi \in \Omega(x_0)$ such that $V(\phi) = \ell$ and $DV(\phi)F(\phi) = 0$. Therefore, $\mathrm{dist}(x_t(x_0), \Omega(x_0) \cap \mathcal{R}) \to 0$ as $t \to +\infty$, and it reaches the largest forward strongly (weakly) invariant subset there. □

These results present the extensions of integral and LaSalle invariance principles, respectively, for retarded differential inclusions.

The definitions and results in this subsection are given for time-delay systems, and they also hold in the context of conventional ordinary differential inclusions (Desch et al., 2001; Filippov, 1988; Ryan, 1998).

3. Problem statement

Let $f(x) = \frac{b}{\kappa}(x - x_0)^\kappa + a$ be a 1D map of interest with $x, x_0 \in \mathbb{R}$, where $\kappa \ge 2$ is an unknown even integer, the parameters $a \in \mathbb{R}$ and $b \in \mathbb{R}$ are unknown, the sign of b is known (for brevity below we will assume that $b > 0$). We will need the following hypotheses:

Assumption 1. There are known constants $a_0 \ge 0$, $x_{\min} > 0$ and $x_{\max} > x_{\min}$ such that $a + a_0 \ge 0$ and $x_{\min} < x_0 < x_{\max}$.

We consider here the case of some knowledge about the map (i.e., “grey box” model) – the information of the interval for the extremum point x_0 and the lower bound on a.

It is required to design an algorithm generating a continuous signal $x(t) \in \mathbb{R}$ and measure

$$y(t) = f(x(t)),$$

providing realization of the seeking goal:

$$\lim_{t\to+\infty} x(t) = x_0. \tag{3}$$

Such a statement corresponds to the conventional extremum seeking problem (Krstić & Wang, 2000; Scheinker & Krstić, 2017; Zhang & Ordóñez, 2012).

4. Preliminary derivative-based solution

Assume that $x(t)$ is a continuously differentiable function of time, then

$$\dot{y}(t) = f'(x(t))\dot{x}(t),$$

where

$$f'(x) = b(x - x_0)^{\kappa-1}$$

is the gradient of f. And clearly, if the signal $f'(x(t))$ can be computed (or the function f' can be calculated) the extremum seeking problem has a simple solution: $\dot{x}(t) = -\gamma f'(x(t))$ for any $\gamma > 0$. Usually, in the extremum seeking literature, additive and/or multiplicative dither signals are used to approximate this gradient (Dochain et al., 2011; Krstić & Wang, 2000; Scheinker & Krstić, 2017), and practical convergence to x_0 is proved via classical averaging or Lie-brackets-based one. Another approach is based on different line search methods (Korovin & Utkin, 1974; Nusawardhana & Zak, 2003; Teixeira & Zak, 1998). In this work we are going to propose a new extremum-seeking algorithm, which does not use auxiliary perturbations focusing on extraction of the information about f' from $\dot{y}(t)$ (or its estimates).

Further, assume that $x(t)$ is twice continuously differentiable, then

$$\ddot{y}(t) = f''(x(t))\dot{x}^2(t) + f'(x(t))\ddot{x}(t),$$

where

$$f''(x) = b(\kappa - 1)(x - x_0)^{\kappa-2}.$$

Therefore, under Assumption 1, for bounded by $\max\{|x_{\min}|, |x_{\max}|\}$ variable $x(t)$ with a respectively bounded $\dot{x}(t)$ and $\ddot{x}(t)$, the signal $\ddot{y}(t)$ will be bounded, which allows us to apply the super-twisting differentiator (1) in order to exactly estimate $\dot{y}(t)$ in a finite time. Then $f'(x(t)) = \dot{x}^{-1}(t)\dot{y}(t)$, which can be used for solution of the extremum seeking problem. For illustration of this idea, consider an optimization algorithm:

$$\begin{aligned}
\dot{x}_1(t) &= x_2(t), \\
\dot{x}_2(t) &= \begin{cases} \sin(\frac{2\pi}{T}t) & \text{if } t \in [0, T], \\ -a_1\dfrac{z_2(t)}{x_2(t)} - a_2(a_0 + y(t))x_2(t) & \text{if } t > T, \end{cases} \\
x(t) &= x_1(t), \\
& x_1(0) \in (x_{\min}, x_{\max}),\ x_2(0) \in \mathbb{R},
\end{aligned} \tag{4}$$

