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SAMPLED-DATA FINITE-DIMENSIONAL OBSERVER-BASED CONTROL OF 1D STOCHASTIC PARABOLIC PDEs*

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Abstract. Sampled-data control of PDEs has become an active research area; however, existing results are confined to deterministic PDEs. Sampled-data controller design of stochastic PDEs is a challenging open problem. In this paper we suggest a solution to this problem for 1D stochastic diffusion-reaction equations under discrete-time nonlocal measurement via the modal decomposition method, where both the considered system and the measurement are subject to nonlinear multiplicative noise. We present two methods: a direct one with sampled-data controller implemented via zero-order hold device, and a dynamic-extension-based one with sampled-data controller implemented via a generalized hold device. For both methods, we provide mean-square L^2 exponential stability analysis of the full-order closed-loop system. We construct a Lyapunov functional V that depends on both the deterministic and stochastic parts of the finite-dimensional part of the closed-loop system. We employ corresponding Itô's formulas for stochastic ODEs and PDEs, respectively, and further combine V with Halanay's inequality with respect to the expected value of V to compensate for sampling in the infinite-dimensional tail. We provide linear matrix inequalities (LMIs) for finding the observer dimension and upper bounds on sampling intervals and noise intensities that preserve the mean-square exponential stability. We prove that the LMIs are always feasible for large enough observer dimension and small enough bounds on sampling intervals and noise intensities. A numerical example demonstrates the efficiency of our methods. The example shows that for the same bounds on noise intensities, the dynamic-extension-based controller allows larger sampling intervals, but this is due to its complexity (generalized hold device for sample-data implementation compared to zero-order hold for the direct method).

Key words. stochastic parabolic PDEs, sampled-data control, observer-based control, boundary control, Lyapunov–Krasovskii method

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1. Introduction.

Stochastic PDEs are natural generalizations of deterministic PDEs and stochastic ODEs, and their theory has motivations coming from both mathematics and natural sciences: physics, chemistry, biology, and mathematical finance [5]. However, control theory for stochastic PDEs is not mature till now compared with the deterministic setting and stochastic finite-dimensional problems [30]. Indeed, many tools and methods which are effective in the deterministic case no longer work in the stochastic setting.

Recently, some scholars extended the spatial decomposition method introduced in [9, 10] to stochastic parabolic PDEs with *linear* multiplicative noise [18, 41]. However, spatial decomposition requires many sensors and actuators, covering the whole spatial domain. Inspired by the finite-dimensional observer-based control for deterministic PDEs via the modal decomposition approach (see, e.g., [1, 4]), Christofides and coworkers studied finite-dimensional state-feedback and output-feedback controllers

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for stochastic PDEs subject to additive noise under nonlocal actuation [3, 15]. In [32], Munteanu presented the first results on finite-dimensional boundary state-feedback stabilization for stochastic heat equations with *nonlinear* multiplicative noise by using a fixed point argument. The latter results were restricted to full state knowledge, and their extension to partial state knowledge seems to be nontrivial (see the conclusion of [32]). Note that the results in [1, 3, 4, 15, 32] for deterministic or stochastic parabolic PDEs by the modal decomposition approach were qualitative, and efficient bounds on the observer or controller dimensions were not provided. In the recent paper [21], the first constructive LMI-based method for a finite-dimensional observer-based controller of deterministic parabolic PDEs was suggested, where the observer dimension was found from simple LMI conditions. Finite-dimensional boundary control of 1D parabolic PDEs under point or boundary measurement was studied in [23, 28] by employing a dynamic extension. In our recent paper [40], by using several stochastic analysis techniques, the results in [21, 23, 28] were extended to stochastic parabolic PDEs with nonlinear multiplicative noise under boundary control and observer.

Sampled-data finite-dimensional controllers for deterministic parabolic PDEs, implemented via zero-order hold devices, were suggested in [9, 10] for distributed static output-feedback control, in [20] for boundary state-feedback control, and in [22] for observer-based control. In [26], a sampled-data implementation of boundary controller via a generalized hold device was developed for 1D parabolic PDEs under discrete-time point measurement. Event-triggered sampled-data control of deterministic parabolic PDEs was studied in [6] for boundary state-feedback control and in [36, 17] for distributed static output control. For sampled-data and delayed control of parabolic PDEs, combinations of Lyapunov functionals with Halanay's inequality appear to be an efficient tool. The combinations were introduced for stabilization via the spatial decomposition method in [10, 36] and via modal decomposition in [22, 25, 26]. However, for stochastic systems there are few results on sample-data control, and all of them are confined to stochastic ODEs (see [42, 43]). Sampled-data control of stochastic PDEs is a challenging open problem. Note that all the results for sampled-data control of deterministic PDEs employed Lyapunov functionals for H^1 stability. However, for stochastic PDEs, Itô's formula is well studied for L^2 stability. This is probably the reason why earlier suggested spatial decomposition methods for stochastic PDEs [18, 41] have not yet been extended to the sampled-data case.

In the present paper, for the first time, we provide a solution for sampled-data control of stochastic parabolic PDEs. We consider finite-dimensional observer-based boundary control of 1D stochastic parabolic PDEs under discrete-time nonlocal measurement, where both the PDE and the measurement are subject to nonlinear multiplicative noise. Inspired by [22] and [26] for the deterministic PDEs, we present two methods: a direct method and a dynamic-extension-based method. For the direct one, we consider the sampled-data via a zero-order hold device, and for the dynamic-extension-based one, we suggest a sampled-data via a generalized hold device. For both methods, we construct an appropriate Lyapunov functional V for mean-square L^2 exponential stability and employ corresponding Itô's formulas for stochastic ODEs and PDEs. We provide LMIs for finding the observer dimension and upper bounds on sampling intervals and noise intensities that preserve the exponential stability. We prove that the LMIs are always feasible for large enough observer dimension and small enough bound on sampling intervals and noise intensities. Numerical simulations demonstrate the efficiency of the two methods and show that for the same bounds on noise intensities, the dynamic-extension-based controller allows larger sampling intervals.

The contribution of this paper is summarized as follows:

- We study, for the first time, the sampled-data implementation of finite-dimensional output-feedback controller for stochastic PDEs and provide mean-square exponential L^2 stability analysis.
- Compared to deterministic sampled-data control in [22, 26], the Lyapunov analysis has the following novel features:
  - (a) We employ a novel stochastic Lyapunov functional V that depends on the deterministic and stochastic terms of the finite-dimensional part of the closed-loop system. Such a construction of V is inspired by [11] for stochastic ODEs.
  - (b) Halanay's inequality in the stochastic case is applied with respect to expected value of V to compensate for sampling in the infinite-dimensional tail.
  - (c) We present novel design LMIs to choose the controller gain, which leads to improved results in the numerical example compared to the design presented in [22, 26] for the deterministic case.

In the conference version of this paper (see [39]) the dynamic-extension-based method was not considered.

Notation and preliminaries. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ of increasing sub-σ-fields of $\mathcal{F}$ (see [5, p. 71]), and let $\mathbb{E}\{\cdot\}$ be the expectation operator. Denote by $L^2(0,1)$ the space of square integrable functions with inner product $\langle f, g\rangle = \int_0^1 f(x)g(x)\mathrm{d}x$ and induced norm $\|f\|_{L^2}^2 = \langle f, f\rangle$. Let $L^2(\Omega; L^2(0,1))$ be the set of all $\mathcal{F}_0$-measurable random variables $z \in L^2(0,1)$ with $\mathbb{E}\|z\|_{L^2}^2 < 0$. $H^1(0,1)$ is the Sobolev space of functions $f : [0,1] \longrightarrow \mathbb{R}$ with a square integrable weak derivative. The norm defined in $H^1(0,1)$ is $\|f\|_{H^1}^2 = \|f\|_{L^2}^2 + \|f'\|_{L^2}^2$. Let $\mathbb{Z}_+$ denote the set of nonnegative integers, and let $\mathbb{N}$ denote the set of positive integers. The Euclidean norm is denoted by $|\cdot|$. For $P \in \mathbb{R}^{n\times n}$, $P > 0$ means that P is positive definite. The symmetric elements of a symmetric matrix will be denoted by $*$. For $0 < P \in \mathbb{R}^{n\times n}$ and $x \in \mathbb{R}^n$, we write $|x|_P^2 = x^{\mathrm{T}}Px$. Let I denote the identity matrix of appropriate size.

