Finite-dimensional boundary control of a wave equation with viscous friction and boundary measurements

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***Abstract*—Recently, a constructive approach to the design of finite-dimensional observer-based controller has been proposed for parabolic partial differential equations (PDEs). This paper extends it to hyperbolic PDEs. Namely, we design a finite-dimensional, output-feedback, boundary controller for a wave equation with in-domain viscous friction. The control-free system is unstable for any friction coefficient due to an external force. Our approach is based on modal decomposition: an observer-based controller is designed for a finite-dimensional projection of the wave equation on N eigenfunctions (modes) of the Sturm–Liouville operator. The danger of this approach is the "spillover" effect: such a controller may have a deteriorating effect on the stability of the unconsidered modes and cause instability of the full system. Our main contribution is an appropriate Lyapunov-based analysis leading to linear matrix inequalities (LMIs) that allow one to find a controller gain and number of modes, N, guaranteeing that the "spillover" effect does not occur. An important merit of the derived LMIs is that their complexity does not change when N grows. Moreover, we show that appropriate N always exists and, if the LMIs are feasible for some N, they remain so for $N + 1$.**

I. Introduction

DYNAMICAL systems described by partial differential equations (PDEs) are infinite-dimensional, i.e., their states belong to infinite-dimensional functional spaces. Controllers for such systems are preferred to be finite-dimensional to ease implementation [1], [2]. A popular way of designing such controllers is modal decomposition also called eigenfunction expansion, Galerkin's method, model reduction, etc. The main idea is to project the PDE state on a finite-dimensional subspace (comprised of modes) and design a controller for the resulting reduced-order model [3]–[10]. The main problem of this approach is the "spillover" effect: a controller that stabilizes the reduced-order system may have deteriorating effect on the stability of the full system [11], [12].

Recently, significant progress has been made in the design of finite-dimensional controllers for *parabolic* PDEs. Modal decomposition was used to establish the input-to-state stability with respect to boundary disturbances [13], [14]. This enabled the design of sampled-data *state*-feedback boundary control [15]. Later, modal decomposition was combined with Lyapunov functionals to design *state*-feedback boundary control for semilinear parabolic PDEs [16]. Finite-dimensional *output* feedback was proposed in [17]. The conditions in [17] are in terms of linear matrix inequalities (LMIs) that are feasible for large enough number of modes, which allows one to find how many modes to consider to avoid the "spillover" phenomenon. The order of these LMIs was subsequently reduced in [18]. This approach was extended to input/output delays [18]–[20] and Kuramoto–Sivashinsky equation [21].

Hyperbolic PDEs (the wave equation in particular) are more difficult to stabilize than parabolic PDEs since they may have infinitely many non-vanishing modes. This is avoided by introducing boundary or in-domain damping [22]–[27]. Controllability and state-feedback boundary control of undamped wave equations were studied in [28]–[30], where boundary friction was introduced to stabilize the system. Modal decomposition has also been used to design finite-dimensional controllers for wave equations with in-domain damping [3], [5], [31]. These works, however, were concerned with bounded control and observation operators and did not specify how many modes one needs to consider to avoid the "spillover" phenomenon.

In this paper, we design a finite-dimensional, output-feedback, boundary controller for the wave equation with in-domain damping. Our results extend the constructive modal decomposition approach to finite-dimensional observer-based control introduced in [17] for parabolic PDEs. The main difference is that the state of a wave equation comprises the displacement and its velocity, which lie in different functional spaces. This makes the Lyapunov-based analysis challenging since the Lyapunov functional has to contain the products of the Fourier coefficients corresponding to functions from different spaces. To manage with boundary control, we employ the dynamic extension approach suggested in [16] for parabolic PDEs. By carefully choosing the weights in the Lyapunov functional, we ensure that it is well-defined and properly treats the "spillover" terms. Eventually, we derive linear matrix inequalities (LMIs) that allow one to find an appropriate controller gain and the number of modes required by the observer and controller to avoid the "spillover" effect.

Notations: $|\cdot|$ is the Euclidean norm, $\|\cdot\|$ or $\|\cdot\|_{L^2}$ is the L^2 norm, $\langle\cdot,\cdot\rangle$ or $\langle\cdot,\cdot\rangle_{L^2}$ is the scalar product in L^2, $\|\cdot\|_{H^1}$ is the H^1 norm defined as $\|f\|_{H^1}^2 = \|f\|^2 + \|f'\|^2$. For a matrix P, the notation $P < 0$ implies that P is negative-definite with the symmetric elements sometimes marked as $*$; $0_{n\times m}$ is the matrix of zeros from $\mathbb{R}^{n\times m}$. Partial derivatives are denoted by indices, e.g., $z_t = \partial z/\partial t$. The other notations are standard.

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II. Plant description

Consider the damped wave equation

$$\begin{aligned} &z_{tt}(x,t)+2\mu z_t(x,t)=c^2 z_{xx}(x,t)+az(x,t),\\ &z_x(0,t)=0, \qquad\qquad\qquad\qquad\qquad\qquad\qquad (1)\\ &z(1,t)=u(t),\end{aligned}$$

where $z\colon [0,1]\times[0,\infty)\to\mathbb{R}$ is the state representing the displacement from equilibrium, $u\colon[0,\infty)\to\mathbb{R}$ is the control input, $2\mu>0$ is the friction coefficient, $c>0$ is the propagation speed, and $a>0$ characterizes the external force acting on the string, which may arise as a result of linearization. The measured output is

$$y(t)=z(0,t). \tag{2}$$

The system is unstable for any friction coefficient if $u\equiv 0$ and $a>(c\pi)^2/4$. The objective is to design a finite-dimensional controller that exponentially stabilizes (1) with any decay rate $\alpha\in[0,\mu)$ using the measurements (2).

III. Design of the dynamic controller

A. Change of variables

Following [16], we employ a change of state variables that transforms the boundary control into a distributed one by extending the system dynamics. First, we introduce m scalar functions

$$\begin{aligned}&\psi_k(x)=(-1)^k\cos(\sqrt{\xi_k}x),\quad \xi_k=\pi^2k^2,\\ &k=m_0,\dots,m_0+m-1,\end{aligned}$$

which satisfy the relations

$$\psi_k''=-\xi_k\psi_k,\quad \psi_k'(0)=0,\quad \psi_k(1)=1. \tag{3}$$

Here, $m_0\in\mathbb{N}$ is such that $\mu^2-(c^2\xi_{m_0}-a)\le 0$, i.e.,

$$m_0\ge\frac{\sqrt{a+\mu^2}}{\pi c}. \tag{4}$$

The reason for this choice of m_0 is explained below (6).

