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Brief paper

Delayed finite-dimensional observer-based control of 2D linear parabolic PDEs☆

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ABSTRACT

Recently, a constructive method was suggested for finite-dimensional observer-based control of 1D linear heat equation, which is robust to input/output delays. In this paper, we aim to extend this method to the 2D case with general time-varying input/output delays or sawtooth delays (that correspond to network-based control). We use the modal decomposition approach and consider boundary or non-local sensing together with non-local actuation, or Neumann actuation with non-local sensing. To compensate the output delay that appears in the infinite-dimensional part of the closed-loop system, for the first time for delayed PDEs we suggest a vector Lyapunov functional combined with the recently introduced vector Halanay inequality. We provide linear matrix inequality (LMI) conditions for finding the observer dimension and upper bounds on delays that preserve the exponential stability. We prove that the LMIs are always feasible for large enough observer dimension and small enough upper bounds on delays. A numerical example demonstrates the efficiency of our method and shows that the employment of vector Halanay's inequality allows for larger delays than the classical scalar Halanay inequality for comparatively large observer dimension.

1. Introduction

Finite-dimensional observer-based controllers for PDEs are attractive in applications. Such controllers were designed by the modal decomposition method and have been extensively studied since the 1980s (Balas, 1988; Christofides, 2001; Curtain, 1982; Grüne & Meurer, 2022; Harkort & Deutscher, 2011), where efficient bound estimate on the observer and controller dimensions is a challenging problem. In recent paper (Katz & Fridman, 2020), the first constructive LMI-based method for finite-dimensional observer-based control of 1D parabolic PDEs was suggested, where the observer dimension was found from simple LMI conditions. The results in Katz and Fridman (2020) were then extended to input/output delay robustness (Katz & Fridman, 2021, 2022a, 2022b), delayed PDEs (Lhachemi & Shorten, 2023a) and delay compensation (Katz & Fridman, 2022a; Lhachemi & Prieur, 2022, 2023; Lhachemi & Shorten, 2023b). However, the above results were confined to 1D parabolic PDEs.

In recent years, control of high-dimensional PDEs became an active research area. Such systems have promising applications in engineering, water heating, metal rolling, sheet forming, medical imaging (see e.g. Meurer (2012)) as well as in multi-agents deployment (Qi, Vazquez, & Krstic, 2015). Sampled-data observers for 2D and ND heat equations with globally Lipschitz nonlinearities have been suggested in Am and Fridman (2014) and Selivanov and Fridman (2019). Observer-based output-feedback controller for a linear parabolic ND PDEs was designed in Wang and Wang (2021). In Kang and Fridman (2021), the sampled-data control of 2D Kuramoto–Sivashinsky equation was explored. The results in Am and Fridman (2014), Kang and Fridman (2021), Selivanov and Fridman (2019) and Wang and Wang (2021) employed spatial decomposition approach where many sensors/actuators should be utilized.

The boundary state-feedback stabilization of ND parabolic PDEs was studied in Barbu (2013) and Munteanu (2019) by modal decomposition approach and in Meurer (2012) and Liu and Xie (2020) by backstepping method. Observer-based boundary control for ND parabolic PDEs under boundary measurement over cubes and balls was explored in Jadachowski, Meurer, and Kugi (2015) and Vazquez and Krstic (2016) by the backstepping method. In Feng, Lang, and Liu (2022) and Meng and Feng (2022), observer-based control via modal decomposition approach was

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designed for ND parabolic PDEs. Note that the observer designs in Feng et al. (2022), Jadachowski et al. (2015), Meng and Feng (2022) and Vazquez and Krstic (2016) are in the from of PDEs. In Lhachemi, Munteanu, and Prieur (2023), for the first time, the finite-dimensional observer-based control was studied for 2D and 3D parabolic PDEs under boundary actuation on an arbitrary subdomain and in-domain pointwise measurement. It was shown in Lhachemi et al. (2023) that the closed-loop system is stable provided the dimension of the controller is large enough. Note that the results in Feng et al. (2022), Jadachowski et al. (2015), Lhachemi et al. (2023), Meng and Feng (2022) and Vazquez and Krstic (2016) are confined to observer-based controller design of ND delay-free PDEs. For ND parabolic PDEs, efficient finite-dimensional observer-based design with a quantitative bound on the observer as well as the input/output delay robustness remained open challenging problems.

In this paper, we aim to study finite-dimensional observer-based control of linear heat equation with input/output delays in Ω, an open and connected subset of $\mathbb{R}^2$. We consider either differentiable time-varying input/output delays or sawtooth delays (that correspond to network-based control). Based on modal decomposition approach, we consider the boundary or non-local sensing together with non-local actuation, or to Neumann actuation with non-local sensing. The novelty of this paper compared to existing results can be formulated as follows:

- Compared with Feng et al. (2022), Jadachowski et al. (2015), Lhachemi et al. (2023), Meng and Feng (2022) and Vazquez and Krstic (2016) for observer-based design of high-dimensional parabolic PDEs, we give efficient finite-dimensional observer-based design and provide LMI conditions for finding observer dimension and upper bounds of delays. We prove that the LMIs are always feasible for large enough observer dimension and small enough upper bounds on delays.
- Differently from Katz and Fridman (2021, 2022a) and Katz and Fridman (2022b) for 1D parabolic PDEs where Lyapunov functional combined with classical scalar Halanay's inequality (see P. 138 in Fridman (2014)) was suggested, we construct vector Lyapunov functional combined with recently introduced vector Halanay's inequality (see Mazenc, Malisoff, and Krstic (2022)). The latter allows to efficiently compensate the fast-varying output delay that appears in the infinite-dimensional part of the closed loop system essentially improving the upper bounds on delays in most of the numerical examples.
- Compared with spatial decomposition approach suggested in Am and Fridman (2014), Kang and Fridman (2021), Selivanov and Fridman (2019) and Wang and Wang (2021) for robust stabilization of ND parabolic PDEs, the modal decomposition approach in this paper allows for fewer actuators and sensors (including single boundary actuator or sensor).

Notations and preliminaries: For any bounded domain $\Omega \subset \mathbb{R}^2$, denote by $L^2(\Omega)$ the space of square integrable functions with inner product $\langle f, g\rangle = \int_\Omega f(x)g(x)\mathrm{d}x$ and induced norm $\|f\|_{L^2}^2 = \langle f, f\rangle$. $H^1(\Omega)$ is the Sobolev space of functions $f : \Omega \longrightarrow \mathbb{R}$ with a square integrable weak derivative. The norm defined in $H^1(\Omega)$ is $\|f\|_{H^1}^2 = \|f\|_{L^2}^2 + \|\nabla f\|_{L^2}^2$, where $\nabla f = [f_{x_1}, f_{x_2}]^{\mathrm{T}}$ and $\|\nabla f\|_{L^2}^2 = \int_\Omega [(f_{x_1})^2 + (f_{x_2})^2]\mathrm{d}x$. The Euclidean norm is denoted by $|\cdot|$. For $P \in \mathbb{R}^{n\times n}$, $P > 0$ means that P is symmetric and positive definite. The symmetric elements of a symmetric matrix will be denoted by $*$. For $0 < P \in \mathbb{R}^{n\times n}$ and $x \in \mathbb{R}^n$, we write $|x|_P^2 = x^{\mathrm{T}}Px$. Denote $\mathbb{N}$ by the set of positive integers.

Let $\Omega \subset \mathbb{R}^2$ be a bounded open connected set. Following Tucsnak and Weiss (2009), we assume that either the boundary $\partial\Omega$ is of class C^2 or Ω is a rectangular domain. Let $\partial\Omega$ be split into two disjoint parts $\partial\Omega = \Gamma_D \cup \Gamma_N$ such that Γ_D and Γ_N have non-zero Lebesgue measurement. Here subscripts D and N stand for Dirichlet and for Neumann boundary conditions respectively. Let

$$\begin{aligned} &\mathcal{A}\phi = -\Delta\phi,\ \mathcal{D}(\mathcal{A}) = \{\phi | \phi \in H^2(\Omega) \cap H^1_\Gamma(\Omega)\},\\ &H^1_\Gamma(\Omega) = \{\phi \in H^1(\Omega) | \phi(x) = 0 \text{ for } x \in \Gamma_D,\\ &\qquad \tfrac{\partial\phi}{\partial\mathbf{n}}(x) = 0 \text{ for } x \in \Gamma_N\}, \end{aligned} \tag{1.1}$$

where $\frac{\partial}{\partial\mathbf{n}}$ is the normal derivative. It follows from Tucsnak and Weiss (2009, Proposition 3.2.12) that the eigenvalues $\{\lambda_n\}_{n=1}^\infty$ of $\mathcal{A}$ are real and we can repeat each eigenvalue according to its finite multiplicity to get

$$\lambda_1 < \lambda_2 \le \cdots \le \lambda_n \le \ldots, \quad \lim_{n\to\infty} \lambda_n = \infty. \tag{1.2}$$

We denote the corresponding eigenfunctions as $\{\phi_n\}_{n=1}^\infty$. Let $\delta > 0$. From (1.2), it follows that there exists $N_0 \in \mathbb{N}$ such that

$$-\lambda_n + q + \delta < 0,\ n > N_0, \tag{1.3}$$

where $q \in \mathbb{R}$ is a constant reaction coefficient, N_0 is the number of modes used for the controller design. Throughout the paper, d will denote the maximum of the geometric multiplicities of λ_n, $n = 1, \ldots, N_0$.

Differently from the 1D case where $\lambda_N = O(N^2)$, $N \to \infty$, for λ_N, we have the following estimate which will be used for the asymptotic feasibility of LMIs:

Lemma 1.1 (*Strauss, 2007, Sec. 11.6*)**.** *For eigenvalues (1.2), we have* $\lim_{N\to\infty} \frac{\lambda_N}{N} = \frac{4\pi}{|\Omega|}$, *where* $|\Omega|$ *is the area of* Ω.

Since $\mathcal{A}$ is strictly positive and diagonalizable, we have (see Proposition 3.4.8 in Tucsnak and Weiss (2009))

$\mathcal{D}(\mathcal{A}^{\frac{1}{2}}) = \{h \in L^2(\Omega) | \sum_{n=1}^\infty \lambda_n |\langle h, \phi_n\rangle|^2 < \infty\}$.

Following Remark 3.4.4 in Tucsnak and Weiss (2009), we can regard $\mathcal{D}(\mathcal{A}^{\frac{1}{2}})$ as the completion of $\mathcal{D}(\mathcal{A})$ with respect to the norm $\|f\|_{\frac{1}{2}} = \sqrt{\langle \mathcal{A}f, f\rangle} = \sqrt{\sum_{n=1}^\infty \lambda_n |\langle f, \phi_n\rangle|^2}, f \in \mathcal{D}(\mathcal{A})$. For $h \in \mathcal{D}(\mathcal{A})$, we have $\|\nabla h\|_{L^2}^2 = \langle h, \mathcal{A}h\rangle = \|h\|_{\frac{1}{2}}^2$, which implies

$$\|\nabla h\|_{L^2}^2 = \sum_{n=1}^\infty \lambda_n h_n^2. \tag{1.4}$$

We have $\|f\|_{L^2}^2 \le C(\Omega)\|\nabla f\|_{L^2}^2$, $f|_{\Gamma_D} = 0$ for some constant $C(\Omega) > 0$ (see Glitch (2021)), which together with (1.4) implies the equivalence of $\|\cdot\|_{\frac{1}{2}}$ and $\|\cdot\|_{H^1}$ subject to $f(x) = 0, x \in \Gamma_D$. We have $\mathcal{D}(\mathcal{A}^{\frac{1}{2}}) = \{h \in H^1(\Omega) | h(x) = 0, x \in \Gamma_D\}$. Finally, density of $\mathcal{D}(\mathcal{A})$ in $\mathcal{D}(\mathcal{A}^{\frac{1}{2}})$ yields that (1.4) holds for any $h \overset{L^2(\Omega)}{=} \sum_{n=1}^\infty h_n\phi_n \in \mathcal{D}(\mathcal{A}^{\frac{1}{2}})$.