where $a_1 > 0$ and $a_2 > 0$ are tuning parameters, $z_2(t)$ is the output of (1), and T comes from Lemma 1 (the system is initially excited for $t \in [0, T)$ while differentiator is converging). As we can conclude, for a proper tuning of the gains λ_1 and λ_2 all conditions of this lemma are satisfied, and $z_2(t) = \dot{y}(t)$ for all $t \ge T$ (in such a case $\frac{z_2(t)}{x_2(t)} = f'(x_1(t))$ and $\ddot{y}(t)$ stays bounded for $t \ge T$ as it is required in Lemma 1). Therefore, introducing the seeking error $\epsilon_1(t) = x_1(t) - x_0$ and its velocity $\epsilon_2(t) = x_2(t)$ we get its dynamics for $t \ge T$:

$$\begin{aligned}
\dot{\epsilon}_1(t) &= \epsilon_2(t), \\
\dot{\epsilon}_2(t) &= -a_1 b\epsilon_1^{\kappa-1}(t) - a_2(a_0 + a + \frac{b}{\kappa}\epsilon_1^\kappa(t))\epsilon_2(t),
\end{aligned} \tag{5}$$

which has a canonical Liénard form, and it admits a Lyapunov function (Aleksandrov, Efimov, & Dashkovskiy, 2023):

$$\begin{aligned}
V(\epsilon_1, \epsilon_2) &= \left(\epsilon_2 + a_2\int_0^{\epsilon_1} a_0 + a + \frac{b}{\kappa}s^\kappa ds\right)^2 + \epsilon_2^2 \\
&+ 4a_1 b\int_0^{\epsilon_1} s^{\kappa-1}ds \\
&= \left(\epsilon_2 + a_2\left((a_0 + a)\epsilon_1 + \frac{b}{\kappa(\kappa+1)}\epsilon_1^{\kappa+1}\right)\right)^2 + \epsilon_2^2 \\
&+ \frac{4a_1 b}{\kappa}\epsilon_1^\kappa,
\end{aligned} \tag{6}$$

which is positive definite and radially unbounded under introduced restrictions, and whose time derivative calculated on the solutions of (5) takes the form:

$$\begin{aligned}
\dot{V} &= -2a_1 a_2 b\left((a_0 + a)\epsilon_1^\kappa + \frac{b}{\kappa(\kappa+1)}\epsilon_1^{2\kappa}\right) \\
&- 2a_2\left(a_0 + a + \frac{b}{\kappa}\epsilon_1^\kappa\right)\epsilon_2^2
\end{aligned}$$

guaranteeing the global asymptotic stability of the origin. Thus, the following result has been proven:

Proposition 1. *Under Assumption 1, for any $T > 0$, $a_1 > 0$ and $a_2 > 0$ there exist the gains $0 < \lambda_1 < \lambda_2$ such that all state variables $z_1(t)$, $z_2(t)$, $x_1(t)$, $x_2(t)$ of the system (1) governed by the algorithm (4) stay bounded for $t \ge 0$ and (3) is realized.*

Increasing the values of the gains a_1 and a_2 it is possible to accelerate the convergence in this system.

The proposed extremum seeking algorithm (1), (4) just illustrates the main approach of this work, but it is difficult to apply it directly since the division in the term $\frac{z_2(t)}{x_2(t)}$ may produce a discontinuity for any numeric or measurement inaccuracy. Introducing the change of coordinates:

$$\xi_1(t) = x_1(t),\ \xi_2(t) = |x_2(t)|x_2(t), \tag{7}$$

another system can be used for implementation of (4) for $t \geq T$:

$$\begin{aligned} \dot{\xi}_1(t) &= \sqrt{|\xi_2(t)|}\operatorname{sign}(\xi_2(t)), \\ \dot{\xi}_2(t) &= \begin{cases} \sin(\frac{2\pi}{T}t) & \text{if } t \in [0,T] \\ -2(a_1 z_2(t)\operatorname{sign}(\xi_2(t)) & \\ +a_2(a_0 + y(t))\xi_2(t)) & \text{if } t > T, \end{cases} \\ x(t) &= \xi_1(t), \\ &\xi_1(0) \in (x_{\min}, x_{\max}),\ \xi_2(0) \in \mathbb{R}, \end{aligned} \tag{8}$$

which has a discontinuous right-hand side similarly to (1), and its Filippov regularization is upper semicontinuous with nonempty, convex and compact values, hence, it admits a nonempty set of solutions for any initial conditions. Note this algorithm does not belong to the class of heavy ball methods (Ghadimi, Feyzmahdavian, & Johansson, 2015) due to the presence of the product $y(t)\xi_2(t) = f(\xi_1(t))\xi_2(t)$. Substituting $z_2(t) = \dot{y}(t)$ in the expression above, for $t \geq T$ we obtain:

$$\dot{\xi}_1(t) = \sqrt{|\xi_2(t)|}\operatorname{sign}(\xi_2(t)),$$
$$\dot{\xi}_2(t) = -2\left(a_1 f'(\xi_1(t))\sqrt{|\xi_2(t)|} + a_2(a_0 + y(t))\xi_2(t)\right),$$

