Received: 27 July 2023 | Revised: 12 January 2024 | Accepted: 16 January 2024

DOI: 10.1002/rnc.7227

RESEARCH ARTICLE

WILEY

# On local ISS of nonlinear second-order time-delay systems without damping

**Alexander Aleksandrov**[1,2] | **Denis Efimov**[3,4] | **Emilia Fridman**[5]

[1]Saint Petersburg State University, Saint Petersburg, Russia

[2]Institute for Problems in Mechanical Engineering, Saint Petersburg, Russia

[3]Inria, Univ. Lille, CNRS, UMR 9189-CRIStAL, Lille, France

[4]ITMO University, Saint Petersburg, Russia

[5]School of Electrical Engineering, Tel-Aviv University, Tel-Aviv, Israel

**Correspondence**
Denis Efimov, Inria, Univ. Lille, CNRS, UMR 9189-CRIStAL, Lille, France.
Email: denis.efimov@inria.fr

**Funding information**
Ministry of Science and Higher Education of the Russian Federation, Grant/Award Number: 075-15-2021-573; Israel Science Foundation, Grant/Award Number: 673/19; Israel Science Foundation-National Natural Science Foundation of China joint research program, Grant/Award Number: 3054/23

**Summary**

For a second-order system with vector position, either constant or time-varying delays, and a power nonlinearity of the degree higher than one, which does not contain a velocity-proportional damping term, the conditions of local input-to-state stability are proposed. The result is based on application of Lyapunov-Razumikhin approach, for which new time estimates on decay of solutions are obtained. The approach is extended to attitude stabilization of a rigid body, and it is illustrated by simulations.

**KEYWORDS**

input-to-state stability, Lyapunov-Razumikhin function, time-delay systems

## 1 | INTRODUCTION

Raising of IoT (Internet of Things) and appearance of numerous applications of control and estimation algorithms for cyber-physical systems result in an intensive workflow for design and analysis of time-delay systems.[1–4] Indeed, any kind of networked communications usually leads to emergence of various lags, samplings or packet losses, then the time delays represent an efficient and common modeling framework.[5,6]

Frequently, the role of delays is considered to be negative since their parasitic inclusion in the loop implies the quality of transients degradation in dynamical systems.[7] Nevertheless, the lags can be artificially introduced in a system to compensate periodic disturbances or approximate derivatives, also improving robustness and convergence rates in the closed-loop dynamics.[8–19] A usual price is an accurate and sophisticated stability analysis of functional differential equations, which is more complex than in the delay-free counterparts.

A benchmark scenario, where introduction of a small delay ensures asymptotic stability for a neutrally stable planar system, was studied in Reference 1: $\ddot{x}(t) = -a_1 x(t) + a_2 x(t-\tau)$ with $a_1, a_2, \tau > 0$ and $x(t) \in \mathbb{R}$, and an extension of that result was obtained in the works,[20,21] where a second-order linear time-varying system is considered including a delayed position term with a negative gain, and it is proven that it can be stabilized by a position feedback with positive gain and with a sufficiently small delay.

In general, different kinds of delayed planar dynamics are omnipresent in modeling the mechanical or power systems controlled by or connected to a network providing communication or resources[22–25] (in this case the delays appear through the lags in control or measurement channels). The contribution of this paper consists in extension of the results of References 20,21 to a nonlinear case with external bounded perturbations. Using Lyapunov-Razumikhin approach we will demonstrate that in the nonlinear setting the local input-to-state stability (ISS) can be guaranteed for any value of the delay (under some mild restrictions) with the attraction domain dependent on the magnitude of delays and other parameters (only qualitative estimates will be given). The method will be extended to attitude regulation of rigid body. The Lyapunov-Razumikhin approach will be complemented by a rate of convergence estimation result.

The outline of this work is as follows. Preliminaries are given in Section 2. The considered analysis problem is described in Section 3. The main stability results are formulated in Section 4. Two illustrative examples are shown in Section 5. A conference version of this paper is restricted to disturbance-free analysis of the scalar systems.[26]

## 2 PRELIMINARIES

### 2.1 Notation

The real numbers are denoted by $\mathbb{R}$, $\mathbb{R}_+ = \{s \in \mathbb{R} : s \geq 0\}$, and $|s|$ is an absolute value for $s \in \mathbb{R}$. A norm for a vector $x \in \mathbb{R}^n$ is defined as $\|x\|_p = \left(\sum_{i=1}^n |x_i|^p\right)^{p^{-1}}$ for any $p \in [1, +\infty)$, then $\|x\| = \|x\|_2$ is the usual Euclidean norm.

For a matrix $A \in \mathbb{R}^{n\times n}$ its induced norm is denoted by $\|A\|$; if it is symmetric, then $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ correspond to minimal and maximal eigenvalues, respectively, and $A > 0$ means that it is positive definite. For $v \in \mathbb{R}^n$, diag$[v]$ denotes a diagonal matrix with the vector $v$ on the main diagonal.

We denote by $\mathbb{C}([a, b], \mathbb{R}^n)$, $-\infty < a < b < +\infty$ the Banach space of continuous functions $\phi : [a, b] \to \mathbb{R}^n$ with the uniform norm $\|\phi\|_{\mathbb{C}} = \sup_{a\leq\varsigma\leq b} \|\phi(\varsigma)\|$. For a (Lebesgue) measurable function $d : \mathbb{R}_+ \to \mathbb{R}^m$ define the norm $\|d\|_\infty = \operatorname{ess\,sup}_{t\geq0} \|d(t)\|$ and the set of $d$ with the property $\|d\|_\infty < +\infty$ we further denote as $\mathcal{L}_\infty^m$.

### 2.2 Inequalities

The *Young's inequality* claims that for any $\mathfrak{a}, \mathfrak{b} \in \mathbb{R}_+$:[27]

$$\mathfrak{a}\mathfrak{b} \leq \frac{1}{p}\mathfrak{a}^p + \frac{p-1}{p}\mathfrak{b}^{\frac{p}{p-1}}$$

for any $p > 1$.

For any $x \in \mathbb{R}_+^n$, *Jensen's inequality* implies $\left(\sum_{i=1}^n x_i\right)^\gamma \leq n^{\gamma-1}\sum_{i=1}^n x_i^\gamma$ if $\gamma > 1$, and $n^{1-\gamma}\left(\sum_{i=1}^n x_i\right)^\gamma \geq \sum_{i=1}^n x_i^\gamma$ if $\gamma \in (0, 1)$.[4]

Using the properties of homogeneous functions the following results can be obtained:

**Lemma 1** (28). *Let $\mathfrak{a}, \mathfrak{b} \in \mathbb{R}_+$ and $\ell > 0$, $\alpha > 0$, $\beta > 0$, $\gamma > 0$, $\delta > 0$ be given, then*

$$\mathfrak{a}^\alpha + \mathfrak{b}^\beta - \ell\mathfrak{a}^\gamma\mathfrak{b}^\delta \geq 0$$

*provided that* $\max\{\mathfrak{a}^\alpha, \mathfrak{b}^\beta\} \leq \ell^{\frac{1}{1-\psi}}$ *and* $\psi = \frac{\gamma}{\alpha} + \frac{\delta}{\beta} > 1$.

**Lemma 2** (29). *For $x, y \in \mathbb{R}^n$ denote*

$$\mathcal{W}(x, y) = \|x\|^\alpha + \|y\|^\beta + c_1\|x\|^\eta\|y\|^\zeta - c_2\|x\|^\gamma\|y\|^\delta,$$

*where $c_1, c_2, \alpha, \beta, \gamma, \delta, \eta, \zeta$ are positive constants. If $\frac{\eta}{\alpha} + \frac{\zeta}{\beta} < 1$, then the function $\mathcal{W}(x, y)$ is positive definite for any values of $c_1$ and $c_2$ if and only if*

$$\gamma + \delta\frac{\alpha-\eta}{\zeta} > \alpha, \quad \gamma\frac{\beta-\zeta}{\eta} + \delta > \beta.$$

## 2.3 Estimation of rates of convergence through Lyapunov-Razumikhin approach

Consider an autonomous functional differential equation of retarded type:[2]

$$dx(t)/dt = f(x_t), \ t \geq 0, \tag{1}$$

where $x(t) \in \mathbb{R}^n$ and $x_t \in \mathbb{C}([-\tau, 0], \mathbb{R}^n)$ is the state function, $x_t(s) = x(t+s)$, $-\tau \leq s \leq 0$ and $\tau > 0$ is a finite delay; $f : \mathbb{C}([-\tau, 0], \mathbb{R}^n) \to \mathbb{R}^n$ is a continuous function, $f(0) = 0$, and is such that solutions in forward time for the system (1) exist and are unique.[2] Denote such a unique solution satisfying the initial condition $x_0 \in \mathbb{C}([-\tau, 0], \mathbb{R}^n)$ by $x(t, x_0)$, which is defined on time interval $[-\tau, T)$ with $0 < T \leq +\infty$ (we will use the notation $x(t)$ to reference $x(t, x_0)$ if the origin of $x_0$ is clear from the context).

For a locally Lipschitz continuous function $V : \mathbb{R}^n \to \mathbb{R}_+$ the upper directional Dini derivative is defined as follows:

$$D^+V(x)v = \limsup_{h \to 0^+} \frac{V(x+hv) - V(x)}{h}$$

for any $x \in \mathbb{R}^n$ and $v \in \mathbb{R}^n$.

A continuous function $\sigma : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to class $\mathcal{K}$ if it is strictly increasing and $\sigma(0) = 0$; it belongs to class $\mathcal{K}_\infty$ if it is also radially unbounded. A continuous function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ belongs to class $\mathcal{KL}$ if $\beta(\cdot, r) \in \mathcal{K}$ and $\beta(r, \cdot)$ is decreasing to zero for any fixed $r > 0$. For any $\alpha \in \mathcal{K}$ we can define $\beta_\alpha \in \mathcal{KL}$, verifying the properties $\beta_\alpha(s, 0) = s$ and $\beta_\alpha(\beta_\alpha(s, t_1), t_2) = \beta_\alpha(s, t_1 + t_2)$ for any $s, t_1, t_2 \in \mathbb{R}_+$, such that $y(t) \leq \beta_\alpha(y(0), t)$ for all $t \geq 0$ and $y(0) \in \mathbb{R}_+$, where $y(t)$ satisfies $\dot{y}(t) \leq -\alpha(y(t))$ for $t \geq 0$.

The following result is an extension of Reference 30, denote $\Omega = \{x \in \mathbb{R}^n : \|x\| < \rho\}$ and $\Omega_\mathbb{C} = \{\varphi \in \mathbb{C}([-\tau, 0], \mathbb{R}^n) : \|\varphi\|_\mathbb{C} < \rho\}$ for some finite $\rho > 0$.

