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Robust stability and stabilization of nonlinear mechanical systems with distributed delay

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***Abstract*—The problems of stability and stabilization are addressed for a class of nonlinear mechanical systems with distributed delays. Assuming that potential and kinetic energy functions are homogeneous of different degrees, it is shown that the global asymptotic stability of the zero solution for an auxiliary delay-free nonlinear system implies the local asymptotic stability for the original model with distributed delay. The influence of additional nonlinear and time-varying perturbations is investigated using the averaging techniques. The results are obtained applying the Lyapunov–Krasovskii approach, and next extended via the Lyapunov–Razumikhin method to the case with negligible dissipation. The efficiency of the proposed theory is illustrated by solving the problem of a rigid body stabilization.**

I. Introduction

The stability analysis of nonlinear systems is a complex problem, further complicated by the presence of time-delays and time-varying perturbations [1], [2]. The rise of the internet of things and cyber-physical systems technologies has led to scenarios where these factors are present simultaneously [3]. One of the main methods for stability analysis of time-delay systems is based on Lyapunov–Krasovskii (LK) functionals [2]. Existence of such properly chosen functionals provides the necessary and sufficient conditions for stability, being extended to verify the input-to-state stability (ISS) property for systems with bounded disturbances [4], [5].

Distributed delays can arise from communication networks, the implementation of control/estimation algorithms [6], [7], or human involvement in the loop [8]. Analyzing the stability of these systems requires specialized extensions of the previously established methods [9], [10]. The complexity of the investigation increases when external perturbations are present, especially for assessing the permissible upper bounds of disturbances in relation to the delayed state. Carefully considering the time-varying nature of the perturbations can lead to less conservative bounds, where the efficiency of the averaging method has been demonstrated in dealing with periodic or almost periodic perturbations [11].

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This paper studies the stability problem for a class of mechanical systems containing the delays, or stabilized by delay-dependent control laws. Starting from earlier works, when the delays were considered as perturbations [12], [13], there are many recent results devoted to introduction of the delay in the control or estimation algorithms in order to improve the transients and robustness of the closed-loop systems [14]. For mechanical systems this is frequently related to applications of PID-controllers with variable kernels [1], [6], [15]–[17]. Moreover, during the last decades, PID-controllers with distributed delays are widely used in formation control problems, see, e.g., [2], [18], [19] and the references therein. In [20], the stability of the trivial equilibrium position of a mechanical system was studied for the delay-free case, and impact of non-stationary perturbations on the stability was analyzed using averaging. Complete-type LK functionals for homogeneous systems with distributed delay were proposed in [7], [21], where stability with respect to time-varying disturbances was investigated using the averaging methods, with application to some mechanical systems.

In this paper, we are going to continue the latter research by skipping the requirement on homogeneity of nominal system. It is assumed that kinetic and potential energy functions are homogeneous of different degrees, i.e., the total system is not homogeneous, then it is impossible to benefit all advantageous properties of homogeneous dynamics (the results of [22], [23] or [7], [21] cannot be used), and the stability analysis problem becomes more complex. New LK functionals are introduced, the robustness margins with respect to nonlinear and time-varying disturbances are evaluated. Through the Lyapunov–Razumikhin method these results are developed to dissipation-free mechanical systems described by Rayleigh equations. The efficiency of our findings is illustrated by integral control of a rigid body.

II. Preliminaries

The real numbers are denoted by $\mathbb{R}$, $\mathbb{R}_+ = \{s \in \mathbb{R} : s \geq 0\}$, and $|s|$ is an absolute value for $s \in \mathbb{R}$. Euclidean norm for a real n-dimensional vector $x \in \mathbb{R}^n$ is defined as $\|x\|$. We denote by $C([-\tau, 0], \mathbb{R}^n)$, $0 < \tau < +\infty$ the Banach space of continuous functions $\phi : [-\tau, 0] \to \mathbb{R}^n$ with the uniform norm $\|\phi\|_\tau = \sup_{-\tau \leq \varsigma \leq 0} \|\phi(\varsigma)\|$, then $C^1([-\tau, 0], \mathbb{R}^n)$ is the set of continuously differentiable functions with the uniform norm $\|\varphi\|_\tau = \sup_{\varsigma \in [-\tau,0]} (\|\varphi(\varsigma)\| + \|\dot{\varphi}(\varsigma)\|)$.

A function $f : \mathbb{R}^n \to \mathbb{R}^p$, for any integers $n, p \geq 1$, is called homogeneous with respect to standard dilation of degree $\nu \in \mathbb{R}_+$ if $f(\lambda x) = \lambda^\nu f(x)$ for all $\lambda > 0$ and $x \in \mathbb{R}^n$.

For $v \in \mathbb{R}^n$, $\mathrm{diag}\{v\}$ corresponds to a diagonal matrix with the components of vector v on the main diagonal.

The Young's inequality claims that for any $\mathfrak{a}, \mathfrak{b} \in \mathbb{R}_+$ [24]:

$$\mathfrak{a}\mathfrak{b} \leq \frac{1}{p}\mathfrak{a}^p + \frac{p-1}{p}\mathfrak{b}^{\frac{p}{p-1}}$$

for any $p > 1$, while Hölder's inequality for any $f, g : I \to \mathbb{R}$ with $I \subset \mathbb{R}$ ensures for any $p > 1$ that [2]:

$$\int_I |f(s)g(s)|ds \leq \left(\int_I |f(s)|^p ds\right)^{\frac{1}{p}} \left(\int_I |g(s)|^{\frac{p}{p-1}} ds\right)^{\frac{p-1}{p}}.$$

Lemma 1. *[22] Let $\mathfrak{a}, \mathfrak{b} \in \mathbb{R}_+$ and $\ell > 0$, $\alpha > 0$, $\beta > 0$, $\gamma > 0$, $\delta > 0$ be given, then*

$$\mathfrak{a}^\alpha + \mathfrak{b}^\beta - \ell\mathfrak{a}^\gamma\mathfrak{b}^\delta \geq 0$$

provided that $\max\{\mathfrak{a}^\alpha, \mathfrak{b}^\beta\} \leq \ell^{\frac{1}{1-\frac{\gamma}{\alpha}-\frac{\delta}{\beta}}}$ *and* $\frac{\gamma}{\alpha} + \frac{\delta}{\beta} > 1$.

The standard definitions of stability and related properties for time-delay systems can be found in [1], [2], [25], and for delay-free dynamics in [26].