As in [16], we consider

$$w(x,t):=z(x,t)-\psi^T(x)\bar u(t),$$

where

$$\begin{aligned}&\bar u(t)=\operatorname{col}\{u_1(t),\dots,u_m(t)\},\\ &\psi(x)=\operatorname{col}\{\psi_{m_0}(x),\dots,\psi_{m_0+m-1}(x)\}.\end{aligned}$$

Then (1) and (3) imply

$$\begin{aligned}&w_{tt}+2\mu w_t=c^2w_{xx}+aw+\psi^T(-\ddot{\bar u}-2\mu\dot{\bar u}-c^2\Xi\bar u+a\bar u)\\ &w_x(0,t)=0,\\ &w(1,t)=u(t)-\psi^T(1)\bar u(t),\end{aligned}$$

where $\Xi:=\operatorname{diag}\{\xi_{m_0},\dots,\xi_{m_0+m-1}\}$. Note that $\bar u$ should be smooth enough for the above to be meaningful. We establish the required smoothness in Section IV. To zero the right-hand boundary condition, we take

$$u(t)=\psi^T(1)\bar u(t)=\textstyle\sum_{k=1}^m u_k(t). \tag{5}$$

Introducing

$$\bar v:=-\ddot{\bar u}-2\mu\dot{\bar u}-c^2\Xi\bar u+a\bar u,$$

we obtain

$$\ddot{\bar u}(t)+2\mu\dot{\bar u}(t)=(aI-c^2\Xi)\bar u(t)-\bar v(t), \tag{6a}$$

$$\begin{aligned}w_{tt}(x,t)+2\mu w_t(x,t)=c^2w_{xx}(x,t)+aw(x,t)&\\ +\psi^T(x)\bar v(t),&\end{aligned} \tag{6b}$$

$$w_x(0,t)=0=w(1,t), \tag{6c}$$

where $\bar v\colon[0,\infty)\to\mathbb{R}^m$ is a new control input. To find the original control, $u(t)$, one needs to solve (6a) for $\bar u(0)=0$ and $\dot{\bar u}(0)=0$, and use (5). Note that, if m_0 satisfies (4) and $\bar v\equiv 0$, then (6a) guarantees $|u(t)|\le C_0e^{-\alpha t}$ and $|\dot u(t)|\le C_1e^{-\alpha t}$ for any $\alpha\in[0,\mu)$ with some C_0 and C_1. Moreover, these bounds remain valid for (6a) if $|\bar v(t)|\le Ce^{-\alpha_0 t}$ with $\alpha_0>\alpha$.

Remark 1: To pass the control from the boundary to the interior, we employed a change of variables that introduced input derivatives. This can be avoided using the results of [32, Section 13.7] (see, e.g., [33]). The extension of this approach to hyperbolic systems may be a topic for future research.

B. Modal decomposition

The Sturm–Liouville problem associated with (6b), (6c) is

$$\varphi''=-\lambda\varphi,\qquad \varphi'(0)=0=\varphi(1).$$

The solutions,

$$\varphi_n=\sqrt2\cos(\sqrt{\lambda_n}x),\quad \lambda_n=\left(n-\tfrac12\right)^2\pi^2,\quad n\in\mathbb{N}, \tag{7}$$

form a complete orthonormal system in $L^2(0,1)$. Therefore,

$$w(\cdot,t)\overset{L^2}{=}\textstyle\sum_{n=1}^\infty w_n(t)\varphi_n,\qquad w_n(t):=\langle w(\cdot,t),\varphi_n\rangle.$$

In Section IV, we show that $w(\cdot,t)\in H^1(0,1)$ subject to $w(1,t)=0$ if $w(\cdot,0)\in H^1(0,1)$, $w(1,0)=0$, $w_t(\cdot,0)\in L^2(0,1)$, and $\bar v$ is as designed below (see (13), (15), (16)). Thus, $w(x,t)=\sum_{n=1}^\infty w_n(t)\varphi_n(x)$ for each $x\in[0,1]$. Substituting this into (6b), we obtain

$$\ddot w_n(t)+2\mu\dot w_n(t)=(a-c^2\lambda_n)w_n(t)+b_n\bar v(t),\quad n\in\mathbb{N}, \tag{8}$$

where $b_n=[\langle\psi_{m_0},\varphi_n\rangle,\dots,\langle\psi_{m_0+m-1},\varphi_n\rangle]\in\mathbb{R}^{1\times m}$. Direct calculations give

$$\langle\psi_k,\varphi_n\rangle=\frac{(-1)^n\sqrt{2\lambda_n}}{\xi_k-\lambda_n}.$$

The characteristic roots of (8) are

$$s_n^\pm=-\mu\pm\sqrt{\mu^2-(c^2\lambda_n-a)}.$$

Let N_0 be such that $\mu^2-(c^2\lambda_{N_0+1}-a)\le 0$, i.e., (cf. (4))

$$N_0\ge\frac{\sqrt{a+\mu^2}}{\pi c}-\frac12. \tag{9}$$

Then the modes corresponding to $n>N_0$ have the fastest possible decay rate $\mu>0$ for $\bar v\equiv 0$. We design a finite-dimensional controller using the first N_0 modes and a finite-dimensional observer using the first $N\ge N_0$ modes. The value of N depends on the desired decay rate $\alpha\in[0,\mu)$ and is found by solving the LMIs of Theorem 1 below. We introduce

$$\begin{aligned}&w^{N_0}:=\operatorname{col}\{w_1,\dots,w_{N_0},\dot w_1,\dots,\dot w_{N_0}\},\\ &w^{N-N_0}:=\operatorname{col}\{w_{N_0+1},\dots,w_N,\dot w_{N_0+1},\dots,\dot w_N\}.\end{aligned} \tag{10}$$

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Here, $w^{N_0} \in \mathbb{R}^{2N_0}$ and $w^{N-N_0} \in \mathbb{R}^{2(N-N_0)}$. From (8),

$$\begin{aligned}\dot{w}^{N_0}(t) &= A_0 w^{N_0}(t) + B_0 \bar{v}(t),\\ \dot{w}^{N-N_0}(t) &= A_1 w^{N-N_0}(t) + B_1 \bar{v}(t),\end{aligned}$$

where

$$\begin{aligned}&A_0 := \begin{bmatrix} 0 & I \\ aI - c^2\Lambda_0 & -2\mu I\end{bmatrix}, \quad A_1 := \begin{bmatrix} 0 & I \\ aI - c^2\Lambda_1 & -2\mu I\end{bmatrix},\\ &\Lambda_0 := \operatorname{diag}\{\lambda_1, \dots, \lambda_{N_0}\}, \qquad \Lambda_1 := \operatorname{diag}\{\lambda_{N_0+1}, \dots, \lambda_N\},\\ &B_0 := \begin{bmatrix} 0_{N_0\times m} \\ b_1 \\ \vdots \\ b_{N_0}\end{bmatrix}, \qquad B_1 := \begin{bmatrix} 0_{(N-N_0)\times m} \\ b_{N_0+1} \\ \vdots \\ b_N\end{bmatrix}.\end{aligned}$$