Given a positive integer N and $h \in L^2(\Omega)$ satisfying $h \overset{L^2}{=} \sum_{n=1}^\infty h_n\phi_n$, where $h_n = \langle h, \phi_n\rangle$, we denote $\|h\|_N^2 = \sum_{n=N+1}^\infty h_n^2$. For $\phi \in L^2(\Omega)$ and $\mathbf{b} = [b_1, \ldots, b_d]^{\mathrm{T}} \in (L^2(\Omega))^d$, we denote $\langle \mathbf{b}, \phi\rangle = [\langle b_1, \phi\rangle, \ldots, \langle b_d, \phi\rangle]^{\mathrm{T}}$.

Lemma 1.2 (*Vector Halanay's Inequality Mazenc et al., 2022*)**.** *Let* $M \in \mathbb{R}^{n\times n}$ *be a Metzler and Hurwitz matrix and* $P \in \mathbb{R}^{n\times n}$ *be a nonnegative matrix. Let* $\tau = \max\{\tau_1, \ldots, \tau_n\}$ *with* $\tau_i > 0$ *and* $V = [V_1, \ldots, V_n]^{\mathrm{T}} : [-\tau, \infty) \to [0, \infty)^n$ *be* C^1 *and*

$\dot{V}(t) \le MV(t) + P \sup_{s\in[t-\tau,t]} V(s)$,

where $\sup_{s\in[t-\tau,t]} V(s) = \mathrm{col}\{\sup_{s\in[t-\tau_i,t]} V_i(s)\}_{i=1}^n$. *If* $M + P$ *is Hurwitz, then* $|V(t)| \le De^{-\delta_0 t}$, $t \ge 0$ *for some* $\delta_0 > 0$ *and* $D > 0$.

2. Non-local actuation and measurement

2.1. System under consideration and controller design

Consider the following heat equation under delayed nonlocal actuation:

$$\begin{aligned} &z_t(x,t)=\Delta z(x,t)+qz(x,t)\\ &\qquad+\mathbf{b}^{\mathrm{T}}(x)u(t-\tau_u(t)),\ \text{in } \Omega\times(0,\infty),\\ &z(x,t)=0,\quad \text{on } \Gamma_D\times(0,\infty),\\ &\tfrac{\partial z}{\partial \mathbf{n}}(x,t)=0,\quad \text{on } \Gamma_N\times(0,\infty),\\ &z(\cdot,0)=z_0(\cdot)\in L^2(\Omega), \end{aligned} \tag{2.1}$$

where $u(t)=[u_1(t),\dots,u_d(t)]^{\mathrm{T}}$ is the control input to be designed later, $\tau_u(t)$ is a known input delay which is upper bounded by $\tau_{M,u}$, $\mathbf{b}=[b_1,\dots,b_d]^{\mathrm{T}}\in(L^2(\Omega))^d$. Assume the following delayed non-local measurement:

$$\begin{aligned} &y(t)=\langle \mathbf{c}, z(\cdot,t-\tau_y(t))\rangle,\quad t-\tau_y(t)\ge 0,\\ &y(t)=0,\ t-\tau_y(t)<0,\ \ \mathbf{c}=[c_1,\dots,c_d]^{\mathrm{T}}\in(L^2(\Omega))^d, \end{aligned} \tag{2.2}$$

where $\tau_y(t)$ is a known measurement delay which is upper bounded by $\tau_{M,y}$. The controller construction will follow Katz and Fridman (2020) for 1D case (where only simple eigenvalues appear), but the single-input and single-output as in Katz and Fridman (2020) are not applicable to the 2D case due to the existence of multiple eigenvalues (the system is uncontrollable and unobservable). Here we introduce multi-input $u(t)$ and multi-output (2.2) with $\mathbf{b}$, $\mathbf{c}$ satisfying Assumption 1 (see below) to manage with the controllability and observability.

We treat two classes of input/output delays: continuously differentiable delays and sawtooth delays that correspond to network-based control. For the case of continuously differentiable delays, we assume that $\tau_u(t)$ and $\tau_y(t)$ are lower bounded by $\tau_m>0$. This assumption is employed for well-posedness only. As in Katz and Fridman (2021) and Liu and Fridman (2014), we assume that there exists a unique $t_*\in[\tau_m,\min\{\tau_{M,y},\tau_{M,u}\}]$ such that $t-\tau(t)<0$ if $t<t_*$ and $t-\tau(t)\ge 0$ if $t\ge t_*$ for $\tau(t)\in\{\tau_u(t),\tau_y(t)\}$. For the case of sawtooth delays, $\tau_y(t)$ and $\tau_u(t)$ are induced by two networks: from sensor to controller and from controller to actuator, respectively (see Section 7.5 in Fridman (2014)). Henceforth the dependence of $\tau_y(t)$ and $\tau_u(t)$ on t will be suppressed to shorten notations.

We present the solution to (2.1) as

$$z(x,t)\overset{L^2}{=}\sum_{n=1}^{\infty}z_n(t)\phi_n(x),\ \ z_n(t)=\langle z(\cdot,t),\phi_n\rangle, \tag{2.3}$$

where $\{\phi_n\}_{n=1}^{\infty}$ are corresponding eigenfunctions of eigenvalues (1.2). Differentiating z_n in (2.3) and applying Green's first identity, we obtain

$$\begin{aligned} &\dot z_n(t)=(-\lambda_n+q)z_n(t)+\mathbf{b}_n^{\mathrm{T}}u(t-\tau_u),\ \ t\ge 0,\\ &z_n(0)=\langle z(\cdot,0),\phi_n\rangle,\quad \mathbf{b}_n=\langle \mathbf{b},\phi_n\rangle\in\mathbb{R}^d. \end{aligned} \tag{2.4}$$

Let $\delta>0$ be a desired decay rate and N_0 be defined by (1.3). We construct a N-dimensional ($N\ge N_0$) observer of the form

$$\hat z(x,t)=\sum_{n=1}^{N}\hat z_n(t)\phi_n(x),\quad N>N_0, \tag{2.5}$$

where $\hat z_n(t)$ satisfy

$$\begin{aligned} &\dot{\hat z}_n(t)=(-\lambda_n+q)\hat z_n(t)+\mathbf{b}_n^{\mathrm{T}}u(t-\tau_u)\\ &\qquad -l_n\left[\langle \mathbf{c},\hat z(\cdot,t-\tau_y)\rangle-y(t)\right],\ \ t>0,\\ &\hat z_n(0)=0,\quad t\le 0, \end{aligned} \tag{2.6}$$

with $y(t)$ in (2.2), observer gains $l_n\in\mathbb{R}^{1\times d}$, $1\le n\le N_0$ being designed later and $l_n=0_{1\times d}$ for $N_0<n\le N$.

Introduce the notations

$$\begin{aligned} &A_0=\mathrm{diag}\{-\lambda_n+q\}_{n=1}^{N_0},\ A_1=\mathrm{diag}\{-\lambda_n+q\}_{n=N_0+1}^{N},\\ &\mathbf{c}_n=\langle \mathbf{c},\phi_n\rangle,\ \mathbf{C}_0=[\mathbf{c}_1,\dots,\mathbf{c}_{N_0}],\ \mathbf{C}_1=[\mathbf{c}_{N_0+1},\dots,\mathbf{c}_N],\\ &\mathbf{B}_0=[\mathbf{b}_1,\dots,\mathbf{b}_{N_0}]^{\mathrm{T}},\quad \mathbf{B}_1=[\mathbf{b}_{N_0+1},\dots,\mathbf{b}_N]^{\mathrm{T}}. \end{aligned} \tag{2.7}$$

We rewrite A_0 as:

$$\begin{aligned} &A_0=\mathrm{diag}\{\tilde A_1,\dots,\tilde A_p\},\\ &\tilde A_j=\mathrm{diag}\{-\lambda_j+q,\dots,-\lambda_j+q\}\in\mathbb{R}^{n_j\times n_j},\\ &\lambda_k\neq\lambda_j\ \text{ iff }\ k\neq j,\quad k,j=1,\dots,p, \end{aligned} \tag{2.8}$$

where $n_1,\dots,n_p$ are positive integers such that $n_1+\cdots+n_p=N_0$. Clearly, $n_j\le d$, $j=1,\dots,p$ and there exists at least one $j\in\{1,\dots,p\}$ such that $n_j=d$. According to the partition of (2.8), we rewrite $\mathbf{B}_0$ and $\mathbf{C}_0$ as

$$\begin{aligned} &\mathbf{B}_0=[B_1^{\mathrm{T}},\dots,B_p^{\mathrm{T}}]^{\mathrm{T}},\quad B_j\in\mathbb{R}^{n_j\times d},\\ &\mathbf{C}_0=[C_1,\dots,C_p],\quad C_j\in\mathbb{R}^{d\times n_j}. \end{aligned}$$

Assumption 1. Let $\mathrm{rank}(B_j)=n_j$ and $\mathrm{rank}(C_j)=n_j$, $j=1,\dots,p$.

Lemma 2.1. *Under Assumption 1, the pair $(A_0,\mathbf{B}_0)$ is controllable and the pair $(A_0,\mathbf{C}_0)$ is observable.*

Proof. The proof is inspired by Lemma 7.2 of Meng and Feng (2022). Assume that the pair $(A_0,\mathbf{C}_0)$ is not observable. By the Hautus test (see Tucsnak and Weiss (2009, Remark 1.5.2)), there exist $0\neq\nu\in\mathbb{R}^{N_0}$ and $j\in\{1,\dots,p\}$ such that $A_0\nu=\lambda_j\nu$, $\mathbf{C}_0\nu=0$. Without loss of generality, we suppose that $\nu=\mathrm{col}\{\nu_1,\dots,\nu_p\}$, where $\nu_j=[\nu_j^{(1)},\dots,\nu_j^{(n_j)}]^{\mathrm{T}}$. Then we have $A_0\nu-\lambda_j\nu=\mathrm{col}\{(\lambda_k-\lambda_j)\nu_k\}_{k=1}^{p}=0$ and $\sum_{k=1}^{p}C_k\nu_k=0$, which implies $\nu_k=0$ for $k\neq j$ and $C_j\nu_j=0$. Since $\mathrm{rank}(C_j)=n_j$, we have $\nu_j=0$. This contradicts to the fact $\nu\neq 0$. Therefore, pair $(A_0,\mathbf{C}_0)$ is observable. The controllability of $(A_0,\mathbf{B}_0)$ follows similarly.