III. Statement of the Problem

Consider a vector Rayleigh equation (see [27])

$$A\ddot{x}(t)+\frac{\partial W(\dot{x}(t))}{\partial \dot{x}}+\frac{\partial \Pi(x(t))}{\partial x}+\int_{t-\tau}^{t} \frac{\partial \widetilde{\Pi}(x(s))}{\partial x}ds = 0 \quad (1)$$

that models dynamics of a mechanical system, where $x(t), \dot{x}(t) \in \mathbb{R}^n$ are vectors of generalized coordinates and velocities, respectively, A is a constant, symmetric and positive definite matrix of inertial characteristics, $W : \mathbb{R}^n \to \mathbb{R}_+$ is a continuously differentiable and positive definite homogeneous of the degree $\nu+1 > 2$ function (with respect to the standard dilation, i.e., $W(\lambda\dot{x}) = \lambda^{\nu+1}W(\dot{x})$ for all $\lambda > 0$ and $\dot{x} \in \mathbb{R}^n$), $\Pi : \mathbb{R}^n \to \mathbb{R}$ and $\widetilde{\Pi} : \mathbb{R}^n \to \mathbb{R}$ are continuously differentiable homogeneous of the degree $\mu+1 > 2$ functions with respect to the standard dilation, $\tau = \mathrm{const} > 0$ is the maximal delay. Hence, the system is under the action of strongly nonlinear dissipative and potential forces, whereas the term $\int_{t-\tau}^{t} \frac{\partial \widetilde{\Pi}(x(s))}{\partial x}ds$ can be interpreted as integral part of a modified PID-controller [28], [29]. It is worth noting that, in numerous models of mechanical systems, the forces of viscous friction and restoring forces are approximated by homogeneous functions with degrees higher than one (see, e.g., [30]–[32] and the references therein), whereas PID-controllers are often used for smoothing transients and vibration suppression [1], [28].

Assume that initial functions for solutions of (1) belong to the space $C^1([-\tau, 0], \mathbb{R}^n)$. Denote by x_t the restriction of a solution $x(t)$ to the segment $[t-\tau, t]$, i.e., $x_t : \xi \mapsto x(t+\xi)$ for $\xi \in [-\tau, 0]$.

The system (1) admits the equilibrium position

$$x = \dot{x} = 0. \quad (2)$$

We will look for conditions ensuring the asymptotic stability of this equilibrium position. Our study will be based on the Lyapunov direct method and constructions of special complete-type LK functionals. Furthermore, the impact of nonlinear time-varying perturbations on the stability of the equilibrium position (2) will be analyzed. In the case where the considered perturbations admit zero mean values, an original technique for the application of the averaging method will be proposed. In addition, with the aid of the developed tools, new conditions for the monoaxial stabilization of a rigid body will be derived.

IV. Stability of Nominal System

Our results are based on the following standard hypothesis:

Assumption 1. *Let the function $\Pi(x) + \tau\widetilde{\Pi}(x)$ be positive definite for all $x \in \mathbb{R}^n$.*

This assumption introduces restrictions on admissible values of the delay τ, and it can be verified using the upper or lower bounds on the delay (the value of τ can be uncertain). It is well known [33] that, under Assumption 1, the equilibrium position (2) of the delay-free counterpart of (1):

$$A\ddot{x}(t) + \frac{\partial W(\dot{x}(t))}{\partial \dot{x}} + \frac{\partial \Pi(x(t))}{\partial x} + \tau\frac{\partial \widetilde{\Pi}(x(t))}{\partial x} = 0,$$

which is obtained after replacing $x(s)$ by $x(t)$, is asymptotically stable.

Theorem 1. *Let Assumption 1 be fulfilled. If*

$$\nu < \frac{3\mu - 1}{\mu + 1}, \quad (3)$$

then the equilibrium position (2) *of the system* (1) *is asymptotically stable.*

Proof. In [34], it was suggested to construct a Lyapunov function for the delay-free system in the form of a sum of the complete energy with an auxiliary cross term:

$$V_1(x, \dot{x}) = \frac{1}{2}\dot{x}^\top A\dot{x} + \varepsilon\|x\|^{\sigma-1}x^\top A\dot{x} + \Pi(x) + \tau\widetilde{\Pi}(x),$$

where $\varepsilon > 0$ and $\sigma \geq 1$ are parameters. Differentiating V_1 along the solutions of (1), we obtain

$$\begin{aligned}\dot{V}_1 = &-\dot{x}^\top(t)\frac{\partial W(\dot{x}(t))}{\partial \dot{x}} - \dot{x}^\top(t)\int_{t-\tau}^{t} \frac{\partial \widetilde{\Pi}(x(s))}{\partial x}ds \\ &+\tau\dot{x}^\top(t)\frac{\partial \widetilde{\Pi}(x(t))}{\partial x} + \varepsilon\dot{x}^\top(t)A\frac{\partial}{\partial x}\left(\|x(t)\|^{\sigma-1}x(t)\right)\dot{x}(t) \\ &-\varepsilon\|x(t)\|^{\sigma-1}x^\top(t)\bigg(\frac{\partial W(\dot{x}(t))}{\partial \dot{x}} + \frac{\partial \Pi(x(t))}{\partial x} \\ &+ \int_{t-\tau}^{t} \frac{\partial \widetilde{\Pi}(x(s))}{\partial x}ds\bigg).\end{aligned}$$

Using the properties of homogeneous functions (see [30]), we arrive at the inequality

$$\dot V_1 \le -a_1\|\dot x(t)\|^{\nu+1} - \big(\dot x(t)+\varepsilon\|x(t)\|^{\sigma-1}x(t)\big)^\top \times \int_{t-\tau}^{t} \frac{\partial\widetilde\Pi(x(s))}{\partial x}ds + \tau\dot x^\top(t)\frac{\partial\widetilde\Pi(x(t))}{\partial x} +\varepsilon a_2\|\dot x(t)\|^2\|x(t)\|^{\sigma-1}+\varepsilon a_3\|x(t)\|^{\sigma}\|\dot x(t)\|^{\nu} -\varepsilon(\mu+1)\|x(t)\|^{\sigma-1}\Pi(x(t)),$$

where a_1,a_2,a_3 are positive coefficients.

Next, according to the approach developed in [7], choose a LK functional candidate as follows:

$$V_2(x_t) = V_1(x(t),\dot x(t))+\int_{t-\tau}^{t}(\lambda+\beta(s+\tau-t))\|x(s)\|^{\mu+\sigma}ds -\big(\dot x(t)+\varepsilon\|x(t)\|^{\sigma-1}x(t)\big)^\top\int_{t-\tau}^{t}(s+\tau-t)\frac{\partial\widetilde\Pi(x(s))}{\partial x}ds, \tag{4}$$

where parameters $\lambda,\beta>0$. The direct computations show:

$$b_1\|\dot x(t)\|^2 + b_2\|x(t)\|^{\mu+1} + \lambda\int_{t-\tau}^{t}\|x(s)\|^{\mu+\sigma}ds -b_4\big(\|\dot x(t)\|+\varepsilon\|x(t)\|^{\sigma}\big)\int_{t-\tau}^{t}\|x(s)\|^{\mu}ds -\varepsilon b_3\|x(t)\|^{\sigma}\|\dot x(t)\| \le V_2(x_t)\le b_5\|\dot x(t)\|^2 +b_6\|x(t)\|^{\mu+1}+\varepsilon b_3\|x(t)\|^{\sigma}\|\dot x(t)\| +b_4\big(\|\dot x(t)\|+\varepsilon\|x(t)\|^{\sigma}\big)\int_{t-\tau}^{t}\|x(s)\|^{\mu}ds +(\lambda+\beta\tau)\int_{t-\tau}^{t}\|x(s)\|^{\mu+\sigma}ds,$$