The output (2) can be presented as

$$\begin{aligned} y(t) &= w(0,t) + \psi^T(0)\bar{u}(t) \\ &= C_0 w^{N_0} + C_1 w^{N-N_0} + \zeta(t) + \psi^T(0)\bar{u}(t),\end{aligned} \tag{11}$$

where

$$\begin{aligned} C_0 &= \left[\sqrt{2}, \dots, \sqrt{2}, 0, \dots, 0\right] \in \mathbb{R}^{1\times 2N_0},\\ C_1 &= \left[\sqrt{2}, \dots, \sqrt{2}, 0, \dots, 0\right] \in \mathbb{R}^{1\times 2(N-N_0)},\\ \zeta(t) &= \sum_{n=N+1}^{\infty} \varphi_n(0) w_n(t) = \sqrt{2}\sum_{n=N+1}^{\infty} w_n(t).\end{aligned}$$

C. Boundary observer

The Hautus lemma [34, Lemma 3.3.7] implies that (A_0, C_0) is observable. Indeed, if $A_0 v = \lambda v$ for some $\lambda \in \mathbb{C}$ and $v = \operatorname{col}\{v_1, v_2\} \neq 0$ with $v_1, v_2 \in \mathbb{C}^{N_0}$, then

$$v_2 = \lambda v_1 \quad \text{and} \quad [(\lambda^2 + 2\mu\lambda - a)I + c^2\Lambda_0]v_1 = 0.$$

Since $v \neq 0$, $v_2 = \lambda v_1$ implies $v_1 \neq 0$. Since $\lambda_n \neq \lambda_m$, the last relation can hold only if exactly one element of v_1 is not zero. Then $C_0 v$ equals this element times $\sqrt{2}$, which is not zero.

Due to the observability of (A_0, C_0), for any $\alpha > 0$, there are $L \in \mathbb{R}^{2N_0\times 1}$ and $0 < P_0 \in \mathbb{R}^{2N_0\times 2N_0}$ such that

$$P_0(A_0 - LC_0) + (A_0 - LC_0)^T P_0 + 2\alpha P_0 < 0. \tag{12}$$

Denote the estimates of (10) as

$$\begin{aligned}\hat{w}^{N_0} &:= \operatorname{col}\{\hat{w}_1, \dots, \hat{w}_{N_0}, \dot{\hat{w}}_1, \dots, \dot{\hat{w}}_{N_0}\},\\ \hat{w}^{N-N_0} &:= \operatorname{col}\{\hat{w}_{N_0+1}, \dots, \hat{w}_N, \dot{\hat{w}}_{N_0+1}, \dots, \dot{\hat{w}}_N\}.\end{aligned}$$

Then the observer is $\hat{w}(x,t) := \sum_{n=1}^{N} \hat{w}_n(t)\varphi_n(x)$, where

$$\begin{aligned}&\dot{\hat{w}}^{N_0} = A_0\hat{w}^{N_0} + B_0\bar{v} - L[\hat{w}(0,t) + \psi^T(0)\bar{u}(t) - y(t)],\\ &\dot{\hat{w}}^{N-N_0} = A_1\hat{w}^{N-N_0} + B_1\bar{v},\\ &\hat{w}^{N_0}(0) = 0, \quad \hat{w}^{N-N_0}(0) = 0\end{aligned} \tag{13}$$

with A_1, A_0, B_1, B_0 given above. A correcting term is used only for the first N_0 coefficients, where N_0 is chosen as the smallest integer satisfying (9). Similarly to [17], we deliberately do not introduce a correcting term in the second equation of (13) to avoid numerical problems. Otherwise, $L \in \mathbb{R}^{2N\times 1}$ with $N \to \infty$ as the desired decay rate gets closer to μ, making the choice of L a challenging numerical problem. In particular, MATLAB fails to perform pole-placement when $N \geq 7$ in the example of Section VI. Furthermore, an additional correcting term will not improve the observer's convergence rate since the decay rate of $w_n(t)$ with $n > N$ is upper-bounded by μ. Finally, an additional correcting term makes it more difficult to prove that the LMIs of Theorem 1 are feasible for a large enough N. Note that due to the choice of N_0, both w^{N-N_0} and $\hat{w}^{N-N_0}$ exponentially go to zero if $\bar{v} \equiv 0$ (see the discussion around (9)).

In view of (11), the correcting term can be expressed as

$$\hat{w}(0,t) + \psi^T(0)\bar{u}(t) - y(t) = -C_0 e^{N_0} - C_1 e^{N-N_0} - \zeta,$$

where $e^{N_0} = w^{N_0} - \hat{w}^{N_0}$ and $e^{N-N_0} = w^{N-N_0} - \hat{w}^{N-N_0}$ are the approximation errors for the first N Fourier coefficients. Therefore,

$$\begin{aligned}\dot{e}^{N_0} &= (A_0 - LC_0)e^{N_0} - LC_1 e^{N-N_0} - L\zeta,\\ \dot{e}^{N-N_0} &= A_1 e^{N-N_0}.\end{aligned}$$

D. Boundary controller

The controllability of (A_0, B_0) can be established in a manner similar to Section III-C, where one also needs to use $\langle \psi_k, \varphi_n\rangle \neq 0$. Then, for any $\alpha > 0$, there are $K \in \mathbb{R}^{m\times 2N_0}$ and $0 < P \in \mathbb{R}^{2N_0\times 2N_0}$ such that

$$P(A_0 - B_0K) + (A_0 - B_0K)^T P + 2\alpha P < 0. \tag{14}$$

Therefore, we take

$$\bar{v}(t) = -K\hat{w}^{N_0}(t). \tag{15}$$

Summarizing, the boundary control is $u(t) = \sum_{k=1}^{m} u_k(t)$ with u_k obtained as the solution of

$$\begin{aligned}&\bar{u} = \operatorname{col}\{u_1, \dots, u_m\},\\ &\ddot{\bar{u}}(t) + 2\mu\dot{\bar{u}}(t) = (aI - c^2\Xi)\bar{u}(t) + K\hat{w}^{N_0}(t),\\ &\bar{u}(0) = \dot{\bar{u}}(0) = 0,\end{aligned} \tag{16}$$

where $\hat{w}^{N_0}(t)$ is the solution of (13).