then it becomes clear that for $x(t) = \xi_1(t)$ the conditions of Lemma 1 are still valid for $t \geq T$, and that the right-hand side of (8) is a function of $\xi_2(t)$. Consequently, the line $\xi_2 = 0$ contains the set of equilibria in this system. However, since the square root function is not Lipschitz continuous at zero, while $f'(\xi_1(t)) \neq 0$ there is always a trajectory that leaves this line (i.e., it is easy to verify that a scalar differential equation $\dot{\zeta}(t) = \alpha\sqrt{|\zeta(t)|}$ with $\zeta(0) = 0$ and $\alpha \in \mathbb{R}$ has a solution $\zeta(t) = \frac{\alpha|\alpha|}{4}t^2$ for $t \geq 0$). Thus, the line $\xi_2 = 0$ is composed by weakly invariant steady states, and the only strongly invariant one corresponds to $\xi_1 = x_0$ due to $f'(x_0) = 0$ by construction.

Our main result in this section is as follows:

Theorem 3. *Under Assumption 1, for any $T > 0$, $a_1 > 0$ and $a_2 > 0$ there exist the gains $0 < \lambda_1 < \lambda_2$ such that all state variables $z_1(t)$, $z_2(t)$, $\xi_1(t)$, $\xi_2(t)$ of the system (1) with the algorithm (8) stay bounded for $t \geq 0$ and (3) is realized weakly (i.e., for any initial conditions there is a solution verifying this property).*

Note that the result is semi-global with respect to the bounds $x_{\min}$ and $x_{\max}$ introduced in Assumption 1 and the parameters of the map a, b and κ: for all their values, there are the gains λ_1 and λ_2.

Proof. There exist $0 < \lambda_1 < \lambda_2$ such that all conditions of Lemma 1 are verified for given T and $t \leq T$, then $z_2(t) = \dot{y}(t)$ for $t \geq T$ as desired provided that the variables $\xi_1(t)$, $\xi_2(t)$ are bounded for $t \geq T$. Indeed, in such a case these gains λ_1 and λ_2 can be further selected to ensure that Lemma 1 holds for $t \geq T$. To prove boundedness and convergence of the state in (8), define the seeking error $\varepsilon_1(t) = \xi_1(t) - x_0$ and denote the signed square of its velocity by $\varepsilon_2(t) = \xi_2(t)$, which for $t \geq T$ obey the differential equations:

$$\dot{\varepsilon}_1(t) = \sqrt{|\varepsilon_2(t)|}\operatorname{sign}(\varepsilon_2(t)),$$
$$\dot{\varepsilon}_2(t) = -2(a_1 b\varepsilon_1^{\kappa-1}(t)\sqrt{|\varepsilon_2(t)|} + a_2(a_0 + a$$
$$+\frac{b}{\kappa}\varepsilon_1^{\kappa}(t))\varepsilon_2(t)),$$

and consider the same Lyapunov function (6) rewritten for these coordinates:

$$V(\varepsilon_1, \varepsilon_2) = |\varepsilon_2| + \frac{4a_1 b}{\kappa}\varepsilon_1^{\kappa}$$
$$+\left(\sqrt{|\varepsilon_2|}\operatorname{sign}(\varepsilon_2) + a_2\left((a_0 + a)\varepsilon_1 + \frac{b}{\kappa(\kappa+1)}\varepsilon_1^{\kappa+1}\right)\right)^2.$$

Note that the dynamics of $\varepsilon_1(t)$ and $\varepsilon_2(t)$ are described by continuous differential equations, which are not locally Lipschitz continuous on the set $\varepsilon_2 = 0$, while the Lyapunov function V is not continuously differentiable for $\varepsilon_2 = 0$. However, as it has been discussed above, away the origin, for any point with $\varepsilon_1 \neq 0$ and $\varepsilon_2 = 0$ always there is a solution that immediately exits the set $\varepsilon_2 = 0$, so outside the origin the time derivative of $V(\varepsilon_1(t), \varepsilon_2(t))$ should be well defined for almost all $t \geq 0$ and it takes the form for $\varepsilon_2 \neq 0$:

$$\dot{V} = -2a_1 a_2 b\left((a_0 + a)\varepsilon_1^{\kappa} + \frac{b}{\kappa(\kappa+1)}\varepsilon_1^{2\kappa}\right)$$
$$-2a_2(a_0 + a + \frac{b}{\kappa}\varepsilon_1^{\kappa})|\varepsilon_2|$$

that is as before. Therefore, for any initial conditions there is a solution such that away the origin $V(\varepsilon_1(t), \varepsilon_2(t))$ is strictly decaying for almost all $t \geq 0$, hence the origin is weakly attracting. □

The system (1), (8) has well-defined solutions, and according to this theorem it solves the extremum seeking problem.