Consider the Sturm–Liouville eigenvalue problem

$$\phi'' + \lambda\phi = 0,\ x \in (0,1), \quad \phi(0) = \phi'(1) = 0. \tag{1.1}$$

This problem induces a sequence of eigenvalues with corresponding eigenfunctions given by

$$\lambda_n = (n - 0.5)^2\pi^2, \quad \phi_n(x) = \sqrt{2}\sin(\sqrt{\lambda_n}x), \qquad x \in [0,1], n \in \mathbb{N}. \tag{1.2}$$

The eigenfunctions $\{\phi_n\}_{n=1}^\infty$ form a complete orthonormal system in $L^2(0,1)$. Given a positive integer N and $h \in L^2(0,1)$ satisfying $h \overset{L^2}{=} \sum_{n=1}^\infty h_n\phi_n$, we denote $\|h\|_N^2 = \sum_{n=N+1}^\infty h_n^2$.

2. Sampled-data control using zero-order hold device.

Consider the following stochastic 1D heat equation subject to nonlinear multiplicative noise under Neumann actuation:

$$\begin{aligned} &\mathrm{d}z(x,t) = [\tfrac{\partial^2}{\partial x^2}z(x,t) + qz(x,t)]\mathrm{d}t + \sigma_1(x, z(x,t))\mathrm{d}\mathcal{W}_1(t), \quad t \geq 0,\\ &z(0,t) = 0, \quad z_x(1,t) = u(t),\\ &z(x,0) = z_0(x), \quad x \in [0,1], \end{aligned} \tag{2.1}$$

where $z_0 \in L^2(\Omega; L^2(0,1))$, $q \in \mathbb{R}$ is the reaction coefficient, $u(t)$ is a control input to be designed, $\sigma_1(x, z(x,t))\mathrm{d}\mathcal{W}_1(t)$ is the nonlinear multiplicative noise which is due to the random parameter variation of $qz(x,t)\mathrm{d}t$ [13], $\mathcal{W}_1(t)$ is a 1D standard Brownian motion defined on $(\Omega,\mathcal{F},\mathbb{P})$, and the nonlinear function $\sigma_1:[0,1]\times\mathbb{R}\to\mathbb{R}$ describes the distribution of noise with respect to the space and state and satisfies

$$\sigma_1(x,0)=0,\ |\sigma_1(x,z_1)-\sigma_1(x,z_2)|\le\bar{\sigma}_1|z_1-z_2|\ \forall z_1,z_2\in\mathbb{R},x\in[0,1], \tag{2.2}$$

where $\bar{\sigma}_1>0$ describes the upper bound of noise intensity.

Following [26], let $0=s_0<\ldots<s_k<\cdots$, and $\lim_{k\to\infty}s_k=\infty$ be the measurement sampling instances. The sampling is variable and subject to $s_{k+1}-s_k\le\tau_{M,y}$ for all $k\in\mathbb{Z}_+$ and some constant $\tau_{M,y}>0$. We consider the nonlocal measurement with nonlinear multiplicative noise (see, e.g., [12])

$$\mathrm{d}y(t)=\langle c,z(\cdot,s_k)\rangle\mathrm{d}t+\sigma_2(\langle c,z(\cdot,s_k)\rangle)\mathrm{d}\mathcal{W}_2(t),\quad t\ge 0, \tag{2.3}$$

where $c\in L^2(0,1)$, the nonlinear function $\sigma_2:\mathbb{R}\to\mathbb{R}$ satisfies

$$\sigma_2(0)=0,\ |\sigma_2(z_1)-\sigma_2(z_2)|\le\bar{\sigma}_2|z_1-z_2|\quad\forall z_1,z_2\in\mathbb{R}, \tag{2.4}$$

with $\bar{\sigma}_2>0$ describing the upper bound of measurement noise intensity, and $\mathcal{W}_2(t)$ is a 1D standard Brownian motion defined on $(\Omega,\mathcal{F},\mathbb{P})$. Note that $\mathcal{W}_1(t)$ and $\mathcal{W}_2(t)$ are assumed to be mutually independent.

Remark 2.1. For simplicity, we impose the globally Lipschitz continuity condition on the nonlinear terms σ_1,σ_2. The globally Lipschitz condition can be relaxed to hold locally, but then the solution may be local. In the scenario, we can only study the regional stabilization as in [24] for the deterministic Kuramoto–Sivashinsky equation.

As in [40], our results can be easily extended to a more general Sturm–Liouville operator $\frac{\partial}{\partial}(p(x)\frac{\partial}{\partial x}z(x,t))+q(x)$ on the right-hand side of (2.1).

In section 2.2 below, we will prove that for (2.1) with initial value $z_0\in\mathcal{D}(\mathcal{A}_1)$ almost surely and $z_0\in L^2(\Omega;L^2(0,1))$, where operator $\mathcal{A}_1$ is defined in (2.30) below, there exists a unique global solution. Therefore, we can present the solution to (2.1) as

$$z(x,t)=\textstyle\sum_{n=1}^{\infty}z_n(t)\phi_n(x),\ z_n(t)=\langle z(\cdot,t),\phi_n\rangle \tag{2.5}$$

with ϕ_n, $n\in\mathbb{N}$, defined as in (1.2). Differentiating z_n in (2.5), integrating by parts, and using (1.2), we obtain

$$\begin{aligned}&\mathrm{d}z_n(t)=[(-\lambda_n+q)z_n(t)+b_nu(t)]\mathrm{d}t+\sigma_{1,n}(t)\mathrm{d}\mathcal{W}_1(t),\ t\ge 0,\\&b_n=\phi_n(1)=(-1)^{n+1}\sqrt{2},\ \ \sigma_{1,n}(t)=\langle\sigma_1(\cdot,\textstyle\sum_{j=1}^{\infty}z_j(t)\phi_j),\phi_n\rangle,\quad n\in\mathbb{N},\\&z_n(0)=\langle z_0,\phi_n\rangle=:z_{0,n}.\end{aligned} \tag{2.6}$$

By (1.2) and the integral convergence test, we have

$$\textstyle\sum_{n=N+1}^{\infty}\frac{b_n^2}{\lambda_n}\le\frac{2}{\pi^2N}(\frac{1}{N}+1)=:\chi_N,\ N\ge 1. \tag{2.7}$$

Let $\delta>0$ be a desired decay rate and let $N_0\in\mathbb{N}$ satisfy

$$-\lambda_n+q+\delta+\tfrac{\bar{\sigma}_1^2}{2}<0,\ \ n>N_0, \tag{2.8}$$

where N_0 means the number of "relatively unstable" modes that need to be stabilized and is used for the controller design. Note that compared with [21, 23, 28] for the deterministic PDEs, the additional term $\bar{\sigma}_1^2/2$ in (2.8) is induced by the stochastic perturbations (see Remark 2.2 in [40]). Let $N \in \mathbb{N}$, $N_0 \leq N$, where N will define the dimension of the observer.

Following [21, 37], we construct a finite-dimensional observer of the form

$$\hat{z}(x,t) = \textstyle\sum_{n=1}^{N} \hat{z}_n(t)\phi_n(x), \tag{2.9}$$

where $\hat{z}_n(t)$ $(1 \leq n \leq N)$ satisfy

$$\begin{aligned} &\mathrm{d}\hat{z}_n(t) = [(-\lambda_n + q)\hat{z}_n(t) + b_n u(t)]\mathrm{d}t - l_n[\langle c, \hat{z}(\cdot, t-\tau_y)\rangle \mathrm{d}t - \mathrm{d}y(t)],\ t \geq 0, \\ &\hat{z}_n(0) = 0,\ 1 \leq n \leq N, \end{aligned} \tag{2.10}$$

with $y(t)$ satisfying (2.3) and scalar observer gains $\{l_n\}_{n=1}^{N}$.

Let

$$A_0 = \mathrm{diag}\{-\lambda_n + q\}_{n=1}^{N_0},\ B_0 = [b_1, \ldots, b_{N_0}]^{\mathrm{T}}, c_n = \langle c, \phi_n\rangle,\ C_0 = [c_1, \ldots, c_{N_0}]. \tag{2.11}$$

We assume that $c_n \neq 0$, $1 \leq n \leq N_0$. Then the pair (A_0, C_0) is observable by the Hautus lemma. Choose $l_1, \ldots, l_{N_0}$ such that $L_0 = [l_1, \ldots, l_{N_0}]^{\mathrm{T}}$ satisfies

$$P_o(A_0 - L_0C_0) + (A_0 - L_0C_0)^{\mathrm{T}} P_o < -2\delta P_o, \tag{2.12}$$

where $0 < P_o \in \mathbb{R}^{N_0 \times N_0}$. Furthermore, choose $l_n = 0$, $n > N_0$.

