Stability of Systems With Uncertain Delays: A New "Complete" Lyapunov–Krasovskii Functional

Emilia Fridman

***Abstract*—Stability of linear systems with uncertain time-varying delay is considered in the case, where the nominal value of the delay is constant and *nonzero*. Recently a new construction of Lyapunov-Krasovskii functionals (LKFs) has been introduced: To a nominal LKF, which is appropriate to the system with nominal delays, terms are added that correspond to the perturbed system and that vanish when the delay perturbations approach 0. In the present note we combine a "complete" nominal LKF, the derivative of which along the trajectories depends on states and their derivatives, with the additional terms depending on the delay perturbation. The new method is applied to the case of multiple uncertain delays with one nonzero nominal value. Numerical examples illustrate the efficiency of the method.**

***Index Terms*—Lyapunov–Krasovskii method, stability, time-varying delay, uncertain delay.**

I. Introduction

It is well-known (see e.g., [4] and [11]) that the choice of an appropriate Lyapunov-Krasovskii functional (LKF) is crucial for deriving stability and bounded real criteria and, as a result, for obtaining a solution to various control problems. The general ("complete") form of this functional that has been used by many authors (see e.g., [1], [7], [17]) leads to a system of partial differential equations. Special (reduced) forms of LKFs lead to simpler delay-independent and (less conservative) *delay-dependent* finite dimensional conditions in terms of Riccati equations or linear matrix inequalities (LMIs).

During the last decade, a considerable amount of attention has been paid to stability and control of linear systems with uncertain delays (either constant or time-varying) lying in the given segment $[0, \mu]$ (see, e.g., [2], [4], [11], [12], [15], [16], and the references therein). This case of delay may be considered as uncertain delay with a *zero nominal value* and a delay perturbation from $[0, \mu]$ and (following [3]) it will be denoted as uncertain *"small"* delay. Delay-dependent stability conditions in the case of uncertain 'small' delay were based in the past on three main model transformations of the original system (see [4] and [11]). Recently, a new *descriptor* model transformation was introduced [2]. Unlike previous transformations, the descriptor model leads to a system without additional dynamics, it does not depend on additional assumptions for stability of the transformed system (as in the case of neutral type transformation) and it requires bounding of fewer cross-terms. Moreover, for the first time by Lyapunov–Krasovskii method, it is applicable to the case of "fast varying" delay, where no constraints are given on the delay derivative [4].

In the descriptor approach both, the state vector and (different from the other LKF methods) *its derivative*, appear in the expression for the *derivative* of the LKF along the trajectories of the system. The dependence of the derivative of LKF on the state derivative makes it possible to treat the delay uncertainty in a less conservative way.

The case of uncertain *"nonsmall"* time-varying delay, where the *nominal delay value is nonzero* and constant appears in different applications such as internet networks, biological systems [10]. Only a few papers have been published on this topic. The stability of linear retarded type systems with one uncertain 'nonsmall' delay, where the *nominal value is nonzero* and constant, has been studied in [8]. Sufficient stability conditions there for the case of time-varying delay have been derived via modification of "complete" LKF which was introduced in [9] and via the first model transformation. This LKF does not depend explicitly on the delay perturbation and, as a result, the conditions obtained are conservative and complicated. A different modification of "complete" LKF for systems with a known differentiable delay $\tau(t)$ has been introduced in [13], where stability conditions were given in terms of the convergence of $\int_v^\infty |\dot{\tau}(t)|dt$ $(v > 0)$. The latter convergence can not be verified in the case of uncertain time-varying delay.

In the recent paper [3] a new construction of the LKF has been introduced: to a nominal LKF, which is appropriate to the nominal system (with nominal delays), the terms are added which correspond to the perturbed system and which vanish when the delay perturbations approach 0. In [3], the descriptor type nominal LKF has been considered and the conditions obtained can be feasible if they are feasible for the nominal system. In the case when the latter assumption does not hold (e.g., when the nondelayed system is not asymptotically stable), the "complete" nominal LKF (which corresponds to necessary and sufficient conditions for stability) should be considered.

It is the objective of this note to derive sufficient conditions for stability of the linear system with uncertain delay via a new "complete" nominal LKF. Unlike the existing "complete" LKFs (see, e.g., [1], [7]–[9], [13], [14], and [17]), the derivative of the new nominal LKF along the trajectories of the nominal system depends on the state and the *state derivative* which allows a less conservative treatment of the *delay perturbation*. The algorithm for stability is given in terms of linear algebraic operations, definite integral and LMIs. Numerical examples illustrate the efficiency of the new method.

Notation: Throughout this note, the superscript "T" stands for matrix transposition, $\mathcal{R}^n$ denotes the n-dimensional Euclidean space with vector norm $|\cdot|$, $\mathcal{R}^{n\times m}$ is the set of all $n \times m$ real matrices, and the notation $P > 0$, for $P \in \mathcal{R}^{n\times n}$ means that P is symmetric and positive definite. We also denote $x_t(\theta) = x(t+\theta)$ $(\theta \in [-h-\mu, 0])$ and $\|x_t\| = \sup_{\theta\in[-h-\mu,0]} |x_t(\theta)|$. The symmetric elements of the symmetric matrix will be denoted by $*$.