$$\dot V_2 \le -a_1\|\dot x(t)\|^{\nu+1} - \varepsilon(\mu+1)\|x(t)\|^{\sigma-1}(\Pi(x(t)) +\tau\widetilde\Pi(x(t))) + a_4\bigg(\|\dot x(t)\|^{\nu}+\varepsilon\|x(t)\|^{\sigma-1}\|\dot x(t)\| + \int_{t-\tau}^{t}\|x(s)\|^{\mu}ds + \|x(t)\|^{\mu}\bigg)\int_{t-\tau}^{t}\|x(s)\|^{\mu}ds +\varepsilon a_2\|\dot x(t)\|^2\|x(t)\|^{\sigma-1}+\varepsilon a_3\|x(t)\|^{\sigma}\|\dot x(t)\|^{\nu} +(\lambda+\tau\beta)\|x(t)\|^{\sigma+\mu} - \lambda\|x(t-\tau)\|^{\sigma+\mu} - \beta\int_{t-\tau}^{t}\|x(s)\|^{\sigma+\mu}ds$$
$$\le -a_1\|\dot x(t)\|^{\nu+1} - (\varepsilon a_5-\lambda-\tau\beta)\|x(t)\|^{\sigma+\mu} -\beta\int_{t-\tau}^{t}\|x(s)\|^{\sigma+\mu}ds + a_4\bigg(\|\dot x(t)\|^{\nu}+\varepsilon\|x(t)\|^{\sigma-1}\|\dot x(t)\| + \int_{t-\tau}^{t}\|x(s)\|^{\mu}ds+\|x(t)\|^{\mu}\bigg)\int_{t-\tau}^{t}\|x(s)\|^{\mu}ds +\varepsilon a_2\|\dot x(t)\|^2\|x(t)\|^{\sigma-1}+\varepsilon a_3\|x(t)\|^{\sigma}\|\dot x(t)\|^{\nu}$$

where $a_4,a_5,b_1,b_2,b_3,b_4,b_5,b_6$ are suitable positive constants. Let $4(\lambda+\tau\beta)<\varepsilon a_5$, then with the aid of Lemma 1, Young's and Hölder's inequalities, it is straightforward to verify that if

$$\max\left\{\frac{\mu}{\nu};\ \frac{(\mu+1)(\nu+1)}{2}-\mu\right\}\le\sigma<\mu \tag{5}$$

and ε is sufficiently small, then there is $\delta>0$ such that

$$\frac12\left(b_1\|\dot x(t)\|^2+b_2\|x(t)\|^{\mu+1}+\lambda\int_{t-\tau}^{t}\|x(s)\|^{\mu+\sigma}ds\right)\le V_2(x_t) \le 2\left(b_5\|\dot x(t)\|^2+b_6\|x(t)\|^{\mu+1}+(\lambda+\beta\tau)\int_{t-\tau}^{t}\|x(s)\|^{\mu+\sigma}ds\right), \tag{6}$$

$$\dot V_2 \le -\frac12\left(a_1\|\dot x(t)\|^{\nu+1}+\varepsilon a_5\|x(t)\|^{\sigma+\mu}+\beta\int_{t-\tau}^{t}\|x(s)\|^{\sigma+\mu}ds\right) \tag{7}$$

for $\|\dot x(t)\|^{\nu+1}+\|x(t)\|^{\sigma+\mu}+\int_{t-\tau}^{t}\|x(s)\|^{\sigma+\mu}ds<\delta$. Hence [35], (4) is a complete-type LK functional for (1) guaranteeing the asymptotic stability of the equilibrium position (2).

To complete the proof, it should be noted that, for the existence of a number σ satisfying (5), it is necessary and sufficient the fulfillment of (3). □

In the case of homogeneous dynamics (1), $\nu=\frac{2\mu}{\mu+1}$ and the restriction (3) is verified [7] (another LK functional can be used).

Extending the proof of Theorem 1 the following estimate on the decay of solutions in (1) is derived:

Corollary 1. *Let all conditions of Theorem 1 be satisfied, then there exist $\varkappa_1,\varkappa_2,\varkappa_3>0$ such that*

$$\|\dot x(t)\|^2+\|x(t)\|^{\mu+1}\le\frac{\varkappa_1}{(1+\varkappa_2 t)^{\frac{1}{\wp-1}}}$$

for all $t\ge0$ and $V_2(x_0)\le\varkappa_3$, where $\wp=\frac{\nu+1}{2}\max\left\{1,\frac{\mu}{\nu}\frac{2}{\mu+1}\right\}$ and V_2 is given in (4).

Proof. The proof follows (6), (7) under the restriction $V_2(x_0)\le\varkappa_3$ with a sufficiently small $\varkappa_3$, which can be rewritten as $\dot V_2\le-\varkappa V_2^{\wp}(x_t)$ for some $\varkappa$, then the required time estimate follows by a comparison principle. □

V. Robust stability conditions

Along with (1), consider the associated perturbed system

$$A\ddot x(t)+\frac{\partial W(\dot x(t))}{\partial\dot x}+\frac{\partial\Pi(x(t))}{\partial x} + \int_{t-\tau}^{t}\frac{\partial\widetilde\Pi(x(s))}{\partial x}ds=\int_{t-\tau}^{t}G(s,x(s),\dot x(s))ds, \tag{8}$$

where a continuous vector function $G(t,x,\dot x)$ that represents the disturbances is defined for

$$t\ge-\tau,\quad \|x\|<H,\quad \|\dot x\|<H\quad(0<H\le+\infty). \tag{9}$$

Remark 1. The term $\int_{t-\tau}^{t}G(s,x(s),\dot x(s))ds$ may characterize control deviations in the integral part of a modified PID-controller due to external perturbations.

We look for conditions under which the following class of perturbations does not disturb the asymptotic stability:

Assumption 2. *The upper bound $\|G(t,x,\dot x)\|\le c\|x\|^{\rho}$ holds in the domain* (9)*, where $c>0$ and $\rho>0$.*

This condition can be verified if G is homogeneous of degree ρ in the variable x and bounded in t and $\dot x$.

Theorem 2. *Let assumptions 1, 2 and the inequality* (3) *be fulfilled. If*

$$\rho > \max\left\{\mu;\ \frac{\nu(\mu+1)}{2}\right\}, \tag{10}$$

then the equilibrium (2) *of* (8) *is asymptotically stable.*

Proof. Differentiating (4) along solutions of (8), we obtain

$$\begin{aligned}\dot V_2 = \Psi(x_t) - \left(\int_{t-\tau}^{t}(s+\tau-t)\frac{\partial\widetilde\Pi(x(s))}{\partial x}ds\right)^{\top}\\ \times A^{-1}\int_{t-\tau}^{t}G(s,x(s),\dot x(s))ds\\ +\big(\dot x(t)+\varepsilon\|x(t)\|^{\sigma-1}x(t)\big)^{\top}\int_{t-\tau}^{t}G(s,x(s),\dot x(s))ds,\end{aligned}$$

where $\Psi(x_t)$ denotes the derivative of (4) along the solutions of (1). Hence,