**Theorem 1.** *Let there exist a locally Lipschitz continuous Lyapunov-Razumikhin function $V : \Omega \to \mathbb{R}_+$ such that*

*(i) for some $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and all $x \in \Omega$:*

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|);$$

*(ii) for some $\gamma, \alpha \in \mathcal{K}$, with $\gamma(s) > s$for all $s \in (0, \alpha_2(\rho))$, and all $\varphi \in \Omega_\mathbb{C}$:*

$$\max_{\theta \in [-\tau, 0]} V(\varphi(\theta)) \leq \gamma(V(\varphi(0))) \Rightarrow D^+V(\varphi(0))f(\varphi) \leq -\alpha(V(\varphi(0))).$$

*If $\exp\left(\frac{\ln \ell}{\tau} t\right) s \neq \beta_\alpha(s, t)$ for all $s \in (0, \alpha_2(\rho))$ and $t > 0$, where $\ell \in (0, 1)$ is the Lipschitz constant of the function $\gamma^{-1}$ at zero (i.e., $|\gamma^{-1}(s)| \leq \ell s$ for all $s \in (0, \alpha_2(\rho))$), then the origin is locally asymptotically stable for the system (1) with the rate of convergence:*

$$\|x(t, x_0)\| \leq \alpha_1^{-1} \circ \max\left\{\exp\left(\frac{\ln \ell}{\tau} t\right) \alpha_2(\|x_0\|_\mathbb{C}), \beta_\alpha(\alpha_2(\|x_0\|_\mathbb{C}), t)\right\}$$

*for all $t \geq 0$ and $x_0 \in \Omega_\mathbb{C}$.*

*Proof.* Since all conditions of the Lyapunov-Razumikhin theorem[2] are satisfied, then the system (1) is asymptotically stable at the origin. Take any $x_0 \in \Omega_\mathbb{C}$, the solution $x(t, x_0)$ is well defined for all $t \geq 0$, and in particular

$$V(x(t)) \leq \max_{\theta \in [-\tau, 0]} V(x_0(\theta))$$

for all $t \geq 0$.[31]

First, assume that the implication given in the formulation of the theorem is not satisfied and

$$\max_{\theta \in [-\tau, 0]} V(x_t(\theta)) > \gamma(V(x(t)))$$

for an interval of time $t \in [0, t_1]$, where $t_1 \geq 0$ is (possibly infinite) instant of time. This inequality implies that for any $t \in [0, t_1]$ there exists

$$\theta_t = \min\left\{\vartheta \in [-\tau, 0] : V(x_t(\vartheta)) = \max_{\theta\in[-\tau,0]} V(x_t(\theta))\right\},$$

where we introduce the minimum over $\vartheta \in [-\tau, 0]$ to resolve the issue of existence of several time instants, where $V(x_t(\vartheta))$ reaches the maximum on the interval $[-\tau, 0]$. Note that the inequality $\theta_t \leq -\varepsilon_{x_0}$ is satisfied for some $\varepsilon_{x_0} \in (0, \tau]$ dependent on initial conditions $x_0$, since the maximum is calculated under the restriction that $\max_{\theta\in[-\tau,0]} V(x_t(\theta)) > \gamma(V(x(t)))$ with $\gamma(s) > s > 0$ and the solution $x(t, x_0)$ is bounded for $t \geq 0$. Note that in such a case there exists $\ell \in (0, 1)$ such that $\gamma^{-1}(s) \leq \ell s$ for all $s \in [0, \alpha_2(\rho)) \supset [0, \max_{\theta\in[-\tau,0]} V(x_0(\theta))]$, thus,

$$V(x(t)) < \gamma^{-1}(V(x_t(\theta_t))) = \exp(\ln \ell)V(x_t(\theta_t)) \leq \exp\left(-\frac{\theta_t}{\tau}\ln \ell\right)V(x_t(\theta_t)).$$

Recursively applying this estimate, that is,

$$V(x_t(\theta_t)) = V(x(t + \theta_t)) < \exp\left(-\frac{\theta_{t+\theta_t}}{\tau}\ln \ell\right)V\left(x_{t+\theta_t}(\theta_{t+\theta_t})\right),$$

we obtain

$$V(x(t)) < \exp\left(-\frac{\theta_t + \theta_{t+\theta_t}}{\tau}\ln \ell\right)V\left(x_{t+\theta_t}(\theta_{t+\theta_t})\right),$$

and by induction,

$$\begin{aligned} V(x(t)) &\leq \exp\left(\frac{\ln \ell}{\tau}t\right)\max_{\theta\in[-\tau,0]} V(x_0(\theta)) \\ &\leq \max\left\{\exp\left(\frac{\ln \ell}{\tau}t\right)\max_{\theta\in[-\tau,0]} V(x_0(\theta)), \beta_\alpha\left(\max_{\theta\in[-\tau,0]} V(x_0(\theta)), t\right)\right\} \end{aligned} \tag{2}$$

for all $t \in [0, t_1]$ (i.e., for $t \geq 0$ sufficiently small it could be $t + \theta_t < 0$ and the sum $-(\theta_t + \theta_{t+\theta_t} + \cdots)$ above belongs to the interval $[t, t + \tau]$).

Now, suppose that for $t \in [t_1, t_2)$ the implication $\max_{\theta\in[-\tau,0]} V(x_t(\theta)) \leq \gamma(V(x(t)))$ holds, where $t_2 \geq t_1$ is (possibly again infinite) the time instant that the relation stated in the theorem fails for the first time higher than $t_1$ (by their definitions, $t_1 + t_2 > 0$). By construction, in such a case

$$D^+V(x(t))f(x_t) \leq -\alpha(V(x(t)))$$

for all $t \in [t_1, t_2)$ and, consequently, using standard comparison theorem result we obtain that

$$V(x(t)) \leq \beta_\alpha(V(x(t_1)), t - t_1).$$

Hence, due to properties of $\beta_\alpha$, we substantiated that the estimate (2) is satisfied for all $t \in [0, t_2)$. Next, the analysis above can be iterated for all $t \geq 0$, and the estimate for $x(t)$ can be derived using the bounds of $V$. Indeed, let us check by contradiction that (2) is valid for all $t \geq 0$. Recall that $V(x(t)) \leq \max_{\theta\in[-\tau,0]} V(x_0(\theta))$ for all $t \geq 0$ and let $t_3 \geq 0$ be a time instant such that

$$V(x(t_3)) = \max\left\{\exp\left(\frac{\ln \ell}{\tau}t_3\right)\max_{\theta\in[-\tau,0]} V(x_0(\theta)), \beta_\alpha\left(\max_{\theta\in[-\tau,0]} V(x_0(\theta)), t_3\right)\right\}$$

and (2) is valid for all $t \in [0, t_3)$. These properties imply that $\max_{\theta\in[-\tau,0]} V\left(x_{t_3}(\theta)\right) \leq \gamma(V(x(t_3)))$ and, consequently, $D^+V(x(t_3))f(x_{t_3}) \leq -\alpha(V(x(t_3)))$, which means that (2) cannot be violated since the argument functions of $\max\{\cdot\}$ above are not intersecting, and the inequality (2) has also to be preserved at any such instant $t_3$. ■

The result is formulated for the case of local asymptotic stability, and its modification for a global/practical/ISS analysis is straightforward. It can be interpreted as follows: the Razumikhin condition implicitly defines two rates of convergence presented in the system: first, the inverse of $\max_{\theta\in[-\tau,0]} V(\varphi(\theta)) \le \gamma(V(\varphi(0)))$ guarantees at least exponential decreasing for $V$ with the time constant $-\frac{\ln \ell}{\tau}$, second, the dynamics $D^+V(\varphi(0))f(\varphi) \le -\alpha(V(\varphi(0)))$ ensures another velocity of decay for $V$; finally, the established by the Lyapunov-Razumikhin approach convergence rate is the minimum of these two.

## 3 | STATEMENT OF THE PROBLEM

In References 20,21, the problem of asymptotic stability for the scalar equation

$$\ddot{x}(t) + ax(t-h) - bx(t-g) = 0 \tag{3}$$

was studied, where $x(t) \in \mathbb{R}$, $a$ and $b$ are constant positive coefficients, $h$ and $g$ are constant positive delays (all these parameters can also be time-varying). The system (3) has no damping proportional to the velocity $\dot{x} \in \mathbb{R}$, and in the delay-free case (when $h = g = 0$) under the restriction $a > b$ it has purely oscillating trajectories. It was proven that if the inequalities

$$a > b, \quad gb > ah$$

are satisfied and the values of $a - b$ and $gb - ah$ are sufficiently small, then the system (3) is asymptotically stable.

In the present paper, we consider a nonlinear vector counterpart of (3):

$$\ddot{x}(t) + \frac{\partial G(x(t-h))}{\partial x} - \frac{\partial Q(x(t-g))}{\partial x} = d(t), \tag{4}$$

where $x(t) \in \mathbb{R}^n$, functions $G, Q : \mathbb{R}^n \to \mathbb{R}^n$ are twice continuously differentiable and homogeneous of the order $\mu + 1$ for $\mu > 1$:

$$G(\lambda x) = \lambda^{\mu+1}G(x), \ Q(\lambda x) = \lambda^{\mu+1}Q(x) \quad \forall x \in \mathbb{R}^n, \ \forall \lambda > 0;$$

$g > 0$ and $h \ge 0$ are constant positive delays; $d \in \mathcal{L}_\infty^n$ is the exogenous disturbance. The instantaneous value of the state vector of (4) is $[x^\top(t)\ \dot{x}^\top(t)]^\top \in \mathbb{R}^{2n}$, then $x_t, \dot{x}_t \in \mathbb{C}([-\tau, 0], \mathbb{R}^n)$ are the components of state function ($\dot{x}_t(s) = \dot{x}(t+s)$ for $-\tau \le s \le 0$), where $\tau = \max\{h, g\}$. Assume that initial functions, $x_0$ and $\dot{x}_0$, for solutions of (4) belong to the space $\mathbb{C}([-\tau, 0], \mathbb{R}^n)$.

Under introduced restrictions, for $\mu > 1$ being a rational number with an odd numerator and denominator, taking $G(x) = a^\top x^{\mu+1}$ and $Q(x) = b^\top x^{\mu+1}$ for some $a, b \in \mathbb{R}^n$ (the power operation is understood elementwise, that is, $x^\mu(t) = [x_1^\mu(t)\ \dots\ x_n^\mu(t)]^\top$) we obtain a nonlinear counterpart of (3) (coinciding with (3) for $n = 1$ and $\mu = 1$).