II. Problem Formulation

Consider the following linear system with uncertain time-varying delay $\tau(t)$:

$$\dot{x}(t) = A_0x(t) + A_1x(t-\tau(t)), \qquad t \geq t_0 \tag{1}$$

$$x(t_0+\theta) = \phi(\theta), \qquad \theta \in [-h-\mu, 0] \tag{2}$$

where $x(t) \in \mathcal{R}^n$ is the system state, ϕ is the initial function and $h+\mu$ is an upper-bound on the time-delay $0 \leq \tau(t) \leq h+\mu$, $t \geq 0$. The uncertain delay $\tau(t)$ is supposed to have the following form:

$$\tau(t) = h + \eta(t) \tag{3}$$

where $h > 0$ is a nominal constant value and $\eta(t)$ is a time-varying perturbation. We assume that $\eta(t)$ satisfies the inequality

$$|\eta(t)| \leq \mu \leq h \tag{4}$$

with the known upper bound μ, i.e., $\tau(t) \in [h-\mu, h+\mu]$.

We assume the following.

A1) Given the constant nominal value of the delay $h > 0$, the nominal system

$$\dot{x}(t) = A_0x(t) + A_1x(t-h) \tag{5}$$

is asymptotically stable.

Manuscript received March 28, 2004; revised August 30, 2004 and November 16, 2005. Recommended by Associate Editor S.-I. Niculescu. This work was supported by the Kamea Fund of Israel.

E. Fridman is with the Department of Electrical Engineering-Systems, Tel Aviv University, Tel-Aviv 69978, Israel (e-mail: emilia@eng.tau.ac.il).

Digital Object Identifier 10.1109/TAC.2006.872769

Similarly to delay-dependent methods and following [8] we represent (1) in the form

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-h) + A_1[x(t-h-\eta(t)) - x(t-h)], \qquad t \geq t_0. \quad (6)$$

As suggested in [3], we consider the following form of LKF:

$$V = V_n + V_a \quad (7)$$

where V_n is a nominal LKF which corresponds to the nominal system (5) and V_a consists of additional terms and depends on μ and $V_a \to 0$ for $\mu \to 0$. Therefore, for $\mu \to 0$ $V \to V_n$. The latter will guarantee that if the conditions for the stability of the nominal system are feasible, then the stability conditions for the perturbed system will be feasible for small enough μ.

As we have already mentioned, for the stability analysis of systems with uncertain delay, the descriptor model transformation appeared to be the less conservative one. By the descriptor approach, the derivative of the nominal LKF depends on both, $x(t)$ and $\dot{x}(t)$. For the nominal system in the present note we choose the "complete" LKF, the derivative of which depends on $x(t)$ and $\dot{x}(t)$. In Section III-D, we generalize our results to the case of multiple uncertain delays, where one delay is nonsmall.

III. Main Results

A. A New "Complete" LKF for the Nominal System

Consider the nominal system (5) with the initial condition $x(t) = \phi(t), t \in [-h, 0]$, where ϕ is continuous. We shall find such a LKF

$$V_n(\phi) = V_{n0}(\phi) + V_{n1}(\phi) \quad (8)$$

that along the trajectories of the nominal system (5) has a form

$$\dot{V}_{n0}(x_t) = -x^T(t) W_0 x(t) \quad (9a)$$
$$\dot{V}_{n1}(x_t) = -\dot{x}^T(t) W_1 \dot{x}(t), \qquad t \geq 0 \quad (9b)$$

where $W_0 > 0$ and $W_1 > 0$ are some constant matrices. The difference between our "complete" LKF and the one considered in [8] is in the term V_{n1}. This term is taken to be zero in [8], but it is exactly this term that makes it possible to treat the perturbed delay in a less conservative way.

Remark 3.1: A different "complete" LKF for systems with a known differentiable delay $\tau(t)$ has been introduced in [13], where stability conditions were given in terms of the convergence of $\int_v^\infty |\dot{\tau}(t)| dt$ $(v > 0)$. The latter conditions can not be verified in the case of *"fast-varying"* delays, which includes nondifferentiable delays.

The term V_{n0} has been constructed in [9]

$$V_{n0}(\phi) = \int_0^\infty x^T(t,\phi) W_0 x(t,\phi) dt \quad (10)$$

where $x(t,\phi)$ is a solution to (5) with $x(t) = \phi(t), t \in [-h, 0]$. This term has the following form:

$$V_{n0}(\phi) = \phi^T(0) U_0(0) \phi(0) + 2\phi^T(0) \int_{-h}^0 U_0(-h-\theta) A_1 \phi(\theta) d\theta + \int_{-h}^0 \int_{-h}^0 \phi^T(\theta_2) A_1^T U_0(\theta_2 - \theta_1) A_1 \phi(\theta_1) d\theta_1 d\theta_2 \quad (11)$$

where the matrix function $U_0(\theta)$ is defined for $\theta \in R$ (and not only for $\theta \in [-h, 0]$) as

$$U_0(\theta) = \int_0^\infty K^T(t) W_0 K(t+\theta) dt. \quad (12)$$

Here, $K(t)$ is a fundamental matrix associated with the nominal system (5), i.e., $K(t)$ is an $n \times n$-matrix function satisfying

$$\dot{K}(t) = A_0 K(t) + A_1 K(t-h), \qquad t \geq 0 \quad (13)$$

with the initial condition $K(0) = I$ and $K(t) = 0$ for $t < 0$. The previous definition of U_0 for all $\theta \in R$ and the resulting symmetry $(U_0(\theta) = U_0^T(-\theta), \theta \geq 0)$ allow to find U_0 from the boundary value problem for ordinary differential equation (see [9] and Section III-C). The derivation of (11) is based on the well-known representation of $x(t,\phi)$ given by (see, e.g., [6])

$$x(t,\phi) = K(t)\phi(0) + \int_{-h}^0 K(t-h-\theta) A_1 \phi(\theta) d\theta. \quad (14)$$