Remark 2: Note that $\hat{w}^{N-N_0}(t)$ is used in the first equation of the observer (it is embedded in the $\hat{w}(0,t)$ term) but is not used in the control law (15). One of the reasons is similar to the one given in Section III-C: growing dimension of K leads to computational difficulties. Moreover, if $\hat{w}^{N-N_0}(t)$ is used in the feedback, this removes the separation of the slow dynamics (with $n \leq N_0$) and fast dynamics (with $n > N$) of the closed-loop system (for details, see [18]).

IV. Well-posedness of the closed-loop system

To establish the well-posedness of the closed-loop system (6b), (6c), (13), (15), we define the following Hilbert spaces

$$\begin{aligned}&H_R^1(0,1) := \{f \in H^1(0,1) \mid f(1) = 0\},\\ &H_{LR}^2(0,1) := \{f \in H^2(0,1) \mid f'(0) = 0 = f(1)\},\\ &X := H_R^1(0,1) \times L^2(0,1) \times \mathbb{R}^{2N}\end{aligned}$$

with the norms

$$\begin{aligned}&\|f\|_{H_R^1} := \|f'\|_{L^2},\\ &\|f\|_{H_{LR}^2} := \|f''\|_{L^2},\\ &\|(f, g, w^N)\|_X := \|f\|_{H_R^1} + \|g\|_{L^2} + |w^N|.\end{aligned}$$

The closed-loop system can be written in the abstract form

$$\dot{\eta}(t) = (\mathcal{A}+\mathcal{B})\eta(t),$$

where $\eta(t) := \mathrm{col}\{w(\cdot,t), w_t(\cdot,t), \hat{w}^{N_0}(t), \hat{w}^{N-N_0}(t)\} \in X$,

$$\mathcal{A} := \begin{bmatrix} \mathcal{A}_0 & 0 \\ 0 & \mathcal{A}_1 \end{bmatrix}, \quad \mathcal{A}_0 := \begin{bmatrix} 0 & I \\ c^2\frac{\partial^2}{\partial x^2} + aI & -2\mu I \end{bmatrix},$$
$$\mathcal{A}_1 := \begin{bmatrix} A_0 - B_0K - LC_0 & -LC_1 \\ -B_1K & A_1 \end{bmatrix},$$
$$\mathcal{B}\eta := \mathrm{col}\{0, -\psi^T K\hat{w}^{N_0}, Lw(0,t), 0\}.$$

The operator $\mathcal{A}_0$ is an infinitesimal generator of a C_0 semigroup with the dense in $H^1_R \times L^2$ domain

$$D(\mathcal{A}_0) = H^2_{LR}(0,1) \times H^1_R(0,1).$$

Therefore, $\mathcal{A}$ with $D(\mathcal{A}) = D(\mathcal{A}_0) \times \mathbb{R}^{2N}$ generates a C_0 semigroup. Since $w(1,t) = 0$, we have

$$|w(0,t)| \le \int_0^1 |w_x(x,t)|\,dx \le \|w(\cdot,t)\|_{H^1_R}.$$

Hence, the operator $\mathcal{B}\colon X \to X$ is bounded:

$$\begin{aligned}\|\mathcal{B}\eta\| &= \|\psi^T K\hat{w}^{N_0}\|_{L^2} + |Lw(0,t)| \\ &\le \|\psi\||K||\hat{w}^{N_0}| + |L|\|w\|_{H^1_R} \le \max\{\|\psi\||K|, |L|\}\|\eta\|_X.\end{aligned}$$

By [35, Theorem 3.1.1], $\mathcal{A}+\mathcal{B}$ is an infinitesimal generator of a C_0 semigroup on X. Therefore, for any

$$w(\cdot,0) \in H^1_R(0,1), \quad w_t(\cdot,0) \in L^2(0,1),$$

there is a unique mild solution

$$w \in C([0,\infty), H^1_R(0,1)) \cap C^1([0,\infty), L^2(0,1)).$$

Moreover, $\hat{w}^{N_0}(t) \in C^1[0,\infty)$ and, by (15), $\bar{u}(t) \in C^1[0,\infty)$. Since $z = w + \psi^T\bar{u}$, (1) has a unique mild solution for

$$z(\cdot,0) \in H^1_R(0,1), \quad z_t(\cdot,0) \in L^2(0,1).$$

V. Stability conditions

In this section, we formulate our main result — the linear matrix inequalities (LMIs) whose feasibility guarantees that the dynamic controller (5), (13), (16) guarantees the stability of (1), (2). It will be shown in the Appendix that the stability of the closed-loop system follows from the stability of

$$\begin{aligned}\dot{\hat{w}}^{N_0} &= (A_0 - B_0K)\hat{w}^{N_0} + L(C_0e^{N_0} + C_1e^{N-N_0} + \zeta),\\ \dot{\hat{w}}^{N-N_0} &= A_1\hat{w}^{N-N_0} - B_1K\hat{w}^{N_0},\\ \dot{e}^{N_0} &= (A_0 - LC_0)e^{N_0} - LC_1e^{N-N_0} - L\zeta,\\ \dot{e}^{N-N_0} &= A_1e^{N-N_0},\\ \dot{\bar{w}}_n &= A_n\bar{w}_n - \bar{b}_nK\hat{w}^{N_0}, \quad n > N,\end{aligned} \tag{17}$$

where $\hat{w}^{N_0}$ and $\hat{w}^{N-N_0}$ represent the observer state, e^{N_0} and e^{N-N_0} represent the observation error, and

$$A_n = \begin{bmatrix} 0 & 1 \\ a - c^2\lambda_n & -2\mu \end{bmatrix}, \; \bar{b}_n = \begin{bmatrix} 0 \\ b_n \end{bmatrix}, \; \bar{w}_n = \begin{bmatrix} w_n \\ \dot{w}_n \end{bmatrix}. \tag{18}$$

Since e^{N-N_0} is decoupled and stable, whereas $\hat{w}^{N-N_0}$ is stable provided $\hat{w}^{N_0}$ is stable, the stability of (17) is reduced to that of the reduced-order (17) without the 2nd and 4th equations. The latter leads to reduced-order LMIs (as in [18] for the parabolic case). The main challenge in establishing the stability of (17) is to bound the destabilizing effect that the boundary feedback has on the high-order modes described by the last equation of (17), and the effect of the infinite-dimensional residue (represented by ζ) on the finite-dimensional states $\hat{w}^{N_0}$ and e^{N_0}. Our main idea is to use

$$V_\infty := \sum_{n=N+1}^{\infty} \lambda_n \bar{w}_n^T P_n \bar{w}_n, \quad P_n = \begin{bmatrix} 1 & \mu\gamma_n^{-1} \\ \mu\gamma_n^{-1} & \gamma_n^{-1} \end{bmatrix}, \tag{19}$$

where $\gamma_n := c^2\lambda_n - a > 0$. The series converges since $w(\cdot,t) \in H^1(0,1)$, $w_t(\cdot,t) \in L^2(0,1)$, and $\lim \lambda_n/\gamma_n < \infty$. We found this P_n by solving

$$P_nA_n + A_n^TP_n = -2\mu P_n, \quad \forall n > N, \tag{20}$$

with A_n from (18). In the Appendix, we establish a convenient lower bound for P_n that, combined with (20), allows us to bound the effect of $\bar{v}$ on $\bar{w}_n$ with $n > N$, which is the key step in the proof of the following theorem.