5. Delay-based derivative estimation

In this section the main contribution of the paper is given.

To this end, note that synthesis of a numerical scheme that discretizes (1) with preservation of the property $z_2(t) = \dot{y}(t)$ for $t \geq T$ is a complicated issue (Acary & Brogliato, 2010; Polyakov, Efimov, & Brogliato, 2019) (discretization is required for implementation of the algorithm on any digital device). Moreover, presence of numerical or measurement noises does not allow the derivative to be calculated exactly in (1), as well as in any other differentiator. Thus, it is more convenient for applications to replace an exact differentiator by another method, which is approximating the derivative with a known error, and analyse how such a modification influences the suggested extremum seeking approach.

In this paper the time-delay framework is used for derivative approximation (Fridman & Shaikhet, 2017; Selivanov & Fridman, 2018): the expression

$$\dot{y}(t) = \frac{y(t) - y(t-\tau) + \dot{R}(t)}{\tau},\ t \geq 0$$

is satisfied for any delay $\tau > 0$, where

$$R(t) = \int_{t-\tau}^{t}(s - t + \tau)\dot{y}(s)ds.$$

Consequently, the system (1), (4) can be replaced with

$$\begin{aligned} \dot{x}_1(t) &= x_2(t), \\ \dot{x}_2(t) &= -a_1\frac{y(t) - y(t-\tau)}{\tau x_2(t)} - a_2(a_0 + y(t))x_2(t), \\ x(t) &= x_1(t), \end{aligned} \tag{9}$$

$$x_1(s) = x_{\min},\ x_2(s) = \text{const} > 0,\ \forall s \in [-\tau, 0],$$

where $a_1 > 0$ and $a_2 > 0$ are the same tuning parameters and $\tau > 0$ is any delay, and the output of the differentiator (1) is replaced with the simplest derivative estimate $\frac{y(t)-y(t-\tau)}{\tau}$, then the state of this system belongs to $C^2_{[-\tau,0]}$. Using the change of coordinates (7) we get an implementable version of the optimization algorithm (9):

$$\begin{aligned} \dot{\xi}_1(t) &= \sqrt{|\xi_2(t)|}\operatorname{sign}(\xi_2(t)), \\ \dot{\xi}_2(t) &= -2(a_1\frac{y(t) - y(t-\tau)}{\tau}\operatorname{sign}(\xi_2(t)) \\ &\quad + a_2(a_0 + y(t))\xi_2(t)), \end{aligned} \tag{10}$$

$x(t) = \xi_1(t)$.

The dynamics of $\xi_2(t)$ can be rewritten as follows:

$$\dot{\xi}_2(t) = -2(a_1 f'(\xi_1(t))\sqrt{|\xi_2(t)|} + a_2(a_0 + a + f(\xi_1(t)))\xi_2(t) - \frac{a_1}{\tau}\dot{R}(t)\mathrm{sign}(\xi_2(t))),$$

then introducing the seeking error $\varepsilon_1(t) = \xi_1(t) - x_0$ and denoting the signed square of its velocity by $\varepsilon_2(t) = \xi_2(t)$, we obtain an equivalent representation of (10):

$$\dot{\varepsilon}_1(t) = \sqrt{|\varepsilon_2(t)|}\mathrm{sign}(\varepsilon_2(t)), \quad (11)$$
$$\dot{\varepsilon}_2(t) = -2(a_1 b\varepsilon_1^{\kappa-1}(t)\sqrt{|\varepsilon_2(t)|} + a_2(a_0 + a + \frac{b}{\kappa}\varepsilon_1^{\kappa}(t))\varepsilon_2(t) - \frac{a_1}{\tau}\dot{R}(t)\mathrm{sign}(\varepsilon_2(t))),$$

where with respect to the error dynamics studied in the previous section we have an additional perturbation term $2\frac{a_1}{\tau}\dot{R}(t)\mathrm{sign}(\varepsilon_2(t))$ that is originated from the derivative estimation by the delayed measurements, and

$$\dot{R}(t) = \tau b\varepsilon_1^{\kappa-1}(t)\sqrt{|\varepsilon_2(t)|}\mathrm{sign}(\varepsilon_2(t)) + \frac{b}{\kappa}(\varepsilon_1^{\kappa}(t-\tau) - \varepsilon_1^{\kappa}(t)).$$

We need now to establish the conditions of boundedness and convergence of a trajectory of (11) to the origin (that implies realization of (3) in (10)).