Similarly to V_{n0}, we find V_{n1} in the form

$$V_{n1}(\phi) = \int_0^\infty \frac{d}{dt} x^T(t,\phi) W_1 \frac{d}{dt} x(t,\phi) dt. \quad (15)$$

Note that $K(t)$ is piecewise-continuous with a jump in $t = 0$, while $\dot{K}(t)$ is piecewise-continuous and has jumps in $t = 0$ and $t = h$

$$\dot{K}(t) = \begin{cases} 0, & \text{if } t < 0 \\ A_0 \exp A_0 t, & \text{if } t \in [0, h) \\ A_0 \exp A_0 h + A_1, & \text{if } t = h^+. \end{cases} \quad (16)$$

Differentiating (14) in t and taking into account that for $0 \leq t \leq h$

$$x(t,\phi) = K(t)\phi(0) + \int_{-h}^{t-h} K(t-h-\theta) A_1 \phi(\theta) d\theta$$

and, thus

$$\frac{d}{dt} x(t,\phi) = \dot{K}(t)\phi(0) + \int_{-h}^{t-h} \dot{K}(t-h-\theta) A_1 \phi(\theta) d\theta + A_1 \phi(t-h) = \dot{K}(t)\phi(0) + \int_{-h}^{0} \dot{K}(t-h-\theta) A_1 \phi(\theta) d\theta + A_1 \phi(t-h), \qquad t \in [0, h]$$

we obtain (17), as shown at the bottom of the page.

$$\frac{d}{dt} x(t,\phi) = \begin{cases} \dot{K}(t)\phi(0) + \int_{-h}^0 \dot{K}(t-h-\theta) A_1 \phi(\theta) d\theta + A_1 \phi(t-h), & \text{if } t \in [0, h) \\ \dot{K}(t)\phi(0) + \int_{-h}^0 \dot{K}(t-h-\theta) A_1 \phi(\theta) d\theta, & \text{if } t \geq h \end{cases} \quad (17)$$

Substituting (17) into (15) and denoting

$$U_1(\theta) = \int_0^\infty \dot{K}^T(t)W_1\dot{K}(t+\theta)dt, \qquad \theta \in R \tag{18}$$

we find

$$\begin{aligned} V_n(\phi) &= V_{n0}(\phi) + V_{n1}(\phi) \\ &= \phi^T(0)U(0)\phi(0) + 2\phi^T(0) \\ &\quad \times \int_{-h}^0 U(-h-\theta)A_1\phi(\theta)d\theta \\ &\quad + \int_{-h}^0\int_{-h}^0 \phi^T(\theta_2)A_1^T U(\theta_2-\theta_1) \\ &\quad \times A_1\phi(\theta_1)d\theta_1 d\theta_2 + \bar{V}_n \end{aligned} \tag{19}$$

where

$$U(\theta) = U_0(\theta) + U_1(\theta), \qquad \theta \in R \tag{20a}$$

$$\bar{V}_n = \int_0^h \phi^T(t-h)A_1^T W_1 \times \left\{ A_1\phi(t-h) + 2\left[\dot{K}(t)\phi(0) + \int_{-h}^0 \dot{K}(t-h-\theta)A_1\phi(\theta)d\theta \right] \right\} dt. \tag{20b}$$

Since

$$\dot{K}(t) = K(t)A_0 + K(t-h)A_1, \qquad t \in R \tag{21}$$

where for $t = 0$ and $t = h$ the right-hand derivative is taken, we have

$$U_1(\theta) = \int_0^\infty \left[A_0^T K^T(t) + A_1^T K^T(t-h)\right] \times W_1\left[K(t+\theta)A_0 + K(t+\theta-h)A_1\right]dt. \tag{22}$$

Denote

$$X(\theta) = \int_0^\infty K^T(t)W_1K(t+\theta))d\theta, \qquad \theta \in R. \tag{23}$$

Then

$$\begin{aligned} U_1(\theta) &= A_0^T X(\theta)A_0 + A_1^T X(\theta+h)A_0 \\ &\quad + A_0^T X(\theta-h)A_1 + A_1^T X(\theta)A_1 \\ &\theta \in R. \end{aligned} \tag{24}$$

Proposition 3.1: Assume **A1)**. Let W_0, W_1 be symmetric matrices. Then U_1 and U given by (18) and (20a) are well-defined for all $\theta \in R$. Moreover, $U_1(\theta) = U_1^T(-\theta)$ and $U(\theta) = U^T(-\theta)$ for $\theta \geq 0$.

Proof: From (24) and the fact that X is well-defined it follows that U_1 and thus U are well defined. Similar to U_0, $X(\theta) = X^T(-\theta)$ for all $\theta \geq 0$ [9]. Then, (24) yields for $\theta \geq 0$

$$\begin{aligned} U_1^T(-\theta) &= A_0^T X^T(-\theta)A_0 + A_0^T X^T(-\theta+h)A_1 \\ &\quad + A_1^T X^T(-\theta-h)A_0 + A_1^T X^T(-\theta)A_1 \\ &= A_0^T X(\theta)A_0 + A_0^T X(\theta-h)A_1 \\ &\quad + A_1^T X(\theta+h)A_0 + A_1^T X(\theta)A_1 = U_1(\theta) \end{aligned}$$

and $U^T(-\theta) = U(\theta)$. This completes the proof.