In the system (4), the term $\frac{\partial G(x(t-h))}{\partial x}$ can be interpreted as a part of its own dynamics, while $\frac{\partial Q(x(t-g))}{\partial x}$ is the stabilizing term introduced by a control, which uses only delayed position measurements.[32] In such a case it is necessary to select the shape of $Q$ and $g$ providing ISS for (4) (see References 4,33 for the basic definitions and results on ISS of time-delay systems).

In this work, we will formulate the conditions of local ISS for (4), that is, that there exist $\psi \in \mathcal{KL}$, $\gamma \in \mathcal{K}$ and $\rho > 0$ such that

$$\|x(t)\| + \|\dot{x}(t)\| \le \psi(\|x_0\|_{\mathbb{C}} + \|\dot{x}_0\|_{\mathbb{C}}, t) + \gamma(\|d\|_\infty)$$

for all $t \ge 0$, all $\|x_0\|_{\mathbb{C}} < \rho$, $\|\dot{x}_0\|_{\mathbb{C}} < \rho$ and $\|d\|_\infty < \rho$. It is worth highlighting that our conditions are less conservative in the nonlinear case than those for (3). In addition, we will extend our result obtained for constant delays to time-varying ones, and next apply it to the rotation stabilization of rigid body using delayed feedback. Note that since $\mu > 1$, the local asymptotic stability of (4) cannot be derived from References 20,21 using the linearization techniques.

# 4 | MAIN RESULTS

For the proof of the results below we will use the approaches proposed in References 32,34.

## 4.1 | General case

**Theorem 2.** *Let the function* $\Pi(x) = G(x) - Q(x)$ *be positive definite, and the matrix*

$$F(x) = g\frac{\partial^2 Q(x)}{\partial x^2} - h\frac{\partial^2 G(x)}{\partial x^2}$$

*be positive definite for every fixed* $x \neq 0$, *then the system* (4) *is locally input-to-state stable.*

*Remark* 1. Compared with the result of References 20,21, in Theorem 2 it is not assumed that the values of $G(y) - Q(y)$ and $g\frac{\partial^2 Q(y)}{\partial x^2} - h\frac{\partial^2 G(y)}{\partial x^2}$ are sufficiently small for $y \in \mathbb{R}^n$ with $\|y\| = 1$ (the analogues of smallness of $a - b$ and $gb - ah$), while the delay $h$ can be zero.

*Proof.* By adding and subtracting the delay-free terms $\frac{\partial G(x(t))}{\partial x}$ and $\frac{\partial Q(x(t))}{\partial x}$, using the Mean Value Theorem,

$$\begin{aligned}
\frac{\partial G(x(t-h))}{\partial x} &= \frac{\partial G(x(t))}{\partial x} - h\frac{\partial^2 G(x(t-\eta_1(t)h))}{\partial x^2}\dot{x}(t-\eta_1(t)h), \\
\frac{\partial Q(x(t-g))}{\partial x} &= \frac{\partial Q(x(t))}{\partial x} - g\frac{\partial^2 Q(x(t-\eta_2(t)g))}{\partial x^2}\dot{x}(t-\eta_2(t)g)
\end{aligned}$$

for $\eta_1(t), \eta_2(t) \in (0,1)^n$ (for brevity of presentation we use the notation $x(t-\eta_1(t)h) = [x_1(t-\eta_{11}(t)h) \ \dots \ x_n(t-\eta_{1n}(t)h)]^\top$), and again by adding and subtracting the delay-free terms $h\frac{\partial^2 G(x(t))}{\partial x^2}\dot{x}(t)$ and $g\frac{\partial^2 Q(x(t))}{\partial x^2}\dot{x}(t)$, the system (4) can be represented in the form

$$\ddot{x}(t) + F(x(t))\dot{x}(t) + \frac{\partial \Pi(x(t))}{\partial x} = d(t) + \Delta(x_t, \dot{x}_t), \tag{5}$$

where $\Delta(x_t, \dot{x}_t) = (\Delta_1(x_t, \dot{x}_t) \ \dots \ \Delta_n(x_t, \dot{x}_t))^\top$ and

$$\begin{aligned}
\Delta_i(x_t, \dot{x}_t) = {} & h\sum_{j=1}^{n}\left(\frac{\partial^2 G(x(t-\eta_{1i}(t)h))}{\partial x_i \partial x_j}\dot{x}_j(t-\eta_{1i}(t)h) - \frac{\partial^2 G(x(t))}{\partial x_i \partial x_j}\dot{x}_j(t)\right) \\
& - g\sum_{j=1}^{n}\left(\frac{\partial^2 Q(x(t-\eta_{2i}(t)g))}{\partial x_i \partial x_j}\dot{x}_j(t-\eta_{2i}(t)g) - \frac{\partial^2 Q(x(t))}{\partial x_i \partial x_j}\dot{x}_j(t)\right).
\end{aligned}$$

Hence, in (5) the nominal part takes a form of the Liénard equation,

$$\ddot{x}(t) + F(x(t))\dot{x}(t) + \frac{\partial \Pi(x(t))}{\partial x} = 0, \tag{6}$$

which is asymptotically stable at the origin,[35–37] and the remaining terms $d(t) + \Delta(x_t, \dot{x}_t)$ are considered as perturbations. For the rest of the proof, all computations are done for the case $h > 0$, and if $h = 0$, then the arguments stay unchanged by imposing the respective terms to be zero.

Let us choose a Lyapunov function for (5) as follows:

$$V(x, \dot{x}) = \frac{1}{2}\dot{x}^\top \dot{x} + \Pi(x) - \delta\|\dot{x}\|^{\lambda-1}x^\top \dot{x} + \varepsilon\|x\|^{\sigma-1}x^\top \dot{x},$$

where $\delta$ and $\varepsilon$ are positive coefficients, $\lambda \geq 1, \sigma \geq 1$ are powers to be properly selected. Note that the full energy $E(x, \dot{x}) = \frac{1}{2}\dot{x}^\top \dot{x} + \Pi(x)$ can be used to establish global asymptotic stability of the Liénard Equation (6), however,

$E$ is not a strict Lyapunov function in this case, then a variant of such a Lyapunov function $V$ was proposed in Reference 37 being strict and guaranteeing asymptotic stability of the origin locally. Using properties of homogeneous functions (i.e., $a_1\|x\|^{\mu+1} \le \Pi(x) \le a_2\|x\|^{\mu+1}$ for all $x \in \mathbb{R}^n$ for some real $a_1 \le a_2$, and $\frac{\partial \Pi(x)}{\partial x}$ is also a homogeneous vector function of degree $\mu$) we obtain

$$\frac{1}{2}\|\dot{x}\|^2 + a_1\|x\|^{\mu+1} - \delta\|\dot{x}\|^{\lambda}\|x\| - \varepsilon\|x\|^{\sigma}\|\dot{x}\| \le V(x,\dot{x}) \le \frac{1}{2}\|\dot{x}\|^2 + a_2\|x\|^{\mu+1} + \delta\|\dot{x}\|^{\lambda}\|x\| + \varepsilon\|x\|^{\sigma}\|\dot{x}\|,$$

$$\begin{aligned}
\dot{V}(t) = & -\dot{x}^{\top}(t)F(x)\dot{x}(t) + \dot{x}^{\top}(t)(d(t) + \Delta(x_t,\dot{x}_t)) - \delta\|\dot{x}(t)\|^{\lambda+1} \\
& + \delta x^{\top}(t)\frac{\partial}{\partial \dot{x}}\left(\|\dot{x}(t)\|^{\lambda-1}\dot{x}(t)\right)\left(F(x(t))\dot{x}(t) + \frac{\partial \Pi(x(t))}{\partial x} - d(t) - \Delta(x_t,\dot{x}_t)\right) \\
& - \varepsilon\|x(t)\|^{\sigma-1}x^{\top}(t)\left(F(x(t))\dot{x}(t) + \frac{\partial \Pi(x(t))}{\partial x} - d(t) - \Delta(x_t,\dot{x}_t)\right) \\
& + \varepsilon\dot{x}^{\top}(t)\frac{\partial}{\partial x}\left(\|x(t)\|^{\sigma-1}x(t)\right)\dot{x}(t) \\
\le & -a_3\|\dot{x}(t)\|^2\|x(t)\|^{\mu-1} - \delta\|\dot{x}(t)\|^{\lambda+1} - \varepsilon a_4\|x(t)\|^{\sigma+\mu} + \varepsilon a_5\|\dot{x}(t)\|^2\|x(t)\|^{\sigma-1} \\
& + a_6\varepsilon\|x(t)\|^{\sigma+\mu-1}\|\dot{x}(t)\| + a_7\delta\|x(t)\|^{\mu}\|\dot{x}(t)\|^{\lambda} + a_8\delta\|x(t)\|^{\mu+1}\|\dot{x}(t)\|^{\lambda-1} \\
& + \varrho(\|x(t)\|, \|\dot{x}(t)\|)\|d(t) + \Delta(x_t,\dot{x}_t)\|, \\
& \varrho(\|x(t)\|, \|\dot{x}(t)\|) = \|\dot{x}(t)\| + \varepsilon\|x(t)\|^{\sigma} + \delta a_9\|x(t)\|\|\dot{x}(t)\|^{\lambda-1},
\end{aligned}$$

where all $a_i$, $i = 1, \dots, 9$ are positive constants ($a_1 > 0$ due to positive definiteness of $\Pi$). In what follows, the main derivations are done for the case $d \equiv 0$, since local input-to-state stability can be concluded from local asymptotic stability for functional differential equations.[38]

With the aid of Lemma 1 and Young's inequality, we obtain that if

$$\lambda \ge \frac{2\mu}{\mu+1}, \quad \sigma \ge \frac{\mu+1}{2}, \tag{7}$$

and in the case where $\lambda = \frac{2\mu}{\mu+1}$ or $\sigma = \frac{\mu+1}{2}$ the values of parameters $\delta$ or $\varepsilon$ are sufficiently small, respectively, there exist positive numbers $a_{10}, a_{11}, D_1$ such that

$$a_{10}(\|x\|^{\mu+1} + \|\dot{x}\|^2) \le V(x,\dot{x}) \le a_{11}(\|x\|^{\mu+1} + \|\dot{x}\|^2) \tag{8}$$

for all $\|x\|^2 + \|\dot{x}\|^2 < D_1$. Using Lemma 2, it is straightforward to show that if

$$\sigma > \mu, \quad \lambda > \frac{3\sigma-1}{\sigma+1}, \tag{9}$$

then there exists a positive number $D_2$ such that

$$\begin{aligned}
\dot{V}(t) \le & -\frac{1}{2}\left(a_3\|\dot{x}(t)\|^2\|x(t)\|^{\mu-1} + \delta\|\dot{x}(t)\|^{\lambda+1} + \varepsilon a_4\|x(t)\|^{\sigma+\mu}\right) \\
& + \varrho(\|x(t)\|, \|\dot{x}(t)\|)\|\Delta(x_t,\dot{x}_t)\|
\end{aligned}$$

for all $\|x(t)\|^2 + \|\dot{x}(t)\|^2 < D_2$. It should be noted that if the inequalities (9) are fulfilled, then the restrictions (7) are also verified.