Since $b_n \neq 0$ (see (2.6)), the pair (A_0, B_0) is controllable. We propose an N_0-dimensional sampled-data controller of the form

$$u(t) = K_0 \hat{z}^{N_0}(t_j),\ t \in [t_j, t_{j+1}),\ \ \hat{z}^{N_0}(t) = [\hat{z}_1(t), \ldots, \hat{z}_{N_0}(t)]^{\mathrm{T}}, \tag{2.13}$$

where $\{t_j\}_{j=1}^{\infty}$ are the controller hold times satisfying $0 = t_0 < \cdots < t_j < \cdots$, $\lim_{j\to\infty} t_j = \infty$. Assume that the sampling is variable and subject to $t_{j+1} - t_j \leq \tau_{M,u}$ for all $j \in \mathbb{Z}_+$ and some constant $\tau_{M,u} > 0$. In (2.13), the controller gain $K_0 \in \mathbb{R}^{1\times N_0}$ will be obtained from the state-feedback controller design (see section 2.1).

Using the time-delay approach to sampled-data measurement and control (see, e.g., [8, 22]), we introduce the following representations of the measurement and input delays:

$$\begin{aligned} &\tau_y(t) = t - s_k,\ \ t \in [s_k, s_{k+1}), \quad \tau_y(t) \leq \tau_{M,y}, \\ &\tau_u(t) = t - t_j,\ \ t \in [t_j, t_{j+1}), \quad \tau_u(t) \leq \tau_{M,u}. \end{aligned} \tag{2.14}$$

Henceforth the dependence of $\tau_y(t)$, $\tau_u(t)$ on t will be suppressed to shorten notation.

2.1. Design of gain K_0.

Since in many applications one cannot a priori know the noise intensity, here we ignore the stochastic noise (i.e., $\sigma_1(x,z) \equiv 0$) and present the state-feedback design for controller gain.

Consider (2.1) with $\sigma_1(x,z) \equiv 0$. By presenting the solution as (2.5) and differentiating z_n in (2.5), we obtain (2.6) with $\sigma_{1,n} \equiv 0$. Let $\delta > 0$ be a desired decay rate, and let $N_0 \in \mathbb{N}$ satisfy $-\lambda_n + q + \delta < 0$, $n > N_0$.

Since $b_n \neq 0$ (see (2.6)), the pair (A_0, B_0) is controllable. We propose an N_0-dimensional sampled-data controller of the form

$$u(t) = K_0 z^{N_0}(t_j),\ t \in [t_j, t_{j+1}),\ \ z^{N_0}(t) = [z_1(t), \ldots, z_{N_0}(t)]^{\mathrm{T}}, \tag{2.15}$$

where $K_0 \in \mathbb{R}^{1\times N_0}$ is obtained from LMIs below (see (2.23) and (2.24)). Using the notation of (2.11) and (2.14), we have the closed-loop system for $t \geq 0$:

$$\begin{aligned} &\dot{z}^{N_0}(t) = A_0 z^{N_0}(t) + B_0 K_0 z^{N_0}(t - \tau_u), \\ &\dot{z}_n(t) = (-\lambda_n + q) z_n(t) + b_n K_0 z^{N_0}(t - \tau_u), \quad n > N_0. \end{aligned} \tag{2.16}$$

For the exponential stability of (2.16), we consider the Lyapunov functional:

$$\begin{aligned}V(t) &= \sum_{n=N_0+1}^{\infty} \rho z_n^2(t) + |z^{N_0}(t)|_P^2 + \int_{t-\tau_{M,u}}^{t} \mathrm{e}^{-2\delta(t-s)} |z^{N_0}(s)|_S^2 \mathrm{d}s \\ &+ \tau_{M,u} \int_{-\tau_{M,u}}^{0} \int_{t+\theta}^{t} \mathrm{e}^{-2\delta(t-s)} |\dot{z}^{N_0}(s)|_R \mathrm{d}s\mathrm{d}\theta,\end{aligned}$$

where $0 < P, S, R \in \mathbb{R}^{N_0 \times N_0}$, and $\rho > 0$. Let $G \in \mathbb{R}^{N_0 \times N_0}$ satisfy

$$\begin{bmatrix} R & G \\ * & R \end{bmatrix} \geq 0. \tag{2.17}$$

Using the descriptor method

$$0 = [(z^{N_0}(t))^{\mathrm{T}} P_2^{\mathrm{T}} + (\dot{z}^{N_0}(t))^{\mathrm{T}} P_3^{\mathrm{T}}][A_0 z^{N_0}(t) + B_0 K_0 z^{N_0}(t - \tau_u) - \dot{z}^{N_0}(t)], \tag{2.18}$$

with some matrices $P_2, P_3 \in \mathbb{R}^{N_0 \times N_0}$ (see [7]) and further employing Jensen's and Park's inequalities (see, e.g, section 3.6.3 of [8]), we arrive at

$$\begin{aligned}\dot{V}(t) + 2\delta V(t) &\leq \eta^{\mathrm{T}}(t) \Xi \eta(t) + 2\rho \sum_{n=N_0+1}^{\infty} (-\lambda_n + q + \delta) z_n^2(t) \\ &+ \rho \sum_{n=N_0+1}^{\infty} 2 z_n(t) b_n K_0 z^{N_0}(t - \tau_u),\end{aligned} \tag{2.19}$$

where $\eta(t) = \mathrm{col}\{z^{N_0}(t), \dot{z}^{N_0}(t), z^{N_0}(t - \tau_{M,u}), z^{N_0}(t - \tau_u)\}$,

$$\begin{aligned}\Xi &= \begin{bmatrix} \Xi_0 & P - P_2^{\mathrm{T}} + A_0^{\mathrm{T}} P_3 & \varepsilon_u G & \varepsilon_u (R - G) + P_2^{\mathrm{T}} B_0 K_0 \\ * & -P_3 - P_3^{\mathrm{T}} + \tau_{M,u}^2 R & 0 & P_3^{\mathrm{T}} B_0 K_0 \\ * & * & -\varepsilon_u (S + R) & \varepsilon_u (R - G^{\mathrm{T}}) \\ * & * & * & \varepsilon_u (-2R + G + G^{\mathrm{T}}) \end{bmatrix}, \\ \Xi_0 &= A_0^{\mathrm{T}} P_2 + P_2^{\mathrm{T}} A_0 + S - \varepsilon_u R + 2\delta P, \ \varepsilon_u = \mathrm{e}^{-2\delta \tau_{M,u}}.\end{aligned} \tag{2.20}$$

By substituting Young's inequality

$$\begin{aligned}&\rho \sum_{n=N_0+1}^{\infty} 2 z_n(t) b_n K_0 z^{N_0}(t - \tau_u) \\ &\leq \alpha \rho^2 \sum_{n=N_0+1}^{\infty} \lambda_n z_n^2(t) + \tfrac{1}{\alpha} \sum_{n=N_0+1}^{\infty} \tfrac{b_n^2}{\lambda_n} |K_0 z^{N_0}(t - \tau_u)|^2 \\ &\overset{(2.7)}{\leq} \alpha \rho^2 \sum_{n=N_0+1}^{\infty} \lambda_n z_n^2(t) + \tfrac{\chi_{N_0}}{\alpha} |K_0 z^{N_0}(t - \tau_u)|^2\end{aligned}$$

into (2.19), we obtain

$$\dot{V}(t) + 2\delta V(t) \leq \eta^{\mathrm{T}}(t) \tilde{\Xi} \eta(t) + \sum_{n=N_0+1}^{\infty} \mu_n z_n^2(t) \leq 0, \tag{2.21}$$

provided $\mu_n := 2\rho(-\lambda_n + q + \delta) + \alpha \rho^2 \lambda_n < 0$, $n > N_0$ and

$$\tilde{\Xi} := \begin{bmatrix} \Xi_0 & P - P_2^{\mathrm{T}} + A_0^{\mathrm{T}} P_3 & \varepsilon_u G & \varepsilon_u (R - G) + P_2^{\mathrm{T}} B_0 K_0 \\ * & -P_3 - P_3^{\mathrm{T}} + \tau_{M,u}^2 R & 0 & P_3^{\mathrm{T}} B_0 K_0 \\ * & * & -\varepsilon_u (S + R) & \varepsilon_u (R - G^{\mathrm{T}}) \\ * & * & * & \varepsilon_u (-2R + G + G^{\mathrm{T}}) + \frac{\chi_{N_0}}{\alpha} K_0^{\mathrm{T}} K_0 \end{bmatrix} < 0. \tag{2.22}$$

Feasibility of (2.21) guarantees the exponential stability of (2.1) with $\sigma_1(x, z) \equiv 0$ under state-feedback controller (2.15).