$$\begin{aligned}\dot V_2 \le -a_1\|\dot x(t)\|^{\nu+1} - (\varepsilon a_5 - \lambda - \tau\beta)\|x(t)\|^{\sigma+\mu}\\ -\beta\int_{t-\tau}^{t}\|x(s)\|^{\sigma+\mu}ds + a_4\bigg(\|\dot x(t)\|^{\nu} + \varepsilon\|x(t)\|^{\sigma-1}\|\dot x(t)\|\\ + \int_{t-\tau}^{t}\|x(s)\|^{\mu}ds + \|x(t)\|^{\mu}\bigg)\int_{t-\tau}^{t}\|x(s)\|^{\mu}ds\\ +\varepsilon a_2\|\dot x(t)\|^2\|x(t)\|^{\sigma-1} + \varepsilon a_3\|x(t)\|^{\sigma}\|\dot x(t)\|^{\nu}\\ +c\big(\|\dot x(t)\| + \varepsilon\|x(t)\|^{\sigma}\big)\int_{t-\tau}^{t}\|x(s)\|^{\rho}ds\\ +\bar a\int_{t-\tau}^{t}\|x(s)\|^{\rho}ds\int_{t-\tau}^{t}\|x(s)\|^{\mu}ds\end{aligned}$$

under (9), where $\bar a = \mathrm{const} > 0$. Similarly to the proof of Theorem 1 it can be shown that if (5) is valid and

$$\rho > \frac{\nu(\sigma+\mu)}{\nu+1},$$

then, for an appropriate choice of parameters $\lambda, \beta, \varepsilon, \delta$, the functional (4) and its derivative with respect to (8) satisfy estimates (6), (7) for $\|\dot x(t)\|^{\nu+1} + \|x(t)\|^{\sigma+\mu} + \int_{t-\tau}^{t}\|x(s)\|^{\sigma+\mu}ds < \delta$. To obtain the largest domain of admissible values of ρ one should take $\sigma = \max\{\mu/\nu;\ (\mu+1)(\nu+1)/2-\mu\}$. As a result, we arrive at (10), and this completes the proof. □

The system (1) can be treated as a nonlinear approximation for (8), and the condition (10) determines how much the order of the perturbations should be greater than those of the functions in the original equations to guarantee the preservation of the asymptotic stability.

VI. Stability analysis via averaging

Next, consider the case when the perturbed system has the form

$$A\ddot x(t) + \frac{\partial W(\dot x(t))}{\partial \dot x} + \frac{\partial \Pi(x(t))}{\partial x} + \int_{t-\tau}^{t}\frac{\partial\widetilde\Pi(x(s))}{\partial x}ds = \int_{t-\tau}^{t}D(s)Q(x(s))ds, \tag{11}$$

where the matrix $D(t)\in\mathbb{R}^{n\times n}$ is continuous and bounded for $t\in[-\tau,+\infty)$, components of a continuously differentiable vector function $Q:\mathbb{R}^n\to\mathbb{R}^n$ are homogeneous of the degree $\rho\ge 1$ with respect to the standard dilation.

Assumption 3. *Let*

$$\frac{1}{T}\int_{t}^{t+T}D(u)du \to 0 \quad \text{as} \quad T\to+\infty$$

uniformly with respect to $t\ge 0$.

Remark 2. From Assumption 3 it follows that the mean values of entries of the matrix $D(t)$ are equal to zero.

Let us note that (11) is a special case of the system (8), and under the introduced restrictions Assumption 2 is verified for (11). Therefore, Theorem 2 provides us sufficient conditions of the asymptotic stability for (11). However, we will show that, taking into account Assumption 3 and applying the averaging method, less conservative stability conditions can be derived than those formulated in Theorem 2:

Theorem 3. *Let assumptions 1, 3 and the inequality* (3) *be fulfilled. If*

$$\rho \ge \max\left\{\mu;\ \frac{\nu(\mu+1)}{2}\right\}, \tag{12}$$

then the equilibrium (2) *of* (11) *is asymptotically stable.*

Proof. Using the approaches developed in [7], [21], construct the following LK functional for (11):

$$\begin{aligned}V_3(t,x_t) = V_2(x_t) + \big(\dot x(t)+\varepsilon\|x(t)\|^{\sigma-1}x(t)\big)^{\top}\\ \times\int_{t-\tau}^{t}(s+\tau-t)D(s)Q(x(s))ds\\ -\tau\big(\dot x(t)+\varepsilon\|x(t)\|^{\sigma-1}x(t)\big)^{\top}\int_{0}^{t}e^{\alpha(u-t)}D(u)du\,Q(x(t)),\end{aligned}$$

where α is a positive parameter and the functional $V_2(x_t)$ is defined by the formula (4) (as we will demonstrate below, this functional is locally positive definite for small enough α under introduced restrictions on D). Differentiating the functional V_3 along the solutions of (11), we obtain

$$\begin{aligned}\dot V_3 = \Psi(x_t) + \alpha\tau\big(\dot x(t)\\ +\varepsilon\|x(t)\|^{\sigma-1}x(t)\big)^{\top}\int_{0}^{t}e^{\alpha(u-t)}D(u)du\,Q(x(t))\\ -\tau\big(\dot x(t)+\varepsilon\|x(t)\|^{\sigma-1}x(t)\big)^{\top}\\ \times\int_{0}^{t}e^{\alpha(u-t)}D(u)du\,\frac{\partial Q(x(t))}{\partial x}\dot x(t)\\ -\left(\int_{t-\tau}^{t}(s+\tau-t)\frac{\partial\widetilde\Pi(x(s))}{\partial x}ds\right)^{\top}A^{-1}\\ \times\int_{t-\tau}^{t}D(s)Q(x(s))ds + \bigg(\varepsilon\frac{\partial}{\partial x}\left(\|x(t)\|^{\sigma-1}x(t)\right)\dot x(t)\end{aligned}$$

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$$-A^{-1}\Big(\frac{\partial W(\dot{x}(t))}{\partial \dot{x}}+\frac{\partial \Pi(x(t))}{\partial x}+\int_{t-\tau}^{t}\frac{\partial \widetilde{\Pi}(x(s))}{\partial x}ds$$
$$-\int_{t-\tau}^{t}D(s)Q(x(s))ds\Big)\Bigg)^{\top}\left(\int_{t-\tau}^{t}(s+\tau-t)D(s)Q(x(s))ds\right.$$
$$\left.-\tau\int_{0}^{t}e^{\alpha(u-t)}D(u)du\,Q(x(t))\right),$$

where $\Psi(x_t)$ is the derivative of $V_2(x_t)$ along the solutions of (1). Hence, the estimates