Let, for a solution of the Equation (5), the inequality $\|x(t)\|^2 + \|\dot{x}(t)\|^2 < D_3$ and the Razumikhin condition

$$V(x(\xi), \dot{x}(\xi)) \le 2V(x(t), \dot{x}(t))$$

be satisfied for all $\xi \in [t-2\tau, t]$ and $t \ge 0$ (we need due to technical reasons to augment the delay value), where $D_3 = \min\{D_1, D_2\}$. Then, recalling the estimates (8),

$$a_{10}(\|x(\xi)\|^{\mu+1} + \|\dot{x}(\xi)\|^2) \le V(x(\xi), \dot{x}(\xi)) \le 2V(x(t), \dot{x}(t)) \le 2a_{11}(\|x(t)\|^{\mu+1} + \|\dot{x}(t)\|^2),$$

which implies:

$$\|x(\xi)\|^{\mu+1} \le 2\frac{a_{11}}{a_{10}}(\|x(t)\|^{\mu+1} + \|\dot{x}(t)\|^2),$$
$$\|\dot{x}(\xi)\|^2 \le 2\frac{a_{11}}{a_{10}}(\|x(t)\|^{\mu+1} + \|\dot{x}(t)\|^2),$$

that leads to

$$\|x(\xi)\| \le c_1\left(\|x(t)\| + \|\dot{x}(t)\|^{\frac{2}{\mu+1}}\right),\ \ \|\dot{x}(\xi)\| \le c_2(\|x(t)\|^{\frac{\mu+1}{2}} + \|\dot{x}(t)\|)$$

for any $\xi \in [t-2\tau, t]$, for some positive constants $c_1$ and $c_2$. Therefore, we obtain

$$\begin{aligned} |\Delta_i(x_t, \dot{x}_t)| \le h\sum_{j=1}^{n}&\left|\frac{\partial^2 G(x(t-\eta_{1i}(t)h))}{\partial x_i \partial x_j}\dot{x}_j(t-\eta_{1i}(t)h) - \frac{\partial^2 G(x(t))}{\partial x_i \partial x_j}\dot{x}_j(t)\right| \\ &+ g\sum_{j=1}^{n}\left|\frac{\partial^2 Q(x(t-\eta_{2i}(t)g))}{\partial x_i \partial x_j}\dot{x}_j(t-\eta_{2i}(t)g) - \frac{\partial^2 Q(x(t))}{\partial x_i \partial x_j}\dot{x}_j(t)\right|. \end{aligned}$$

and considering the first term, an upper bound can be derived:

$$\begin{aligned} &\left|\frac{\partial^2 G(x(t-\eta_{1i}(t)h))}{\partial x_i \partial x_j}\dot{x}_j(t-\eta_{1i}(t)h) - \frac{\partial^2 G(x(t))}{\partial x_i \partial x_j}\dot{x}_j(t)\right| \\ &= \left|\frac{\partial^2 G(x(t-\eta_{1i}(t)h))}{\partial x_i \partial x_j}[\dot{x}_j(t-\eta_{1i}(t)h) + \dot{x}_j(t) - \dot{x}_j(t)] - \frac{\partial^2 G(x(t))}{\partial x_i \partial x_j}\dot{x}_j(t)\right| \\ &\le |\dot{x}_j(t)|\left|\frac{\partial^2 G(x(t-\eta_{1i}(t)h))}{\partial x_i \partial x_j} - \frac{\partial^2 G(x(t))}{\partial x_i \partial x_j}\right| \\ &\quad + \left|\frac{\partial^2 G(x(t-\eta_{1i}(t)h))}{\partial x_i \partial x_j}\right||\dot{x}_j(t-\eta_{1i}(t)h) - \dot{x}_j(t)| \\ &\le |\dot{x}_j(t)|\left|\frac{\partial^2 G(x(t-\eta_{1i}(t)h))}{\partial x_i \partial x_j} - \frac{\partial^2 G(x(t))}{\partial x_i \partial x_j}\right| \\ &\quad + h\left|\frac{\partial^2 G(x(t-\eta_{1i}(t)h))}{\partial x_i \partial x_j}\right||\ddot{x}_j(t-\eta_{3ij}(t)h)| \\ &\le |\dot{x}_j(t)|\left|\frac{\partial^2 G(x(t-\eta_{1i}(t)h))}{\partial x_i \partial x_j} - \frac{\partial^2 G(x(t))}{\partial x_i \partial x_j}\right| + h\left|\frac{\partial^2 G(x(t-\eta_{1i}(t)h))}{\partial x_i \partial x_j}\right| \\ &\quad \times c_3[\|x(t-\eta_{3ij}(t)h-h)\|^{\mu} + \|x(t-\eta_{3ij}(t)h-g)\|^{\mu}] \\ &\le |\dot{x}_j(t)|\left|\frac{\partial^2 G(x(t-\eta_{1i}(t)h))}{\partial x_i \partial x_j} - \frac{\partial^2 G(x(t))}{\partial x_i \partial x_j}\right| + hc_4(\|x(t)\| + \|\dot{x}(t)\|^{\frac{2}{\mu+1}})^{2\mu-1} \end{aligned}$$

where as before $\eta_{3ij}(t) \in (0,1)$, $c_3, c_4 > 0$, and the Razumikhin condition was used on the last step. A similar estimate can be derived for another term:

$$\begin{aligned} &\left|\frac{\partial^2 Q(x(t-\eta_{2i}(t)g))}{\partial x_i \partial x_j}\dot{x}_j(t-\eta_{2i}(t)g) - \frac{\partial^2 Q(x(t))}{\partial x_i \partial x_j}\dot{x}_j(t)\right| \\ &\le |\dot{x}_j(t)|\left|\frac{\partial^2 Q(x(t-\eta_{2i}(t)g))}{\partial x_i \partial x_j} - \frac{\partial^2 Q(x(t))}{\partial x_i \partial x_j}\right| + gc_4(\|x(t)\| + \|\dot{x}(t)\|^{\frac{2}{\mu+1}})^{2\mu-1}. \end{aligned}$$

Applying Lemma 2, Young's and Jensen's inequalities, it can be shown that if

$$\sigma < 2\mu - 1, \quad \lambda < \frac{4\mu-2}{\mu+1}, \tag{10}$$

the inequalities (9) are fulfilled and the value of $D_3$ is sufficiently small, then

$$\begin{aligned}\dot V(t) \le{}& -\frac{1}{3}\left(a_3\|\dot x(t)\|^2\|x(t)\|^{\mu-1}+\delta\|\dot x(t)\|^{\lambda+1}+\varepsilon a_4\|x(t)\|^{\sigma+\mu}\right)\\ &+\varrho(\|x(t)\|,\|\dot x(t)\|)\sum_{i=1}^{n}\sum_{j=1}^{n}|\dot x_j(t)|\left[h\left|\frac{\partial^2 G(x(t-\eta_{1i}(t)h))}{\partial x_i\partial x_j}-\frac{\partial^2 G(x(t))}{\partial x_i\partial x_j}\right|\right.\\ &\left.+g\left|\frac{\partial^2 Q(x(t-\eta_{2i}(t)g))}{\partial x_i\partial x_j}-\frac{\partial^2 Q(x(t))}{\partial x_i\partial x_j}\right|\right].\end{aligned}$$

Finally, consider the terms $\left|\frac{\partial^2 G(x(t-\eta_{1i}(t)h))}{\partial x_i\partial x_j}-\frac{\partial^2 G(x(t))}{\partial x_i\partial x_j}\right|$ and $\left|\frac{\partial^2 Q(x(t-\eta_{2i}(t)g))}{\partial x_i\partial x_j}-\frac{\partial^2 Q(x(t))}{\partial x_i\partial x_j}\right|$. For the former one we obtain:

$$\begin{aligned}&\left|\frac{\partial^2 G(x(t-\eta_{1i}(t)h))}{\partial x_i\partial x_j}-\frac{\partial^2 G(x(t))}{\partial x_i\partial x_j}\right|\\ &=\|x(t)\|^{\mu-1}\left|\frac{\partial^2 G}{\partial x_i\partial x_j}\left(\frac{x(t)+x(t-\eta_{1i}(t)h)-x(t)}{\|x(t)\|}\right)-\frac{\partial^2 G}{\partial x_i\partial x_j}\left(\frac{x(t)}{\|x(t)\|}\right)\right|\\ &=\|x(t)\|^{\mu-1}\left|\frac{\partial^2 G}{\partial x_i\partial x_j}\left(\frac{x(t)}{\|x(t)\|}-\eta_{1i}(t)h\frac{\dot x(t-\eta_{4i}(t)h)}{\|x(t)\|}\right)-\frac{\partial^2 G}{\partial x_i\partial x_j}\left(\frac{x(t)}{\|x(t)\|}\right)\right|\end{aligned}$$

where $\eta_{4i}(t)\in(0,1)^n$, and applying the Razumikhin condition as above we get an upper bound:

$$\frac{\|\dot x(t-\eta_{4i}(t)h)\|}{\|x(t)\|}\le\frac{c_2(\|x(t)\|^{\frac{\mu+1}{2}}+\|\dot x(t)\|)}{\|x(t)\|}=c_2\left(\|x(t)\|^{\frac{\mu-1}{2}}+\frac{\|\dot x(t)\|}{\|x(t)\|}\right).$$