Similarly to the controller design in section 5.2.1 of [8], we choose $P_3 = \varepsilon P_2$ with a tuning scalar $\varepsilon > 0$ and denote

$$\bar{P}_2 = P_2^{-1}, \ Y = K_0 \bar{P}_2, \ [\bar{P}, \bar{R}, \bar{S}, \bar{G}] = \bar{P}_2^{\mathrm{T}} [P, R, S, G] \bar{P}_2, \ \bar{\rho} = 1/\rho.$$

We multiply $\tilde{\Xi}$ in (2.22) by $\mathrm{diag}\{\bar{P}_2, \bar{P}_2, \bar{P}_2, \bar{P}_2\}$ and its transpose, from the right and the left, respectively. By the Schur complement, we find that (2.22) holds iff

$$\begin{bmatrix} \hat{\Xi}_0 & \bar{P} - \bar{P}_2 + \varepsilon \bar{P}_2^{\mathrm{T}} A_0^{\mathrm{T}} & \varepsilon_u \bar{G} & \varepsilon_u(\bar{R} - \bar{G}) + B_0 Y & 0 \\ * & -\varepsilon(\bar{P}_2 + \bar{P}_2^{\mathrm{T}}) + \tau_{M,u}^2 \bar{R} & 0 & \varepsilon B_0 Y & 0 \\ * & * & -\varepsilon_u(\bar{S} + \bar{R}) & \varepsilon_u(\bar{R} - \bar{G}^{\mathrm{T}}) & 0 \\ * & * & * & \varepsilon_u(-2\bar{R} + \bar{G} + \bar{G}^{\mathrm{T}}) & Y^{\mathrm{T}} \\ * & * & * & * & -\frac{\alpha}{\chi_{N_0}} \end{bmatrix} < 0, \tag{2.23}$$

$$\hat{\Xi}_0 = \bar{P}_2^{\mathrm{T}} A_0^{\mathrm{T}} + A_0 \bar{P}_2 + \bar{S} - \varepsilon_u \bar{R} + 2\delta \bar{P}.$$

Similarly, multiplying (2.17) by $\mathrm{diag}\{\bar{P}_2, \bar{P}_2\}$ and its transpose, from the right and the left, respectively, we conclude that (2.17) holds iff

$$\begin{bmatrix} \bar{R} & \bar{G} \\ * & \bar{R} \end{bmatrix} \geq 0. \tag{2.24}$$

From the monotonicity of λ_n, we find that $\mu_n < 0$ for all $n > N$ iff

$$2\bar{\rho}(-\lambda_{N_0+1} + q + \delta) + \alpha \lambda_{N_0+1} < 0. \tag{2.25}$$

In particular, (2.23), (2.24), and (2.25) are LMIs that depend on the tuning parameter ε and decision variables $\bar{P}$, $\bar{P}_2$, $\bar{R}$, $\bar{S}$, $\bar{G}$, Y, $\bar{\rho}$, and α. If LMIs (2.23) and (2.24) hold, the controller gain is obtained by $K_0 = Y \bar{P}_2^{-1}$.

Remark 2.2. In [21, 22, 28], the controller gain $K_0 \in \mathbb{R}^{1 \times N_0}$ is chosen such that

$$P_c(A_0 + B_0 K_0) + (A_0 + B_0 K_0)^{\mathrm{T}} P_c \leq -2\delta P_c \tag{2.26}$$

holds with $0 < P_c \in \mathbb{R}^{N_0 \times N_0}$. Choose $\bar{P}_c = P_c^{-1}$ and $Y = K_0 \bar{P}_c$. Multiplying (2.26) by $\bar{P}_c$ from the right and left, respectively, we find that (2.26) holds iff

$$A_0 \bar{P}_c + \bar{P}_c A_0^{\mathrm{T}} + B_0 Y + Y^{\mathrm{T}} B_0^{\mathrm{T}} + 2\delta \bar{P}_c \leq 0, \tag{2.27}$$

where $\bar{P}_c$ and Y are decision variables. If (2.27) is feasible, then the controller gain is obtained by $K_0 = Y \bar{P}_c^{-1}$. In this section, the controller gain design depends on $\tau_{M,u}$. This gain should satisfy (2.26), but it is difficult to find an efficient K_0 from (2.27) (see the numerical example in section 4).

2.2. Well-posedness of the closed-loop system. For well-posedness of system (2.1), (2.10) with control input (2.13), we introduce the change of variables

$$w(x,t) = z(x,t) - r(x)u(t), \quad r(x) = x, \tag{2.28}$$

which leads to the equivalent stochastic PDEs

$$\begin{aligned} \mathrm{d}w(x,t) = &\left[\tfrac{\partial^2}{\partial x^2} w(x,t) + q w(x,t) + q r(x) u(t)\right] \mathrm{d}t \\ &+ \sigma_1(x, w(x,t) + r(x)u(t)) \mathrm{d}\mathcal{W}_1(t), \quad t \geq 0, \\ w(0,t) = 0,\ & w_x(1,t) = 0, \quad t \geq 0, \\ w(x,0) = z_0&(x), \quad x \in [0,1]. \end{aligned} \tag{2.29}$$

We define an operator

$$\begin{aligned} &\mathcal{A}_1 : \mathcal{D}(\mathcal{A}_1) \subseteq L^2(0,1) \to L^2(0,1),\ \mathcal{A}_1 w = w'', \\ &\mathcal{D}(\mathcal{A}_1) = \{w \in H^2(0,1) : w(0) = w'(1) = 0\}, \end{aligned} \tag{2.30}$$

and the notation

$$
\begin{aligned}
&A_1 = \mathrm{diag}\{-\lambda_{n+1}+q\}_{n=N_0+1}^{N},\ B_1 = [b_{N_0+1},\dots,b_N]^{\mathrm{T}},\ B = \mathrm{col}\{B_0, B_1\},\\
&K_1 = [K_0, 0_{1\times(N-N_0)}],\ C = [c_1,\dots,c_N],\ \tilde{L}_0 = \mathrm{col}\{L_0, 0_{(N-N_0)\times 1}\}.
\end{aligned} \tag{2.31}
$$

Let $\mathcal{H} = L^2(0,1)\times\mathbb{R}^N$ be a Hilbert space with norm $\|\cdot\|_{\mathcal{H}}^2 = \|\cdot\|_{L^2}^2 + |\cdot|^2$. Consider $\mathcal{V} := H_L^1(0,1)\times\mathbb{R}^N$ with norm $\|\cdot\|_{\mathcal{V}}^2 = \|\cdot\|_{H^1}^2 + |\cdot|^2$, where $H_L^1(0,1) := \{w \in H^1(0,1) : w(0) = 0\}$. From page 183 in [38], $H^{-1}(0,1) = (H_L^1(0,1))'$ is the dual of $H_L^1(0,1)$ with respect to the pivot space $L^2(0,1)$. Let $\mathcal{V}' := H^{-1}(0,1)\times\mathbb{R}^N$. Hence, $\mathcal{V}\subset\mathcal{H}\subset\mathcal{V}'$. The duality scalar product between $\mathcal{V}'$ and $\mathcal{V}$ is denoted by $\langle\cdot,\cdot\rangle_{\mathcal{V}',\mathcal{V}}$. We have $\langle\varsigma, v\rangle_{\mathcal{V}',\mathcal{V}} = \langle\varsigma, v\rangle_{\mathcal{H}}$ for all $\varsigma\in\mathcal{H}, v\in\mathcal{V}$ (see [35, p. 71]). Consider the state $\xi(t) = \mathrm{col}\{w(\cdot,t), \hat{z}^N(t)\}$ with $\hat{z}^N(t) = [\hat{z}_1(t),\dots,\hat{z}_N(t)]^{\mathrm{T}}$. We present the closed-loop system as

$$
\mathrm{d}\xi(t) = [\mathcal{A}\xi(t) + f(t,\xi(t))]\mathrm{d}t + g(t,\xi(t))\mathrm{d}\mathcal{W}(t), \quad t\ge 0, \tag{2.32}
$$

with $\mathcal{W}(t) = [\mathcal{W}_1(t), \mathcal{W}_2(t)]^{\mathrm{T}}$ and

$$
\begin{aligned}
&\mathcal{A} = \mathrm{diag}\{\mathcal{A}_1, \mathcal{A}_2\},\ \mathcal{A}_2 = \mathrm{diag}\{A_0, A_1\},\ f(t,\xi(t)) = \begin{bmatrix} qw(\cdot,t) + qr(\cdot)K_1\hat{z}^N(t-\tau_u) \\ f_1(t)\end{bmatrix},\\
&f_1(t) = BK_1\hat{z}^N(t-\tau_u) - \tilde{L}_0 C\hat{z}^N(t-\tau_y) + \tilde{L}_0\langle c, w(\cdot,t-\tau_y) + r(\cdot)K_1\hat{z}^N(t-\tau_u)\rangle,\\
&g(t,\xi(t)) = \mathrm{diag}\{g_1, g_2\},\quad g_1 = \sigma_1(\cdot, w(\cdot,t) + r(\cdot)K_1\hat{z}^N(t-\tau_u)),\\
&g_2 = \tilde{L}_0\sigma_2(\langle c, w(\cdot,t-\tau_y) + r(\cdot)K_1\hat{z}^N(t-\tau_u)\rangle).
\end{aligned}
$$