$$b_1\|\dot{x}(t)\|^2+b_2\|x(t)\|^{\mu+1}-\varepsilon b_3\|x\|^{\sigma}\|\dot{x}(t)\|$$
$$+\lambda\int_{t-\tau}^{t}\|x(s)\|^{\mu+\sigma}ds-b_4\big(\|\dot{x}(t)\|+\varepsilon\|x(t)\|^{\sigma}\big)$$
$$\times\left(\int_{t-\tau}^{t}\|x(s)\|^{\mu}ds+\int_{t-\tau}^{t}\|x(s)\|^{\rho}ds+\frac{1}{\alpha}\|x(t)\|^{\rho}\right)$$
$$\le V_3(t,x_t)\le b_5\|\dot{x}(t)\|^2+b_6\|x(t)\|^{\mu+1}+\varepsilon b_3\|x\|^{\sigma}\|\dot{x}(t)\|$$
$$+(\lambda+\beta\tau)\int_{t-\tau}^{t}\|x(s)\|^{\mu+\sigma}ds+b_4\big(\|\dot{x}(t)\|+\varepsilon\|x(t)\|^{\sigma}\big)$$
$$\times\left(\int_{t-\tau}^{t}\|x(s)\|^{\mu}ds+\int_{t-\tau}^{t}\|x(s)\|^{\rho}ds+\frac{1}{\alpha}\|x(t)\|^{\rho}\right),$$

$$\dot{V}_3\le -a_1\|\dot{x}(t)\|^{\nu+1}+\varepsilon a_2\|\dot{x}(t)\|^2\|x(t)\|^{\sigma-1}$$
$$+\varepsilon a_3\|x(t)\|^{\sigma}\|\dot{x}(t)\|^{\nu}+a_4\Bigg(\|\dot{x}(t)\|^{\nu}+\varepsilon\|x(t)\|^{\sigma-1}\|\dot{x}(t)\|$$
$$+\|x(t)\|^{\mu}+\int_{t-\tau}^{t}\|x(s)\|^{\mu}ds+\int_{t-\tau}^{t}\|x(s)\|^{\rho}ds\Bigg)$$
$$\times\int_{t-\tau}^{t}\|x(s)\|^{\mu}ds-(\varepsilon a_5-\lambda-\tau\beta)\|x(t)\|^{\sigma+\mu}$$
$$-\beta\int_{t-\tau}^{t}\|x(s)\|^{\sigma+\mu}ds+\alpha a_6\left\|\int_{0}^{t}e^{\alpha(u-t)}D(u)du\right\|$$
$$\times\big(\|\dot{x}(t)\|+\varepsilon\|x(t)\|^{\sigma}\big)\|x(t)\|^{\rho}$$
$$+\frac{a_7}{\alpha}\big(\|\dot{x}(t)\|+\varepsilon\|x(t)\|^{\sigma}\big)\|x(t)\|^{\rho-1}\|\dot{x}(t)\|$$

$$+a_8\left(\int_{t-\tau}^{t}\|x(s)\|^{\rho}ds+\frac{1}{\alpha}\|x(t)\|^{\rho}\right)\Bigg(\|\dot{x}(t)\|^{\nu}$$
$$+\varepsilon\|x(t)\|^{\sigma-1}\|\dot{x}(t)\|+\int_{t-\tau}^{t}\|x(s)\|^{\mu}ds$$
$$+\|x(t)\|^{\mu}+\int_{t-\tau}^{t}\|x(s)\|^{\rho}ds\Bigg)$$

hold, where $a_j>0$, $b_k>0$, $j=1,\dots,8$, $k=1,\dots,6$.

It is known (see [11]) that, under Assumption 3,

$$\alpha\left\|\int_{0}^{t}e^{\alpha(u-t)}D(u)du\right\|\to 0\quad\text{as}\quad\alpha\to 0$$

uniformly with respect to $t\in[0,+\infty)$. Taking into account this result, in a similar way as in the proof of Theorem 1, it can be verified that if the condition (5) holds and

$$\rho\ge\max\left\{\mu;\ \frac{\nu(\sigma+\mu)}{\nu+1}\right\},\tag{13}$$

then, for sufficiently small values of tuning parameters $\alpha,\lambda,\beta,\varepsilon$, there exists a number $\delta>0$ such that estimates of the form (6) and (7) are verified for the functional V_3 (we need to substitute there V_2 by V_3) for $\|\dot{x}(t)\|^{\nu+1}+\|x(t)\|^{\sigma+\mu}+\int_{t-\tau}^{t}\|x(s)\|^{\sigma+\mu}ds<\delta$.

To complete the proof, it should be noted that (13) defines the largest domain of admissible values of ρ in the case where $\sigma=\max\{\mu/\nu;\ (\mu+1)(\nu+1)/2-\mu\}$. As a result, we arrive at (12). □

Remark 3. In comparison with Theorem 2, Theorem 3 guarantees the preservation of the asymptotic stability in the case where the strict inequality (10) is replaced by the non-strict one (12). Moreover, if $\nu\le 2\mu/(\mu+1)$, then from Theorem 3 it follows that the asymptotic stability take place for $\rho=\mu$, i.e., for perturbations whose order coincides with that of positional forces in the original system. It is worth noting that we assume that the matrix $D(t)$ is bounded, but we do not impose any constraints on the magnitudes of entries of the matrix.

Next, consider the case where, instead of Assumption 3, the following condition is fulfilled:

Assumption 4. *Let* $\left\|\int_0^t D(u)du\right\|\le M$ *for* $t\ge 0$, *where* M *is a positive constant.*

For instance, this assumption is satisfied if entries of $D(t)$ are periodic functions with zero mean values.

It should be noted that, under Assumption 4, while constructing the functional V_3, one can take $\alpha=0$. Then, similarly to the proof of Theorem 3, we obtain the result:

Corollary 2. *Let assumptions 1, 4 and the inequality* (3) *be fulfilled. If*

$$\rho>\frac{\nu+1}{4}\max\left\{\mu+1;\ \frac{2\mu}{\nu}\right\},\tag{14}$$

then the equilibrium (2) *of* (11) *is asymptotically stable.*

Corollary 2 ensures the asymptotic stability even in the case when $\rho<\mu$.

Example 1. Consider the system

$$\ddot{x}(t)+a\|\dot{x}(t)\|^{1/2}\dot{x}(t)-\begin{pmatrix}x_1^3(t)\\x_2^3(t)\\x_3^3(t)\end{pmatrix}=U(t),$$

where $x(t)\in\mathbb{R}^3$, $a=\text{const}>0$, $U(t)\in\mathbb{R}^3$ is a control vector. It is known (see [27]) that if $U\equiv 0$, then the equilibrium position (2) of this system is unstable. Let

$$U=-b\int_{t-\tau}^{t}\|x(s)\|^2x(s)ds,$$

where $b>0$, $\tau>0$. Then we arrive at the closed-loop system in the form (1):

$$\ddot{x}(t)+a\sqrt{\|\dot{x}(t)\|}\dot{x}(t)-\begin{bmatrix}x_1^3(t)\\x_2^3(t)\\x_3^3(t)\end{bmatrix}+b\int_{t-\tau}^{t}\|x(s)\|^2x(s)ds=0,$$

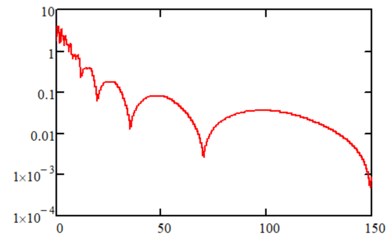

Figure 1. Behavior of $\|x(t)\| + \|\dot{x}(t)\|$ in logarithmic scale, $t \in [0, 150]$

where $A = I$, $W(\dot{x}) = 2a\|\dot{x}\|^{5/2}/5$, $\Pi(x) = -(x_1^4 + x_2^4 + x_3^4)/4$, $\widetilde{\Pi}(x) = b\|x\|^4/4$, $\nu = 3/2$, $\mu = 3$. In this case, the inequality (3) is satisfied. Consider the function

$$\Pi(x) + \tau\widetilde{\Pi}(x) = \frac{1}{4}\left(\tau b\|x\|^4 - (x_1^4 + x_2^4 + x_3^4)\right),$$

which is positive definite if and only if $\tau b > 1$. Hence, if the delay τ is sufficiently large, then the equilibrium position (2) of the system is asymptotically stable.