Under Assumption 1, we can let $L_0=\mathrm{col}\{l_1,\dots,l_{N_0}\}\in\mathbb{R}^{N_0\times d}$ and $K_0\in\mathbb{R}^{d\times N_0}$ satisfy

$$P_o(A_0-L_0\mathbf{C}_0)+(A_0-L_0\mathbf{C}_0)^{\mathrm{T}}P_o<-2\delta P_o, \tag{2.9a}$$

$$P_c(A_0-\mathbf{B}_0K_0)+(A_0-\mathbf{B}_0K_0)^{\mathrm{T}}P_c\le-2\delta P_c, \tag{2.9b}$$

for $0<P_o,P_c\in\mathbb{R}^{N_0\times N_0}$. We propose a controller of the form

$$u(t)=-K_0\hat z^{N_0}(t),\quad \hat z^{N_0}=[\hat z_1,\dots,\hat z_{N_0}]^{\mathrm{T}}. \tag{2.10}$$

For well-posedness of closed-loop system (2.1), (2.6) with control input (2.10), we consider the state $\xi(t)=\mathrm{col}\{z(\cdot,t),\hat z^N(t)\}$, where $\hat z^N(t)=\mathrm{col}\{\hat z_n(t)\}_{n=1}^{N}$. The closed-loop system can be presented as

$$\begin{aligned} &\tfrac{\mathrm{d}}{\mathrm{d}t}\xi(t)+\mathrm{diag}\{\mathcal{A},\mathcal{A}_0\}\xi(t)=\begin{bmatrix} qz(\cdot,t)+f_1(t-\tau_u)\\ f_2(t-\tau_u)+f_3(t-\tau_y)\end{bmatrix},\\ &\mathcal{A}_0=\mathrm{diag}\{-A_0,-A_1\},\ f_1(t)=-\mathbf{b}^{\mathrm{T}}(\cdot)K_0\hat z^{N_0}(t),\\ &f_2(t)=-\mathbf{B}K_0\hat z^{N_0}(t),\ \mathbf{B}=[\mathbf{B}_0^{\mathrm{T}},\mathbf{B}_1^{\mathrm{T}}]^{\mathrm{T}},\ \mathbf{C}=[\mathbf{C}_0,\mathbf{C}_1],\\ &f_3(t)=-\begin{bmatrix} L_0\\ 0_{(N-N_0)\times d}\end{bmatrix}[\mathbf{C}\hat z^N(t)-\langle \mathbf{c},z(\cdot,t)\rangle], \end{aligned} \tag{2.11}$$

where $\mathcal{A}$ is defined in (1.1). We begin with continuously differentiable delays. By using Theorems 6.1.2 and 6.1.5 in Pazy (1983) together with the step method on intervals $[0,t_*]$, $[t_*,(s+1)\tau_m]$, $[(s+1)\tau_m,(s+2)\tau_m]$, …, where $s\in\mathbb{N}$ satisfies $s\tau_m\le t_*<(s+1)\tau_m$ (see arguments similar to the well-posedness in Section 3 of Katz and Fridman (2021)), we obtain that for any initial value $\xi(0)=[z_0(\cdot),0]^{\mathrm{T}}\in\mathcal{D}(\mathcal{A})\times\mathbb{R}^N$, the closed-loop system (2.11) has a unique classical solution

$$\begin{aligned} &\xi\in C([0,\infty),L^2(\Omega)\times\mathbb{R}^N)\cap C^1([0,\infty)\setminus J,L^2(\Omega)\times\mathbb{R}^N),\\ &\xi(t)\in\mathcal{D}(\mathcal{A})\times\mathbb{R}^N,\quad \forall t\ge 0, \end{aligned} \tag{2.12}$$

where $J=\{t_*,(s+1)\tau_m,(s+2)\tau_m,\dots\}$. The well-posedness for sawtooth delays follows similarly.

2.2. Stability analysis and main results

Let $e_n(t)=z_n(t)-\hat{z}_n(t)$, $1\le n\le N$ be the estimation error. The last term on the right-hand side of (2.6) can be presented as

$$\begin{aligned}&\textstyle\sum_{n=1}^N \mathbf{c}_n\hat{z}_n(t-\tau_y)-y(t)\\&=-\textstyle\sum_{n=1}^N \mathbf{c}_n e_n(t-\tau_y)-\zeta(t-\tau_y),\\&\zeta(t)=\textstyle\sum_{n=N+1}^\infty \mathbf{c}_n z_n(t).\end{aligned}\tag{2.13}$$

From (2.4), (2.6), (2.13), the error system has the form

$$\begin{aligned}\dot{e}_n(t)=(-\lambda_n+q)e_n(t)-l_n\textstyle\sum_{i=1}^N \mathbf{c}_i e_i(t-\tau_y)\\ -l_n\zeta(t-\tau_y),\ \ 1\le n\le N.\end{aligned}\tag{2.14}$$

Denote

$$\begin{aligned}&\hat{z}^{N-N_0}(t)=[\hat{z}_{N_0+1}(t),\dots,\hat{z}_N(t)]^{\mathrm T},\\&e^{N_0}(t)=[e_1(t),\dots,e_{N_0}(t)]^{\mathrm T},\ \ \mathcal{C}_0=[\mathbf{C}_0,0_{d\times N_0}]\\&e^{N-N_0}(t)=[e_{N_0+1}(t),\dots,e_N(t)]^{\mathrm T},\\&X_0(t)=\mathrm{col}\{\hat{z}^{N_0}(t),e^{N_0}(t)\},\ \ \mathcal{K}_0=[K_0,0_{d\times N_0}],\\&F_0=\begin{bmatrix}A_0-\mathbf{B}_0K_0 & L_0\mathbf{C}_0\\ 0 & A_0-L_0\mathbf{C}_0\end{bmatrix},\ \ \mathcal{L}_0=\begin{bmatrix}L_0\\-L_0\end{bmatrix},\\&\nu_{\tau_u}(t)=\hat{z}^{N_0}(t)-\hat{z}^{N_0}(t-\tau_u),\ \ \mathcal{B}_0=\begin{bmatrix}\mathbf{B}_0\\0_{N_0\times d}\end{bmatrix},\\&\nu_{\tau_y}(t)=X_0(t)-X_0(t-\tau_y).\end{aligned}\tag{2.15}$$

We follow Katz and Fridman (2022a) and consider the reduced-order closed-loop system. First, from (2.14) and $l_n=0$ for $N_0+1\le n\le N$, we have $\dot{e}^{N-N_0}(t)=A_1e^{N-N_0}(t)$, $t\ge 0$, which is exponentially decaying (since A_1 defined in (2.7) is stable due to (1.3)). It follows

$$e^{N-N_0}(t-\tau_y)=\mathrm{e}^{-A_1\tau_y}e^{N-N_0}(t).\tag{2.16}$$

By (2.6), (2.10), (2.14), and (2.16), we obtain the reduced-order closed-loop system

$$\begin{aligned}\dot{X}_0(t)=F_0X_0(t)+\mathcal{B}_0K_0\nu_{\tau_u}(t)-\mathcal{L}_0\mathcal{C}_0\nu_{\tau_y}(t)\\ +\mathcal{L}_0\zeta(t-\tau_y)+\mathcal{L}_0\mathbf{C}_1\mathrm{e}^{-A_1\tau_y}e^{N-N_0}(t),\end{aligned}\tag{2.17a}$$

$$\dot{z}_n(t)=(-\lambda_n+q)z_n(t)-\mathbf{b}_n^{\mathrm T}\mathcal{K}_0X_0(t-\tau_y),\ \ n>N,\tag{2.17b}$$

where $\zeta(t)$ is defined in (2.13). Note that $\zeta(t)$ does not depend on $\hat{z}^{N-N_0}(t)$ which satisfies

$$\dot{\hat{z}}^{N-N_0}(t)=A_1\hat{z}^{N-N_0}(t)-\mathbf{B}_1\mathcal{K}_0X_0(t-\tau_u),\tag{2.18}$$

and is exponentially decaying provided $X_0(t)$ is exponentially decaying. Therefore, for stability of (2.1) under the control law (2.10), it is sufficient to show the stability of the reduced-order system (2.17). The latter can be considered as a singularly perturbed system with the slow sate $X_0(t)$ and the fast infinite-dimensional state $z_n(t)$, $n>N$.

For exponential L^2-stability of the closed-loop system (2.17), we consider the following vector Lyapunov functional

$$\begin{aligned}&V(t)=[V_0(t),\ \ V_{\mathrm{tail}}(t)]^{\mathrm T},\ V_{\mathrm{tail}}(t)=\textstyle\sum_{n=N+1}^\infty z_n^2(t),\\&V_0(t)=V_P(t)+V_y(t)+V_u(t)+V_e(t),\\&V_P(t)=|X_0(t)|_P^2,\quad V_e(t)=p_e|e^{N-N_0}(t)|^2,\\&V_y(t)=\textstyle\int_{t-\tau_{M,y}}^t \mathrm{e}^{2\delta(s-t)}|X_0(s)|_{S_y}^2\mathrm{d}s,\\&\quad+\tau_{M,y}\textstyle\int_{-\tau_{M,y}}^0\int_{t+\theta}^t \mathrm{e}^{2\delta(s-t)}|\dot{X}_0(s)|_{R_y}^2\mathrm{d}s\mathrm{d}\theta,\\&V_u(t)=\textstyle\int_{t-\tau_{M,u}}^t \mathrm{e}^{2\delta(s-t)}|\mathcal{K}_0X_0(s)|_{S_u}^2\mathrm{d}s\\&\quad+\tau_{M,u}\textstyle\int_{-\tau_{M,u}}^0\int_{t+\theta}^t \mathrm{e}^{2\delta(s-t)}|\mathcal{K}_0\dot{X}_0(s)|_{R_u}^2\mathrm{d}s\mathrm{d}\theta,\end{aligned}\tag{2.19}$$

where $0<P,S_y,R_y\in\mathbb{R}^{2N_0\times 2N_0}$ and $0<S_u,R_u\in\mathbb{R}^{d\times d}$. Here $V_y(t)$ is used to compensate $\nu_{\tau_y}(t)$, $V_u(t)$ is used to compensate $\nu_{\tau_u}(t)$, and $V_e(t)$ is used to compensate $e^{N-N_0}(t)$. To compensate $\zeta(t-\tau_y)$ we will use vector Halanay's inequality and the following Cauchy–Schwarz inequality:

$$\begin{aligned}&|\zeta(t)|^2\le\|\mathbf{c}\|_N^2\textstyle\sum_{n=N+1}^\infty z_n^2(t),\\&\|\mathbf{c}\|_N^2:=\textstyle\sum_{j=1}^d\|c_j\|_N^2=\sum_{n=N+1}^\infty|\mathbf{c}_n|^2.\end{aligned}\tag{2.20}$$

As explained in Remark 2.1, compared to the classical Halanay's inequality, the vector one allows to use smaller δ in V_y and V_u in the stability analysis essentially improving results in the numerical examples for comparatively large N.

Differentiation of $V_{\mathrm{tail}}(t)$ along (2.17b) gives

$$\begin{aligned}\dot{V}_{\mathrm{tail}}(t)=\textstyle\sum_{n=N+1}^\infty 2(-\lambda_n+q)z_n^2(t)\\ -\textstyle\sum_{n=N+1}^\infty 2z_n(t)\mathbf{b}_n^{\mathrm T}\mathcal{K}_0X(t-\tau_u).\end{aligned}\tag{2.21}$$

Let $\alpha>0$. Applying Young's inequality we arrive at

$$\begin{aligned}&-\textstyle\sum_{n=N+1}^\infty 2z_n(t)\mathbf{b}_n^{\mathrm T}\mathcal{K}_0X(t-\tau_u)\\&\le\tfrac{\|\mathbf{b}\|_N^2}{\alpha}X^{\mathrm T}(t-\tau_u)\mathcal{K}_0^{\mathrm T}\mathcal{K}_0X(t-\tau_u)\\&\quad+\alpha\textstyle\sum_{n=N+1}^\infty z_n^2(t),\quad\|\mathbf{b}\|_N^2:=\sum_{i=1}^d\|b_i\|_N^2.\end{aligned}\tag{2.22}$$

From (2.21) and (2.22), we have

$$\begin{aligned}&\dot{V}_{\mathrm{tail}}(t)+[2\lambda_{N+1}-2q-\alpha]V_{\mathrm{tail}}(t)\\&\le\tfrac{\|\mathbf{b}\|_N^2}{\alpha}|\mathcal{K}_0X(t-\tau_u)|^2\le\beta V_0(t-\tau_u)\end{aligned}\tag{2.23}$$

provided

$$\tfrac{\|\mathbf{b}\|_N^2}{\alpha}\mathcal{K}_0^{\mathrm T}\mathcal{K}_0<\beta P.\tag{2.24}$$

Let $\beta_0=\alpha\beta$. By Schur complement, we find that (2.24) holds iff

$$\begin{bmatrix}-P & \mathcal{K}_0^{\mathrm T}\\ * & -\frac{\beta_0}{\|\mathbf{b}\|_N^2}I\end{bmatrix}<0.\tag{2.25}$$

Let

$$\begin{aligned}&\varepsilon_y=e^{-2\delta\tau_{M,y}},\ \ \theta_{\tau_y}(t)=e^{N_0}(t-\tau_y)-e^{N_0}(t-\tau_{M,y}),\\&\varepsilon_u=e^{-2\delta\tau_{M,u}},\ \ \theta_{\tau_u}(t)=\hat{z}^{N_0}(t-\tau_u)-\hat{z}^{N_0}(t-\tau_{M,u}).\end{aligned}$$