Hence, for any $\tilde\varepsilon>0$, there exists $\tilde\delta>0$ such that if $\|x(t)\|<\tilde\delta$, $\|\dot x(t)\|<\tilde\delta\|x(t)\|<\tilde\delta^2$, then

$$\left|\frac{\partial^2 G(x(t-\eta_{1i}(t)h))}{\partial x_i\partial x_j}-\frac{\partial^2 G(x(t))}{\partial x_i\partial x_j}\right|\le\tilde\varepsilon\|x(t)\|^{\mu-1},$$

while inversely, when $\tilde\delta\|x(t)\|\le\|\dot x(t)\|<\tilde\delta^2$, we obtain

$$\begin{aligned}&\left|\frac{\partial^2 Q(x(t-\eta_{2i}(t)g))}{\partial x_i\partial x_j}-\frac{\partial^2 Q(x(t))}{\partial x_i\partial x_j}\right|\\ &\le\left|\frac{\partial^2 G(x(t)+x(t-\eta_{1i}(t)h)-x(t))}{\partial x_i\partial x_j}-\frac{\partial^2 G(x(t))}{\partial x_i\partial x_j}\right|\\ &\le c_5\tilde\delta^{1-\mu}\|\dot x(t)\|^{\mu-1}+c_6\|\dot x(t-\eta_{5i}(t)h)\|^{\mu-1}\\ &\le c_5\tilde\delta^{1-\mu}\|\dot x(t)\|^{\mu-1}+c_6c_2^{\mu-1}\left(\|x(t)\|^{\frac{\mu+1}{2}}+\|\dot x(t)\|\right)^{\mu-1}\\ &\le\|\dot x(t)\|^{\mu-1}\left(c_5\tilde\delta^{1-\mu}+c_6c_2^{\mu-1}\left[1+\frac{\|x(t)\|^{\frac{\mu+1}{2}}}{\|\dot x(t)\|}\right]^{\mu-1}\right)\\ &\le\|\dot x(t)\|^{\mu-1}\left(c_5\tilde\delta^{1-\mu}+c_6c_2^{\mu-1}\left(1+\tilde\delta^{-1}\|x(t)\|^{\frac{\mu-1}{2}}\right)^{\mu-1}\right)\le c_7\|\dot x(t)\|^{\mu-1},\end{aligned}$$

where $\eta_{5i}(t)\in(0,1)^n$, and $c_5, c_6, c_7$ are positive constants. Therefore, the following upper estimate has been derived:

$$\left|\frac{\partial^2 G(x(t-\eta_{1i}(t)h))}{\partial x_i\partial x_j}-\frac{\partial^2 G(x(t))}{\partial x_i\partial x_j}\right|\le\tilde\varepsilon\|x(t)\|^{\mu-1}+c_7\|\dot x(t)\|^{\mu-1}$$

for all $\|x(t)\| < \tilde{\delta}$ and $\|\dot{x}(t)\| < \tilde{\delta}^2$. The same approach can be used to calculate an estimate for the term $\left|\frac{\partial^2 Q(x(t-\eta_{2i}(t)g))}{\partial x_i \partial x_j} - \frac{\partial^2 Q(x(t))}{\partial x_i \partial x_j}\right|$.

As a result, we obtain that if the value of $D_3$ is sufficiently small, the conditions (9) and (10) are fulfilled and

$$\lambda < \mu, \tag{11}$$

then

$$\begin{aligned}
&\varrho(\|x(t)\|, \|\dot{x}(t)\|)\sum_{i=1}^{n}\sum_{j=1}^{n}|\dot{x}_j(t)|\left[h\left|\frac{\partial^2 G(x(t-\eta_{1i}(t)h))}{\partial x_i \partial x_j} - \frac{\partial^2 G(x(t))}{\partial x_i \partial x_j}\right|\right.\\
&\left.+g\left|\frac{\partial^2 Q(x(t-\eta_{2i}(t)g))}{\partial x_i \partial x_j} - \frac{\partial^2 Q(x(t))}{\partial x_i \partial x_j}\right|\right]\\
&\leq (h+g)c_8(\|x(t)\|^{\mu-1} + \|\dot{x}(t)\|^{\mu-1})\sum_{i=1}^{n}\sum_{j=1}^{n}|\dot{x}_j(t)|\\
&\leq \frac{1}{12}\left(a_3\|\dot{x}(t)\|^2\|x(t)\|^{\mu-1} + \delta\|\dot{x}(t)\|^{\lambda+1} + \varepsilon a_4\|x(t)\|^{\sigma+\mu}\right)
\end{aligned} \tag{12}$$

for some $c_8 > 0$ implying that

$$\dot{V}(t) \leq -\frac{1}{4}\left(a_3\|\dot{x}(t)\|^2\|x(t)\|^{\mu-1} + \delta\|\dot{x}(t)\|^{\lambda+1} + \varepsilon a_4\|x(t)\|^{\sigma+\mu}\right) + \varkappa_{\max}\|d(t)\|,$$

where $\varkappa_{\max} > 0$ locally exists by continuity of the system and $V$. Consequently, the conditions of the Lyapunov-Razumikhin theorem[4] are satisfied: if

$$\max\left\{\frac{1}{2}V(x(\xi), \dot{x}(\xi)), \gamma\|d(t)\|^{1/\max\left\{\frac{\mu+\sigma}{\mu+1}, \frac{\lambda+1}{2}\right\}}\right\} \leq V(x(t), \dot{x}(t))$$

for all $\xi \in [t-2\tau, t]$ and some sufficiently small $\gamma > 0$, then

$$\begin{aligned}
\dot{V}(t) &\leq -\frac{1}{8}(\delta\|\dot{x}(t)\|^{\lambda+1} + \varepsilon a_4\|x(t)\|^{\sigma+\mu})\\
&\leq -c_9 V^{\max\left\{\frac{\mu+\sigma}{\mu+1}, \frac{\lambda+1}{2}\right\}}(x(t), \dot{x}(t))
\end{aligned}$$

for all $\|x(t)\|^2 + \|\dot{x}(t)\|^2 < D_3$ and some $c_9 > 0$.

It is easy to verify that the domain of admissible values of parameters $\sigma$ and $\lambda$ defined by the inequalities (7), (9), (10), (11) is not empty for any $\mu > 1$. Indeed, let us take $\sigma = \mu + \epsilon$, $\lambda = \frac{3\sigma-1}{\sigma+1} + \chi$, where $0 < \epsilon < \mu - 1$ such that $\mu^2 + \epsilon(\mu-5) > 1$, $(\mu-1)^2 > (3-\mu)\epsilon$ and $\chi > 0$ is sufficiently small, then all constraints on $\sigma$ and $\lambda$ can be satisfied by tuning $\epsilon$ and $\chi$: the inequalities (9) correspond directly to the choice of $\sigma$ and $\lambda$; (10) are verified due to the upper limit on $\epsilon$ and for a sufficiently small $\chi < \frac{\mu^2-1+\epsilon(\mu-5)}{(\mu+1)(\mu+\epsilon+1)}$, where the right hand side is positive provided that $\mu^2 + \epsilon(\mu-5) > 1$; (7) are valid due to $\mu > 1$ and for $\chi \geq \frac{1-\mu^2-\epsilon(\mu+3)}{(\mu+1)(\mu+\epsilon+1)}$, that is true for any positive $\chi$ since $\mu^2 + \epsilon(\mu+3) > 1$ follows from $\mu^2 + \epsilon(\mu-5) > 1$; finally, (11) holds for a sufficiently small $\chi < \frac{(\mu-1)^2+(\mu-3)\epsilon}{\mu+\epsilon+1}$, where the right-hand side is positive provided that $(\mu-1)^2 > (3-\mu)\epsilon$.

This completes the proof. ■

Note that (4) is $\mathbf{r}$-homogeneous for $\mathbf{r} = [1 \ \dots \ 1 \ \frac{\mu+1}{2} \ \dots \ \frac{\mu+1}{2}]$ of degree $\frac{\mu-1}{2}$ in the sense of Reference 39 where it has been proven that if a homogeneous system of positive degree is locally asymptotically stable in a given vicinity of the origin for any $\tau > 0$, then it is globally asymptotically stable independently of the delay value. In Theorem 2, due to (12) the estimate on the domain, where $V(x(t), \dot{x}(t)) \geq 0$ and $\dot{V}(t) \leq 0$, determined by $D_3$ is inversely proportional to $\tau$ (depending as well on other parameters), hence, the global result from Reference 39 cannot be applied. In addition, the delay free model (4),

$$\ddot{x}(t) + \Pi(x(t)) = 0,$$

has purely oscillating trajectories, therefore, it does not confirm the asymptotically stable behavior of the system for vanishing delays, whose appearance changes qualitatively the kind of stability in (4).

*Remark* 2. In the proof of Theorem 2 many positive constants are introduced ($a_i$ for $i = 1 \dots 11$, $c_j$ for $j = 1 \dots 9$ and $D_k$ for $k = 1, 2, 3$) without providing estimations of their values. The reason is that often, in nonlinear setting, such estimations are very conservative, and only qualitative bounds can be established indicating the kind of behavior presented in the system. To illustrate this claim, let us consider how the constants $a_{10}$ and $a_{11}$ can be evaluated. Therefore, we need to show, for example, that

$$\frac{1}{8}\|\dot{x}\|^2 + \frac{a_1}{4}\|x\|^{\mu+1} - \delta\|\dot{x}\|^{\lambda}\|x\| \geq 0,$$
$$\frac{1}{8}\|\dot{x}\|^2 + \frac{a_1}{4}\|x\|^{\mu+1} - \varepsilon\|x\|^{\sigma}\|\dot{x}\| \geq 0$$

under (7) and a proper choice of $D_1$, then $a_{10} = \min\{\frac{1}{4}, \frac{a_1}{2}\}$. The inequalities above are implied by

$$\|\dot{x}\|^2 + \|x\|^{\mu+1} - \frac{2\delta}{a_{10}}\|\dot{x}\|^{\lambda}\|x\| \geq 0,$$
$$\|\dot{x}\|^2 + \|x\|^{\mu+1} - \frac{2\varepsilon}{a_{10}}\|x\|^{\sigma}\|\dot{x}\| \geq 0,$$

whose validity follows from Lemma 1 provided that $\frac{1}{\mu+1} + \frac{\lambda}{2} > 1$ and $\frac{\sigma}{\mu+1} + \frac{1}{2} > 1$, respectively, which are verified if (7) holds with strict signs of inequalities. If (7) takes the form of equalities, then using Young's inequality we can show that the desired relations hold for sufficiently small $\delta$ and $\varepsilon$. To derive the upper bounds in (8), the Young's inequality can be applied with posterior utilization of (7) and smallness of $D_1$:

$$\|\dot{x}\|^{\lambda}\|x\| \leq \frac{1}{\mu+1}\|x\|^{\mu+1} + \frac{\mu}{\mu+1}\|\dot{x}\|^{\lambda\frac{\mu+1}{\mu}} \leq \frac{1}{\mu+1}\|x\|^{\mu+1} + \frac{\mu}{\mu+1}\|\dot{x}\|^2,$$
$$\|x\|^{\sigma}\|\dot{x}\| \leq \frac{1}{2}\|x\|^{2\sigma} + \frac{1}{2}\|\dot{x}\|^2 \leq \frac{1}{2}\|x\|^{\mu+1} + \frac{1}{2}\|\dot{x}\|^2,$$

then $a_{11} = \max\{\frac{1+\varepsilon}{2} + \frac{\delta\mu}{\mu+1}, a_2 + \frac{\delta}{\mu+1} + \frac{\varepsilon}{2}\}$.