The results of simulation of the closed-loop system are shown in Fig. 1 for $a = 1$, $b = 5$ and $\tau = 0.5$ (the explicit Euler method was used with step $\Delta t = 0.01$).

VII. Analysis without velocity-dependent terms

Our aim here is to extend the proposed approaches to the case of mechanical systems without dissipation.

Consider a variant of the system (11) without velocity-dependent terms (i.e., with $\frac{\partial W(\dot{x}(t))}{\partial \dot{x}} = 0$):

$$A\ddot{x}(t) + \frac{\partial \Pi(x(t))}{\partial x} + \int_{t-\tau}^{t} \frac{\partial \widetilde{\Pi}(x(s))}{\partial x} ds = 0, \tag{15}$$

where as before $x(t), \dot{x}(t) \in \mathbb{R}^n$ are vectors of generalized coordinates and velocities, respectively, A is a constant, symmetric and positive definite matrix of inertial characteristics, $\Pi(x)$ and $\widetilde{\Pi}(x)$ are twice continuously differentiable for $x \in \mathbb{R}^n$ homogeneous of the degree $\mu + 1 > 2$ functions with respect to the standard dilation, $\tau = \text{const} > 0$. The system (15) admits the equilibrium position (2).

We will look for conditions ensuring the asymptotic stability of the equilibrium position. The model (15) corresponds to the case when the system itself has no dissipation (or it is almost negligible), and the control utilizes only the measurements of position x.

Note that under Assumption 1, the delay-free system will be just stable, but not asymptotically stable, and the previously used approach cannot be applied directly. Analysis of this kind of systems was done in [14], [36], [37] for the pointwise delay case, and here we will consider the distributed one.

Applying the Mean value theorem, let us rewrite the system (15) as follows:

$$A\ddot{x}(t) + \frac{\partial \Pi(x(t))}{\partial x} + \tau\frac{\partial \widetilde{\Pi}(x(t))}{\partial x} - \frac{\tau^2}{2}\frac{\partial^2 \widetilde{\Pi}(x(t))}{\partial x^2}\dot{x}(t) + \int_{t-\tau}^{t} \Delta(s,t)\,(s-t)ds = 0, \tag{16}$$

where $\Delta(s,t) = \frac{\partial^2 \widetilde{\Pi}(x(t+\vartheta(s,t)(s-t)))}{\partial x^2}\dot{x}(t + \vartheta(s,t)(s-t)) - \frac{\partial^2 \widetilde{\Pi}(x(t))}{\partial x^2}\dot{x}(t)$ for $\vartheta(s,t) \in (0,1)^n$ (for brevity of presentation we use the notation $x(t+\vartheta(s,t)(s-t)) = [x_1(t+\vartheta_1(s,t)(s-t)) \dots x_n(t+\vartheta_n(s,t)(s-t))]^\top$).

Theorem 4. *Let Assumption 1 be verified, and the matrix $\partial^2\widetilde{\Pi}(x)/\partial x^2$ be negative definite for any $x \in \mathbb{R}^n \setminus \{0\}$. Then the equilibrium position (2) of the system (15) is asymptotically stable.*

Proof. The proof follows the main steps of [37], starting with observation that the delay-free version of (16) is in the Liénard form with known Lyapunov function, then the Lyapunov–Razumikhin approach can be applied with taking care of the multipliers $\int_{t-\tau}^{t}(s-t)ds$. □

A similar problem for linear mechanical systems having a distributed delay with variable kernels was solved in [16], [17], and the stability is achieved by imposing restrictions on the kernels. Moreover, for constant kernels, as in Theorem 4, the approach of that papers cannot be applied.

VIII. Applications

In the present section we apply the new theoretical outcomes to a problem of the attitude control of a rigid body.

Assume that a rigid body rotates around its mass center O with angular velocity $\omega(t) \in \mathbb{R}^3$. Let the axes $Oxyz$ be principal central axes of inertia of the body. The Euler equations modeling the attitude motion of the body under the action of a control torque $M(t) \in \mathbb{R}^3$ have the form

$$J\dot{\omega}(t) + \omega(t) \times (J\omega(t)) = M(t), \tag{17}$$

where $J = \text{diag}\{[A_1\ A_2\ A_3]^\top\} \in \mathbb{R}_+^{3\times3}$ is a body inertia tensor in the axes $Oxyz$ [38], [39].

Let two unit vectors $\eta(t) \in \mathbb{R}^3$ and $r \in \mathbb{R}^3$ be given. The vector $\eta(t)$ is constant in the inertial space, whereas the vector r is constant in the body-fixed frame. Then vector $\eta(t)$ rotates with respect to the system $Oxyz$ with the angular velocity $-\omega(t)$, hence,

$$\dot{\eta}(t) = -\omega(t) \times \eta(t). \tag{18}$$

As a result, we obtain the system composed by the Euler dynamic equations (17) and the Poisson kinematic equations (18).

Consider the problem of monoaxial stabilization of the body [39]. It is required to design control torque $M(t)$ providing the existence and the asymptotic stability of the equilibrium position

$$\omega = 0, \qquad \eta = r \tag{19}$$

for the corresponding closed-loop system.

It is worth mentioning that this problem is of significant practical importance due to its applications in control of Earth-pointing satellites, space missions with telescopes, remote sensing, etc., see [38], [40], [41]. It is known [20], [39] that the desired control torque can be chosen as follows

$$M = -\frac{\partial W(\omega)}{\partial \omega} - p\|\eta - r\|^{\mu-1}\eta \times r,$$

where $W(\omega)$ is a continuously differentiable for $\omega \in \mathbb{R}^3$ positive definite homogeneous of the degree $\nu + 1 \geq 2$ function with respect to the standard dilation, $p = \text{const} > 0$, $\mu \geq 1$. In [20], the impact of time-varying perturbations with zero mean values on the stability of the equilibrium position (19) was analyzed. In this paper, in line with disturbances, we will take into account terms with distributed delay in control and perturbed torques.