Differentiation of $V_0(t)$ along (2.17a) gives

$$\begin{aligned}&\dot{V}_0(t)+2\delta V_0(t)\le X_0^{\mathrm T}(t)[PF_0+F_0^{\mathrm T}P+2\delta P]X_0(t)\\&+2X_0^{\mathrm T}(t)P[\mathcal{B}_0K_0\nu_{\tau_u}(t)-\mathcal{L}_0\mathcal{C}_0\nu_{\tau_y}(t)+\mathcal{L}_0\zeta(t-\tau_y)]\\&+2X_0^{\mathrm T}(t)P\mathcal{L}_0\mathbf{C}_1\mathrm{e}^{-A_1\tau_y}e^{N-N_0}(t)\\&+|X_0(t)|_{S_y}^2-\varepsilon_y|X_0(t)-\nu_{\tau_y}(t)-\theta_{\tau_y}(t)|_{S_y}^2\\&+\tau_{M,y}^2|\dot{X}_0(t)|_{R_y}^2-\varepsilon_y\tau_{M,y}\textstyle\int_{t-\tau_{M,y}}^t|\dot{X}_0(s)|_{R_y}^2\mathrm{d}s\\&+|\mathcal{K}_0X_0(t)|_{S_u}^2-\varepsilon_u|\mathcal{K}_0X_0(t)-K_0\nu_{\tau_u}(t)-K_0\theta_{\tau_u}(t)|_{S_u}^2\\&+\tau_{M,u}^2|\mathcal{K}_0\dot{X}_0(t)|_{R_u}^2-\varepsilon_u\tau_{M,u}\textstyle\int_{t-\tau_{M,u}}^t|\mathcal{K}_0\dot{X}_0(s)|_{R_u}^2\mathrm{d}s\\&+2p_e(e^{N-N_0}(t))^{\mathrm T}[A_1+\delta I]e^{N-N_0}(t).\end{aligned}\tag{2.26}$$

Let $G_y\in\mathbb{R}^{2N_0\times 2N_0}$ and $G_u\in\mathbb{R}^{d\times d}$ satisfy

$$\begin{bmatrix}R_y & G_y\\ * & R_y\end{bmatrix}\ge 0,\ \ \begin{bmatrix}R_u & G_u\\ * & R_u\end{bmatrix}\ge 0.\tag{2.27}$$

Applying Jensen's and Park's inequalities (see, e.g., Fridman (2014, Section 3.6.3)), we obtain for $\xi_y(t)=\mathrm{col}\{\nu_{\tau_y}(t),\theta_{\tau_y}(t)\}$, $\xi_u(t)=\mathrm{col}\{K_0\nu_{\tau_u}(t),K_0\theta_{\tau_u}(t)\}$,

$$\begin{aligned}&-\tau_{M,y}\textstyle\int_{t-\tau_{M,y}}^t|\dot{X}_0(s)|_{R_y}^2\mathrm{d}s\le-\xi_y^{\mathrm T}(t)\begin{bmatrix}R_y & G_y\\ * & R_y\end{bmatrix}\xi_y(t),\\&-\tau_{M,u}\textstyle\int_{t-\tau_{M,u}}^t|\mathcal{K}_0\dot{X}_0(s)|_{R_u}^2\mathrm{d}s\le-\xi_u^{\mathrm T}(t)\begin{bmatrix}R_u & G_u\\ * & R_u\end{bmatrix}\xi_u(t).\end{aligned}\tag{2.28}$$

Let $\eta(t) = \mathrm{col}\{X_0(t), \zeta(t-\tau_y), \xi_y(t), \xi_u(t), e^{N-N_0}(t)\}$. Substituting (2.28) into (2.26), we get for $\delta_1 > 0$,

$$\begin{aligned} &\dot{V}_0(t) + 2\delta V_0(t) - 2\delta_1 V_{\mathrm{tail}}(t-\tau_y) \\ &\overset{(2.20)}{\le} \dot{V}_X(t) + 2\delta V_X(t) - \tfrac{2\delta_1}{\|\mathbf{c}\|_N^2}|\zeta(t-\tau_y)|^2 \\ &\le \eta^{\mathrm{T}}(t)\Phi\eta(t) \le 0 \end{aligned} \tag{2.29}$$

provided

$$\begin{aligned} \Phi = &\left[\begin{array}{c|c} \Phi_0 & P\mathcal{L}_0\mathbf{C}_1 e^{-A_1\tau_y} \\ \hline * & 2p_e(A_1+\delta I) \end{array}\right] \\ &+ \Lambda^{\mathrm{T}}[\tau_{M,y}^2 R_y + \tau_{M,u}^2 \mathcal{K}_0^{\mathrm{T}} R_u \mathcal{K}_0]\Lambda \le 0, \end{aligned} \tag{2.30}$$

where

$$\Phi_0 = \left[\begin{array}{cc|cc|cc} \Omega_0 & P\mathcal{L}_0 & \Omega_1 & \varepsilon_y S_y & \Omega_2 & \varepsilon_u \mathcal{K}_0^{\mathrm{T}} S_u \\ * & -\frac{2\delta_1}{\|\mathbf{c}\|_N^2} I & 0 & 0 & 0 & 0 \\ \hline \multicolumn{2}{c|}{*} & \multicolumn{2}{c|}{\Omega_y} & \multicolumn{2}{c}{0} \\ \hline \multicolumn{2}{c|}{*} & \multicolumn{2}{c|}{*} & \multicolumn{2}{c}{\Omega_u} \end{array}\right],$$

$$\Omega_0 = PF_0 + F_0^{\mathrm{T}}P + 2\delta P + (1-\varepsilon_y)S_y + (1-\varepsilon_u)\mathcal{K}_0^{\mathrm{T}} S_u \mathcal{K}_0,$$

$$\Omega_1 = \varepsilon_y S_y - P\mathcal{L}_0\mathcal{C}_0, \quad \Omega_2 = P\mathcal{B}_0 + \varepsilon_u \mathcal{K}_0^{\mathrm{T}} S_u,$$

$$\begin{aligned} &\Lambda = [\Lambda_0,\ \mathcal{L}_0\mathbf{C}_1 e^{-A_1\tau_y}],\ \Lambda_0 = [F_0, \mathcal{L}_0, -\mathcal{L}_0\mathcal{C}_0, 0, \mathcal{B}_0, 0], \\ &\Omega_J = \begin{bmatrix} -\varepsilon_J(S_J+R_J) & -\varepsilon_J(S_J+G_J) \\ * & -\varepsilon_J(S_J+R_J) \end{bmatrix}, J \in \{y,u\}. \end{aligned} \tag{2.31}$$

We now show the feasibility of (2.30) for large N. Since $A_1 + \delta I < 0$ due to (1.3), by Schur complement for $p_e \to \infty$, we obtain that the feasibility of (2.30) holds iff

$$\Phi_0 + \Lambda_0^{\mathrm{T}}[\tau_{M,y}^2 R_y + \tau_{M,u}^2 \mathcal{K}_0^{\mathrm{T}} R_u \mathcal{K}_0]\Lambda_0 \le 0. \tag{2.32}$$

From (2.23) and (2.29), we have

$$\begin{aligned} \dot{V}(t) \le &\begin{bmatrix} -2\delta & 0 \\ 0 & -2\lambda_{N+1} + 2q + \frac{1}{\alpha} \end{bmatrix} V(t) \\ &+ \begin{bmatrix} 0 & 2\delta_1 \\ 0 & 0 \end{bmatrix} V(t-\tau_y) + \begin{bmatrix} 0 & 0 \\ \beta & 0 \end{bmatrix} V(t-\tau_u). \end{aligned} \tag{2.33}$$

By vector Halanay's inequality (see Lemma 1.2) we have

$$|V(t)| \le De^{-2\delta_0 t}, \quad t \ge 0 \tag{2.34}$$

for some $\delta_0 > 0$ and $D > 0$, provided

$$\begin{bmatrix} -2\delta & 2\delta_1 \\ \beta & -2\lambda_{N+1} + 2q + \alpha \end{bmatrix} \text{ is Hurwitz.} \tag{2.35}$$

By Parseval's equality, we obtain from (2.34) that

$$\|z(\cdot,t)\|_{L^2}^2 + \|z(\cdot,t) - \hat{z}(\cdot,t)\|_{L^2}^2 \le \tilde{D}e^{-\delta_0 t}, \ t \ge 0 \tag{2.36}$$

for some $\tilde{D} > 0$. Recalling that $\beta_0 = \alpha\beta$, we find that (2.35) holds iff

$$\begin{aligned} &-2(\lambda_{N+1} - q + \delta) + \alpha < 0, \\ &\begin{bmatrix} -2\alpha(\lambda_{N+1} - q) + \frac{\delta_1}{\delta}\beta_0 & \alpha \\ * & -1 \end{bmatrix} < 0. \end{aligned} \tag{2.37}$$

For asymptotic feasibility of LMIs (2.25), (2.27), (2.32), and (2.37) with large N and small $\tau_{M,y}, \tau_{M,u} > 0$, let $S_i = 0, G_i = 0$ for $i \in \{y,u\}$. Taking $\tau_{M,y}, \tau_{M,u} \to 0^+$, it is sufficient to show (2.25), (2.37) and

$$\left[\begin{array}{cc|cc} PF_0 + F_0^{\mathrm{T}}P + 2\delta P & P\mathcal{L}_0 & -P\mathcal{L}_0\mathcal{C}_0 & P\mathcal{B}_0 \\ * & -\frac{2\delta_1}{\|\mathbf{c}\|_N^2} I & 0 & 0 \\ \hline * & * & -R_y & 0 \\ * & * & * & -R_u \end{array}\right] < 0. \tag{2.38}$$

Take $\alpha = \delta = 1$, $\delta_1 = \beta_0 = N^{\frac{1}{3}}$, $R_y = NI$, $R_u = NI$. Let $0 < P \in \mathbb{R}^{2N_0 \times 2N_0}$ be the solution to the Lyapunov equation $P(F_0 + \delta I) + (F_0 + \delta I)^{\mathrm{T}}P = -I$. We have $\|P\| = O(1)$, $N \to \infty$. Substituting above values into (2.25), (2.37), (2.38) and using Schur complement and the fact that $\lambda_N = O(N)$ (see Lemma 1.1), $\|\mathcal{L}_0\| = O(1)$, $\|\mathcal{B}_0\| = O(1)$ for $N \to \infty$, we obtain the feasibility of (2.25), (2.37) and (2.38) for large enough N. Fixing such N and using continuity, we have that (2.25), (2.27), (2.30) and (2.37) are feasible for small enough $\tau_{M,y}, \tau_{M,u} > 0$. Summarizing, we arrive at:

Theorem 2.1. *Consider (2.1) with control law (2.10) and measurement (2.2). For $\delta > 0$, let $N_0 \in \mathbb{N}$ satisfy (1.3) and $N \in \mathbb{N}$ satisfy $N \ge N_0$. Let Assumption 1 hold and L_0, K_0 be obtained from (2.9). Given $\tau_{M,y}, \tau_{M,u} > 0$ and $\delta_1 > 0$, let there exist $0 < P, S_y, R_y \in \mathbb{R}^{2N_0 \times 2N_0}$, $0 < S_u, R_u \in \mathbb{R}^{d \times d}$, $G_y \in \mathbb{R}^{2N_0 \times 2N_0}$, $G_u \in \mathbb{R}^{d \times d}$ and scalars $\alpha, \beta_0 > 0$ such that LMIs (2.25), (2.27), (2.32) with Φ_0 and Λ_0 given in (2.31), and (2.37) hold. Then the solution $z(x,t)$ to (2.1) subject to the control law (2.6), (2.10) and the corresponding observer $\hat{z}(x,t)$ given by (2.5) satisfy (2.36) for some $\tilde{D} > 0$ and $\delta_0 > 0$. Moreover, LMIs (2.25), (2.27), (2.32), and (2.37) are always feasible for large enough N and small enough $\tau_{M,y}, \tau_{M,u} > 0$.*

Remark 2.1. Multiplying decision variables P, S_i, R_i, G_i ($i \in \{y,u\}$) in (2.25), (2.27), (2.32) by δ_1 and changing β_0 in (2.25) and (2.37) to $\frac{\beta_0}{\delta_1}$, we find that the feasibility of LMIs (2.25), (2.27), (2.32), and (2.37) is independent of $\delta_1 > 0$. The fact also holds true for Theorems 3.1 and 4.1. This is different from the classical Halanay inequality (see Remark 2.3) where $\delta_1 \le \delta$ should not be small to compensate $\zeta(t-\tau_y)$. However, compared to the classical Halanay inequality, the vector one needs constraint (2.24) (i.e., (2.25) which is usually more difficult to meet for larger N_0) whose feasibility requires $\|\mathbf{b}\|_N^2$ or $\frac{1}{\alpha}$ to be very small. This together with (2.37) implies that N should be very large.