Again, the right-hand side of (11) is multiplied by $\varepsilon_2(t)$ (or its sign), hence the set $\varepsilon_2 = 0$ contains the equilibria of the system. However, it is important to recall that the perturbation term $\dot{R}(t)$ includes the delayed variables, then the Filippov's convex embedding of (11) is a functional differential inclusion as (2). Moreover, the presence of $\mathrm{sign}(\varepsilon_2(t))$ and multiplication of other items by $\varepsilon_2(t)$ in different powers implies that the set $\varepsilon_2 = 0$ can be reached only if $\dot{R}(t) \le 0$, which follows by $\varepsilon_1^{\kappa}(t-\tau) - \varepsilon_1^{\kappa}(t) \le 0$. As a consequence, once a trajectory of (11) enters the set $\varepsilon_2 = 0$, it stays there during a time interval less or equal to τ (in such a case $\dot{\varepsilon}_1(t) = 0$ and at least after the delay period τ the equality $\varepsilon_1^{\kappa}(t-\tau) = \varepsilon_1^{\kappa}(t)$ should be verified), next $\dot{R}(t) = 0$ and for $\varepsilon_1(t) \ne 0$ the term $-2a_1 b\varepsilon_1^{\kappa-1}(t)\sqrt{|\varepsilon_2(t)|}$ guarantees existence of a trajectory leaving the set $\varepsilon_2 = 0$ as in the previous section. So, the only difference with the case of the system (1), (8) is that trajectories of (11) for $\varepsilon_1 \ne 0$ leave the set $\varepsilon_2 = 0$ not obligatory immediately, and may stay in this set a time interval of the maximal length $\tau > 0$. Therefore, zero is the strongly invariant steady-state solution of (11), all other equilibria in the set $\varepsilon_1 \ne 0, \varepsilon_2 = 0$ are weak (there is a trajectory that quits these points).

Unfortunately, the Lyapunov function (6) cannot be incorporated in the analysis of the error dynamics (11) with $\dot{R}(t) \ne 0$, i.e., such a Lyapunov function is not an input-to-state stable one for this system. Therefore, to analyse stability properties of (11), since it is a time-delay system, a locally Lipschitz continuous Lyapunov-Krasovskii functional is proposed:

$$U(\varepsilon_{1t}, \varepsilon_{2t}) = W_1(\varepsilon_1(t), \varepsilon_2(t)) + qW_2(\varepsilon_{1t}, \varepsilon_{2t}) - \frac{a_1}{\tau}R(t),$$

where

$$W_1(\varepsilon_1, \varepsilon_2) = \frac{a_1 b}{\kappa}\varepsilon_1^{\kappa} + \frac{1}{2}|\varepsilon_2|,$$
$$W_2(\varepsilon_{1t}, \varepsilon_{2t}) = \int_{t-\tau}^{t} \mathbf{e}^{\omega(s-t)}(s-t+\tau)^2\dot{y}^2(s)ds,$$

$q > 0$ and $\omega > 0$ are tuning parameters. Note that $q\mathbf{e}^{-\omega\tau}\zeta^2 - \frac{a_1}{\tau}\zeta \ge -\left(\frac{a_1}{2\tau}\right)^2\frac{\mathbf{e}^{\omega\tau}}{q}$ for any $\zeta \in \mathbb{R}$, then

$$U(\varepsilon_{1t}, \varepsilon_{2t}) \ge W_1(\varepsilon_1(0), \varepsilon_2(0)) + \int_{t-\tau}^{t} q\mathbf{e}^{-\omega\tau}(s-t+\tau)^2\dot{y}^2(s) - \frac{a_1}{\tau}(s-t+\tau)\dot{y}(s)ds$$
$$\ge W_1(\varepsilon_1(0), \varepsilon_2(0)) - \left(\frac{a_1}{2}\right)^2\frac{\mathbf{e}^{\omega\tau}}{\tau q}$$

and U has a constant lower limit since $W_1(\varepsilon_1, \varepsilon_2) \ge 0$ for any $(\varepsilon_1, \varepsilon_2) \in \mathbb{R}^2$. Next, for brevity denote $U(t) = U(\varepsilon_{1t}, \varepsilon_{2t})$, $W_1(t) = W_1(\varepsilon_1(t), \varepsilon_2(t))$ and $W_2(t) = W_2(\varepsilon_{1t}, \varepsilon_{2t})$, direct computations show (note that $W_1(\varepsilon_1(t), \varepsilon_2(t)) = const$ while the trajectory is on the set $\varepsilon_2(t) = 0$):

$$\dot{W}_1(t) = -a_2(a_0 + a + \frac{b}{\kappa}\varepsilon_1^{\kappa}(t))|\varepsilon_2(t)| + \frac{a_1}{\tau}\dot{R}(t)\mathrm{sign}^2(\varepsilon_2(t)),$$
$$\dot{W}_2(t) = -\omega W_2(t) - 2\int_{t-\tau}^{t}\mathbf{e}^{\omega(s-t)}(s-t+\tau)\dot{y}^2(s)ds + \tau^2\dot{y}^2(t)$$
$$\le -\omega W_2(t) - 2\frac{\mathbf{e}^{-\omega\tau}}{\tau^2}R^2(t) + \tau^2\dot{y}^2(t),$$