Setting $t - h = \theta_2$ in $\bar{V}_n$ and applying (16), we find

$$\bar{V}_n = \int_{-h}^0 \phi^T(\theta_2)A_1^T W_1 \times \left\{ A_1\phi(\theta_2) + 2\left[A_0 e^{A_0(\theta_2+h)}\phi(0) + \int_{-h}^{\theta_2} A_0 e^{A_0(\theta_2-\theta_1)}A_1\phi(\theta_1)d\theta_1 \right] \right\} d\theta_2. \tag{25}$$

Lemma 3.1: Assume A1. Given $n \times n$ matrices $W_0 > 0, W_1 > 0$, LKF (19) with U defined by (20a), (18) and (12) satisfies the following conditions:

$$\begin{aligned} \frac{d}{dt}V(x(t+\cdot,\phi)) &= -x^T(t,\phi)W_0x(t,\phi) \\ &\quad - \dot{x}^T(t,\phi)W_1\dot{x}(t,\phi) \end{aligned} \tag{26a}$$

$$V_n(\phi) \geq \varepsilon|\phi(0)|^2. \tag{26b}$$

Proof: Substituting (12), (18) into (19) and applying (14), (17) we finally obtain

$$V_n(\phi) = \int_0^\infty \left[x^T(t,\phi)W_0x(t,\phi) + \frac{d}{dt}x^T(t,\phi)W_1\frac{d}{dt}x^T(t,\phi) \right] dt.$$

Since for the autonomous system (5) $x(s+t,\phi) = x(s, x(t+\cdot,\phi))$, we have

$$\begin{aligned} &V_n(x(t+\cdot,\phi)) \\ &= \int_0^\infty \left[x^T(s+t,\phi)W_0x(s+t,\phi) + \frac{d}{dt}x^T(s+t,\phi)W_1\frac{d}{dt}x(s+t,\phi) \right] ds \\ &= \int_t^\infty \left[x^T(\theta,\phi)W_0x(\theta,\phi) + \frac{d}{d\theta}x^T(\theta,\phi)W_1\frac{d}{d\theta}x(\theta,\phi) \right] d\theta. \end{aligned}$$

Differentiating in t the latter equation we derive (26a).

Define functional $V_\varepsilon(\phi) = V_n(\phi) - \varepsilon\phi^T(0)\phi(0)$. Then for small enough $\varepsilon > 0$

$$\begin{aligned} V_\varepsilon(x(t+\cdot,\phi)) &= -x^T(t,\phi)W_0x(t,\phi) \\ &\quad - \frac{d}{dt}x^T(t,\phi)W_1\frac{d}{dt}x(t,\phi) \\ &\quad - 2\varepsilon x^T(t,\phi)\frac{d}{dt}x(t,\phi) \\ &\triangleq w_\varepsilon(x(t+\cdot,\phi) \geq 0. \end{aligned}$$

We have

$$V_\varepsilon(\phi) = \int_0^\infty w_\varepsilon(x(t+\cdot,\phi)dt \geq 0$$

and thus $V_n(\phi) \geq \varepsilon\phi^T(0)\phi(0)$, i.e., (26b) is valid, which completes the proof.

B. Stability of the System With Uncertain Non-Small Delay

Our LKF for (1) will depend on x and $\dot{x}$. Thus, (see [10, Th. 1.6, p. 337]) the initial functions in (2) are restricted to be absolutely continuous with a square-integrable derivative. We note that uniform (with respect to t_0) asymptotic stability of linear retarded type system (1) with differentiable initial functions implies uniform asymptotic stability of (1) with a wider class of continuous initial functions. This follows from the fact that the solutions $x(t, t_0, \phi)$ to (1), (2) with the continuous functions ϕ become differentiable for $t \geq t_0 + h + \mu$, while for $t \in [t_0, t_0 + h + \mu]$ they satisfy the uniform bound $|x(t, t_0, \phi)| \leq m\|\phi\|$ with some constant $m > 0$ [6].

We represent (6) (with differentiable initial functions) in the equivalent form

$$\dot{x}(t) = A_0x(t) + A_1x(t-h) - A_1\int_{t-h-\eta(t)}^{t-h}\dot{x}(s)ds \quad t \geq t_0. \tag{27}$$

Differentiating V_n defined by (19) along the trajectories of (27), we find similar to [8]

$$\dot{V}_n(x_t) = -x^T(t)W_0x(t) - \dot{x}^T(t)W_1\dot{x}(t) + \Delta(t) \tag{28}$$

where

$$\Delta(t) = -2\int_{t-h-\eta}^{t-h}\dot{x}^T(s)A_1^T \times \left[U(0)x(t) + \int_{-h}^{0}Q^T(h+\theta) \times A_1x(t+\theta)d\theta\right]ds \tag{29a}$$

$$Q^T(h+\theta) = U^T(h+\theta) + e^{A_0^T(h+\theta)}A_0^TW_1. \tag{29b}$$

By standard bounding

$$\begin{aligned}|\Delta(t)| \leq & \sum_{i=1}^{2}\left|\int_{t-h-\eta(t)}^{t-h}\dot{x}^T(s)A_1^TR_i^{-1}A_1\dot{x}(s)ds\right| \\ & + \left|\int_{t-h-\eta(t)}^{t-h}x^T(t)U(0)R_1U(0)x(t)ds\right| \\ & + \left|\int_{t-h-\eta(t)}^{t-h}\int_{-h}^{0}x^T(t+\theta)A_1^TQ(h+\theta) \times R_2Q^T(h+\theta)A_1x(t+\theta)d\theta ds\right| \\ \leq & \sum_{i=1}^{2}\int_{t-h-\mu}^{t-h+\mu}\dot{x}^T(s)A_1^TR_i^{-1}A_1\dot{x}(s)ds \\ & + \mu x^T(t)U(0)R_1U(0)x(t) \\ & + \mu\int_{-h}^{0}x^T(t+\theta)A_1^TQ(h+\theta) \\ & \times R_2Q^T(h+\theta)A_1x(t+\theta)d\theta. \end{aligned} \tag{30}$$