Theorem 1: Let $\alpha \in [0,\mu)$ be a desired decay rate for system (1) with output (2). Then, for any $m \in \mathbb{N}$, $m_0 \ge 0$ satisfying (4), N_0 satisfying (9), $L \in \mathbb{R}^{2N_0\times 1}$ satisfying (12), and large enough $N \ge N_0$, there exist $Q \in \mathbb{R}^{m\times 2N_0}$, $0 < \bar{P} \in \mathbb{R}^{2N_0\times 2N_0}$, $0 < P_0 \in \mathbb{R}^{2N_0\times 2N_0}$, such that

$$\Phi = \begin{bmatrix} \Phi_{11} & LC_0 & L & \Phi_{14} \\ * & \Phi_{22} & -P_0L & 0 \\ * & * & \Phi_{33} & 0 \\ * & * & * & \Phi_{44} \end{bmatrix} < 0,$$

where

$$\begin{aligned}\Phi_{11} &= A_0\bar{P} + \bar{P}A_0^T + 2\alpha\bar{P} - B_0Q - Q^TB_0^T,\\ \Phi_{14} &= Q^T\tfrac{\lambda_{N+1}}{c^2\lambda_{N+1}-a}\left[\tfrac{1}{2}I - \textstyle\sum_{n=1}^N b_n^Tb_n\right],\\ \Phi_{22} &= P_0(A_0 - LC_0) + (A_0 - LC_0)^TP_0 + 2\alpha P_0,\\ \Phi_{33} &= -(\mu-\alpha)\left[1 - \tfrac{\mu^2}{c^2\lambda_{N+1}-a}\right],\\ \Phi_{44} &= -(\mu-\alpha)\tfrac{\lambda_{N+1}}{c^2\lambda_{N+1}-a}\left[\tfrac{1}{2}I - \textstyle\sum_{n=1}^N b_n^Tb_n\right].\end{aligned}$$

In this case, the dynamic boundary controller (13), (15), (16) with $K = Q\bar{P}^{-1}$ guarantees global exponential stability of (1), (2) with the decay rate α, i.e., $\exists C > 0$:

$$\|z(\cdot,t)\|_{H^1} + \|z_t(\cdot,t)\|_{L^2} \le Ce^{-\alpha t}(\|z(\cdot,0)\|_{H^1} + \|z_t(\cdot,0)\|_{L^2}).$$

The proof is given in the Appendix.

Remark 3: Since (A_0, B_0) is controllable, one could simply choose K such that $A_0 - B_0K$ is stable. In this case, however, the control signal (15) may destabilize the higher-order modes described by the last equation of (17). Theorem 1 provides the value of K such that this does not happen.

Remark 4: As in [18] for the heat equation, here the size of Φ does not change when N grows. This is due to the choice of L and K (see Remark 2). Moreover, both Φ_{33} and $-\Phi_{14}\Phi_{44}^{-1}\Phi_{14}^T$ monotonically decrease with N. Therefore, if the LMIs of Theorem 1 are feasible for N, then they are feasible for $N+1$.

Remark 5: An alternative approach is to consider modal decomposition with respect to the Riesz basis comprising

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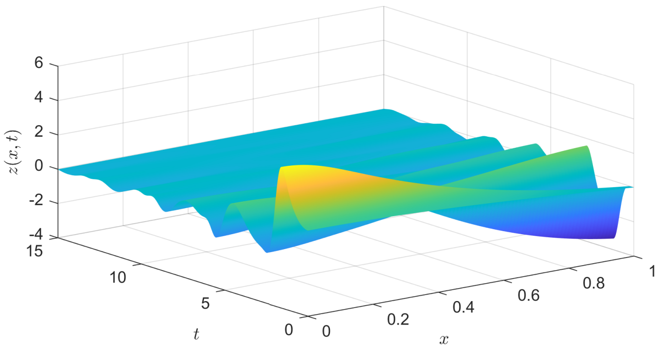

Fig. 1. Evolution of the state $z(x,t)$

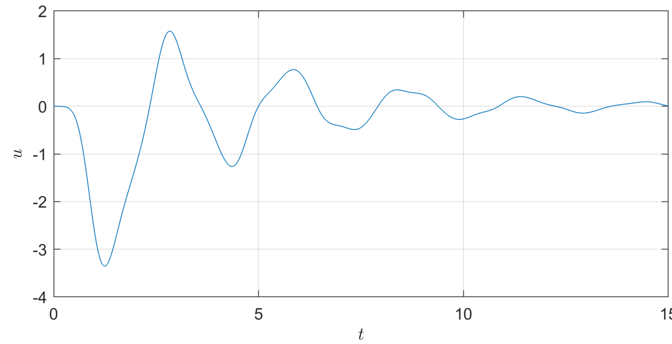

Fig. 2. Boundary control input $u(t)$

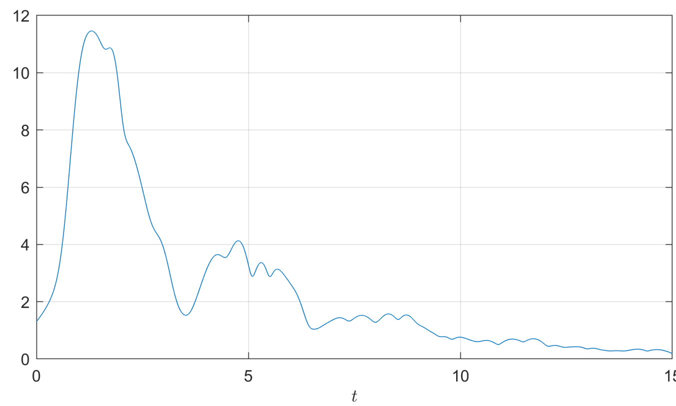

Fig. 3. The performance measure $\|z(\cdot,t)\|_{H^1} + \|z_t(\cdot,t)\|_{L^2}$

the complex-valued eigenfunctions of $\mathcal{A}_0$, introduced in Section IV [10], [30]. We consider Sturm–Liouville eigenfunctions (7) that simplify modal decomposition but make the residue analysis more challenging. This challenge is overcome by the judicious choice of V_∞ and P_n given in (19).