where on the last step Jensen's inequality was used. Therefore, since $\dot{y}(t) = b\varepsilon_1^{\kappa-1}(t)\sqrt{|\varepsilon_2(t)|}\mathrm{sign}(\varepsilon_2(t))$ we have the following estimate:

$$\dot{U}(t) \le -a_2(a_0 + a + \frac{b}{\kappa}\varepsilon_1^{\kappa}(t))|\varepsilon_2(t)| + \frac{a_1}{\tau}\dot{R}(t)\left(\mathrm{sign}^2(\varepsilon_2(t)) - 1\right) - q\omega W_2(t) - 2q\frac{\mathbf{e}^{-\omega\tau}}{\tau^2}R^2(t) + q\tau^2 b^2\varepsilon_1^{2\kappa-2}(t)|\varepsilon_2(t)|.$$

Consider the set $\mathcal{X} = \{(\varepsilon_1, \varepsilon_2) \in \mathbb{R}^2 : \varepsilon_1^{\kappa-2} \le \frac{1}{\tau^2}\frac{a_2}{\kappa q b}\}$ and the related functional extension

$$\mathbb{X} = \{(\phi_1, \phi_2) \in C_{[-\tau,0]}^2 : \phi_1^{\kappa-2}(s) \le \frac{1}{\tau^2}\frac{a_2}{\kappa q b},\ s \in [-\tau, 0]\}, \quad (12)$$

which is nonempty for any $\kappa \ge 2$ if $a_2 > q\tau^2\kappa b$ (this constraint can be always satisfied for any a_2 and τ by the choice of q), then $-a_2\frac{b}{\kappa}\varepsilon_1^{\kappa}(t)|\varepsilon_2(t)| + q\tau^2 b^2\varepsilon_1^{2\kappa-2}(t)|\varepsilon_2(t)| \le 0$ for $(\varepsilon_{1t}, \varepsilon_{2t}) \in \mathbb{X}$, hence,

$$\dot{U}(t) \le -a_2(a_0 + a)|\varepsilon_2(t)| + \frac{a_1}{\tau}\dot{R}(t)\left(\mathrm{sign}^2(\varepsilon_2(t)) - 1\right) - q\omega W_2(t) - 2q\frac{\mathbf{e}^{-\omega\tau}}{\tau^2}R^2(t)$$

and $\dot{U}(t) < 0$ for $\varepsilon_2(t) \ne 0$ into the set $\mathbb{X}$.

As it has been explained above, any non-zero trajectory of (11) may stay at the set $\varepsilon_2 = 0$ on a time interval, which we define by $[t_{1k}, t_{2k}) \subseteq \mathbb{R}_+$ for $k \in \mathbb{Z}_+$:

$$\varepsilon_2(t) = 0\ \forall t \in [t_{1k}, t_{2k}), \quad (13)$$

with the length $t_{2k} - t_{1k}$ less or equal than τ while $\dot{R}(t) < 0$ (and it leaves this set once $\dot{R}(t_{2k}) = 0$), then

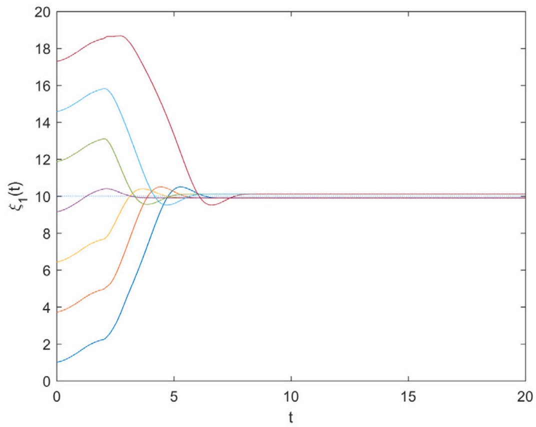
$$\dot{U}(t) \le -q\omega W_2(t) - 2q\frac{\mathbf{e}^{-\omega\tau}}{\tau^2}R^2(t) - \frac{a_1}{\tau}\dot{R}(t),$$

where the right-hand side may be sign indefinite. Note that, while a trajectory belongs to the set $\varepsilon_2 = 0$, the function $W_1(t)$ is constant for $t \in [t_{1k}, t_{2k})$ and in $U(t)$ only the value of the term $qW_2(\varepsilon_{1t}, \varepsilon_{2t}) - \frac{a_1}{\tau}R(t)$ is varying. Hence, let us require that this discrepancy be smaller at the instant t_{2k} of leaving the set $\varepsilon_2 = 0$ than at the instant of entrance t_{1k}: this condition can be formulated as occurrence of $(\varepsilon_{1t_{1k}}, \varepsilon_{2t_{1k}})$ into the set