We choose

$$\begin{aligned} V(x_t) &= V_n(x_t) + V_a(x_t) \\ V_a(x_t) &= \int_{-\mu}^{\mu}\int_{t+\theta-h}^{t}\dot{x}^T(s)A_1^T\left(R_1^{-1}+R_2^{-1}\right)A_1\dot{x}(s)dsd\theta \\ &+ \mu\int_{-h}^{0}\int_{t+\theta}^{t}x^T(s)A_1^TQ(h+\theta) \\ &\times R_2Q^T(h+\theta)A_1x(s)dsd\theta \end{aligned} \tag{31}$$

where $R_1 > 0$, $R_2 > 0$, and V_n is defined by (19), (20a), (25). By Lemma 3.1, if the nominal system (5) is asymptotically stable, then for some $\varepsilon > 0$ $V(x_t) \geq V_n(x_t) \geq \varepsilon|x(t)|^2$. Differentiating V along the trajectories of (27) and taking into account (28) and (30), we find

$$\begin{aligned}\dot{V}(x_t) \leq & -x^T(t) \times \Bigg[W_0 - \mu U(0)R_1U(0) \\ & - \mu\int_{-h}^{0}A_1^TQ(\theta+h)R_2Q^T(\theta+h)A_1d\theta\Bigg]x(t) \\ & - \dot{x}^T(t)\left[W_1 - 2\mu A_1^T\left(R_1^{-1}+R_2^{-1}\right)A_1\right]\dot{x}(t). \end{aligned} \tag{32}$$

Hence, if the following inequalities hold:

$$W_0 - \mu U(0)R_1U(0) - \mu A_1^T\int_0^hQ(s)R_2Q^T(s)dsA_1 > 0 \tag{33a}$$

$$Q(s) = U(s) + W_1A_0e^{A_0s} \tag{33b}$$

$$\begin{bmatrix}-W_1 & 2\mu A_1^T & 2\mu A_1^T \\ * & -2\mu R_1 & 0 \\ * & * & -2\mu R_2\end{bmatrix} < 0 \tag{33c}$$

then $\dot{V} < -c|x(t)|^2$ and (1) is uniformly asymptotically stable [10].

We have proved the following.

Theorem 3.1: Under **A1)**, the system (1) is uniformly (with respect to t_0) asymptotically stable for all piecewise-continuous delays $0 \leq \tau(t) \in [h-\mu, h+\mu]$ if there exist $n \times n$ matrices W_0, W_1, R_1, R_2 that satisfy (33), where U is defined by (12), (18), and (20a).

C. Computation of U and Algorithm for Stability

It was shown in [9] (see also [14] and the references therein) that $X(t)$ given by (23) satisfies the following differential equation and boundary value condition:

$$\begin{aligned}\dot{X}(\theta) &= X(\theta)A_0 + X(\theta-h)A_1, \qquad \theta \geq 0 \\ W_1 &+ X(0)A_0 + A_0^TX(0) + X^T(h)A_1 + A_1^TX(h) = 0.\end{aligned} \tag{34}$$

Denoting $Y(\theta) = X^T(-\theta+h) = X(\theta-h), \theta \geq 0$ we represent (34) in the form of the following boundary value problem for ordinary differential equations:

$$\dot{X}(\theta) = X(\theta)A_0 + Y(\theta)A_1 \tag{35a}$$

$$\dot{Y}(\theta) = -A_1^TX(\theta) - A_0^TY(\theta), \qquad \theta \geq 0 \tag{35b}$$

$$-W_1 = X(0)A_0 + A_0^TX(0) + Y(0)A_1 + A_1^TX(h) \tag{35c}$$

$$Y(h) = X(0). \tag{35d}$$

For computation of $U_1(\theta)$ we apply Kronecker products of matrices. We remind that given $n \times m$ matrix A with elements a_{ij}, $1 \leq i \leq n$, $1 \leq j \leq m$ and $p \times q$ matrix B, their Kronecker product $A \otimes B$ is the $np \times mq$ matrix with the block structure

$$A \otimes B = \begin{bmatrix}a_{11}B & \dots & a_{1m}B \\ \dots & \dots & \dots \\ a_{n1}B & \dots & a_{nm}B\end{bmatrix}.$$

The stack of A is the vector formed by stacking the columns of A into $nm \times 1$ vector shown in the equation at the bottom of the next page. The following holds $(ABD)^S = (D^T \otimes A)B^S$.

We represent (35) and (24) in the form

$$\begin{bmatrix}\dot{X}^S(\theta) \\ \dot{Y}^S(\theta)\end{bmatrix} = \mathcal{A}\begin{bmatrix}X^S(\theta) \\ Y^S(\theta)\end{bmatrix} \tag{36a}$$

$$\mathcal{A} = \begin{bmatrix}A_0^T \otimes I_n & A_1^T \otimes I_n \\ -I_n \otimes A_1^T & -I_n \otimes A_0^T\end{bmatrix} \tag{36b}$$

$$\begin{bmatrix}-W_1^S \\ 0_{n^2\times 1}\end{bmatrix} = \mathcal{B}\begin{bmatrix}X^S(0) \\ Y^S(0)\end{bmatrix} \tag{36c}$$

$$\mathcal{B} = \begin{bmatrix}\left(A_0^T \otimes I_n\right) + \left(I_n \otimes A_0^T\right) & A_1^T \otimes I_n \\ I_{n^2} & 0_{n^2\times n^2}\end{bmatrix} + \begin{bmatrix}I_n \otimes A_1^T & 0_{n^2\times n^2} \\ 0_{n^2\times n^2} & -I_{n^2}\end{bmatrix}e^{\mathcal{A}h} \tag{36d}$$

and

$$U_1^S(\theta) = \mathcal{C}e^{\mathcal{A}\theta}\begin{bmatrix}X^S(0) \\ Y^S(0)\end{bmatrix} \tag{37a}$$

$$\mathcal{C} = \left[\left(A_0^T \otimes A_0^T\right) + \left(A_1^T \otimes A_1^T\right),\ A_1^T \otimes A_0^T\right] + \left(A_0^T \otimes A_1^T\right)\left[I_{n^2} \quad 0\right]e^{\mathcal{A}h}. \tag{37b}$$