Then $\mathcal{A}:\mathcal{V}\to\mathcal{V}'$ is a closed linear operator with domain $\mathcal{D}(\mathcal{A})$ dense in $\mathcal{H}$. For any $\xi_i\in\mathcal{V}$, $i = 1,2$, by arguments similar to those in [40] (see (2.29), (2.30) therein), we can check from (2.2) and (2.4) that there exist constants $\alpha,\beta > 0$ and γ such that

$$
|\langle\mathcal{A}\xi_1,\xi_2\rangle_{\mathcal{V}',\mathcal{V}}| \le \alpha\|\xi_1\|_{\mathcal{V}}\|\xi_2\|_{\mathcal{V}},\ \langle\mathcal{A}\xi_1,\xi_1\rangle_{\mathcal{V}',\mathcal{V}} \le -\beta\|\xi_1\|_{\mathcal{V}}^2 + \gamma\|\xi_1\|_{\mathcal{H}}^2.
$$

For simplicity, we define $\{\mathcal{T}_i\}_{i=0}^{\infty} := \{s_k\}_{k=0}^{\infty}\cup\{t_j\}_{j=0}^{\infty}$ as follows: $0 = \mathcal{T}_0 < \mathcal{T}_1 < \cdots$, $\lim_{i\to\infty}\mathcal{T}_i = \infty$. Clearly, $\mathcal{T}_{i+1} - \mathcal{T}_i \le \max\{\tau_{M,u}, \tau_{M,y}\}$ for all $i\in\mathbb{Z}_+$. Let $\hat{f}(t,\xi) := f(t,\xi) - f(t,0)$ and $\hat{g}(t,\xi) := g(t,\xi) - g(t,0)$. For any $\xi_1,\xi_2\in\mathcal{H}$, by (2.2) and (2.4), we obtain that there exist positive constants κ_1, κ_2 such that

$$
\begin{aligned}
&\|\hat{f}(t,\xi_1)\|_{\mathcal{H}}^2 + \mathrm{tr}\{\hat{g}^{\mathrm{T}}(t,\xi_1)\hat{g}(t,\xi_1)\} \le \kappa_1(1 + \|\xi_1\|_{\mathcal{H}}^2),\\
&\mathrm{tr}\{[g(t,\xi_2) - g(t,\xi_1)]^{\mathrm{T}}[g(\xi_2) - g(\xi_1)]\} + \|f(t,\xi_2) - f(t,\xi_1)\|_{\mathcal{H}}^2 \le \kappa_2\|\xi_2 - \xi_1\|_{\mathcal{H}}^2,
\end{aligned}
$$

where $\mathrm{tr}\{g^{\mathrm{T}}(t,\xi)g(t,\xi)\} := \|g_1\|_{L^2}^2 + |g_2|^2$. We first consider $t\in[0,\mathcal{T}_1]$. Since $f(t,0)$ and $g(t,0)$ depend on z_0 only for $t\in[0,\mathcal{T}_1]$, we have $\mathbb{E}\int_{\mathcal{T}_0}^{\mathcal{T}_1}[\|f(s,0)\|_{\mathcal{H}}^2 + \mathrm{tr}\{g^{\mathrm{T}}(s,0)g(s,0)\}]\mathrm{d}s < M_0$ for some constant $M_0 > 0$. By Theorem 6.7.4 of [2], for any initial value $\xi_0\in\mathcal{D}(\mathcal{A})$ almost surely, (2.32) has a unique strong solution

$$
\xi\in L^2(\Omega; C([0,\mathcal{T}_1];\mathcal{H}))\cap L^2([0,\mathcal{T}_1]\times\Omega;\mathcal{V}], \tag{2.33}
$$

such that

$$
\xi(t)\in\mathcal{D}(\mathcal{A}) = \mathcal{D}(\mathcal{A}_1)\times\mathbb{R}^N\ \ \forall t\in[0,\mathcal{T}_1] \tag{2.34}
$$

almost surly and is adapted to $\mathcal{F}_t$, $t\in[0,\mathcal{T}_1]$. The latter follows from the definition of a strong solution in [29] (see Definition 1.3.3 therein). Next, consider $t\in[\mathcal{T}_1,\mathcal{T}_2)$. In this regard, $f(t,0)$ and $g(t,0)$ depend on $\xi(\mathcal{T}_1)$ only. From (2.33), it follows that there exists a positive constant M_1 such that $\mathbb{E}\int_{\mathcal{T}_1}^{\mathcal{T}_2}[\|f(s,0)\|_{\mathcal{H}}^2+\mathrm{tr}\{g^{\mathrm{T}}(s,0)g(s,0)\}]\mathrm{d}s<M_1$. Since $\xi(\mathcal{T}_1)\in\mathcal{D}(\mathcal{A})$ almost surely, there exists a unique strong solution ξ satisfying (2.33) and (2.34) on $[\mathcal{T}_1,\mathcal{T}_2]$ almost surely. Using these arguments step by step on $[\mathcal{T}_i,\mathcal{T}_{i+1}]$ $(i\in\mathbb{Z}_+)$ with initial conditions $\xi(\mathcal{T}_i)\in\mathcal{D}(\mathcal{A})$, we obtain, for $z_0\in\mathcal{D}(\mathcal{A}_1)$ almost surely, existence of a unique solution $\xi\in L^2(\Omega;C([0,\infty);\mathcal{H}))\cap L^2(\Omega\times[0,\infty)\backslash\mathcal{J};\mathcal{V})$, where $\mathcal{J}=\{\mathcal{T}_i\}_{i=0}^{\infty}$, such that $\xi(t)\in\mathcal{D}(\mathcal{A})$, $t\geq 0$, almost surely. By the change of valuables (2.28), we have that the solutions to (2.1) satisfy $z\in L^2(\Omega;C([0,\infty);L^2(0,1)))\cap L^2(\Omega\times[0,\infty)\backslash\mathcal{J};H^1(0,1))$.

2.3. Mean-square L^2 stability analysis. Let

$$e_n(t)=z_n(t)-\hat{z}_n(t). \tag{2.35}$$

By using (2.3) and (2.9), we write the last term on the right-hand side of (2.10) as

$$\begin{aligned}[\langle c,\hat{z}(\cdot,t-\tau_y)\rangle\mathrm{d}t-\mathrm{d}y(t)]&=-[\textstyle\sum_{n=1}^{N}c_ne_n(t-\tau_y)+\zeta(t-\tau_y)]\mathrm{d}t\\&\quad-\sigma_2(\hat{\zeta}(t-\tau_y))\mathrm{d}\mathcal{W}_2(t),\\ \zeta(t)=\textstyle\sum_{n=N+1}^{\infty}c_nz_n(t),\ \hat{\zeta}(t)&=\zeta(t)+\textstyle\sum_{n=1}^{N}c_n[\hat{z}_n(t)+e_n(t)].\end{aligned} \tag{2.36}$$

By the Cauchy–Schwarz inequality, we have

$$\zeta^2(t)\leq\|c\|_N^2\textstyle\sum_{n=N+1}^{\infty}z_n^2(t). \tag{2.37}$$

Then the error equations have the form

$$\begin{aligned}\mathrm{d}e_n(t)&=\{(-\lambda_n+q)e_n(t)-l_n[\textstyle\sum_{n=1}^{N}c_ne_n(t-\tau_y)+\zeta(t-\tau_y)]\}\mathrm{d}t\\&\quad+\sigma_{1,n}(t)\mathrm{d}\mathcal{W}_1(t)-l_n\sigma_2(\hat{\zeta}(t-\tau_y))\mathrm{d}\mathcal{W}_2(t),\ \ t\geq 0.\end{aligned} \tag{2.38}$$

Recall the notation in (2.11), (2.13), (2.31) and let

$$\begin{aligned}&e^{N_0}(t)=\mathrm{col}\{e_n(t)\}_{n=1}^{N_0},\ \ e^{N-N_0}(t)=\mathrm{col}\{e_n(t)\}_{n=N_0+1}^{N},\\&\hat{z}^{N-N_0}(t)=\mathrm{col}\{\hat{z}_n(t)\}_{n=N_0+1}^{N},\ \ C_1=[c_{N_0+1},\ldots,c_N],\\&X(t)=\mathrm{col}\{\hat{z}^{N_0}(t),e^{N_0}(t),\hat{z}^{N-N_0}(t),e^{N-N_0}(t)\},\\&\mathbb{C}=[C_0,C_0,C_1,C_1],\ \ \mathbb{K}_0=[K_0,0_{1\times(2N-N_0)}],\\&\mathbb{L}_0=\mathrm{col}\{L_0,-L_0,0_{2(N-N_0)\times 1}\},\\&F_0=\begin{bmatrix}A_0+B_0K_0 & L_0C_0 & 0 & L_0C_1\\ 0 & A_0-L_0C_0 & 0 & -L_0C_1\\ B_1K_0 & 0 & A_1 & 0\\ 0 & 0 & 0 & A_1\end{bmatrix},\\&F_1=\mathbb{L}_0\cdot[0,C_0,0,C_1],\ \ F_2=\mathrm{col}\{B_0,0,B_1,0\},\\&\nu_y(t)=X(t)-X(t-\tau_y),\ \ \nu_u(t)=X(t)-X(t-\tau_u).\end{aligned} \tag{2.39}$$