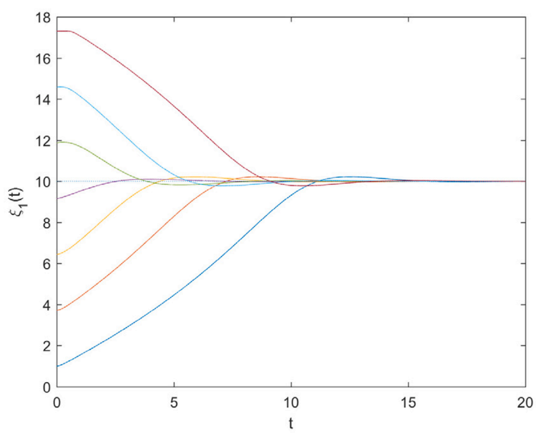
$$\tilde{\mathbb{X}} = \{(\varepsilon_{1t}, \varepsilon_{2t}) \in \mathbb{X} : \int_{t-\tau'(\varepsilon_{1t})}^{t}\mathbf{e}^{\omega(s-t)}(s-t+\tau)^2 \times b\varepsilon_1^{2\kappa-2}(s)|\varepsilon_2(s)|ds \ge \int_{t-\tau'(\varepsilon_{1t})}^{t}(s-t+\tau)\varepsilon_1^{\kappa-1}(s)\sqrt{|\varepsilon_2(s)|}\mathrm{sign}(\varepsilon_2(s))ds,\ \varepsilon_2(0) = 0\}, \quad (14)$$

where

$$\tau'(\phi) = \begin{cases} -\min \mathcal{S}_\phi & \text{if } \mathcal{S}_\phi \ne \emptyset \\ \tau & \text{otherwise,} \end{cases}$$
$$\mathcal{S}_\phi = \{s \in [-\tau, 0] : |\phi(s)| = |\phi(0)|\},$$

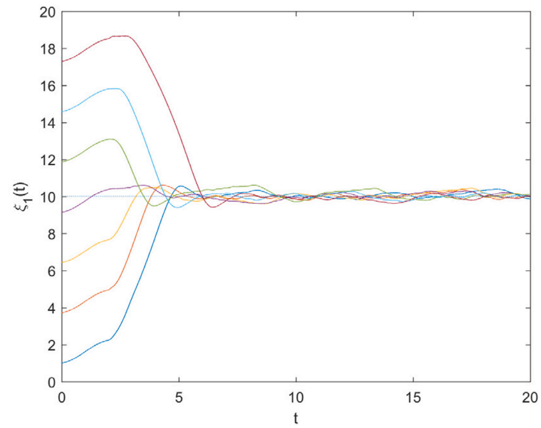


(a) Algorithm (1), (8)

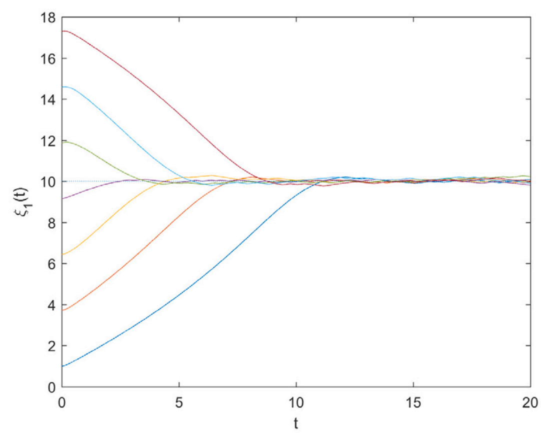


(b) Algorithm (10)

Fig. 1. The results of simulation without noise, $h = 0.001$.



(a) Algorithm (1), (8)



(b) Algorithm (10)

Fig. 2. The results of simulation with noise, $h = 0.001$.

then $U(t_{2k}) \leq U(t_{1k})$ and U does not grow after the stay on the set $\varepsilon_2 = 0$. In other words, the set of equilibria $\varepsilon_2 = 0$, $\varepsilon_1 \neq 0$ is weakly invariant, and the trajectory is just constant there for $t \in [t_{1k}, t_{2k})$, despite that $\dot{U}(t)$ may be sign-varying on this time interval, for non-increasing of $U(t)$ it is enough to provide the property $U(t_{2k}) \leq U(t_{1k})$, which follows from $(\varepsilon_{1t_{1k}}, \varepsilon_{2t_{1k}}) \in \tilde{\mathbb{X}}$. Assume that $(\varepsilon_{1t}, \varepsilon_{2t}) \in \mathbb{X}$ for all $t \geq 0$ and $(\varepsilon_{1t_{1k}}, \varepsilon_{2t_{1k}}) \in \tilde{\mathbb{X}}$ for all $k \in \mathbb{Z}_+$, then $U(t) \leq U(0)$ for all $t \geq 0$, hence,