VI. Numerical Example

Consider system (1) with $\mu = 0.25$, $c = 1$, and $a = 5$. It is unstable without control since $a > (c\pi)^2/4$. The smallest values satisfying (4) and (9) are $m_0 = 1$ and $N_0 = 1$. Let the desired decay rate be $\alpha = 0.1$. Using pole placement, we find

$$L \approx \begin{bmatrix} 0.7071 \\ 1.7908 \end{bmatrix}.$$

The minimum required number of modes to observe, N, depends on the number of functions, m, used in control (5). We found N for different m by solving the LMIs of Theorem 1:

m	1	2	3	4	5	6
N	49,522	10	5	4	4	3

Further increase of m does not change N. The order of the dynamic controller (13), (15), (16) is $2(m+N)$. To minimize it, we take $m = 3$ and $N = 5$. We prefer this to $m = 4$ and $N = 4$ because the LMIs of Theorem 1 are of lower order when m is smaller. In this case, Theorem 1 gives

$$K \approx \begin{bmatrix} -55.9059 & -32.6574 \\ 79.9051 & 46.6752 \\ -24.2195 & -14.1467 \end{bmatrix} \in \mathbb{R}^{m\times 2N_0}.$$

Figure 1 shows $z(x,t)$ for the initial conditions

$$z(x,0) = \cos\left(\frac{\pi x}{2}\right), \quad z_t(x,0) = 0.$$

The control input, calculated using (5), (13), (16), is shown in Fig. 2. The performance measure, $\|z(\cdot,t)\|_{H^1} + \|z_t(\cdot,t)\|_{L^2}$, used in Theorem 1 is shown in Fig 3.

VII. Conclusions

This paper is the first one to provide a constructive method of designing a finite-dimensional, output-feedback controller for a hyperbolic PDE. Namely, we derived linear matrix inequalities that tell how many modes one should take and what gain one should use to exponentially stabilize an unstable wave equation with damping. In future, the suggested method can be extended to other classes of hyperbolic PDEs, such as wave equations with strong damping, nonlinearities, and input/output delays.

Appendix
Proof of Theorem 1

The proof is based on the direct Lypaunov approach that uses the following functional

$$V = \hat{V} + V_0 + V_1 + V_\infty$$

with V_∞ given in (19) and

$$\hat{V} := (\hat{w}^{N_0})^T P \hat{w}^{N_0}, \quad V_0 := (e^{N_0})^T P_0 e^{N_0},$$
$$V_1 := (e^{N-N_0})^T P_1 e^{N-N_0},$$

where $P := \bar{P}^{-1}$ and P_1 is defined below. For convenience, the proof is divided into sections.

A. An upper bound on $\dot{V}_\infty + 2\alpha V_\infty$

Let us show that, for small enough $\nu_0 > 0$ and $\nu_1 > 0$,

$$P_n > \begin{bmatrix} \nu_0 & 0 \\ 0 & \frac{\nu_1}{\lambda_n} \end{bmatrix}, \quad \forall n > N. \tag{21}$$

Indeed, by the Schur complement lemma, it is equivalent to

$$\frac{1}{\gamma_n}\left(1 - \frac{\gamma_n}{\lambda_n}\nu_1 - \frac{\mu^2}{\gamma_n(1-\nu_0)}\right) > 0, \quad \forall n > N.$$

Since $\gamma_n/\lambda_n \to c^2 < \infty$, this holds for a small $\nu_1 > 0$ if it holds for $\nu_1 = 0$. Since γ_n grows monotonically, it suffices to ensure that

$$1 - \frac{\mu^2}{\gamma_{N+1}(1-\nu_0)} > 0 \quad \iff \quad \nu_0 < 1 - \frac{\mu^2}{\gamma_{N+1}}.$$

Such $\nu_0 > 0$ exists because $N \geq N_0$ and $\mu^2 \overset{(9)}{<} \gamma_{N_0+1}$. Note that this also implies

$$P_n \geq \begin{bmatrix} 1-\mu^2/\gamma_{N+1} & 0 \\ 0 & 0 \end{bmatrix}. \tag{22}$$

For $V_n := \lambda_n \bar{w}_n^T P_n \bar{w}_n$, we have

$$\dot{V}_n + 2\alpha V_n \overset{(17)}{=} \lambda_n \bar{w}_n^T [P_n A_n + A_n^T P_n + 2\alpha P_n]\bar{w}_n - 2\lambda_n \bar{w}_n^T P_n \bar{b}_n K \hat{w}^{N_0}.$$

By Young's inequality,

$$-2\lambda_n \bar{w}_n^T P_n \bar{b}_n K \hat{w}^{N_0} \leq \lambda_n(\mu-\alpha)\bar{w}_n^T P_n \bar{w}_n + \lambda_n(\mu-\alpha)^{-1}(\hat{w}^{N_0})^T K^T \bar{b}_n^T P_n \bar{b}_n K \hat{w}^{N_0}.$$

Together with (18), (20), (19), and (22), this implies

$$\begin{aligned}\dot{V}_n + 2\alpha V_n &\leq \lambda_n \bar{w}_n^T [P_n A_n + A_n^T P_n + (\mu+\alpha)P_n]\bar{w}_n \\ &\quad + \lambda_n(\mu-\alpha)^{-1}(\hat{w}^{N_0})^T K^T \bar{b}_n^T P_n \bar{b}_n K \hat{w}^{N_0} \\ &\leq -(\mu-\alpha)\left[1-\frac{\mu^2}{\gamma_{N+1}}\right]\lambda_n w_n^2 \\ &\quad + (\mu-\alpha)^{-1}(\hat{w}^{N_0})^T K^T \left[\frac{\lambda_n}{\gamma_n} b_n^T b_n\right] K \hat{w}^{N_0}.\end{aligned}$$

By Parseval's theorem,

$$\sum_{n=1}^{\infty} b_n^T b_n = \int_0^1 \psi^T(x)\psi(x)\,dx = \frac{1}{2}I.$$