Consider the case where the closed-loop Euler equations can be rewritten as follows:

$$J\dot{\omega}(t) + \omega(t) \times (J\omega(t)) = -\frac{\partial W(\omega(t))}{\partial \omega}$$
$$-a\|\eta(t) - r\|^{\mu-1}\eta(t) \times r - b\int_{t-\tau}^{t} \|\eta(s) - r\|^{\mu-1}\eta(s) \times r ds$$
$$+\int_{t-\tau}^{t} D(s)Q(\eta(s) - r)ds, \qquad (20)$$

where a, b are constant coefficients, $W(\omega)$ is a continuously differentiable for $\omega \in \mathbb{R}^3$ positive definite homogeneous of the degree $\nu + 1 > 2$ function with respect to the standard dilation, the matrix $D(t) \in \mathbb{R}^{3\times3}$ is continuous and bounded for $t \in [-\tau, +\infty)$, components of the vector function $Q(\zeta)$ are continuously differentiable for $\zeta \in \mathbb{R}^3$ homogeneous of the order $\rho \geq 1$ functions with respect to the standard dilation, $\tau > 0$ is the delay, $\mu > 1$. In such a case, the terms that are given in the second line represent the delay-dependent PID-like part of the control and a delayed nonlinear perturbation, respectively.

Assumption 5. *Let* $a + b\tau > 0$.

Theorem 5. *If assumptions 3, 5 and the inequalities* (3), (12) *are fulfilled, then the equilibrium position* (19) *of the system* (18), (20) *is asymptotically stable.*

Proof. On the basis of approaches developed in the previous sections, construct a LK functional for (18), (20) in the form

$$\widetilde{V}(t, \omega(t), \eta_t) = \frac{1}{2}\omega^\top(t)J\omega(t) + \frac{a + b\tau}{\mu + 1}\|\eta(t) - r\|^{\mu+1}$$
$$+\varepsilon\|\eta(t) \times r\|^{\sigma-1}\omega^\top(t)J(\eta(t) \times r)$$
$$-b\big(\omega(t) + \varepsilon\|\eta(t) \times r\|^{\sigma-1}(\eta(t) \times r)\big)^\top$$
$$\cdot\int_{t-\tau}^{t} (s + \tau - t)\|\eta(s) - r\|^{\mu-1}\eta(s) \times r ds$$
$$+\big(\omega(t) + \varepsilon\|\eta(t) \times r\|^{\sigma-1}(\eta(t) \times r)\big)^\top$$
$$\cdot\int_{t-\tau}^{t} (s + \tau - t)D(s)Q(\eta(s) - r)ds$$
$$-\tau\big(\omega(t) + \varepsilon\|\eta(t) \times r\|^{\sigma-1}(\eta(t) \times r)\big)^\top$$
$$\cdot\int_{0}^{t} e^{\alpha(u-t)}D(u)du\, Q(\eta(t) - r)$$
$$+\int_{t-\tau}^{t} (\lambda + \beta(s + \tau - t))\|\eta(s) - r\|^{\mu+\sigma}ds,$$

where $\varepsilon, \alpha, \beta, \lambda$ are positive tuning parameters, $\sigma \geq 1$. In a similar way as in the proof of Theorem 3, it can be verified that if the inequality (13) holds then, for an appropriate choice of values of $\varepsilon, \alpha, \beta, \lambda, \sigma$, there exist positive constants δ, c_1, c_2, c_3 such that

$$c_1\Big(\|\omega(t)\|^2 + \|\eta(t) - r\|^{\mu+1} + \int_{t-\tau}^{t} \|\eta(s)$$
$$-r\|^{\mu+\sigma}ds\Big) \leq \widetilde{V}(t, \omega(t), \eta_t) \leq c_2\Big(\|\omega(t)\|^2 + \|\eta(t) - r\|^{\mu+1}$$
$$+\int_{t-\tau}^{t} \|\eta(s) - r\|^{\mu+\sigma}ds\Big)$$
$$\dot{\widetilde{V}} \leq -c_3\Big(\|\omega(t)\|^{\nu+1} + \|\eta(t) - r\|^{\mu+\sigma}$$
$$+\int_{t-\tau}^{t} \|\eta(s) - r\|^{\mu+\sigma}ds\Big)$$

for $\|\omega(t)\|^2 + \|\eta(t) - r\|^{\mu+1} + \int_{t-\tau}^{t} \|\eta(s) - r\|^{\mu+\sigma}ds < \delta$.

Corollary 3. *If assumptions 4, 5 and the inequalities* (3), (14) *are fulfilled, then the equilibrium position* (19) *of the system* (18), (20) *is asymptotically stable.*

Example 2. Let

$$J = \text{diag}\{[20\ 30\ 20]^\top\},\ a = 0.5,\ b = -1,\ r = [1\ 0\ 0]^\top,$$
$$\nu = 1.5,\ \mu = 3.3,\ \rho = 3.4,\ \tau = 0.25,$$
$$W(\omega) = |\omega_1|^{\nu+1} + 0.25\omega_1\omega_3^\nu + |\omega_2|^{\nu+1} + |\omega_3|^{\nu+1},$$
$$D(s) = \begin{bmatrix} \sin(s) & \cos(s) & \cos(2s) \\ -\sin(2s) & \sin(s) & \cos(s) \\ \cos(s) & \cos(2s) & \sin(s) \end{bmatrix},$$
$$Q(\zeta) = [|\zeta_2|^{\rho-1}\zeta_2\ |\zeta_1|^{\rho-1}\zeta_3\ |\zeta_3|^{\rho-1}\zeta_1]^\top,$$

then all conditions of the last corollary are verified. The results of simulation of the closed-loop system are shown in Fig. 2 (the explicit Euler method was used with step $\Delta t = 0.01$).

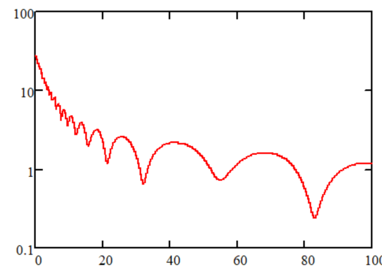

Figure 2. Behavior of $\|\omega(t)\|+\|\eta(t)-r\|$ in logarithmic scale, $t \in [0, 100]$

IX. Conclusion

For a class of nonlinear mechanical systems described by Rayleigh equation with distributed delays, the problem of stability analysis was considered. It was demonstrated that the local asymptotic stability of the zero solution for this class of models, with potential and kinetic energy functions being homogeneous of different degrees, is implied by the global asymptotic stability for an auxiliary delay-free nonlinear system provided that a relation between degrees is verified. The robustness with respect to delayed nonlinear and time-varying disturbances was assessed using the averaging techniques. Several new Lyapunov–Krasovskii functionals were proposed. Using the Lyapunov–Razumikhin approach, these results were developed to stabilization of dissipative-free models. The problem of rigid body stabilization was considered for an illustration of the efficiency of the presented approach.

References

[1] V. Kolmanovskii and A. Myshkis, *Introduction to the Theory and Applications of Functional Differential Equations*, ser. Mathematics and Its Applications. Dordrecht: Springer, 1999, vol. 463.

[2] E. Fridman, *Introduction to Time-Delay Systems: Analysis and Control*. Basel: Birkhäuser, 2014.

[3] L. Hetel, C. Fiter, H. Omran, A. Seuret, E. Fridman, J.-P. Richard, and S. Niculescu, "Recent developments on the stability of systems with aperiodic sampling: An overview," *Automatica*, vol. 76, pp. 309–335, 2017.