Remark 2.2. Note that for $N_0 > 1$, it is difficult to find efficient L_0, K_0 from (2.9) (see numerical example in Section 4). Here for $N_0 > 1$ we can use the following steps to find more efficient L_0 and K_0:

Step 1: We find L_0 from the following inequality:

$$\left[\begin{array}{c|c} P_0(A_0 - L_0\mathbf{C}_0) + (A_0 - L_0\mathbf{C}_0)^{\mathrm{T}}P_0 + 2\delta P_0 & -P_0L_0 \\ \hline * & -\frac{2\delta}{\|\mathbf{c}\|_N^2} I \end{array}\right] < 0. \tag{2.39}$$

The additional terms compared to (2.9) are from the compensation of infinite-tail term of closed-loop system.

Step 2: Based on the L_0 obtained from (2.39), we design the controller gain $K_0 \in \mathbb{R}^{d \times N_0}$ from the delay-free case (i.e., $\tau_u \equiv 0$ and $\tau_y \equiv 0$). In this case, the closed-loop system (2.17) becomes

$$\begin{aligned} &\dot{X}_0(t) = F_0X_0(t) + \mathcal{L}_0\zeta(t) + \mathcal{L}_0\mathbf{C}_1 e^{N-N_0}(t), \\ &\dot{z}_n(t) = (-\lambda_n + q)z_n(t) - B_n\mathcal{K}_0X_0(t),\ n > N. \end{aligned}$$

We consider vector Lyapunov function

$$\begin{aligned} &V(t) = [V_0(t),\ V_{\mathrm{tail}}(t)]^{\mathrm{T}}, \\ &V_0(t) = |\hat{z}^{N_0}(t)|_{P_z}^2 + |e^{N_0}(t)|_{P_e}^2 + p_e|e^{N-N_0}(t)|^2, \end{aligned} \tag{2.40}$$

where $0 < P_z, P_e \in \mathbb{R}^{N_0 \times N_0}$, $p_e > 0$ and $V_{\mathrm{tail}}(t)$ is defined in (2.19). By arguments similar to (2.21)–(2.37), we have (2.36) for some $\tilde{D} > 0$ provided

$$\begin{aligned} &\tfrac{1}{\alpha}K_0^{\mathrm{T}}\Lambda_b K_0 < \beta P_z, \quad \left[\begin{array}{c|c|c} \Phi_z & P_zL_0\mathbf{C}_0 & P_zL_0 \\ \hline * & \Phi_e & -P_eL_0 \\ \hline * & * & -\frac{2\delta_1}{\|c\|_N^2} I \end{array}\right] < 0, \\ &2\delta + 2\lambda_{N+1} - 2q - \alpha > 0, \\ &\delta(2\lambda_{N+1} - 2q - \alpha) - \beta\delta_1 > 0, \end{aligned} \tag{2.41}$$

where

$$\Phi_z = P_z(A_0 - \mathbf{B}_0K_0) + (A_0 - \mathbf{B}_0K_0)^{\mathrm{T}}P_z + 2\delta P_z,$$
$$\Phi_e = P_e(A_0 - L_0\mathbf{C}_0) + (A_0 - L_0\mathbf{C}_0)^{\mathrm{T}}P_e + 2\delta P_e.$$

Let $\beta_0 = \alpha\beta$, $Q_z = P_z^{-1}$ and $Y_z = K_0Q_z$. By Schur complement, we find that (2.41) hold iff

$$\begin{aligned}
&\begin{bmatrix} -Q_z & Y_z^{\mathrm{T}} \\ * & -\frac{\beta_0}{\|b\|_N^2} \end{bmatrix} < 0, \quad \left[\begin{array}{c|c|c} \tilde{\Phi}_z & L_0\mathbf{C}_0 & L_0 \\ \hline * & \Phi_e & -P_eL_0 \\ \hline * & * & -\frac{2\delta_1}{\|c\|_N^2}I \end{array}\right] < 0, \\
&\tilde{\Phi}_z = A_0Q_z + Q_zA_0^{\mathrm{T}} - B_0Y_z - Y_z^{\mathrm{T}}B_0^{\mathrm{T}} + 2\delta Q_z, \\
&-2(\lambda_{N+1} - q + \delta) + \alpha < 0, \\
&\begin{bmatrix} -2\alpha(\lambda_{N+1} - q) + \frac{\delta_1}{\delta}\beta_0 & \alpha \\ * & -1 \end{bmatrix} < 0.
\end{aligned} \tag{2.42}$$

In particular, (2.42) are LMIs that depend on decision variables $0 < Q_z, P_e \in \mathbb{R}^{N_0\times N_0}$, $Y_z \in \mathbb{R}^{d\times N_0}$ and scalars $\alpha, \beta_0 > 0$. If LMIs (2.42) hold, the controller gain is given by $K_0 = Q_z^{-1}Y_z$.

Remark 2.3 (*Stability Analysis Via Classical Halanay's Inequality*). Consider Lyapunov functional

$$V(t) = V_0(t) + V_{\mathrm{tail}}(t) \tag{2.43}$$

with $V_0(t)$ and $V_{\mathrm{tail}}(t)$ in (2.19). To compensate $\zeta(t-\tau_y)$, the following bound is used for $0 < \delta_1 < \delta$:

$$\begin{aligned}
&-2\delta_1 \sup_{t-\tau_{M,y}\le\theta\le t} V(\theta) \le -2\delta_1[V_P(t-\tau_y) + V_{\mathrm{tail}}(t-\tau_y)] \\
&\overset{(2.20)}{\le} -2\delta_1|X_0(t) - \nu_{\tau_y}(t)|_P^2 - \tfrac{2\delta_1}{\|\mathbf{c}\|_N^2}|\zeta(t-\tau_y)|^2.
\end{aligned} \tag{2.44}$$

By arguments similar to (2.21), (2.26)–(2.29), (2.44), and the following Young inequality for $\alpha_1, \alpha_2 > 0$,

$$\begin{aligned}
&-\textstyle\sum_{n=N+1}^{\infty} 2z_n(t)\mathbf{b}_n^{\mathrm{T}}\mathcal{K}_0X(t-\tau_u) \\
&\le \alpha_1\|\mathbf{b}\|_N^2|\mathcal{K}_0X_0(t)|^2 + \alpha_2\|\mathbf{b}\|_N^2|K_0\nu_{\tau_u}(t)|^2 \\
&\quad + (\tfrac{1}{\alpha_1} + \tfrac{1}{\alpha_2})\textstyle\sum_{n=N+1}^{\infty} z_n^2(t),
\end{aligned} \tag{2.45}$$

we have

$$\dot{V}(t) + 2\delta V(t) - 2\delta_1 \sup_{t-\tau_{M,y}\le\theta\le t} V(\theta) \le 0 \tag{2.46}$$

provided (2.27) and the following inequalities hold:

$$\begin{aligned}
&\left[\begin{array}{c|cc} -\lambda_{N+1} + q + \delta & 1 & 1 \\ \hline * & \multicolumn{2}{c}{\mathrm{diag}\{-2\alpha_1, -2\alpha_2\}} \end{array}\right] < 0, \\
&\Phi_0 + \Lambda_0^{\mathrm{T}}[\tau_{M,y}^2R_y + \tau_{M,u}^2\mathcal{K}_0^{\mathrm{T}}R_u\mathcal{K}_0]\Lambda_0 < 0,
\end{aligned} \tag{2.47}$$

where Λ_0 is defined in (2.31) and

$$\begin{aligned}
&\Phi_0 = \left[\begin{array}{cc|cc|cc} \Omega_0 & P\mathcal{L}_0 & \Omega_1 & \varepsilon_yS_y & P\mathcal{B}_0 + \varepsilon_u\mathcal{K}_0^{\mathrm{T}}S_u & \varepsilon_u\mathcal{K}_0^{\mathrm{T}}S_u \\ * & -\frac{2\delta_1}{\|\mathbf{c}\|_N^2}I & 0 & 0 & 0 & 0 \\ \hline \multicolumn{2}{c|}{*} & \multicolumn{2}{c|}{\Omega_y} & \multicolumn{2}{c}{0} \\ \hline \multicolumn{2}{c|}{*} & \multicolumn{2}{c|}{*} & \multicolumn{2}{c}{\Omega_u} \end{array}\right], \\
&\Omega_0 = PF_0 + F_0^{\mathrm{T}}P + 2(\delta-\delta_1)P + (1-\varepsilon_u)\mathcal{K}_0^{\mathrm{T}}S_u\mathcal{K}_0 \\
&\qquad + (1-\varepsilon_y)S_y + \alpha_1\|\mathbf{b}\|_N^2\mathcal{K}_0^{\mathrm{T}}\mathcal{K}_0,
\end{aligned} \tag{2.48}$$

$$\Omega_1 = 2\delta_1P - P\mathcal{L}_0\mathcal{C}_0 + \varepsilon_yS_y,$$

$$\Omega_y = \begin{bmatrix} -2\delta_1P - \varepsilon_y(S_y + R_y) & -\varepsilon_y(S_y + G_y) \\ * & -\varepsilon_y(S_y + R_y) \end{bmatrix},$$

$$\Omega_u = \begin{bmatrix} \alpha_2\|\mathbf{b}\|_N^2I - \varepsilon_u(S_u + R_u) & -\varepsilon_u(S_u + G_u) \\ * & -\varepsilon_u(S_u + R_u) \end{bmatrix}.$$

Then classical Halanay's inequality (see P. 138 in Fridman (2014)) and (2.46) imply (2.36), where $\delta_0 > 0$ is the unique solution of $\delta_0 = \delta - \delta_1 e^{2\delta_0\tau_{M,y}}$.

3. Non-local actuation and boundary measurement

Consider system (2.1) with $\mathbf{b} \in (H^1(\Omega))^d$, $\mathbf{b}(x) = 0$ for $x \in \Gamma_D$. We assume the following delayed boundary measurement:

$$\begin{aligned}
&y(t) = \textstyle\int_{\Gamma_N} \mathbf{c}(x)z(x, t-\tau_y)dx, \quad t - \tau_y \ge 0, \\
&y(t) = 0, \ t - \tau_y < 0, \ \mathbf{c} = [c_1, \ldots, c_d]^{\mathrm{T}} \in (L^2(\Gamma_N))^d.
\end{aligned} \tag{3.1}$$

Note that (3.1) is actually a weighted averaged boundary measurement with $\mathbf{c}$ representing the weighted coefficient. We present the solution to (2.1) as (2.3) with z_n satisfying (2.4). Let $\delta > 0$, N_0 satisfy (1.3) and $N \ge N_0$. We construct a N-dimensional observer of the form (2.5), where $\hat{z}_n(t)$ $(1 \le n \le N)$ satisfy

$$\begin{aligned}
&\dot{\hat{z}}_n(t) = (-\lambda_n + q)\hat{z}_n(t) + b_nu(t) \\
&\qquad - l_n[\textstyle\sum_{i=1}^{N} \mathbf{c}_i\hat{z}_i(t-\tau_y) - y(t)], \quad t > 0, \\
&\hat{z}_n(0) = 0, \quad t \le 0, \quad \mathbf{c}_i = \textstyle\int_{\Gamma_N} \mathbf{c}(x)\phi_i(x)dx,
\end{aligned} \tag{3.2}$$

with $y(t)$ in (3.1) and observer gains $\{l_n\}_{n=1}^N$, $l_n \in \mathbb{R}^{1\times d}$. In this section, all notations are the same as in Section 2 except of $\mathbf{c}_n$ which are defined in (3.2). Let $\mathbf{B}_0$ and $\mathbf{C}_0$ satisfy Assumption 1. From Lemma 2.1, we let $L_0 = \mathrm{col}\{l_1, \ldots, l_{N_0}\} \in \mathbb{R}^{N_0\times d}$ satisfy (2.9a). Define $u(t)$ in (2.10) with $K_0 \in \mathbb{R}^{d\times N_0}$ satisfying (2.9b). By (2.10), (2.13), (2.14), (3.2), and $X_0(t)$ defined in (2.15), we obtain the closed-loop system (2.17).