$$W_1(\varepsilon_1(t), \varepsilon_2(t)) \leq U(0) + \left(\frac{a_1}{2}\right)^2 \frac{\mathbf{e}^{\omega\tau}}{\tau q}, \ \forall t \geq 0 \tag{15}$$

and the trajectory is bounded. Next, recalling the argumentation of Theorems 1 or 2, since $U(t)$ has lower and upper limits, the trajectory has to converge to a forward invariant subset belonging to the set $\varepsilon_2 = 0$, where there is the only strongly invariant equilibrium at the origin.

The following result has been proven, which establishes that for suitable initial conditions, the algorithm (10) finds the extremum:

Theorem 4. *Under Assumption 1, consider the estimation error system (11). For any $\tau > 0$, $a_1 > 0$ and $a_2 > 0$, if $(\varepsilon_{1t}, \varepsilon_{2t}) \in \mathbb{X}$ for all $t \geq 0$ and $(\varepsilon_{1t_{1k}}, \varepsilon_{2t_{1k}}) \in \tilde{\mathbb{X}}$ for all $k \in \mathbb{Z}_+$, where t_{1k} is defined in (13), $\mathbb{X}$ and $\tilde{\mathbb{X}}$ are given by (12) and (14), respectively, then (3) is realized weakly.*

Note that due to (15), if the initial error $(\varepsilon_{10}, \varepsilon_{20})$ is sufficiently small and the parameters a_1 and q are also chosen sufficiently small, then it is ensured that $(\varepsilon_{1t}, \varepsilon_{2t}) \in \mathbb{X}$ for all $t \geq 0$. Moreover, by construction, from (9), we have $(\varepsilon_{1t_{10}}, \varepsilon_{2t_{10}}) \in \tilde{\mathbb{X}}$.

Let us illustrate the efficiency of this scheme in simulations.

6. Simulations

For simulations, let us take

$$a = -1, \ b = 1, \ \kappa = 2, \ x_0 = 10,$$

then Assumption 1 is satisfied for

$$a_0 = 5, \ x_{\min} = 1, \ x_{\max} = 20.$$

For the algorithm (1), (8) take

$$\lambda_1 = 10, \ \lambda_2 = 30, \ a_1 = 5, \ a_2 = 0.5,$$

then simulations show that $T = 2$ is a reasonable choice. For (10) select (arbitrary values):

$$a_1 = 15, \ a_2 = 0.25, \ \tau = 0.05.$$

The results of simulation of $x(t)$ for the algorithms (1), (8) and (10) are shown in Fig. 1 for the noise-free setting, and in Fig. 2 with a uniformly distributed in the interval $[-0.1, 0.1]$ measurement noise $v(t)$ in $y(t)$. For simulation, different initial conditions were selected for $\xi_1(0)$, the explicit Euler method was used with the discretization step $h = 0.001$, and $\xi_2(0) = 0.1$. As we can conclude from these results, the signal $x(t)$ converges to the extremum value x_0 in the absence of noise, and for the noisy experiments, a vicinity of the extremum is reached.

7. Conclusion

A new extremum-seeking approach is presented, which is based on time derivative of the measured optimized function signal, and which is capable to provide an asymptotic convergence to the extremum. For time derivative estimation, two methods are tested: the exact super-twisting differentiator and the time-delay framework. The stability and convergence analyses are based on Lyapunov function and Lyapunov-Krasovskii functional. Since the implementable versions of the algorithm are described by differential inclusions, for the time-delay case, extensions of invariance principle are formulated. Our numeric experiments demonstrate a nice robustness of the presented approach to the measurement noises, but a rigorous proof of this property is left for future research. Other possible challenging extensions deal with development of tuning rules and consideration of multidimensional maps.

CRediT authorship contribution statement

Denis Efimov: Writing – original draft, Formal analysis. **Emilia Fridman:** Methodology, Formal analysis.

Declaration of competing interest

The authors state that they have no conflict of interests.

Acknowledgments

The authors acknowledge the support of the Program to support young researchers at STU under the project AIOperator: eXplainable Intelligence for Industry, as well as the contribution of the Scientific Grant Agency of the Slovak Republic under the grants 1/0490/23 and 1/0297/22. This research is funded by the Horizon Europe under the grant no. 101079342 (Fostering Opportunities Towards Slovak Excellence in Advanced Control for Smart Industries). RP acknowledges the financial support by the European Commission under the grant scheme NextGenerationEU project no. 09I03–03–V04–00530.

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