Note that $Y(0) = X^T(h)$. Similarly, we derive

$$\begin{bmatrix} -W_0^S \\ 0_{n^2\times 1} \end{bmatrix} = \mathcal{B} \begin{bmatrix} U_0^S(0) \\ \left(U_0^T(h)\right)^S \end{bmatrix} \quad U_0^S(\theta) = e^{\mathcal{A}\theta} \begin{bmatrix} U_0^S(0) \\ \left(U_0^T(h)\right)^S \end{bmatrix}. \quad (38)$$

If $\mathcal{B}$ is nonsingular we finally obtain

$$U^S(\theta) = \mathcal{C}e^{\mathcal{A}\theta}\mathcal{B}^{-1} \begin{bmatrix} -W_1^S \\ 0_{n^2\times 1} \end{bmatrix} + [I \quad 0]\, e^{\mathcal{A}\theta}\mathcal{B}^{-1} \begin{bmatrix} -W_0^S \\ 0_{n^2\times 1} \end{bmatrix}$$
$$\theta \in [0, h]. \quad (39)$$

Remark 3.2: As it was mentioned in [9], the matrix $\mathcal{A}$ has symmetric with respect to the imaginary axis eigenvalues. In the case that there exist an eigenvalue with nonzero real part, the matrix $\mathcal{B}$ becomes ill-conditioned for large h and (38) leads to inaccurate results with nonsymmetric $U_0(0)$.

If the nominal system is asymptotically stable and the matrix $\mathcal{B}$ is neither singular nor ill-conditioned, then (39) uniquely defines $U(0) \geq 0$, since (12) and (18) are well-defined and satisfy (39). Moreover, if the matrix $\mathcal{B}$ is nonsingular and not ill-conditioned, and (38) leads to matrix $U_0(0)$ which is not semipositive definite, then the nominal system is not asymptotically stable.

Choosing $R_2 = r_2 I_n$, where $r_2 > 0$ is a scalar, we obtain the following *algorithm* for asymptotic stability of (1) provided (5) is asymptotically stable.

1) Choose $n \times n$-matrices $W_0 > 0, W_1 > 0$ and find $U(0)$ from (39). Check that $U(0) > 0$ and, thus, (39) leads to accurate results.
2) For $U(\theta)$ given by (39) find

$$\mathcal{Q} = \int_0^h Q(\theta)Q^T(\theta)d\theta \quad Q(\theta) = U(\theta) + W_1 A_0 e^{A_0\theta}. \quad (40)$$

3) Given $\mu > 0$, verify that there exist $n \times n$-matrix R_1 and scalar r_2 that satisfy the following LMIs:

$$W_0 - \mu U(0)R_1U(0) - \mu r_2 A_1^T \mathcal{Q} A_1 > 0 \quad (41a)$$
$$\begin{bmatrix} -W_1 & 2\mu A_1^T & 2\mu A_1^T \\ * & -2\mu R_1 & 0 \\ * & * & -2\mu r_2 I \end{bmatrix} < 0. \quad (41b)$$

D. Extension to the Case of Multiple Delays

In the case of multiple delays in the nominal system, the computation of $U(\theta)$ becomes complicated and may be reduced to the boundary value problem for ordinary differentiable equation only if these delays are commensurate. To derive simple stability conditions for linear systems with multiple uncertain delays we consider the case, where the nominal system has one delay. Consider the system

$$\dot{x}(t) = Ax(t) + A_1x(t - h - \eta(t)) + A_2x(t - \eta_1(t)) \quad (42)$$

where $h > 0$, $\eta(t)$ and $\eta_1(t)$ are piecewise-continuous delays satisfying $|\eta(t)| \leq \mu \leq h$, $0 \leq \eta_1(t) \leq \mu_1$. For simplicity only we consider one small delay η_1. The generalization to the finite number of small delays is straightforward.

We represent (42) in the form

$$\dot{x}(t) = A_0x(t) + A_1x(t-h) - A_1\int_{t-h-\eta(t)}^{t-h} \dot{x}(s)ds - A_2\int_{t-\eta_1(t)}^{t} \dot{x}(s)ds$$
$$A_0 = A + A_2. \quad (43)$$

Differentiating V_n defined by (19), (20a), and (25) along the trajectories of (43), we find

$$\dot{V}_n(x_t) = -x^T(t)W_0x(t) - \dot{x}^T(t)W_1\dot{x}(t) + \Delta(t) + \Delta_1(t) \quad (44)$$

where $\Delta(t)$ is given by (29a) and

$$\Delta_1(t) = -2\int_{t-\eta_1(t)}^{t} \dot{x}^T(s)A_2^T \times \left[U(0)x(t) + \int_{-h}^{0} Q^T(h+\theta)A_1x(t+\theta)d\theta\right]ds. \quad (45)$$

Similar to Theorem 3.1, we obtain the following.