Since $\frac{\lambda_n}{\gamma_n}$ decreases monotonically when $n > N$, we have

$$\sum_{n=N+1}^{\infty} \frac{\lambda_n}{\gamma_n} b_n^T b_n \leq \frac{\lambda_{N+1}}{\gamma_{N+1}}\left[\frac{1}{2}I - \sum_{n=1}^{N} b_n^T b_n\right].$$

Note that (7) form an orthogonal basis in H_R^1 with $\langle f,g\rangle_{H_R^1} = \langle f', g'\rangle_{L^2}$. Therefore, $\|w_x(\cdot,t)\|^2 = \sum_{n=1}^{\infty} \lambda_n w_n^2(t)$ and, since $\varphi_n(1) = 0 = w(1,t)$, we have

$$\begin{aligned}\zeta^2(t) &= \left[\sum_{n=N+1}^{\infty} \varphi_n(0) w_n(t)\right]^2 = \left[w(0,t) - \sum_{n=1}^{N} \varphi_n(0) w_n(t)\right]^2 \\ &= \left[-\int_0^1 \left(w_x(x,t) - \sum_{n=1}^{N} \varphi_n'(x) w_n(t)\right) dx\right]^2 \\ &\leq \left\| w_x(\cdot,t) - \sum_{n=1}^{N} \varphi_n'(\cdot) w_n(t)\right\|^2 = \sum_{n=N+1}^{\infty} \lambda_n w_n^2(t).\end{aligned}$$

Therefore,

$$\begin{aligned}\dot{V}_\infty + 2\alpha V_\infty &\leq -(\mu-\alpha)\left[1-\frac{\mu^2}{\gamma_{N+1}}\right]\zeta^2(t) \\ &+ (\mu-\alpha)^{-1}\frac{\lambda_{N+1}}{\gamma_{N+1}}(\hat{w}^{N_0})^T K^T \left[\frac{1}{2}I - \sum_{n=1}^{N} b_n^T b_n\right] K \hat{w}^{N_0}.\end{aligned} \tag{23}$$

B. Proof of $\dot{V} \leq -2\alpha V$

In view of (17),

$$\begin{aligned}\dot{\hat{V}} + 2\alpha\hat{V} &= (\hat{w}^{N_0})^T [P(A_0 - B_0K) + (A_0 - B_0K)^T P]\hat{w}^{N_0} \\ &+ 2(\hat{w}^{N_0})^T PL(C_0 e^{N_0} + C_1 e^{N-N_0} + \zeta) + 2\alpha(\hat{w}^{N_0})^T P \hat{w}^{N_0},\end{aligned}$$

$$\begin{aligned}\dot{V}_0 + 2\alpha V_0 &= (e^{N_0})^T [P_0(A_0 - LC_0) + (A_0 - LC_0)^T P_0] e^{N_0} \\ &\quad - 2(e^{N_0})^T P_0 L[C_1 e^{N-N_0} + \zeta] + 2\alpha(e^{N_0})^T P_0 e^{N_0}, \\ \dot{V}_1 + 2\alpha V_1 &= (e^{N-N_0})^T [P_1 A_1 + A_1^T P_1 + 2\alpha P_1] e^{N-N_0}.\end{aligned}$$

Summing that up and using (23), we obtain

$$\dot{V} + 2\alpha V \leq \upsilon^T \Upsilon \upsilon,$$

where $\upsilon = \mathrm{col}\{\hat{w}^{N_0}, e^{N_0}, \zeta, e^{N-N_0}\}$ and $\Upsilon = \{\Upsilon_{ij}\}_{i,j=1}^4$ is a symmetric matrix composed from the blocks

$$\begin{aligned}\Upsilon_{11} &= P(A_0 - B_0K) + (A_0 - B_0K)^T P + 2\alpha P \\ &\quad + (\mu-\alpha)^{-1}\frac{\lambda_{N+1}}{\gamma_{N+1}} K^T \left[\frac{1}{2}I - \sum_{n=1}^{N} b_n^T b_n\right] K, \\ \Upsilon_{12} &= PLC_0, \quad \Upsilon_{13} = PL, \quad \Upsilon_{14} = PLC_1, \\ \Upsilon_{22} &= P_0(A_0 - LC_0) + (A_0 - LC_0)^T P_0 + 2\alpha P_0, \\ \Upsilon_{23} &= -P_0 L, \quad \Upsilon_{24} = -P_0 LC_1, \\ \Upsilon_{33} &= -(\mu-\alpha)\left[1 - \frac{\mu^2}{\gamma_{N+1}}\right], \quad \Upsilon_{34} = 0, \\ \Upsilon_{44} &= P_1 A_1 + A_1^T P_1 + 2\alpha P_1.\end{aligned}$$

If $\bar{\Upsilon} = \{\Upsilon_{ij}\}_{i,j=1}^3 < 0$, then Υ can be made negative by scaling $P_1 > 0$. Multiplying $\bar{\Upsilon}$ by $\mathrm{diag}\{P^{-1}, I\}$ from left and right, using the Schur complement lemma, and recalling that $Q := KP^{-1}$, we obtain that $\Phi < 0$ implies $\bar{\Upsilon} < 0$. Therefore, $\dot{V} \leq -2\alpha V$, which implies

$$V(t) \leq e^{-2\alpha t} V(0), \quad \forall t \geq 0. \tag{24}$$

C. Exponential stability

Since $P = \bar{P}^{-1} > 0$, there is $c_1 > 0$ such that

$$|\hat{w}^{N_0}(t)|^2 \leq c_1 V(t) \leq c_1 e^{-2\alpha t} V(0). \tag{25}$$

Therefore,

$$|\bar{v}(t)|^2 = |-K\hat{w}^{N_0}(t)|^2 \leq |K|^2 c_1 e^{-2\alpha t} V(0). \tag{26}$$

Relation (9) guarantees that the real parts of the eigenvalues of A_1 are $-\mu$. Therefore, (13) and (26) imply

$$|\hat{w}^{N-N_0}(t)|^2 \leq c_2 e^{-2\alpha t} V(0) \tag{27}$$

for some $c_2 > 0$. By Parseval's identity,

$$\|w_x(\cdot,t)\|^2 + \|w_t(\cdot,t)\|^2 = \sum_{n=1}^{\infty} \lambda_n w_n^2(t) + \sum_{n=1}^{\infty} \dot{w}_n^2(t).$$