Characterizing the input-to-state stability property, the shape of the asymptotic gain of the system, $\gamma\|d(t)\|^{1/\max\left\{\frac{\mu+\sigma}{\mu+1}, \frac{\lambda+1}{2}\right\}}$, has been found in the proof of Theorem 2, while evaluation of a qualitative rate of convergence follows from the result of Theorem 1:

**Corollary 1.** *Let the all conditions of Theorem 2 be satisfied, then*

$$\|x(t)\|^{\mu+1} + \|\dot{x}(t)\|^2 \leq \frac{a_{10}^{-1}(\|x_0\|_{\mathbb{C}}^{\mu+1} + \|\dot{x}_0\|_{\mathbb{C}}^2)}{\left(a_{11}^{1-q} + c_9(q-1)(\|x_0\|_{\mathbb{C}}^{\mu+1} + \|\dot{x}_0\|_{\mathbb{C}}^2)^{q-1}t\right)^{\frac{1}{q-1}}}$$

*for all $t \geq 0$, where $q = \max\left\{\frac{\mu+\sigma}{\mu+1}, \frac{\lambda+1}{2}\right\}$.*

*Proof.* Since $\mu > 1$ and $\lambda > 1$ due to (9), then $q > 1$, and the dynamics $\dot{V} \leq -c_9V^q$ has a slower convergence than any exponential close to the origin, which gives the estimate of the corollary. ∎

The obtained result can be extended to the time-varying case:

$$\ddot{x}(t) + \frac{\partial G(x(t-h(t)))}{\partial x} - \frac{\partial Q(x(t-g(t)))}{\partial x} = d(t) \tag{13}$$

with continuous and bounded for $t \geq 0$ delays $h(t) \geq 0$ and $g(t) > 0$ with $\tau = \sup_{t\geq 0}\{h(t), g(t)\}$.

**Corollary 2.** *Let the function $\Pi(x) = G(x) - Q(x)$ be positive definite and the there exist $R > 0$ such that*

$$\lambda_{\min}\left(g(t)\frac{\partial^2 Q(x)}{\partial x^2} - h(t)\frac{\partial^2 G(x)}{\partial x^2}\right) \geq R\|x\|^{\mu-1}$$

*for all $x \in \mathbb{R}^n$ and $t \geq 0$, then the system (13) is locally input-to-state stable.*

*Proof.* The proof repeats all steps of Theorem 2 (application of the Lyapunov-Razumikhin approach is not much influenced by dependence of the delays on time). ∎

## 4.2 | Application to the problem of the attitude stabilization of a rigid body

In this subsection, we will extend the developed approach to the design of a control torque ensuring the triaxial stabilization of a rigid body.

Consider a rigid body that rotates around its mass center $O$ with an angular velocity $\omega \in \mathbb{R}^3$. Let $Oxyz$ be the principal central axes of inertia of the body. The attitude motion of the body under the action of a control torque $M$ is modeled by the Euler equations (for vectors $u, w \in \mathbb{R}^3$, $u \times v$ denotes its vector product)

$$J\dot{\omega}(t) + \omega(t) \times (J\omega(t)) = M(t) + d(t), \tag{14}$$

where diagonal matrix $0 < J \in \mathbb{R}^{3\times 3}$ is inertia tensor of the body in the axes $Oxyz$,[40] $d(t) \in \mathbb{R}^3$ is the perturbation.

Let two right triples of mutually orthogonal unit vectors $s_1, s_2, s_3$ and $r_1, r_2, r_3$ be given. Vectors $s_1, s_2, s_3$ are fixed in the inertial frame, whereas vectors $r_1, r_2, r_3$ are fixed in the frame connected with the body. Hence, vectors $s_1, s_2, s_3$ rotate with respect to the system $Oxyz$ with the angular velocity $-\omega$, and we arrive at the Poisson kinematic equations

$$\dot{s}_i(t) = -\omega(t) \times s_i(t), \quad i = 1, 2, 3. \tag{15}$$

Our objective is the robust triaxial stabilization of the body. This means that we should design a control torque $M$ for which the system (14), (15) admits the asymptotically stable equilibrium position for $d(t) = 0$:

$$\omega = 0, \qquad s_i = r_i, \quad i = 1, 2, 3. \tag{16}$$

It is known (see References 41,42) that, to solve the stated problem, we can use the torque of the form $M = D(\omega) + F(s_1, s_2)$, where the dissipative component $D : \mathbb{R}^3 \to \mathbb{R}^3$ is a continuous vector function such that the product $\omega^\top D(\omega)$ is negative definite, while the restoring component $F(s_1, s_2)$ is defined as follows:

$$F(s_1, s_2) = \psi^\nu(s_1, s_2)(a_1 r_1 \times s_1 + a_2 r_2 \times s_2),$$

where

$$\psi(s_1, s_2) = \frac{1}{2}\left(a_1\|s_1 - r_1\|^2 + a_2\|s_2 - r_2\|^2\right),$$

$a_1, a_2$ are positive constants, and $\nu \geq 0$.

However, it is worth noticing that creating damping forces and constructing special damping devices for practical mechanical systems may be difficult problems, especially for the attitude stabilization of satellites due to the limited resources of reactive control systems.[40,43,44] To overcome this difficulty, in Reference 42, it was proposed to use attitude control systems in which dissipative torques tend to zero as time increases. On the over hand, the approach developed in the present paper permits us to provide the triaxial stabilization with the aid of a control that does not contain a dissipative component at all.

Denote $F(t) = F(s_1(t), s_2(t))$ and select $M(t) = c_1 F(t) - c_2 F(t - \tau)$, where $c_1, c_2$ are positive constants, $\tau$ is a positive delay. Then the Euler equations (14) take a form corresponding to (4) with $h = 0$ and $g = \tau$, but containing an additional rotation torque in the left-hand side with a more complex shape of the potential terms and having two equilibria ((16)

and its inverse):

$$J\dot{\omega}(t) + \omega(t) \times (J\omega(t)) = c_1 F(t) - c_2 F(t-\tau) + d(t). \tag{17}$$

**Theorem 3.** *Let $c_1 > c_2 > 0$ and $\nu > 0$. Then the equilibrium position (16) of the system (15), (17) is locally input-to-state stable.*

*Proof.* The equations (17) can be rewritten as follows:

$$J\dot{\omega}(t) + \omega(t) \times (J\omega(t)) = (c_1 - c_2)F(t) + c_2\tau\dot{F}(t) + d(t) + \Delta(\omega_t, s_{1t}, s_{2t}),$$

where $\Delta(\omega_t, s_{1t}, s_{2t}) = [\Delta_1(\omega_t, s_{1t}, s_{2t}), \Delta_2(\omega_t, s_{1t}, s_{2t}), \Delta_3(\omega_t, s_{1t}, s_{2t})]^\top$ and $\Delta_i(\omega_t, s_{1t}, s_{2t}) = \tau c_2\big(\dot{F}_i(t-\tau\eta_i(t)) - \dot{F}_i(t)\big)$, $i = 1, 2, 3$, with $\eta(t) \in (0,1)^3$. Construct a Lyapunov function candidate in the form

$$\begin{aligned} V(\omega, s_1, s_2) = {} & \frac{1}{2}\omega^\top J\omega + \frac{c_1 - c_2}{\nu+1}\psi^{\nu+1}(s_1, s_2) + \delta\|\omega\|^{\lambda-1}\omega^\top(a_1 r_1 \times s_1 + a_2 r_2 \times s_2) \\ & -\varepsilon\psi^{\gamma-1}(s_1, s_2)\omega^\top J(a_1 r_1 \times s_1 + a_2 r_2 \times s_2), \end{aligned} \tag{18}$$

where $\delta > 0$, $\varepsilon > 0$, $\lambda \geq 1$, $\gamma \geq 1$, then

$$\begin{aligned} & \beta_1\|\omega\|^2 + \frac{c_1 - c_2}{\nu+1}\psi^{\nu+1}(s_1, s_2) - \delta\beta_2\|\omega\|^\lambda(\|r_1 - s_1\| + \|r_2 - s_2\|) \\ & \quad - \varepsilon\beta_3\psi^{\gamma-1}(s_1, s_2)\|\omega\|(\|r_1 - s_1\| + \|r_2 - s_2\|) \leq V(\omega, s_1, s_2) \\ & \leq \beta_4\|\omega\|^2 + \frac{c_1 - c_2}{\nu+1}\psi^{\nu+1}(s_1, s_2) + \delta\beta_2\|\omega\|^\lambda(\|r_1 - s_1\| + \|r_2 - s_2\|) \\ & \quad + \varepsilon\beta_3\psi^{\gamma-1}(s_1, s_2)\|\omega\|(\|r_1 - s_1\| + \|r_2 - s_2\|) \end{aligned}$$

for $\beta_j > 0, j = 1, 2, 3, 4$.

As previously, considering $d \equiv 0$ and differentiating the Lyapunov function (18) along the solutions of (15), (17), we obtain

$$\begin{aligned} \dot{V}(t) = {} & \tau c_2\omega^\top(t)\dot{F}(t) + \omega^\top(t)\Delta(\omega_t, s_{1t}, s_{2t}) \\ & - \delta\|\omega(t)\|^{\lambda-1}\omega^\top(t)(a_1 r_1 \times (\omega(t) \times s_1(t)) + a_2 r_2 \times (\omega(t) \times s_2(t))) \\ & + \delta(a_1 r_1 \times s_1(t) + a_2 r_2 \times s_2(t))^\top \frac{\partial\big(\|\omega(t)\|^{\lambda-1}\omega(t)\big)}{\partial\omega}\dot{\omega}(t) \\ & - \varepsilon\psi^{\gamma-1}(s_1(t), s_2(t))(a_1 r_1 \times s_1(t) + a_2 r_2 \times s_2(t))^\top\big(-\omega(t) \times (J\omega(t)) \\ & +(c_1 - c_2)F(t) + c_2\tau\dot{F}(t) + \Delta(\omega_t, s_{1t}, s_{2t})\big) \\ & - \varepsilon\omega^\top(t)J\frac{d}{dt}\big(\psi^{\gamma-1}(s_1(t), s_2(t))(a_1 r_1 \times s_1(t) + a_2 r_2 \times s_2(t))\big). \end{aligned}$$

Let us note that

$$\begin{aligned} \omega^\top(t)\dot{F}(t) = {} & \omega^\top(t)\big(\nu\psi^{\nu-1}(s_1(t), s_2(t))\big(a_1 r_1^\top(\omega(t) \times s_1(t)) \\ & +a_2 r_2^\top(\omega(t) \times s_2(t))\big)(a_1 r_1 \times s_1(t) + a_2 r_2 \times s_2(t)) \\ & - \psi^\nu(s_1(t), s_2(t))(a_1 r_1 \times (\omega(t) \times s_1(t)) + a_2 r_2 \times (\omega(t) \times s_2(t)))\big) \\ = {} & -\nu\psi^{\nu-1}(s_1(t), s_2(t))\big(a_1 r_1^\top(\omega(t) \times s_1(t)) + a_2 r_2^\top(\omega(t) \times s_2(t))\big)^2 \\ & - \psi^\nu(s_1(t), s_2(t))\omega^\top(t)(a_1 r_1 \times (\omega(t) \times r_1) + a_2 r_2 \times (\omega(t) \times r_2)) \\ & - \psi^\nu(s_1(t), s_2(t))\omega^\top(t)(a_1 r_1 \times (\omega(t) \times (s_1(t) - r_1)) + a_2 r_2 \times (\omega(t) \times (s_2(t) - r_2))) \\ \leq {} & -\beta_5\psi^\nu(s_1(t), s_2(t))\|\omega(t)\|^2 \\ & + \beta_6\psi^\nu(s_1(t), s_2(t))(\|s_1(t) - r_1\| + \|s_2(t) - r_2\|)\|\omega(t)\|^2, \end{aligned}$$

where $\beta_5, \beta_6$ are positive constants.