Theorem 3.2: Assume **A1)**. Let $U(\theta), Q(\theta), \theta \in [0, h]$ and $\mathcal{Q}$ be given by (39) and (40). Then, the system (42) is uniformly asymptotically stable for all piecewise-continuous delays $|\eta(t)| \leq \mu \leq h$, $0 \leq \eta_1(t) \leq \mu_1$ if there exist $n \times n$ matrices W_0, W_1, R_1, R_{11} and scalars r_2, r_{12} that satisfy

$$W_0 - U(0)[\mu R_1 + \mu_1 R_{11}]U(0) - [\mu r_2 + \mu_1 r_{12}]A_1^T \mathcal{Q} A_1 > 0 \quad (46a)$$

$$\begin{bmatrix} -W_1 & 2\mu A_1^T & 2\mu A_1^T & \mu_1 A_2^T & \mu_1 A_2^T \\ * & -2\mu R_1 & 0 & 0 & 0 \\ * & * & -2\mu r_2 I & 0 & 0 \\ * & * & * & -\mu_1 R_{11} & 0 \\ * & * & * & * & -\mu_1 r_{12} I \end{bmatrix} < 0. \quad (46b)$$

E. Examples

Example 3.1: Consider the system

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix} x(t - \eta_1(t)) + \begin{bmatrix} 0 & 0 \\ -0.4 & 0 \end{bmatrix} x(t - 4 - \eta(t)) \quad (47)$$

which was analyzed in [8] for $\eta_1 = 0$. The nondelayed system (i.e., (47) with $\eta_1 = 0, 4 + \eta = 0$) is not asymptotically stable and thus the simple nominal LKFs are not applicable. For the case of constant delay the following asymptotic stability interval was found by the frequency domain analysis [8]: $\eta_1 = 0$, $-0.6209 < \eta(t) < 0.7963$. Verifying time-domain conditions of [8] for $h = 4$, $W_1 = W_2 = I$, we found (for the case of constant delay and $\eta_1 = 0$) the maximum value of $|\eta| \leq 0.000\,001$ which guarantees the asymptotic stability of (47).

For $h = 4$, $W_0 = W_1 = I$ we obtain from (38)–(40)

$$U_0(0) = \begin{bmatrix} 77.5886 & -0.5000 \\ -0.5000 & 32.6290 \end{bmatrix}$$
$$U(0) = \begin{bmatrix} 262.6477 & -1 \\ -1 & 110.2176 \end{bmatrix}$$
$$\mathcal{Q} = 10^5 \times \begin{bmatrix} 1.8783 & -0.0147 \\ -0.0147 & 0.7982 \end{bmatrix}.$$

LMIs (41) are feasible and, thus, (47) is asymptotically stable for $\eta_1 = 0$, $|\eta(t)| \leq 0.011$. By Theorem 3.2, (47) is uniformly asymptotically stable for $0 \leq \eta_1(t) \leq 0.002$, $|\eta(t)| \leq 0.002$.

Example 3.2: [8] Consider the system

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix} x(t - \tau). \quad (48)$$

For $\tau = h + \eta(t)$, where $h = 1$ and $\eta(t)$ is a differentiable function satisfying $|\eta| \leq \mu$, $|\dot{\eta}| \leq \dot{\mu} < 1$ the following values of μ and $\dot{\mu}$ for uniform asymptotic stability of (48) were found in [8]: $\dot{\mu} < 0.8$ and $\mu < (1/25\,600) < 0.000\,04$. By representing conditions of [8] in the

$$A^S \triangleq [\,a_{11} \quad \ldots \quad a_{n1} \quad a_{12} \quad \ldots \quad a_{n2} \quad \ldots \quad a_{m1} \quad \ldots \quad a_{mn}\,]^T$$

form of LMIs for the same values of $W_1 = W_2 = I$ we obtained a larger value of $\mu = 0.000\,08$.

Applying Theorem 3.1 and choosing $W_0 = W_1 = I$, we find from (38)–(40)

$$U_0(0) = \begin{bmatrix} 4 & 1 \\ 1 & 1 \end{bmatrix} \quad U(0) = \begin{bmatrix} 7 & 2 \\ 2 & 3 \end{bmatrix}$$
$$\mathcal{Q} = \begin{bmatrix} 42.8234 & 2.6938 \\ 2.6938 & 0.6103 \end{bmatrix}$$

and for $\mu = 0.12$ (41a) and (41b) are feasible. Hence, the system is asymptotically stable for essentially larger interval $[0.88, 1.12]$ for a wider class of delays (which may be not differentiable).

By descriptor approach of [3], the resulting interval is wider: $\tau(t) \in [0.73, 1.27]$ with $\mu = 0.27$. By descriptor approach the system is stable and thus conditions of [3] can be applied for $h \leq 254$. In this example, the conditions of [8] and of Theorem 3.1 give reliable results till $h \leq 22$, while for greater values of h matrix $\mathcal{B}$ becomes ill-conditioned and the resulting $U_0(0)$ is not symmetric.

IV. Conclusion

A new Lyapunov–Krasovskii technique is developed for stability of linear system with uncertain time-varying delay in the case when the nominal value of the delay is constant and nonzero: To a "complete" nominal LKF, which is appropriate to the system with the nominal value of the delay, terms are added that correspond to the perturbed system and that vanish when the delay perturbation approaches 0. The nominal "complete" LKF is considered, the derivative of which along the trajectories of the nominal system depends on both, the state and the state derivative. Given matrices W_0 and W_1, the stability sufficient conditions are reduced to linear algebraic operations, definite integral and to LMIs. The new method is applied to the case of multiple uncertain delays with one nonsmall delay. Similarly to "complete" LKF of [8], the new "complete" LKF can be applied in the case where the nondelayed system is not asymptotically stable, but it leads to simpler and less conservative conditions. Feasibility of the latter conditions is guaranteed for small perturbations of the delay.