Note that we need (2.20) to compensate $\zeta(t-\tau_y)$ in (2.17a) by Halanay inequality. However, differently from the non-local measurement where $\sum_{n=N+1}^{\infty}|\mathbf{c}_n|^2 < \infty$, for the boundary measurement with $\mathbf{c}_n$ defined in (3.2), we do not have this property. Here we assume

$$\sum_{n=N+1}^{\infty} \frac{|\mathbf{c}_n|^2}{\lambda_n} \le \varrho_N \le \varrho, \tag{3.3}$$

for some $\varrho_N > 0$, where $\varrho > 0$ is independent of N. For $\zeta(t)$ defined in (2.13), by Cauchy–Schwarz inequality, we have

$$\begin{aligned}
&|\zeta(t)|^2 \le \textstyle\sum_{n=N+1}^{\infty} \frac{|\mathbf{c}_n|^2}{\lambda_n} \sum_{n=N+1}^{\infty} \lambda_nz_n^2(t) \\
&\overset{(3.3)}{\le} \varrho_N \textstyle\sum_{n=N+1}^{\infty} \lambda_nz_n^2(t).
\end{aligned} \tag{3.4}$$

Remark 3.1. Note that (3.3) holds for rectangular domain $\Omega = (0, a_1) \times (0, a_2)$ with the following boundary

$$\Gamma = \Gamma_D \cup \Gamma_N, \quad \Gamma_N = \{(x_1, 0), \ x_1 \in (0, a_1)\}. \tag{3.5}$$

The eigenvalues and corresponding eigenfunctions of $\mathcal{A}$ (see (1.1)) are given by:

$$\begin{aligned}
&\lambda_{m,k} = \pi^2[\tfrac{m^2}{a_1^2} + \tfrac{(k-\frac{1}{2})^2}{a_2^2}], \quad m, k \in \mathbb{N}, \\
&\phi_{m,k}(x_1, x_2) = \tfrac{2}{\sqrt{a_1a_2}}\sin(\tfrac{m\pi x_1}{a_1})\cos(\tfrac{(k-\frac{1}{2})\pi x_2}{a_2}).
\end{aligned} \tag{3.6}$$

We reorder the eigenvalues (3.6) to a non-decreasing sequence (1.2) and denote the corresponding eigenfunctions as $\{\phi_n\}_{n=1}^{\infty}$. Let the corresponding relationship between (1.2) and (3.6) be $n \sim (m, k)$. We have $|\mathbf{c}_n|^2 = |\mathbf{c}_{m,k}|^2 = \sum_{j=1}^d |\int_0^{a_1} c_j(x_1)\phi_{m,k}(x_1, 0)dx_1|^2 = \frac{2}{a_2}\sum_{j=1}^d c_{j,m}^2$ where $c_{j,m} = \int_0^{a_1} c_j(x_1) \cdot \frac{\sqrt{2}}{\sqrt{a_1}}\sin(\frac{m\pi x_1}{a_1})dx_1$ satisfying $\|c_j\|_{L^2(\Gamma_N)}^2 = \sum_{m=1}^{\infty} c_{j,m}^2$. Therefore, we have

$$\begin{aligned}
&\textstyle\sum_{n=1}^{\infty} \frac{|\mathbf{c}_n|^2}{\lambda_n} = \frac{2}{a_2}\sum_{j=1}^d \sum_{m,k=1}^{\infty} \frac{c_{j,m}^2}{\lambda_{m,k}} \\
&\le \textstyle\sum_{j=1}^d \sum_{m=1}^{\infty} c_{j,m}^2 \sum_{k=1}^{\infty} \frac{2a_2}{(k-\frac{1}{2})^2\pi^2} \\
&= a_2 \textstyle\sum_{j=1}^d \|c_j\|_{L^2(0,a_1)}^2 =: \varrho,
\end{aligned} \tag{3.7}$$

where we use $\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = \sum_{k=1}^{\infty} \frac{1}{k^2} - \sum_{k=1}^{\infty} \frac{1}{(2k)^2} = \frac{3}{4}\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{8}$ (see Dyke (2001, P. 99)). Clearly, ϱ is independent of N. From (3.7) it follows

$$\sum_{n=N+1}^{\infty} \frac{|\mathbf{c}_n|^2}{\lambda_n} \le \varrho - \sum_{n=1}^{N} \frac{|\mathbf{c}_n|^2}{\lambda_n} =: \varrho_N. \tag{3.8}$$

Taking into account (3.4), for exponential H^1-stability we consider the vector Lyapunov functional (2.19) with $V_{\mathrm{tail}}(t)$ therein replaced by

$$V_{\mathrm{tail}}(t) = \sum_{n=N+1}^{\infty} \lambda_n z_n^2(t). \tag{3.9}$$

Differentiation of $V_{\mathrm{tail}}(t)$ in (3.9) along (2.17b) gives

$$\begin{aligned} \dot V_{\mathrm{tail}}(t) &= \textstyle\sum_{n=N+1}^{\infty} 2(-\lambda_n + q)\lambda_n z_n^2(t) \\ &\quad - \textstyle\sum_{n=N+1}^{\infty} 2\lambda_n z_n(t)\mathbf{b}_n^{\mathrm{T}}\mathcal{K}_0 X(t-\tau_u) \\ &\le \textstyle\sum_{n=N+1}^{\infty} 2(-\lambda_n + q + \alpha)\lambda_n z_n^2(t) \\ &\quad + \tfrac{1}{\alpha}\|\nabla \mathbf{b}\|_N^2 |\mathcal{K}_0 X(t-\tau_u)|^2 \end{aligned} \tag{3.10}$$

for some $\alpha > 0$, where $\|\nabla \mathbf{b}\|_N^2 = \sum_{j=1}^{d} \|\nabla b_j\|_N^2 \overset{(1.4)}{=} \sum_{j=1}^{d}\sum_{n=N+1}^{\infty} \lambda_n \langle b_j, \phi_n\rangle^2$. By arguments similar to (2.21)–(2.37) and using (3.4), (3.10), we obtain

$$\|z(\cdot,t)\|_{H^1}^2 + \|z(\cdot,t) - \hat z(\cdot,t)\|_{H^1}^2 \le \tilde D e^{-\delta_0 t}, \ \ t \ge 0 \tag{3.11}$$

for some $\tilde D > 0$ and $\delta_0 > 0$ provided LMIs (2.25) (where $\|\mathbf{b}\|_N^2$ is changed to $\|\nabla \mathbf{b}\|_N^2$), (2.27), (2.30) with Φ_0 (where $\|\mathbf{c}\|_N^2$ is changed to ϱ_N) and Λ_0 given in (2.31), and (2.37) hold. The asymptotic feasibility of above LMIs for large enough N and small enough $\tau_{M,y}, \tau_{M,u} > 0$ can be obtained by arguments similar to Theorem 2.1. Summarizing, we arrive at:

Theorem 3.1. *Consider (2.1) with control law (2.10) where $\mathbf{b} \in (H^1(\Omega))^d$, $\mathbf{b}(x) = 0$ for $x \in \Gamma_D$, measurement (3.1), and $z_0 \in \mathcal{D}(\mathcal{A})$. Given $\delta, \delta_1 > 0$, let $N_0 \in \mathbb{N}$ satisfy (1.3) and $N \in \mathbb{N}$ satisfy $N \ge N_0$. Let Assumption 1 and (3.3) hold and L_0, K_0 be obtained from (2.9). Given $\tau_{M,y}, \tau_{M,u} > 0$, let there exist $0 < P, S_y, R_y \in \mathbb{R}^{2N_0\times 2N_0}$, $0 < S_u, R_u \in \mathbb{R}^{d\times d}$, scalars $\alpha, \beta_0 > 0$, $G_y \in \mathbb{R}^{2N_0\times 2N_0}$ and $G_u \in \mathbb{R}^{d\times d}$ such that LMIs (2.25) (where $\|\mathbf{b}\|_N^2$ is changed to $\|\nabla \mathbf{b}\|_N^2$), (2.27), (2.30) with Φ_0 (where $\|\mathbf{c}\|_N^2$ is changed to ϱ_N) and Λ_0 given in (2.31), and (2.37) hold. Then the solution $z(x,t)$ to (2.1) subject to the control law (2.6), (2.10) and the corresponding observer $\hat z(x,t)$ given by (2.5) satisfy (3.11). Moreover, the above LMIs always hold for large enough N and small enough $\tau_{M,y}, \tau_{M,u} > 0$.*

Remark 3.2 (*Stability Analysis Via Classical Halanay's Inequality*)**.** Consider Lyapunov functional (2.43) with $V_0(t)$ in (2.19) and $V_{\mathrm{tail}}(t)$ in (3.9). By arguments similar to (2.21)–(2.37) and using following bound for $0 < \delta_1 < \delta$:

$$\begin{aligned} -2\delta_1 \sup_{t-\tau_{M,y}\le\theta\le t} V(\theta) &\le -2\delta_1[V_P(t-\tau_y) + V_{\mathrm{tail}}(t-\tau_y)] \\ &\overset{(3.4)}{\le} -2\delta_1 |X_0(t) - \nu_{\tau_y}(t)|_P^2 - \tfrac{2\delta_1}{\varrho_N}|\zeta(t-\tau_y)|^2, \end{aligned}$$

and the following Young inequality for $\alpha_1, \alpha_2 > 0$,

$$\begin{aligned} &-\textstyle\sum_{n=N+1}^{\infty} 2\lambda_n z_n(t)\mathbf{b}_n^{\mathrm{T}}\mathcal{K}_0 X(t-\tau_u) \\ &\le \alpha_1 \|\nabla \mathbf{b}\|_N^2 |\mathcal{K}_0 X_0(t)|^2 + \alpha_2 \|\nabla \mathbf{b}\|_N^2 |K_0 \nu_{\tau_u}(t)|^2 \\ &\quad + (\tfrac{1}{\alpha_1} + \tfrac{1}{\alpha_2}) \textstyle\sum_{n=N+1}^{\infty} \lambda_n z_n^2(t), \end{aligned}$$

we obtain (3.11) provided (2.27) and (2.47) hold with Λ_0 in (2.31) and Φ_0, Ω_y, Ω_u in (2.48) (where $\|\mathbf{b}\|_N^2$ and $\|\mathbf{c}\|_N^2$ are changed to $\|\nabla \mathbf{b}\|_N^2$ and ϱ_N, respectively).

4. Boundary actuation and non-local measurement

Consider the delayed Neumann actuation

$$\begin{aligned} &z_t(x,t) = \Delta z(x,t) + qz(x,t), \ \text{in } \Omega \times (0,\infty), \\ &z(x,t) = 0, \quad \text{on } \Gamma_D \times (0,\infty), \\ &\tfrac{\partial z}{\partial \mathbf{n}}(x,t) = \mathbf{b}^{\mathrm{T}}(x)u(t-\tau_u), \quad \text{on } \Gamma_N \times (0,\infty), \\ &z(x,0) = z_0(x), \quad x \in \Omega, \end{aligned} \tag{4.1}$$

where $\mathbf{b} = [b_1,\dots,b_d]^{\mathrm{T}} \in (L^2(\Gamma_N))^d$ and $u(t) = [u_1(t),\dots,u_d(t)]^{\mathrm{T}}$ is the control input to be designed. We consider the delayed non-local measurement (2.2) with $\mathbf{c} \in (L^2(\Omega))^d$. We present the solution to (4.1) as (2.3) and obtain (2.4) with $\mathbf{b}_n = \int_{\Gamma_D} \mathbf{b}(x)\phi_n(x)dx$.