In Reference 42, it was proven that, for any $\tilde{\varepsilon} \in (0,1)$, there exists $\tilde{\delta} > 0$ such that

$$\|a_1 s_1(t) \times r_1 + a_2 s_2 \times r_2\|^2 \geq \tilde{\varepsilon}\left(a_1^2\|s_1(t) - r_1\|^2 + a_2^2\|s_2(t) - r_2\|^2\right)$$

for $t \geq 0$ and $\|s_1(t) - r_1\|^2 + \|s_2(t) - r_2\|^2 < \tilde{\delta}^2$. Therefore, for an appropriate choice of $\tilde{\delta}$, we obtain

$$\begin{aligned}
&-\varepsilon(c_1 - c_2)\psi^{\gamma-1}(s_1(t), s_2(t))(a_1 r_1 \times s_1(t) + a_2 r_2 \times s_2(t))^{\top} F(t) \\
&= -\varepsilon(c_1 - c_2)\psi^{\nu+\gamma-1}(s_1(t), s_2(t))\|a_1 r_1 \times s_1(t) + a_2 r_2 \times s_2(t)\|^2 \\
&\leq -\frac{1}{2}\varepsilon(c_1 - c_2)\psi^{\nu+\gamma-1}(s_1(t), s_2(t))\left(a_1^2\|s_1(t) - r_1\|^2 + a_2^2\|s_2(t) - r_2\|^2\right)
\end{aligned}$$

for $t \geq 0$ and $\|s_1(t) - r_1\|^2 + \|s_2(t) - r_2\|^2 < \tilde{\delta}^2$. Moreover, the estimate

$$\begin{aligned}
&-\delta\|\omega(t)\|^{\lambda-1}\omega^{\top}(t)(a_1 r_1 \times (\omega(t) \times s_1(t)) + a_2 r_2 \times (\omega(t) \times s_2(t))) \\
&\leq -\delta\beta_7\|\omega(t)\|^{\lambda+1} + \delta\beta_8\|\omega(t)\|^{\lambda+1}(\|r_1 - s_1(t)\| + \|r_2 - s_2(t)\|)
\end{aligned}$$

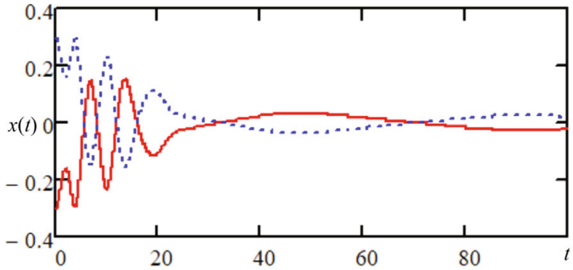
is valid for some positive constants $\beta_7, \beta_8$. As a result, if $\tilde{\delta}$ is sufficiently small and $\|s_1(t) - r_1\|^2 + \|s_2(t) - r_2\|^2 < \tilde{\delta}^2$, then

$$\begin{aligned}
\dot{V}(t) \leq & -\frac{1}{2}\varepsilon(c_1 - c_2)\psi^{\nu+\gamma-1}(s_1(t), s_2(t))\left(a_1^2\|s_1(t) - r_1\|^2 + a_2^2\|s_2(t) - r_2\|^2\right) \\
& - \frac{1}{2}\tau c_2\beta_5\psi^{\nu}(s_1(t), s_2(t))\|\omega(t)\|^2 - \frac{1}{2}\delta\beta_7\|\omega(t)\|^{\lambda+1} \\
& + \varepsilon\beta_9\psi^{\gamma-1}(s_1(t), s_2(t))(\|s_1(t) - r_1\| + \|s_2(t) - r_2\|)\|\omega(t)\|^2 \\
& + \varepsilon\beta_{10}\|\omega(t)\|^2(\|s_1(t) - r_1\| + \|s_2(t) - r_2\|)^{2\gamma-2} \\
& + \varepsilon\beta_{11}\psi^{\nu+\gamma-1}(s_1(t), s_2(t))(\|s_1(t) - r_1\| + \|s_2(t) - r_2\|)^2\|\omega(t)\| \\
& + \delta\beta_{12}(\|s_1(t) - r_1\| + \|s_2(t) - r_2\|)\|\omega(t)\|^{\lambda+1} \\
& + \delta\beta_{13}\psi^{\nu}(s_1(t), s_2(t))(\|s_1(t) - r_1\| + \|s_2(t) - r_2\|)^2\|\omega(t)\|^{\lambda-1} \\
& + \delta\beta_{14}\psi^{\nu-1}(s_1(t), s_2(t))(\|s_1(t) - r_1\| + \|s_2(t) - r_2\|)^3\|\omega(t)\|^{\lambda} \\
& + \beta_{15}\big(\varepsilon\psi^{\gamma-1}(s_1(t), s_2(t))(\|s_1(t) - r_1\| + \|s_2(t) - r_2\|) \\
& + \delta(\|s_1(t) - r_1\| + \|s_2(t) - r_2\|)\|\omega(t)\|^{\lambda-1} + \|\omega(t)\|\big)\|\Delta(\omega_t, s_{1t}, s_{2t})\|,
\end{aligned}$$

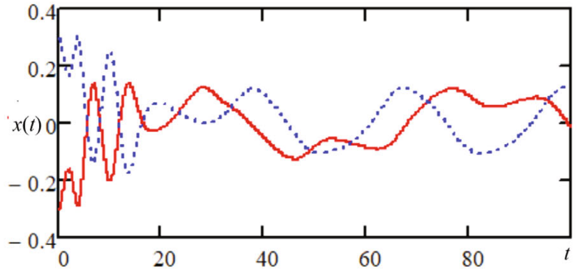
where $\beta_j > 0, j = 9, \ldots, 15$.

The remaining part of the proof is similar to that of Theorem 2. ■

For brevity, only one delay is considered in (17), but the result can be easily extended to the case of two delays in the restoring term.




**(A)** Noise-free case




**(B)** The disturbed case

FIGURE 1 The illustration for Theorem 2. (A) Noise-free case. (B) The disturbed case.

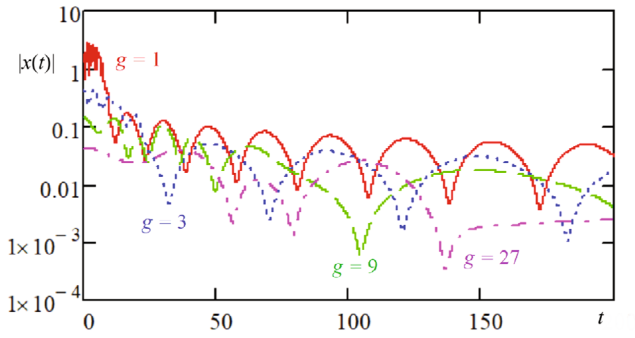



**FIGURE 2** The illustration of dependence of the domain of convergence for different values of delay.

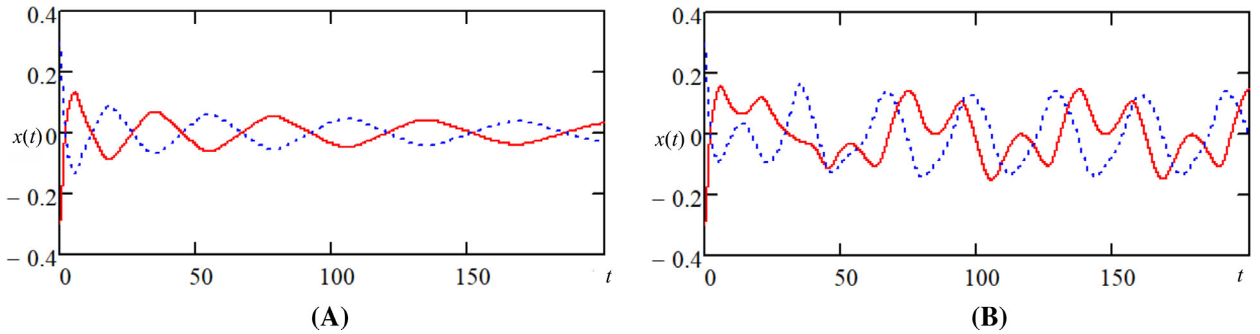



**FIGURE 3** The illustration for Corollary 2. (A) Noise-free case. (B) The disturbed case.

## 5 | EXAMPLES

In this section we restrict ourselves by $n = 2$ and $\mu = 3$.