By (2.6), (2.10), and (2.38), we have the closed-loop system for $t\geq 0$,

$$\mathrm{d}X(t)=F(t)\mathrm{d}t+\Sigma_1(t)\mathrm{d}\mathcal{W}_1(t)+\Sigma_2(t)\mathrm{d}\mathcal{W}_2(t), \tag{2.40a}$$

$$\mathrm{d}z_n(t)=[(-\lambda_n+q)z_n(t)+b_n\mathbb{K}_0[X(t)-\nu_u(t)]]\mathrm{d}t+\sigma_{1,n}(t)\mathrm{d}\mathcal{W}_1(t),\ n>N, \tag{2.40b}$$

where

$$\begin{aligned}&F(t)=F_0X(t)-F_1\nu_y(t)-F_2\mathbb{K}_0\nu_u(t)+\mathbb{L}_0\zeta(t-\tau_y),\\&\Sigma_1(t)=\mathrm{col}\{0_{N_0\times 1},\sigma^{N_0}(t),0_{(N-N_0)\times 1},\sigma^{N-N_0}(t)\},\\&\sigma^{N_0}(t)=\mathrm{col}\{\sigma_{1,n}(t)\}_{n=1}^{N_0},\ \sigma^{N-N_0}(t)=\mathrm{col}\{\sigma_{1,n}(t)\}_{n=N_0+1}^{N},\\&\Sigma_2(t)=\mathbb{L}_0\sigma_2(\zeta(t-\tau_y)+\mathbb{C}X(t)-\mathbb{C}\nu_y(t)).\end{aligned} \tag{2.41}$$

For mean-square L^2 exponential stability of the closed-loop system (2.40), we define the Lyapunov functional

$$
\begin{aligned}
&V(t)=\sum_{i=1}^{2}[V_{S_i}(t)+V_{R_i}(t)+V_{Q_{1i}}(t)+V_{Q_{2i}}(t)]+V_{\mathrm{nom}}(t),\\
&V_{\mathrm{nom}}(t)=V_P(t)+\rho\sum_{n=N+1}^{\infty}z_n^2(t),\quad V_P(t)=|X(t)|_P^2,\\
&V_{S_1}(t)=\int_{t-\tau_{M,y}}^{t}\mathrm{e}^{2\delta_0(s-t)}|X(s)|_{S_1}^2\mathrm{d}s,\\
&V_{S_2}(t)=\int_{t-\tau_{M,u}}^{t}\mathrm{e}^{2\delta_0(s-t)}|\mathbb{K}_0X(s)|_{S_2}^2\mathrm{d}s,\\
&V_{R_1}(t)=\tau_{M,y}\int_{-\tau_{M,y}}^{0}\int_{t+\theta}^{t}\mathrm{e}^{2\delta_0(s-t)}|F(s)|_{R_1}^2\mathrm{d}s\mathrm{d}\theta,\\
&V_{R_2}(t)=\tau_{M,u}\int_{-\tau_{M,u}}^{0}\int_{t+\theta}^{t}\mathrm{e}^{2\delta_0(s-t)}|\mathbb{K}_0F(s)|_{R_2}^2\mathrm{d}s\mathrm{d}\theta,\\
&V_{Q_{1i}}(t)=\int_{-\tau_{M,y}}^{0}\int_{t+\theta}^{t}\mathrm{e}^{2\delta_0(s-t)}|\Sigma_i(s)|_{Q_{1i}}^2\mathrm{d}s\mathrm{d}\theta,\\
&V_{Q_{2i}}(t)=\int_{-\tau_{M,u}}^{0}\int_{t+\theta}^{t}\mathrm{e}^{2\delta_0(s-t)}|\mathbb{K}_0\Sigma_i(s)|_{Q_{2i}}^2\mathrm{d}s\mathrm{d}\theta,\ i=1,2,
\end{aligned}
\tag{2.42}
$$

with $0<P,S_1,R_1,Q_{11},Q_{12}\in\mathbb{R}^{2N\times 2N}$ and positive scalars S_2, R_2, Q_{21}, Q_{22}, ρ. Without loss of generality we assume that $z(\cdot,t)=z_0(\cdot)$ and $\hat{z}(\cdot,t)=0$ for $t<0$. In this regard, the solution of the closed-loop system (2.40) for $t<0$ is well-defined. Functional $V(t)$ is a stochastic extension of the Lyapunov functional in [22, 25]. The terms V_P, V_{S_1}, and V_{S_2} have the same form as the deterministic case. The terms V_{R_1} and V_{R_2} are stochastic extensions of the state-derivative-dependent double integral terms, whereas $V_{Q_{1i}}$ and $V_{Q_{2i}}$ compensate for the stochastic parts of (2.40a) (see [11, 43]). By Parseval's equality and (2.28), we present $V_{\mathrm{nom}}(t)$ in (2.42) as

$$
\begin{aligned}
&V_{\mathrm{nom}}(t)=V_P(t)-V_1(t)+V_2(t,w(\cdot,t)),\\
&V_1(t)=\rho|\mathbb{I}X(t)|^2,\ \mathbb{I}=\begin{bmatrix} I_{N_0} & I_{N_0} & 0 & 0\\ 0 & 0 & I_{N-N_0} & I_{N-N_0}\end{bmatrix},\\
&V_2(t,w(\cdot,t))=\rho\|w(\cdot,t)+r(\cdot)u(t)\|_{L^2}^2.
\end{aligned}
\tag{2.43}
$$

For functions V_P and V_1, calculating the generator $\mathcal{L}$ along stochastic ODE (2.40a) (see [27, p. 149]), we have

$$
\begin{aligned}
&\mathcal{L}V_P(t)+2\delta_0V_P(t)\overset{(2.4)}{\le} X^{\mathrm{T}}(t)[PF_0+F_0^{\mathrm{T}}P+2\delta_0P]X(t)\\
&\quad-2X^{\mathrm{T}}(t)PF_1\nu_y(t)+\Sigma_1^{\mathrm{T}}(t)P\Sigma_1(t)\\
&\quad-2X^{\mathrm{T}}(t)PF_2\mathbb{K}_0\nu_u(t)+2X^{\mathrm{T}}(t)P\mathbb{L}_0\zeta(t-\tau_y)\\
&\quad+\bar{\sigma}_2^2\mathbb{L}_0^{\mathrm{T}}P\mathbb{L}_0[\zeta(t-\tau_y)+\mathbb{C}X(t)-\mathbb{C}\nu_y(t)]^2,\quad t\ge 0,
\end{aligned}
\tag{2.44}
$$

$$
\begin{aligned}
&\mathcal{L}V_1(t)+2\delta_0V_1(t)=2\rho X^{\mathrm{T}}(t)\mathbb{I}^{\mathrm{T}}\mathbb{I}F(t)+\rho\Sigma_1^{\mathrm{T}}(t)\mathbb{I}^{\mathrm{T}}\mathbb{I}\Sigma_1(t)\\
&\qquad+\rho\Sigma_2^{\mathrm{T}}(t)\mathbb{I}^{\mathrm{T}}\mathbb{I}\Sigma_2(t)+2\delta_0\rho|\mathbb{I}X(t)|^2\\
&\quad=2\rho\sum_{n=1}^{N}(-\lambda_n+q+\delta_0)z_n^2(t)+\rho|\Sigma_1(t)|^2\\
&\qquad+2\rho\sum_{n=1}^{N}z_n(t)b_n\mathbb{K}_0[X(t)-\nu_u(t)],\quad t\ge 0.
\end{aligned}
\tag{2.45}
$$