Since $e^{N_0} = w^{N_0} - \hat{w}^{N_0}$ and $e^{N-N_0} = w^{N-N_0} - \hat{w}^{N-N_0}$, we have

$$\begin{aligned}\textstyle\sum_{n=1}^{N} (\lambda_n w_n^2(t) + \dot{w}_n^2(t)) &\leq \lambda_N(|w^{N_0}|^2 + |w^{N-N_0}|^2) \\ &\leq 2\lambda_N(|\hat{w}^{N_0}|^2 + |e^{N_0}|^2 + |\hat{w}^{N-N_0}|^2 + |e^{N-N_0}|^2) \\ &\leq c_3(\hat{V}(t) + V_0(t) + V_1(t) + |\hat{w}^{N-N_0}(t)|^2)\end{aligned}$$

for some $c_3 > 0$. For $c_4 = \max\{\nu_0^{-1}, \nu_1^{-1}\}$, we have

$$\begin{aligned}&\textstyle\sum_{n=N+1}^{\infty}(\lambda_n w_n^2(t) + \dot{w}_n^2(t)) \\ &\qquad \le \; c_4 \textstyle\sum_{n=N+1}^{\infty}(\nu_0\lambda_n w_n^2(t) + \nu_1\dot{w}_n^2(t)) \\ &\qquad \overset{(21)}{<} c_4 \textstyle\sum_{n=N+1}^{\infty} \lambda_n \bar{w}_n^T(t)P_n\bar{w}_n(t) = c_4 V_\infty(t).\end{aligned}$$

Summing up the above and using (24) with (27), we obtain that, for some $c_5 > 0$,

$$\begin{aligned}\|w_x(\cdot,t)\|^2 + \|w_t(\cdot,t)\|^2 &\le \max\{c_3, c_4\}(V(t) + |\hat{w}^{N-N_0}(t)|^2) \\ &\le c_5 e^{-2\alpha t}V(0). \quad (28)\end{aligned}$$

Now we upper-bound $V(0)$ in terms of $w_x(\cdot,0)$ and $w_t(\cdot,0)$. By the Schur complement lemma,

$$\lambda_n P_n < c_6 \begin{bmatrix}\lambda_n & 0\\ 0 & 1\end{bmatrix}, \quad \forall n > N, \tag{29}$$

iff $c_6 > 1$ and

$$\frac{\lambda_n}{\gamma_n} - c_6 - \frac{\lambda_n\mu^2}{\gamma_n^2(1-c_6)} < 0, \quad \forall n > N.$$

Since $\lambda_n/\gamma_n \to 1/c^2 < \infty$, the latter holds for a large enough c_6. Since $\hat{w}^{N_0}(0) = 0$ and $\hat{w}^{N-N_0}(0) = 0$, we have $e^{N_0}(0) = w^{N_0}(0)$ and $e^{N-N_0}(0) = w^{N-N_0}(0)$. Then, (29) implies

$$\begin{aligned}V(0) &\le c_7\Big[|w^{N_0}(0)|^2 + |w^{N-N_0}(0)|^2 \\ &\qquad + \sum_{n=N+1}^{\infty}(\lambda_n w_n^2(0) + \dot{w}_n^2(0))\Big] \qquad (30)\\ &\le c_8 \sum_{n=1}^{\infty}(\lambda_n w_n^2(0) + \dot{w}_n^2(0)) \\ &= c_8(\|w_x(\cdot,0)\|^2 + \|w_t(\cdot,0)\|^2)\end{aligned}$$

for some c_7 and c_8. Combining this with (28), we obtain

$$\begin{aligned}&\|w_x(\cdot,t)\|^2 + \|w_t(\cdot,t)\|^2 \\ &\qquad \le c_5c_8e^{-2\alpha t}(\|w_x(\cdot,0)\|^2 + \|w_t(\cdot,0)\|^2). \quad (31)\end{aligned}$$

Now we rewrite this estimate in terms of $z(x,t)$. Since $z(1,t) = \psi^T(1)\bar{u}(t)$, [36, Lemma 2] with $\nu = 2$ implies

$$\begin{aligned}\|z(\cdot,t)\|^2 &\le 2z^2(1,t) + \frac{8}{\pi^2}\|z_x(\cdot,t)\|^2 \\ &\le 2|\psi(1)|^2|\bar{u}(t)|^2 + \frac{8}{\pi^2}\|z_x(\cdot,t)\|^2.\end{aligned}$$

Therefore, there is $c_9 > 0$ such that

$$\|z(\cdot,t)\|_{H^1}^2 \le c_9(|\bar{u}(t)|^2 + \|z_x(\cdot,t)\|^2). \tag{32}$$

Relation (4) guarantees that the real parts of the characteristic values of (16) are $-\mu$. Therefore, (25) implies

$$\begin{aligned}|\bar{u}(t)|^2 + |\dot{\bar{u}}(t)|^2 &\le c_{10}e^{-2\alpha t}V(0) \\ &\overset{(30)}{\le} c_{11}e^{-2\alpha t}(\|w_x(\cdot,0)\|^2 + \|w_t(\cdot,0)\|^2) \quad (33)\end{aligned}$$

with some $c_{10} > 0$ and $c_{11} > 0$. Combining (31)–(33),

$$\begin{aligned}&\|z(\cdot,t)\|_{H^1} + \|z_t(\cdot,t)\|_{L^2} \\ &\quad \le c_{12}(\|z_x(\cdot,t)\| + |\bar{u}(t)|) + \|z_t(\cdot,t)\| \\ &\quad \le c_{13}(\|w_x(\cdot,t)\| + \|\psi'\||\bar{u}(t)| + \|w_t(\cdot,t)\| + \|\psi\||\dot{\bar{u}}(t)|) \\ &\quad \le Ce^{-\alpha t}(\|w_x(\cdot,0)\| + \|w_t(\cdot,0)\|) \\ &\quad = Ce^{-\alpha t}(\|z(\cdot,0)\|_{H^1} + \|z_t(\cdot,0)\|)\end{aligned}$$

for some positive c_{12}, c_{13}, and C.

D. Feasibility for large N

There exist $P > 0$ satisfying (14) and $P_0 > 0$ satisfying (12). By scaling P and P_0, we can guarantee that

$$\begin{bmatrix}P(A_0 - B_0K) + (A_0 - B_0K)^TP + 2\alpha P & \Upsilon_{12}\\ \Upsilon_{12}^T & \Upsilon_{22}\end{bmatrix} < 0.$$

Replacing P and P_0 with $\mu_0 P$ and $\mu_0 P_0$ with a small enough $\mu_0 > 0$, and P_1 with $\mu_1 P_1$ with a large enough $\mu_1 > 0$, we can guarantee that $\Upsilon < 0$ as $N \to \infty$. By continuity, it remains negative for a large enough $N > 0$. As explained above (24), $\Phi < 0$ is equivalent to $\Upsilon < 0$. Therefore, $\Phi < 0$ for a large enough $N > 0$.

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