For an illustration of the result of Theorem 2, consider the case with $h = 0$, $g = 3$, and

$$G(x) = a\|x\|^{1+\mu},\ Q(x) = b\|x\|^{1+\mu},$$
$$a = 2,\ b = 1.$$

In the case $g = 0$ (or $b = 0$) the system (4) takes the form of a nonlinear mechanical oscillator. The appearance of delayed part for $g \neq 0$ can be interpreted as a stabilizing control that uses delayed position measurements. By Theorem 2, the system is locally ISS. The results of simulation of $x(t)$ for the initial conditions $x(s) = [-0.3\ 0.3]^\top$ and $\dot{x}(s) = [0\ 0]^\top$ for all $s \in [-\tau, 0]$ are shown in Figure 1, without and with the disturbance

$$d(t) = 0.005 \begin{bmatrix} \sin(0.1t) \\ \cos(0.2t) \end{bmatrix}.$$

In order to evaluate the dependence of the domain of convergence on the value of the delay $\tau$, the simulations are performed for $h = 0$ and $g \in \{1, 3, 9, 27\}$ with initial conditions $x(s) = \varpi[1\ 1]^\top$ and $\dot{x}(s) = [0\ 0]^\top$ for all $s \in [-\tau, 0]$, where the parameter $\varpi > 0$ was tuned to obtain the maximal value providing the convergent trajectories in the disturbance-free case. The behavior of the norm of the state for different values of delays and the respective values of $\varpi$ is presented in Figure 2. As we can conclude, in these simulations there is an approximately linear inverse dependence of $\varpi$ in $\tau$, which is aligned with the theoretical observations given in the previous section.

Selecting

$$h(t) = 0.2(1 + \cos(0.5t)),\ g(t) = 1 + \sin^2(10t)$$

and keeping the values of all other parameters unchanged, the conditions of Corollary 2 are verified. The respective results of simulations for the same initial conditions and disturbances as in Figure 1 are given in Figure 3. As we can remark, the time variation of delays does not destroy stability under the proposed restrictions.

These results confirm the conclusions of the theorems that the systems are locally input-to-state stable under introduced mild restrictions (for bigger initial conditions or amplitude of the input, the trajectories become unbounded).

## 6 | CONCLUSION

For a second-order vector system with either constant or time-varying delays, smooth power nonlinearity and a bounded external disturbance, without a velocity damping term, the conditions of local input-to-state stability were formulated. These conditions are simple for checking. It is seen from the proof and the examples that the domain of attraction depends on the value of delays and parameters, but there is no restriction on admissible maximal value of delays. The proposed method was developed for attitude stabilization of a rigid body. New estimates on the decreasing of solutions were derived for the Lyapunov-Razumikhin approach. Development of these results to the case with the power $\mu \in (0, 1)$ can be considered as a direction for future research.

### ACKNOWLEDGMENTS
Section 4 was obtained in Institute for Problems in Mechanical Engineering RAS and supported by the Ministry of Science and Higher Education of the Russian Federation (Project No. 075-15-2021-573), Israel Science Foundation 673/19, Israel Science Foundation-National Natural Science Foundation of China Joint Research Program 3054/23.

### CONFLICT OF INTEREST STATEMENT
The authors declare no conflicts of interest.

### DATA AVAILABILITY STATEMENT
Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

### ORCID
*Alexander Aleksandrov* https://orcid.org/0000-0001-7186-7996
*Denis Efimov* https://orcid.org/0000-0001-8847-5235

### REFERENCES
1. Hale J. *Theory of Functional Differential Equations*. Springer-Verlag; 1977.
2. Kolmanovsky V, Nosov V. *Stability of Functional Differential Equations*. Academic; 1986.
3. Gu K, Kharitonov V, Chen J. *Stability of Time-Delay Systems, Control Engineering*. Birkhäuser; 2003.
4. Fridman E. *Introduction to Time-Delay Systems: Analysis and Control*. Birkhäuser; 2014.
5. Hetel L, Fiter C, Omran H, et al. Recent developments on the stability of systems with aperiodic sampling: an overview. *Automatica*. 2017;76:309-335. doi:10.1016/j.automatica.2016.10.023
6. Liu K, Selivanov A, Fridman E. Survey on time-delay approach to networked control. *Ann Rev Control*. 2019;48:57-79. doi:10.1016/j.arcontrol.2019.06.005
7. Richard JP. Time-delay systems: an overview of some recent advances and open problems. *Automatica*. 2003;39:1667-1694.
8. Abdallab C, Dorato P, Benites-Read J. Delayed-positive feedback canstabilize oscillatory systems. Proc. American Control Conference. 1993 3106–3107.
9. Borne P, Kolmanovskii V, Shaikhet L. Stabilization of inverted pendulum by control with delay. *Dyn Syst Appl*. 2000;9(4):501-514.
10. Niculescu SI, Michiels W. Stabilizing a chain of integrators using multiple delays. *IEEE Trans Automat Contr*. 2004;49(5):802-807.
11. Kharitonov V, Niculescu SI, Moreno J, Michiels W. Static output feedback stabilization: necessary conditionsfor multiple delay controllers. *IEEE Trans Automat Contr*. 2005;50(1):82-86.
12. Karafyllis I. Robust global stabilization by means of discrete-delay output feedback. *Syst Control Lett*. 2008;57(12):987-995.
13. Wang JM, Lv XW, Zhao DX. Exponential stability and spectral analysis of the pendulum system under position and delayed position feedbacks. *Int J Control*. 2011;84:904-915.
14. Fridman E, Shaikhet L. Delay-induced stability of vector second-order systems via simple Lyapunov functionals. *Automatica*. 2016;74:288-296.
15. Fridman E, Shaikhet L. Stabilization by using artificial delays: an LMI approach. *Automatica*. 2017;81:429-437.

16. Ramírez A, Sipahi R, Mondie S, Garrido R. An analytical approach to tuning of delay-based controllers for LTI-SISO systems. *SIAM J Control Optim*. 2017;55(1):397-412.
17. Selivanov A, Fridman E. An improved time-delay implementation of derivative-dependent feedback. *Automatica*. 2018;98:269-276. doi:10.1016/j.automatica.2018.09.035
18. Efimov D, Fridman E, Perruquetti W, Richard JP. Homogeneity of neutral systems and accelerated stabilization of a double integrator by measurement of its position. *Automatica*. 2020;118:109023. doi:10.1016/j.automatica.2020.109023
19. Nekhoroshikh AN, Efimov D, Fridman E, Perruquetti W, Furtat IB, Polyakov A. Practical fixed-time ISS of neutral time-delay systems with application to stabilization by using delays. *Automatica*. 2022;143:110455. doi:10.1016/j.automatica.2022.110455
20. Domoshnitsky A. Nonoscillation, maximum principles and exponential stability of second order delay differential equations without damping term. *J Inequal Appl*. 2014;361:1-26.
21. Berezansky L, Domoshnitsky A, Gitman M, Stolbov V. Exponential stability of a second order delay differential equation without damping term. *Appl Math Comput*. 2015;258:483-488.
22. Schiffer J, Fridman E, Ortega R, Raisch J. Stability of a class of delayed port-Hamiltonian systems with application to microgrids with distributed rotational and electronic generation. *Automatica*. 2016;74:71-79. doi:10.1016/j.automatica.2016.07.022
23. Efimov D, Schiffer J, Ortega R. Robustness of delayed multistable systems with application to droop-controlled inverter-based microgrids. *Int J Control*. 2016;89(5):909-918. doi:10.1080/00207179.2015.1104555
24. Schiffer S, Dörfler F, Fridman E. Robustness of distributed averaging control in power systems: time delays and dynamic communication topology. *Automatica*. 2017;80:261-271. doi:10.1016/j.automatica.2017.02.040
25. Nguyen D, Khazaei J. Multiagent time-delayed fast consensus design for distributed battery energy storage systems. *IEEE Trans Sustain Energy*. 2018;9(3):1397-1406. doi:10.1109/TSTE.2017.2785311
26. Aleksandrov A, Efimov D, Fridman E. On stability of second-order nonlinear time-delay systems without damping. Proc. 62nd IEEE Conference on Decision and Control (CDC), Singapore. 2023.
27. Bacciotti A, Rosier L. *Liapunov Functions and Stability in Control Theory*. Lecture Notes in Control and Inform. Sci. Vol 267. Springer; 2001.
28. Efimov D, Aleksandrov A. Analysis of robustness of homogeneous systems with time delays using Lyapunov-Krasovskii functionals. *Int J Robust Nonlinear Control*. 2021;31:3730-3746.
29. Aleksandrov AY. Stability analysis and synthesis of stabilizing controls for a class of nonlinear mechanical systems. *Nonlinear Dyn*. 2020;100(4):3109-3119. doi:10.1007/s11071-020-05709-0
30. Efimov D, Aleksandrov A. On estimation of rates of convergence in Lyapunov-Razumikhin approach. *Automatica*. 2020;116:108928. doi:10.1016/j.automatica.2020.108928
31. Kolmanovskii V, Myshkis A. *Introduction to the Theory and Applications of Functional Differential Equations*. Mathematics and its Applications. Vol 463. Springer; 1999.
32. Aleksandrov A, Zhabko A, Zhabko I. Time-delayed feedback stabilisation of nonlinear potential systems. *Int J Control*. 2015;88(10):2066-2073.
33. Teel A. Connections between Razumikhin-type theorems and the ISS nonlinear small gain theorem. *IEEE Trans Automat Contr*. 1998;43(7):960-964.
34. Aleksandrov A, Hu GD, Zhabko A. Delay-independent stability conditions for some classes of nonlinear systems. *IEEE Trans Automat Contr*. 2014;59(8):2209-2214.
35. Barbashin E. *Lyapunov Functions*. Nauka; 1970.
36. Rouche N, Mawhin J. *Ordinary Differential Equations: Stability and Periodic Solutions*. Pitman Advanced Pub. Program; 1980.
37. Aleksandrov A, Zhabko A. On the asymptotic stability of equilibria of nonlinear mechanical systems with delay. *Diff Equ*. 2013;49:143-150.
38. Palumbo P, Pepe P, Panunzi S, De Gaetano A. Observer-based closed-loop control for the glucose-insulin system: local input-to-state stability with respect to unknown meal disturbances. Proc. American Control Conference. 2013 1751–1756.
39. Efimov D, Polyakov A, Perruquetti W, Richard JP. Weighted homogeneity for time-delay systems: finite-time and independent of delay stability. *IEEE Trans Automat Contr*. 2016;61(1):210-215.
40. Beletsky V. *Motion of an Artificial Satellite about its Center of Mass*. Israel Program for Scientific Translation; 1966.
41. Zubov V. *Dynamics of Controlled Systems*. Vysshaya Shkola; 1982.
42. Aleksandrov A, Tikhonov A. Attitude stabilization of a rigid body in conditions of decreasing dissipation. *Vestnik StPetersb Univ Math*. 2017;50:384-391. doi:10.3103/S1063454117040021
43. Wertz J. *Spacecraft Attitude Determination and Control*. D. Reidel Publishing Co; 1985.
44. Hughes P. *Spacecraft Attitude Dynamics*. Wiley; 1986.

**How to cite this article:** Aleksandrov A, Efimov D, Fridman E. On local ISS of nonlinear second-order time-delay systems without damping. *Int J Robust Nonlinear Control*. 2024;1-17. doi: 10.1002/rnc.7227