Recalling the operator $\mathcal{A}_1$ given by (2.30), we can write stochastic PDE (2.29) as

$$
\begin{aligned}
\mathrm{d}w(t)&=[\mathcal{A}_1w(t)+qw(t)+qr(\cdot)u(t)]\mathrm{d}t\\
&\qquad+\sigma_1(\cdot,w(t)+r(\cdot)u(t))\mathrm{d}\mathcal{W}_1(t),\ t\in[t_j,t_{j+1}),
\end{aligned}
\tag{2.46}
$$

where $w(t)=w(\cdot,t)$, $r=r(\cdot)$. Since $w(t)$ is a strong solution to (2.46) (see section 2.2), for function $V_2(t,w(\cdot,t))$, we estimate the generator $\mathcal{L}$ of (2.46) as follows (see [2, p. 228]):

$$\begin{aligned}\mathcal{L}V_2(t,w(\cdot,t)) = 2\rho\int_0^1[w(x,t)+r(x)u(t)]\Big[\tfrac{\partial^2 w(x,t)}{\partial x^2}+qw(x,t)+qr(x)u(t)\Big]\mathrm{d}x\\ +\rho\int_0^1\sigma_1^2(x,w(x,t)+r(x)u(t))^2\mathrm{d}x.\end{aligned}$$

By Parseval's equality (see [33, Proposition 10.29]) and (2.2), (2.28), we arrive at

$$\begin{aligned}\mathcal{L}V_2(t,w(\cdot,t)) &\le \rho\textstyle\sum_{n=1}^\infty\int_0^1\phi_n(x)[w(x,t)+r(x)u(t)]\mathrm{d}x\\ &\quad\times\textstyle\int_0^1\phi_n(x)\Big[\tfrac{\partial^2 w(x,t)}{\partial x^2}+qw(x,t)+qr(x)u(t)\Big]\mathrm{d}x+\rho\bar\sigma_1^2\int_0^1 z^2(x,t)\mathrm{d}x\\ &= \rho\textstyle\sum_{n=1}^\infty z_n(t)[(-\lambda_n+q)w_n(t)+qr_nu(t)]+\rho\bar\sigma_1^2\sum_{n=1}^\infty z_n^2(t).\end{aligned}\tag{2.47}$$

where $r_n=\langle r,\phi_n\rangle$. By (2.13) and (2.28), we have

$$w_n(t)=z_n(t)-r_nu(t)=z_n(t)-r_n\mathbb{K}_0[X(t)-\nu_u(t)].\tag{2.48}$$

Substituting (2.48) into (2.47) we get

$$\begin{aligned}&\mathcal{L}V_2(t,w(\cdot,t))+2\delta_0V_2(t,w(\cdot,t))\\ &\quad\le 2\rho\textstyle\sum_{n=1}^\infty(-\lambda_n+q+\delta_0+\frac{\bar\sigma_1^2}{2})z_n^2(t)+2\rho\sum_{n=1}^\infty z_n(t)\lambda_nr_n\mathbb{K}_0[X(t)-\nu_u(t)].\end{aligned}\tag{2.49}$$

Note that

$$b_n=\phi_n(1)=\int_0^1\phi_n'(x)\mathrm{d}x=-\int_0^1 x\phi_n''(x)\mathrm{d}x=\lambda_nr_n.\tag{2.50}$$

Combining (2.43), (2.44), (2.45), (2.49), and (2.50) gives us

$$\begin{aligned}&\mathcal{L}V_{\mathrm{nom}}(t)+2\delta_0V_{\mathrm{nom}}(t)\le X^{\mathrm{T}}(t)[PF_0+F_0^{\mathrm{T}}P+2\delta_0P+\rho\bar\sigma_1^2\mathbb{I}^{\mathrm{T}}\mathbb{I}]X(t)\\ &-2X^{\mathrm{T}}(t)PF_2\mathbb{K}_0\nu_u(t)-2X^{\mathrm{T}}(t)PF_1\nu_y(t)+\Sigma_1^{\mathrm{T}}(t)(P-\rho I)\Sigma_1(t)\\ &+\bar\sigma_2^2\mathbb{L}_0^{\mathrm{T}}P\mathbb{L}_0[\zeta(t-\tau_y)+\mathbb{C}X(t)-\mathbb{C}\nu_y(t)]^2+2X^{\mathrm{T}}(t)P\mathbb{L}_0\zeta(t-\tau_y)\\ &+\rho\textstyle\sum_{n=N+1}^\infty 2z_n(t)b_n\mathbb{K}_0[X(t)-\nu_u(t)]\\ &+\rho\textstyle\sum_{n=N+1}^\infty 2(-\lambda_n+q+\delta_0+\frac{\bar\sigma_1^2}{2})z_n^2(t).\end{aligned}\tag{2.51}$$

Let $\alpha>0$. Applying Young's inequality, we arrive at

$$\begin{aligned}&\rho\textstyle\sum_{n=N+1}^\infty 2z_n(t)b_n\mathbb{K}_0[X(t)-\nu_u(t)]\\ &\quad=\textstyle\sum_{n=N+1}^\infty 2\Big[\frac{\sqrt{2\lambda_n}}{\sqrt\alpha}\rho z_n(t)\Big]\Big[\frac{\sqrt\alpha}{\sqrt{2\lambda_n}}b_n\mathbb{K}_0(X(t)-\nu_u(t))\Big]\\ &\quad\le\textstyle\frac{2\rho^2}{\alpha}\sum_{n=N+1}^\infty\lambda_nz_n^2(t)+\alpha\sum_{n=N+1}^\infty\frac{b_n^2}{\lambda_n}\big[|\mathbb{K}_0X(t)|^2+|\mathbb{K}_0\nu_u(t)|^2\big]\\ &\quad\overset{(2.7)}{\le}\textstyle\frac{2\rho^2}{\alpha}\sum_{n=N+1}^\infty\lambda_nz_n^2(t)+\frac{2\alpha}{\pi^2N}(\frac1N+1)\big[|\mathbb{K}_0X(t)|^2+|\mathbb{K}_0\nu_u(t)|^2\big].\end{aligned}\tag{2.52}$$

We further consider compensation of terms with delays via Lyapunov analysis. Let

$$\begin{aligned}\varepsilon_y&=\mathrm{e}^{-2\delta_0\tau_{M,y}},\quad \theta_y(t)=X(t-\tau_y)-X(t-\tau_{M,y}),\\ \varepsilon_u&=\mathrm{e}^{-2\delta_0\tau_{M,u}},\quad \theta_u(t)=X(t-\tau_u)-X(t-\tau_{M,u}).\end{aligned}\tag{2.53}$$

For functionals $V_{S_i}(t)$, $V_{R_i}(t)$, $i=1,2$, calculation of the generator $\mathcal{L}$ along stochastic ODE (2.40a) (see [27, p. 149]) gives

$$\begin{aligned}\mathcal{L}V_{S_1}(t)+2\delta_0V_{S_1}(t)&=|X(t)|^2_{S_1}-\varepsilon_y|X(t)-\nu_y(t)-\theta_y(t)|^2_{S_1},\\ \mathcal{L}V_{S_2}(t)+2\delta_0V_{S_2}(t)&=|\mathbb{K}_0X(t)|^2_{S_2}-\varepsilon_u|\mathbb{K}_0X(t)-\mathbb{K}_0\nu_u(t)-\mathbb{K}_0\theta_u(t)|^2_{S_2},\end{aligned}\tag{2.54}$$

and

$$\begin{aligned}&\mathcal{L}V_{R_1}(t)+2\delta_0 V_{R_1}(t)\le \tau_{M,y}^2|F(t)|_{R_1}^2-\varepsilon_y\tau_{M,y}\textstyle\int_{t-\tau_{M,y}}^t|F(s)|_{R_1}^2\mathrm{d}s,\\&\mathcal{L}V_{R_2}(t)+2\delta_0 V_{R_2}(t)\le \tau_{M,u}^2|\mathbb{K}_0F(t)|_{R_2}^2-\varepsilon_u\tau_{M,u}\textstyle\int_{t-\tau_{M,u}}^t|\mathbb{K}_0F(s)|_{R_2}^2\mathrm{d}s,\end{aligned}\tag{2.55}$$

where $F(t)$ is defined as in (2.41). Let $G_1\in\mathbb{R}^{2N\times 2N}$ and $G_2\in\mathbb{R}$ satisfy

$$\begin{bmatrix}R_1 & G_1\\ * & R_1\end{bmatrix}\ge 0,\ \begin{bmatrix}R_2 & G_2\\ * & R_2\end{bmatrix}\ge 0.\tag{2.56}$$

Applying Jensen's and Park's inequalities (see, e.g., section 3.6.3 of [8]), we obtain