In this section, all notations are the same as in Section 2 except of $\mathbf{b}_n$ that are defined above. We construct a N-dimensional observer of the form (2.5), where $N \ge N_0$, $\hat z_n(t)$ satisfy (2.6). Let $\mathbf{B}_0$ and $\mathbf{C}_0$ satisfy Assumption 1. From Lemma 2.1, let $L_0 = \mathrm{col}\{l_1,\dots,l_{N_0}\} \in \mathbb{R}^{N_0\times d}$ satisfy (2.9a). Define $u(t)$ in (2.10) with $K_0 \in \mathbb{R}^{d\times N_0}$ satisfying (2.9b).

For the well-posedness of closed-loop system (4.1) and (2.6), with control input (2.10), we introduce the change of variables

$$w(x,t) = z(x,t) - \mathbf{r}^{\mathrm{T}}(x)u(t-\tau_u), \tag{4.2}$$

where $\mathbf{r}(x) = [r_1(x),\dots,r_d(x)]^{\mathrm{T}}$ with $r_j(x)$, $j = 1,\dots,d$ being the solution to the following Laplace equation:

$$\begin{aligned} &\Delta r_j(x) = 0, \quad x \in \Omega, \\ &r_j(x) = 0, \ \ x \in \Gamma_D, \quad \tfrac{\partial r_j}{\partial \mathbf{n}}(x) = b_j(x), \ \ x \in \Gamma_N. \end{aligned} \tag{4.3}$$

Since $b_j \in L^2(\Gamma_N)$, from Feng et al. (2022, Lemma 2.1) we have $r_j \in L^2(\Omega)$. By (4.1), (4.2), and (4.3), we get the equivalent evolution equation:

$$\begin{aligned} &\dot w(t) + \mathcal{A}w(t) = qw(t) - \mathbf{r}^{\mathrm{T}}(\cdot)\dot u(t-\tau_u)(1-\dot\tau_u) \\ &\quad + q\mathbf{r}^{\mathrm{T}}(\cdot)u(t-\tau_u), \qquad w(0) = z(\cdot,0). \end{aligned} \tag{4.4}$$

Define the state $\xi(t) = \mathrm{col}\{w(t), \hat z^N(t)\}$, where $\hat z^N(t) = [\hat z_1(t),\dots,\hat z_N(t)]^{\mathrm{T}}$. By (2.6), (2.10), and (4.4), we present the closed-loop system as

$$\begin{aligned} &\tfrac{d}{dt}\xi(t) + \mathrm{diag}\{\mathcal{A}, \mathcal{A}_0\}\xi(t) = \begin{bmatrix} qw(t) + f_1(t-\tau_u) \\ f_2(t-\tau_u) + f_3(t-\tau_y) \\ 0_{(N-N_0)\times 1} \end{bmatrix}, \\ &f_3(t) = -L_0[\mathbf{C}\hat z^N(t) - \langle \mathbf{c}, w(\cdot,t)\rangle + \langle \mathbf{c}, \mathbf{r}^{\mathrm{T}}(\cdot)K_0\hat z^{N_0}(t-\tau_u)\rangle], \\ &f_1(t) = \mathbf{r}^{\mathrm{T}}(\cdot)(1-\dot\tau_u)K_0[A_0\hat z^{N_0}(t) + f_3(t-\tau_y) \\ &\qquad - B_0K_0\hat z^{N_0}(t-\tau_u)] - q\mathbf{r}^{\mathrm{T}}(\cdot)\hat z^{N_0}(t), \end{aligned} \tag{4.5}$$

where $\mathcal{A}_0$, $\mathbf{C}$, and $f_2(t)$ are defined in (2.11). By arguments similar to the well-posedness in Section 2, we obtain that (4.5) has a unique solution satisfying (2.12). From (4.2), it follows (4.1), subject to (2.6), (2.10), has a unique classical solution such that $z \in C([0,\infty), L^2(\Omega)) \cap C^1((0,\infty), L^2(\Omega))$ and $z(\cdot,t) \in H^2(\Omega)$ with $z(x,t) = 0$, $x \in \Gamma_D$ and $\frac{\partial}{\partial \mathbf{n}}z(x,t) = \mathbf{b}^{\mathrm{T}}(x)u(t-\tau_u)$, $x \in \Gamma_N$, for $t \in [0,\infty)$.

With notations (2.15), the closed-loop system has a form:

$$\begin{aligned} &\dot X_0(t) = F_0X_0(t) - \mathcal{L}_0\mathcal{C}\nu_{\tau_y}(t) + \mathcal{B}\mathcal{K}_0\nu_{\tau_u}(t) + \mathcal{L}_0\zeta(t-\tau_y), \\ &\dot z_n(t) = (-\lambda_n + q)z_n(t) - \mathbf{b}_n^{\mathrm{T}}\mathcal{K}_0X_0(t-\tau_u), \ n > N. \end{aligned} \tag{4.6}$$

For non-local actuation case in Section 2, we employ Young's inequality (2.22) to split the finite- and infinite-dimensional parts, where $\sum_{n=N+1}^{\infty} |\mathbf{b}_n|^2 < \infty$ is used. However, for the boundary actuation with $\mathbf{b}_n$ defined below (4.1), we do not have such property. Here we assume

$$\sum_{n=N+1}^{\infty} \frac{|\mathbf{b}_n|^2}{\lambda_n} \le \rho_N \le \rho, \tag{4.7}$$

Table 1
Chosen gains L_0 and K_0.

$q=3$, $N_0=1$	Theorem 2.1	Theorem 3.1	Theorem 4.1
b_1	f_1	f_3	f_5
c_1	g_1	g_3	g_1
L_0 from (2.9a)	1.6349	2.1837	4.1634
K_0 from (2.9b)	1.2696	47.3821	1.6349
$q=8.1$, $N_0=3$	Theorem 2.1	Theorem 3.1	Theorem 4.1
b_1, b_2	f_1, f_2	f_3, f_4	f_5, f_6
c_1, c_2	g_1, g_2	g_3, g_4	g_1, g_2
L_0 from (2.39)	$\begin{bmatrix} 8.428 & 6.036 \\ -0.295 & -0.424 \\ 0.204 & 0.150 \end{bmatrix}$	$\begin{bmatrix} 9.964 & 58.153 \\ 0.161 & -0.416 \\ 0.927 & -0.188 \end{bmatrix}$	$\begin{bmatrix} 7.108 & 4.841 \\ -0.133 & -0.525 \\ 0.709 & 0.085 \end{bmatrix}$
K_0 from (2.42)	$\delta=0.04$ $\begin{bmatrix} 5.260 & 0.029 & -0.034 \\ -0.094 & 0.253 & -0.097 \end{bmatrix}$	$\delta=0.02$ $\begin{bmatrix} 11.033 & 0.026 & 0 \\ 0 & 0 & -0.040 \end{bmatrix}$	$\delta=0.05$ $\begin{bmatrix} 7.886 & -0.280 & 0.385 \\ -8.444 & 0.039 & 0.518 \end{bmatrix}$

for some $\rho_N > 0$, where $\rho > 0$ is independent of N. Then we use the following Young inequality for $\alpha > 0$:

$$\begin{aligned} &-\textstyle\sum_{n=N+1} 2z_n(t)\mathbf{b}_n^{\mathrm{T}}\mathcal{K}_0X_0(t-\tau_u) \\ &\le \tfrac{1}{\alpha}\textstyle\sum_{n=N+1}^{\infty} \frac{|\mathbf{b}_n|^2}{\lambda_n}|\mathcal{K}_0X_0(t-\tau_u)|^2 + \sum_{n=N+1}^{\infty}\alpha\lambda_n z_n^2(t) \\ &\overset{(4.7)}{\le} \tfrac{\rho_N}{\alpha}|\mathcal{K}_0X_0(t-\tau_u)|^2 + \textstyle\sum_{n=N+1}^{\infty}\alpha\lambda_n z_n^2(t). \end{aligned} \tag{4.8}$$

Remark 4.1. Note that (4.7) holds for rectangular domain. Consider the rectangular domain introduced in Remark 3.1. Similar to estimates (3.7) and (3.8), we have $\sum_{n=N+1}^{\infty}\frac{|\mathbf{b}_n|^2}{\lambda_n} \le \rho - \sum_{n=1}^{N}\frac{|\mathbf{b}_n|^2}{\lambda_n} =: \rho_N$ with $\rho = a_2\sum_{j=1}^{d}\|b_j\|^2_{L^2(0,a_1)}$ which is independent of N.

According to (4.8), we consider the following Cauchy–Schwarz inequality:

$$\begin{aligned} |\zeta(t)|^2 &\le \textstyle\sum_{n=N+1}^{\infty}\frac{|\mathbf{c}_n|^2}{\lambda_n}\sum_{n=N+1}^{\infty}\lambda_n z_n^2(t) \\ &\le \tfrac{\|\mathbf{c}\|_N^2}{\lambda_N}\textstyle\sum_{n=N+1}^{\infty}\lambda_n z_n^2(t), \end{aligned} \tag{4.9}$$

where $\|\mathbf{c}\|_N^2$ is defined in (2.20). Consider the vector Lyapunov functional (2.19) with $V_{\mathrm{tail}}(t)$ therein replaced by (3.9). By arguments similar to (2.23)–(2.37), (3.10), and using (4.8) and (4.9), we conclude that the solutions to (4.1), (2.6), (2.10) satisfy (3.11) for some $\tilde{D} > 0$ and $\delta_0 > 0$ provided (2.27), (2.32) with Φ_0, Λ_0 in (2.31) (where $\|\mathbf{c}\|_N^2$ is changed to $\frac{1}{\lambda_N}\|\mathbf{c}\|_N^2$), and the following inequalities hold:

$$\begin{bmatrix} -P & \mathcal{K}_0^{\mathrm{T}} \\ * & -\frac{\beta_0}{\rho_N}I \end{bmatrix} < 0, \quad \begin{bmatrix} -2\alpha(\lambda_{N+1}-q)+\frac{\delta_1}{\delta}\beta_0 & \alpha \\ * & -1 \end{bmatrix} < 0. \tag{4.10}$$

The asymptotic feasibility of above LMIs for large enough N and small enough $\tau_{M,y}, \tau_{M,u} > 0$ can be obtained by arguments similar to Theorem 2.1. Summarizing, we have:

Theorem 4.1. *Consider (4.1) with control law (2.10) and delayed non-local measurement (2.2). Given $\delta > 0$, let $N_0 \in \mathbb{N}$ satisfy (1.3) and $N \in \mathbb{N}$ satisfy $N \ge N_0$. Let Assumption 1 hold and $L_0 \in \mathbb{R}^{N_0\times d}$, $K_0 \in \mathbb{R}^{d\times N_0}$ be obtained from (2.9). Given $\tau_{M,y}, \tau_{M,u} > 0$, let there exist $0 < P, S_y, R_y \in \mathbb{R}^{2N_0\times 2N_0}$, $0 < S_u, R_u \in \mathbb{R}^{d\times d}$, $G_y \in \mathbb{R}^{2N_0\times 2N_0}$ and $G_u \in \mathbb{R}^{d\times d}$, scalars $\alpha, \beta_0 > 0$ such that LMIs (2.27), (2.30) with Φ_0 and Λ_0 given in (2.31) (where $\|\mathbf{c}\|_N^2$ is changed to $\|\mathbf{c}\|_N^2/\lambda_N$) and (4.10) hold. Then the solution $z(x,t)$ to (4.1) subject to the control law (2.6), (2.10) and the corresponding observer $\hat{z}(x,t)$ given by (2.5) satisfy (3.11) for some $\tilde{D} > 0$ and $\delta_0 > 0$. Moreover, the above inequalities always hold for large enough N and small enough $\tau_{M,y}, \tau_{M,u} > 0$.*

Remark 4.2 (*Stability Analysis Via Classical Halanay's Inequality*)**.** Consider Lyapunov functional (2.43) with $V_0(t)$ in (2.19) and $V_{\mathrm{tail}}(t)$ in (3.9). By arguments similar to (2.21)–(2.37) and using following bound for $0 < \delta_1 < \delta$:

$$\begin{aligned} &-2\delta_1 \sup_{t-\tau_{M,y}\le\theta\le t} V(\theta) \le -2\delta_1[V_P(t-\tau_y)+V_{\mathrm{tail}}(t-\tau_y)] \\ &\overset{(4.9)}{\le} -2\delta_1|X_0(t)-\nu_{\tau_y}(t)|_P^2 - \tfrac{2\delta_1\lambda_N}{\|\mathbf{c}\|_N}|\zeta(t-\tau_y)|^2, \end{aligned}$$

and the following Young inequality for $\alpha_1, \alpha_2 > 0$,

$$\begin{aligned} &-\textstyle\sum_{n=N+1}^{\infty} 2\lambda_n z_n(t)\mathbf{b}_n^{\mathrm{T}}\mathcal{K}_0X(t-\tau_u) \\ &\overset{(4.7)}{\le} \alpha_1\rho_N|\mathcal{K}_0X_0(t)|^2 + \alpha_2\rho_N|K_0\nu_{\tau_u}(t)|^2 \\ &\quad + (\tfrac{1}{\alpha_1}+\tfrac{1}{\alpha_2})\textstyle\sum_{n=N+1}^{\infty}\lambda_n z_n^2(t), \end{aligned}$$

we obtain (3.11) provided (2.27) and (2.47) hold with Λ_0 in (2.31) and Φ_0, Ω_y, Ω_u in (2.48) (where $\|\mathbf{b}\|_N^2$ and $\|\mathbf{c}\|_N^2$ are changed to ρ_N and $\|\mathbf{c}\|_N^2/\lambda_N$, respectively).