[4] E. Fridman, M. Dambrine, and N. Yeganefar, "On input-to-state stability of systems with time-delay: A matrix inequalities approach," *Automatica*, vol. 44, no. 9, pp. 2364–2369, 2008.

[5] D. Efimov and E. Fridman, "Converse Lyapunov-Krasovskii theorem for ISS of neutral systems in Sobolev spaces," *Automatica*, vol. 118, no. 8, p. 109042, 2020.

[6] I. Anan'evskii and V. Kolmanovskii, "Stability of some control systems with aftereffect," *Differ. Equations*, vol. 25, no. 11, pp. 1287–1290, 1989.

[7] A. Aleksandrov, D. Efimov, and E. Fridman, "Stability of homogeneous systems with distributed delay and time-varying perturbations," *Automatica*, vol. 153, no. 7, p. 111058, 2023.

[8] W. Michiels, C.-I. Morărescu, and S.-I. Niculescu, "Consensus problems with distributed delays, with application to traffic flow models," *SIAM Journal on Control and Optimization*, vol. 48, no. 1, pp. 77–101, 2009.

[9] L. Xie, E. Fridman, and U. Shaked, "A robust H_∞ control of distributed delay systems with application to combustion control," *IEEE Trans. Automatic Control*, vol. 46, no. 12, pp. 1930–1935, 2001.

[10] O. Solomon and E. Fridman, "New stability conditions for systems with distributed delays," *Automatica*, vol. 49, no. 11, pp. 3467–3475, 2013.

[11] N. N. Bogoliubov and Y. A. Mitropolsky, *Asymptotic Methods in the Theory of Non-Linear Oscillations*. New York: Gordon and Breach, 1961.

[12] H. Hu and Z. Wang, *Dynamics of Controlled Mechanical Systems with Delayed Feedback*. Berlin: Springer-Verlag, 2002.

[13] P. Ramachandran and Y. Ram, "Stability boundaries of mechanical controlled system with time delay," *Mechanical Systems and Signal Processing*, vol. 27, pp. 523–533, 2012.

[14] E. Fridman and L. Shaikhet, "Stabilization by using artificial delays: An LMI approach," *Automatica*, vol. 81, pp. 429–437, 2017.

[15] I. M. Anan'evskii and V. B. Kolmanovskii, "On stabilization of some control systems with an aftereffect," *Automation and Remote Control*, vol. 9, pp. 1174–1181, 1989.

[16] S. Pavlikov, "On motion stabilization of controlled mechanical systems with a delayed controller," *Dokl. Ross. Akad. Nauk*, vol. 412, no. 2, pp. 176–178, 2007.

[17] ——, "Constant-sign Lyapunov functionals in the problem of the stability of a functional differential equation," *Journal of Applied Mathematics and Mechanics*, vol. 71, no. 3, pp. 339–350, 2007.

[18] L. Marton, "Control of multi-agent systems with distributed delay," *IFAC-PapersOnLine*, vol. 56, no. 2, pp. 8542–8547, 2023.

[19] R. Sipahi, F. Atay, and S.-I. Niculescu, "Stability of traffic flow behavior with distributed delays modeling the memory effects of the drivers," *SIAM J. Applied Mathematics*, vol. 68, pp. 738–759, 2007.

[20] A. Aleksandrov and D. Efimov, "Averaging method for the stability analysis of strongly nonlinear mechanical systems," *Automatica*, vol. 146, p. 110576, 2022.

[21] A. Aleksandrov, D. Efimov, and E. Fridman, "On stability of nonlinear homogeneous systems with distributed delays having variable kernels," *Systems & Control Letters*, vol. 190, p. 105853, 2024.

[22] D. Efimov and A. Aleksandrov, "Analysis of robustness of homogeneous systems with time delays using Lyapunov-Krasovskii functionals," *Int. J. Robust and Nonlinear Control*, vol. 31, pp. 3730–3746, 2021.

[23] G. Portilla, I. Alexandrova, S. Mondié, and A. Zhabko, "Estimates for solutions of homogeneous time-delay systems: comparison of Lyapunov-Krasovskii and Lyapunov-Razumikhin techniques," *International Journal of Control*, vol. 95, no. 11, pp. 3002–3011, 2022.

[24] A. Bacciotti and L. Rosier, *Liapunov Functions and Stability in Control Theory*, ser. Lecture Notes in Control and Inform. Sci. Berlin: Springer, 2001, vol. 267.

[25] J. Hale, *Theory of Functional Differential Equations*. Springer-Verlag, 1977.

[26] H. Khalil, *Nonlinear Systems*. Upper Saddle River NJ: Prentice-Hall, 2002.

[27] N. Rouche, P. Habets, and M. Laloy, *Stability Theory by Liapunov's Direct Method*. New York: Springer, 1977.

[28] A. Formal'sky, "On a modification of the PID controller," *Dyn. Control*, vol. 7, no. 3, pp. 269–277, 1997. [Online]. Available: https://doi.org/10.1023/A:1008202618580

[29] S. Radaideh and M. Hayajneh, "A modified PID controller (PII$\sigma\beta$D)," *Journal of the Franklin Institute*, vol. 339, no. 6, pp. 543–553, 2002.

[30] D. Efimov and A. Polyakov, "Finite-time stability tools for control and estimation," *Foundations and Trends in Systems and Control*, vol. 9, no. 2-3, pp. 171–364, 2021.

[31] O. Gendelman and C. Lamarque, "Dynamics of linear oscillator coupled to strongly nonlinear attachment with multiple states of equilibrium," *Chaos, Solitons and Fractals*, vol. 24, pp. 501–509, 2005.

[32] I. Kovacic and M. Zukovic, "Coupled purely nonlinear oscillators: normal modes and exact solutions for free and forced responses," *Nonlinear Dynamics*, vol. 87, pp. 713–726, 2017.

[33] D. R. Merkin, *Introduction to the Theory of Stability*. New York: Springer, 1997.

[34] A. Aleksandrov, "On the stability of equilibrium of unsteady systems," *Journal of Applied Mathematics and Mechanics*, vol. 60, no. 2, pp. 205–209, 1996, in Russian.

[35] V. Kharitonov, *Time-Delay Systems. Lyapunov Functionals and Matrices*. Basel: Birkhauser, 2013.

[36] A. Domoshnitsky, "Nonoscillation, maximum principles and exponential stability of second order delay differential equations without damping term," *J. Inequal. Appl.*, vol. 361, pp. 1–26, 2014.

[37] A. Aleksandrov, D. Efimov, and E. Fridman, "On local ISS of nonlinear second-order time-delay systems without damping," *International Journal of Robust and Nonlinear Control*, vol. 34, no. 10, pp. 6329–6345, 2024.

[38] V. Beletsky, *Motion of an Artificial Satellite about its Center of Mass*. Jerusalem: Israel Program for Scientific Translation, 1966.

[39] V. Zubov, *Theorie de la Commande (in French)*. Moscow: Mir, 1978.

[40] J. Wertz, *Spacecraft Attitude Determination and Control*. Dordrecht: D. Reidel Publishing Co., 1985.

[41] P. Hughes, *Spacecraft Attitude Dynamics*. New York: Wiley, 1986.