The conditions derived are conservative since one have to choose first W_0 and W_1 in order to verify their feasibility. Less conservative conditions may be derived by choosing $\dot{V}_n$ to be a general negative–definite quadratic form of $x(t)$ and $\dot{x}(t)$.

References

[1] R. Datko, "An algorithm for computing Lyapunov functionals for some differential-difference equations," in *Ordinary Differential Equations*, L. Weiss, Ed. New York: Academic, 1971, pp. 387–398.

[2] E. Fridman, "New Lyapunov-Krasovskii functionals for stability of linear retarded and neutral type systems," *Syst. Control Lett.*, vol. 43, pp. 309–319, 2001.

[3] ——, "A new Lyapunov technique for robust control of systems with uncertain nonsmall delays," *IMA J. Math. Control Inform.*, vol. 23, no. 2, pp. 165–179, 2006.

[4] E. Fridman and U. Shaked, "Delay dependent stability and H_∞ control: Constant and time-varying delays," *Int. J. Control*, vol. 76, no. 1, pp. 48–60, 2003.

[5] K. Gu, "Discretized LMI set in the stability problem of linear time delay systems," *Int. J. Control*, vol. 68, pp. 923–934, 1997.

[6] J. K. Hale and S. M. V. Lunel, *Introduction to Functional Differ ential Equations*. New York: Springer-Verlag, 1993.

[7] W. Huang, "Generalization of Lyapunov's theorem in a linear delay system," *J. Math. Anal. Appl.*, vol. 142, pp. 83–94, 1989.

[8] V. Kharitonov and S. Niculescu, "On the stability of linear systems with uncertain delay," *IEEE Trans. Autom. Control*, vol. 48, no. 1, pp. 127–132, Jan. 2002.

[9] V. Kharitonov and A. Zhabko, "Lyapunov-Krasovskii approach to the robust stability analysis of time-delay systems," *Automatica*, vol. 39, pp. 15–20, 2003.

[10] V. Kolmanovskii and A. Myshkis, *Applied Theory of Functional Differential Equations*. Norwell, MA: Kluwer, 1999.

[11] V. Kolmanovskii and J.-P. Richard, "Stability of some linear systems with delays," *IEEE Trans. Autom. Control*, vol. 44, no. 5, pp. 984–989, May 1999.

[12] X. Li and C. de Souza, "Criteria for robust stability and stabilization of uncertain linear systems with state delay," *Automatica*, vol. 33, pp. 1657–1662, 1997.

[13] J. Louisell, "A stability analysis for a class of differential-delay equations having time-varying delays," in *Delay Differential Equations and Dynamical Systems*. ser. Lecture Notes in Mathematics, no. 1475, S. Busenberg and M. Martelli, Eds. Berlin, Germany: Springer-Verlag, 1991, pp. 225–242.

[14] ——, "Numerics of the stability exponent and eigenvalue abcissas of a matrix delay system," in *Stability and Control of Time-Delay Systems*, ser. Lecture Notes in Control and Information Sciences. London, U.K.: Springer-Verlag, 1997, vol. 228, pp. 140–157.

[15] S. I. Niculescu, *Delay Effects on Stability: A Robust Control Approach*. London, U.K.: Springer-Verlag, 2001, vol. 269, Lecture Notes in Control and Information Sciences.

[16] P. Park, "A delay-dependent stability criterion for systems with uncertain time-invariant delays," *IEEE Trans. Autom. Control*, vol. 44, no. 4, pp. 876–877, Apr. 1999.

[17] Y. M. Repin, "Quadratic Lyapunov functionals for systems with delay" (in Russian), *Prikladnaya Matematika Mehanika*, vol. 29, pp. 564–566, 1965.

Descriptor Discretized Lyapunov Functional Method: Analysis and Design

Emilia Fridman

***Abstract*—Stability and state-feedback stabilization of linear systems with uncertain coefficients and uncertain time-varying delays are considered. The system under consideration may be unstable without delay, but it becomes asymptotically stable for positive values of the delay. A new *descriptor discretized* Lyapunov–Krasovskii functional (LKF) method is introduced, which combines the application of the complete LKF and the discretization method of K. Gu with the descriptor model transformation. For the first time, the new method allows to apply the discretized LKF method to *synthesis* problems. Moreover, the analysis of systems with polytopic time-invariant uncertainties is less restrictive by the new discretized method. Sufficient conditions for robust stability and stabilization of uncertain neutral type systems are derived in terms of linear matrix inequalities (LMIs) via input–output approach to stability. Numerical examples illustrate the efficiency of the new method.**

***Index Terms*—Linear matrix inequality (LMI), Lyapunov–Krasovskii functional (LKF), robust stability, stabilization, time-delay.**

I. Introduction

It is well known that the choice of an appropriate Lyapunov–Krasovskii functional (LKF) is crucial for deriving stability criteria and for obtaining a solution to various robust control problems

Manuscript received April 7, 2005; revised August 18, 2005. Recommended by Associate Editor I. Kolmanovsky. This work was supported by the Kamea Fund of Israel.

The author is with the Department of Electrical Engineering-Systems, Tel Aviv University, Tel-Aviv 69978, Israel (e-mail: emilia@eng.tau.ac.il).

Digital Object Identifier 10.1109/TAC.2006.872828