5. Numerical examples

In this section, we consider a rectangular domain $\Omega = (0,a_1) \times (0,a_2)$ with $a_1 = \frac{4\sqrt{3}}{3}$, $a_2 = \frac{4\sqrt{3}}{3}$ and boundary (3.5). We consider $q = 3$ which results in an unstable open-loop system with 1 unstable mode (in this case, $N_0 = 1$ and $d = 1$) and $q = 8.1$ which results in an unstable open-loop system with 3 unstable modes with $\lambda_1 < \lambda_2 = \lambda_3$ (in this case, $N_0 = 3$ and $d = 2$), respectively. We consider three cases corresponding to Sections 2–4. For all cases we take $\tau_{M,y} = \tau_{M,u} = \tau_M$. In each case, functions $\mathbf{b} = b_1$, $\mathbf{c} = c_1$ for $d = 1$ and $\mathbf{b} = [b_1, b_2]^{\mathrm{T}}$, $\mathbf{c} = [c_1, c_2]^{\mathrm{T}}$ for $d = 2$ are chosen according to Table 1, where

$$f_1(x) = 20x_1(x_2 - x_2^2)\chi_{[0,\frac{a_1}{2}]\times[0,\frac{a_2}{2}]}(x),$$
$$f_2(x) = x_1(x_2 - x_2^2)\chi_{[\frac{a_1}{2},\frac{3a_1}{4}]\times[\frac{a_2}{2},a_2]}(x),$$
$$f_3(x) = (x_1^2 - a_1x_1)(x_2^3 - a_2x_2^2),$$
$$f_4(x) = (x_2 - a_2)\sin(\tfrac{2\pi x_1}{a_1}),$$
$$f_5(x_1) = \sin(\tfrac{2x_1\pi}{a1})\chi_{[0,\frac{a_1}{2}]}, \quad f_6(x_1) = \sin(\tfrac{3x_1\pi}{a1})\chi_{[\frac{a_1}{3},\frac{2a_1}{3}]},$$

and

$$g_1(x) = \chi_{[0,a_1]\times[0,\frac{a_2}{2}]}(x), \quad g_2(x) = \chi_{[\frac{a_1}{2},a_1]\times[0,a_2]}(x),$$
$$g_3(x_1) = 0.2\chi_{[0,\frac{a_1}{4}]}(x_1), \quad g_4(x_1) = 0.2\chi_{[\frac{a_1}{4},a_1]}(x_1).$$

Here χ is an indicator function. We see that $f_1, f_2, g_1, g_2 \in L^1(\Omega)$, $f_3, f_4 \in H^1(\Omega)$, $f_3(x) = f_4(x) = 0$ for $x \in \Gamma_D$, and $g_3, g_4, f_5, f_6 \in L^2(\Gamma_N)$. It can be checked that for each case, Assumption 1 is satisfied.

Table 2
Max τ_M for feasibility of LMIs ($q = 3$, $N_0 = 1$): Theorems 2.1, 3.1 and 4.1 (vector Halanay's inequality) vs. Remarks 2.3, 3.2 and 4.2 (classical scalar Halanay's inequality).

N	2		3		4		5		6		7	
	δ	τ_M	δ	τ_M	δ	τ_M	δ	τ_M	δ	τ_M	δ	τ_M
Theorem 2.1	0.35	0.237	0.12	0.292	0.1	0.303	0.07	0.311	0.05	0.318	0.05	0.39
Remark 2.3	1	0.196	1	0.225	1	0.236	0.95	0.247	0.9	0.256	0.8	0.259
Theorem 3.1	–	–	–	–	0.48	0.137	0.45	0.175	0.3	0.248	0.25	0.259
Remark 3.2	–	–	–	–	3	0.033	2.5	0.041	1.2	0.107	1.1	0.123
Theorem 4.1	0.18	0.276	0.06	0.312	0.06	0.319	0.03	0.323	0.03	0.328	0.02	0.329
Remark 4.2	0.9	0.222	0.8	0.257	0.6	0.266	0.6	0.275	0.5	0.281	0.4	0.285

Table 3
Max τ_M for feasibility of LMIs ($q = 8.1$, $N_0 = 3$): Theorems 2.1, 3.1 and 4.1 (Vector Halanay's inequality) vs. Remarks 2.3, 3.2 and 4.2 (Classical Scalar Halanay's inequality).

N	20		25		30		35		40	
	δ	τ_M	δ	τ_M	δ	τ_M	δ	τ_M	δ	τ_M
Theorem 2.1	0.051	0.0104	0.049	0.0342	0.048	0.0414	0.047	0.0454	0.045	0.0481
Remark 2.3	4.5	0.0267	4	0.0330	3	0.0357	3	0.0376	2.8	0.0395
N	30		35		40		45		50	
	δ	τ_M	δ	τ_M	δ	τ_M	δ	τ_M	δ	τ_M
Theorem 3.1	0.019	0.0168	0.018	0.0219	0.018	0.0271	0.017	0.0301	0.017	0.0311
Remark 3.2	6	0.0206	6	0.0215	5	0.0230	4.5	0.0238	4	0.0240
N	7		8		9		10		15	
	δ	τ_M	δ	τ_M	δ	τ_M	δ	τ_M	δ	τ_M
Theorem 4.1	–	–	0.15	0.0112	0.15	0.0242	0.15	0.0291	0.14	0.0467
Remark 4.2	7	0.0036	6	0.0106	5	0.0136	5	0.0151	2.5	0.0254

For the case that $q = 3$ and $N_0 = 1$, the gains L_0 and K_0 are found from (2.9) with $\delta = 1$ and are given in Table 1. The LMIs of Theorems 2.1, 3.1, and 4.1 as well as their counterparts by classical Halanay's inequality (Remarks 2.3, 3.2, and 4.2) were verified, respectively, for $N = 2, \dots, 8$ to obtain maximal values of τ_M ($\delta = \delta_1 > 0$ is chosen optimally) that preserve the feasibility of LMIs. The results are given in Table 2. From Table 2, it is seen that the vector Halanay inequality always leads to larger delays than the classical scalar Halanay inequality.

For the case that $q = 8.1$ and $N_0 = 3$, we found that the L_0 and K_0 obtained from (2.9) were not efficient for the feasibility of LMIs of Theorems 2.1, 3.1, 4.1 and Remarks 2.3, 3.2, 4.2 even for $\tau_{M,y} = \tau_{M,u} = 0$. We design L_0 ($\delta = \delta_1 = 0.01$, $N = 20$) and K_0 ($N = 30$) from (2.39) and (2.42) in Remark 2.2 and give the values in Table 1. The LMIs of Theorems 2.1, 3.1, and 4.1 as well as their counterparts by classical Halanay's inequality (Remarks 2.3, 3.2, and 4.2) were verified, respectively, for different N to obtain maximal values of τ_M ($\delta = \delta_1 > 0$ is chosen optimally) that preserve the feasibility of LMIs. The results are given in Table 3. From Table 3, it is seen that the vector Halanay inequality leads to larger delays than the classical scalar one for comparatively large N, whereas for comparatively small N, the classical scalar Halanay inequality leads to larger delays. This phenomenon corresponds to Remark 2.1.

For simulation of closed-loop systems studied in Sections 2–4, we consider the case $q = 3$, $N_0 = 1$ and fix $N = 5$. Consider time-varying delays $\tau_y(t) = \frac{\tau_M}{2}[1 + \sin^2 t]$ and $\tau_u(t) = \frac{\tau_M}{2}[1 + \cos^2 t]$ (corresponding maximal values of τ_M are chosen as 0.311, 0.175, and 0.323, respectively according to Table 3). We approximate the solution norm using 150 modes as $\|z(\cdot,t)\|^2_{L^2} \approx \sum_{n=1}^{150} z_n^2(t)$ and $\|\nabla z(\cdot,t)\|^2_{L^2} \approx \sum_{n=1}^{150} \lambda_n z_n^2(t)$. Take initial conditions $z_0(x) = x_1(a_1 - x_1)\cos(\frac{\pi}{2a_2}x_2)$. The closed-loop systems (with the tail ODEs truncated after 150 modes) are simulated using MATLAB. The simulations are presented in Fig. 1. The numerical simulations validate the theoretical results. Stability of the closed-loop systems in simulations was preserved for $\tau_M = 0.48$ for Theorem 2.1, $\tau_M = 0.38$ for Theorem 3.1, and $\tau_M = 0.42$ for Theorem 4.1, which may indicate that our approach is somewhat conservative in this example.

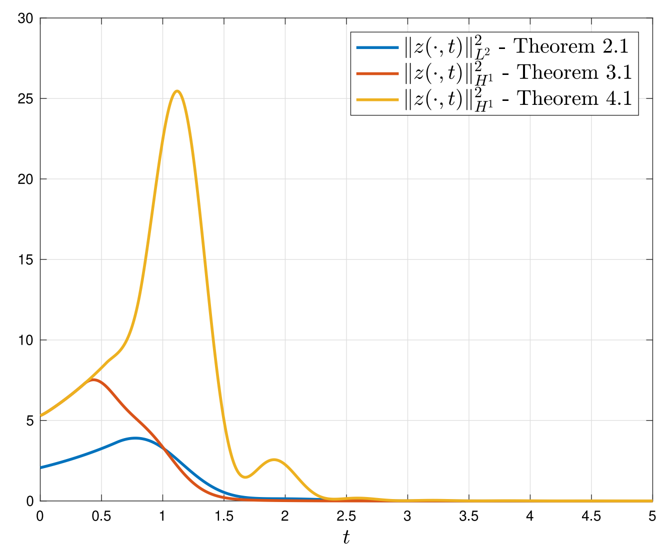

Fig. 1. Evolutions $\|z(\cdot,t)\|^2_{L^2}$ (Theorem 2.1), $\|\nabla z(\cdot,t)\|^2_{L^2}$ (Theorem 3.1), and $\|z(\cdot,t)\|^2_{L^2}$ (Theorem 4.1) vs. t.

6. Conclusions

We considered the finite-dimensional observer-based control of 2D linear heat equation with fast-varying input and output delays. To compensate the output delay that appears in the infinite-dimensional part of the closed-loop system, we suggested a vector Lyapunov functional combined with vector Halanay's inequality. In the numerical examples, the vector Halanay inequality led to larger delays for larger dimensions of the observer that preserve the stability than the classical one. Improvements and extension of the results to various high-dimensional PDEs may be topics for future research.

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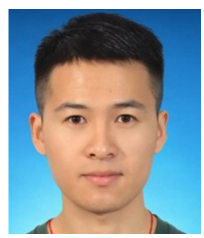

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