(2.57)

$$\begin{aligned}&\tau_{M,y}\textstyle\int_{t-\tau_{M,y}}^t|F(s)|_{R_1}^2\mathrm{d}s\ge\begin{bmatrix}\int_{t-\tau_y}^t F(s)\mathrm{d}s\\ \int_{t-\tau_{M,y}}^{t-\tau_y}F(s)\mathrm{d}s\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}R_1 & G_1\\ * & R_1\end{bmatrix}\begin{bmatrix}\int_{t-\tau_y}^t F(s)\mathrm{d}s\\ \int_{t-\tau_{M,y}}^{t-\tau_y}F(s)\mathrm{d}s\end{bmatrix},\\&\tau_{M,u}\textstyle\int_{t-\tau_{M,u}}^t|\mathbb{K}_0F(s)|_{R_2}^2\mathrm{d}s\ge\begin{bmatrix}\int_{t-\tau_u}^t \mathbb{K}_0F(s)\mathrm{d}s\\ \int_{t-\tau_{M,u}}^{t-\tau_u}\mathbb{K}_0F(s)\mathrm{d}s\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}R_2 & G_2\\ * & R_2\end{bmatrix}\begin{bmatrix}\int_{t-\tau_u}^t \mathbb{K}_0F(s)\mathrm{d}s\\ \int_{t-\tau_{M,u}}^{t-\tau_u}\mathbb{K}_0F(s)\mathrm{d}s\end{bmatrix}.\end{aligned}$$

From (2.40a) we have

$$\begin{aligned}&\textstyle\int_{t-\tau_y}^t F(s)\mathrm{d}s=\nu_y(t)-\xi_{11}^y(t)-\xi_{21}^y(t),\ \int_{t-\tau_{M,y}}^{t-\tau_y}F(s)\mathrm{d}s=\theta_y(t)-\xi_{12}^y(t)-\xi_{22}^y(t),\\&\xi_{i1}^y(t)=\textstyle\int_{t-\tau_y}^t\Sigma_i(s)\mathrm{d}\mathcal{W}_i(s),\ \xi_{i2}^y(t)=\int_{t-\tau_{M,y}}^{t-\tau_y}\Sigma_i(s)\mathrm{d}\mathcal{W}_i(s),\ i=1,2,\\&\textstyle\int_{t-\tau_u}^t\mathbb{K}_0F(s)\mathrm{d}s=\mathbb{K}_0\nu_u(t)-\mathbb{K}_0\xi_{11}^u(t)-\mathbb{K}_0\xi_{21}^u(t),\\&\textstyle\int_{t-\tau_{M,u}}^{t-\tau_u}\mathbb{K}_0F(s)\mathrm{d}s=\mathbb{K}_0\theta_u(t)-\mathbb{K}_0\xi_{12}^u(t)-\mathbb{K}_0\xi_{22}^u(t),\\&\xi_{i1}^u(t)=\textstyle\int_{t-\tau_u}^t\Sigma_i(s)\mathrm{d}\mathcal{W}_i(s),\ \xi_{i2}^u(t)=\int_{t-\tau_{M,u}}^{t-\tau_u}\Sigma_i(s)\mathrm{d}\mathcal{W}_i(s),\ i=1,2.\end{aligned}$$

By employing the Itô integral properties (see [31, (5.27) on p. 28]), we have

$$\begin{aligned}&\mathbb{E}[(\xi_{1i}^y(t))^{\mathrm{T}}R_1\xi_{2i}^y(t)]=0,\ \mathbb{E}[(\xi_{j1}^y(t))^{\mathrm{T}}G_1\xi_{i2}^y(t)]=0,\\&\mathbb{E}[(\mathbb{K}_0\xi_{1i}^u(t))^{\mathrm{T}}R_2\mathbb{K}_0\xi_{2i}^u(t)]=0,\ \mathbb{E}[(\mathbb{K}_0\xi_{j1}^u(t))^{\mathrm{T}}G_2\mathbb{K}_0\xi_{i2}^u(t)]=0,\ i,j=1,2.\end{aligned}\tag{2.58}$$

Let

$$\begin{aligned}&\eta_1(t)=\mathrm{col}\{\nu_y(t),\theta_y(t),\xi_y(t)\},\ \eta_2(t)=\mathrm{col}\{\mathbb{K}_0\nu_u(t),\mathbb{K}_0\theta_u(t),\xi_u(t)\},\\&\xi_y(t)=\mathrm{col}\{\xi_{11}^y(t),\xi_{12}^y(t),\xi_{21}^y(t),\xi_{22}^y(t)\},\\&\xi_u(t)=\mathrm{col}\{\mathbb{K}_0\xi_{11}^u(t),\mathbb{K}_0\xi_{12}^u(t),\mathbb{K}_0\xi_{21}^u(t),\mathbb{K}_0\xi_{22}^u(t)\}.\end{aligned}\tag{2.59}$$

Taking expectation on both sides of (2.55) and using (2.57) as well as (2.58), we get

$$\begin{aligned}&\mathbb{E}[\mathcal{L}V_{R_1}(t)+2\delta_0V_{R_1}(t)]\le\tau_{M,y}^2\mathbb{E}|F(t)|_{R_1}^2\\&\quad+\mathbb{E}\left[\eta_1^{\mathrm{T}}(t)\left[\begin{array}{cc|c}-\varepsilon_yR_1 & -\varepsilon_yG_1 & \varepsilon_y[R_1,G_1,R_1,G_1]\\ * & -\varepsilon_yR_1 & \varepsilon_y[G_1^{\mathrm{T}},R_1,G_1^{\mathrm{T}},R_1]\\ \hline * & * & -\varepsilon_y\mathrm{diag}\{R_1,R_1,R_1,R_1\}\end{array}\right]\eta_1(t)\right],\\&\mathbb{E}[\mathcal{L}V_{R_2}(t)+2\delta_0V_{R_2}(t)]\le\tau_{M,u}^2\mathbb{E}|\mathbb{K}_0F(t)|_{R_2}^2\\&\quad+\mathbb{E}\left[\eta_2^{\mathrm{T}}(t)\left[\begin{array}{cc|c}-\varepsilon_uR_2 & -\varepsilon_uG_2 & \varepsilon_u[R_2,G_2,R_2,G_2]\\ * & -\varepsilon_uR_2 & \varepsilon_u[G_2,R_2,G_2,R_2]\\ \hline * & * & -\varepsilon_u\mathrm{diag}\{R_2,R_2,R_2,R_2\}\end{array}\right]\eta_2(t)\right].\end{aligned}\tag{2.60}$$

For functionals $V_{Q_{1i}}(t)$, $V_{Q_{2i}}(t)$, $i=1,2$, calculating the generator $\mathcal{L}$ along stochastic ODE (2.40a) and taking expectation, we get for $i=1,2$

$$
\begin{aligned}
&\mathbb{E}[\mathcal{L}V_{Q_{1i}}(t)+2\delta_0 V_{Q_{1i}}(t)] \leq \tau_{M,y}\mathbb{E}|\Sigma_i(t)|^2_{Q_{1i}} - \varepsilon_y[\mathbb{E}|\xi^y_{i1}(t)|^2_{Q_{1i}} + \mathbb{E}|\xi^y_{i2}(t)|^2_{Q_{1i}}],\\
&\mathbb{E}[\mathcal{L}V_{Q_{2i}}(t)+2\delta_0 V_{Q_{2i}}(t)] \leq \tau_{M,u}\mathbb{E}|\mathbb{K}_0\Sigma_i(t)|^2_{Q_{2i}}\\
&\qquad\qquad -\varepsilon_u[\mathbb{E}|\mathbb{K}_0\xi^u_{i1}(t)|^2_{Q_{2i}} + \mathbb{E}|\mathbb{K}_0\xi^u_{i2}(t)|^2_{Q_{2i}}],
\end{aligned}
\tag{2.61}
$$

where the following Itô integral properties are used (see, e.g., [27, 31]):

$$
\begin{aligned}
&\mathbb{E}\int_{t-\tau_{M,y}}^t |\Sigma_i(s)|^2_{Q_{1i}}\mathrm{d}s = \mathbb{E}|\xi^y_{i1}(t)|^2_{Q_{1i}} + \mathbb{E}|\xi^y_{i2}(t)|^2_{Q_{1i}},\\
&\mathbb{E}\int_{t-\tau_{M,u}}^t |\mathbb{K}_0\Sigma_i(s)|^2_{Q_{2i}}\mathrm{d}s = \mathbb{E}|\mathbb{K}_0\xi^u_{i1}(t)|^2_{Q_{2i}} + \mathbb{E}|\mathbb{K}_0\xi^u_{i2}(t)|^2_{Q_{2i}}.
\end{aligned}
$$

We will compensate for the $\zeta(t-\tau_y)$ that appears in (2.51) by employing Halanay's inequality. For this we will use the following bound:

$$
\begin{aligned}
-2\delta_1 \sup_{s_k\leq\theta\leq t}\mathbb{E}V(\theta) &\overset{(2.14),(2.42)}{\leq} -2\delta_1\mathbb{E}V_{\mathrm{nom}}(t-\tau_y)\\
&= -2\delta_1\mathbb{E}|X(t)-\nu_y(t)|^2_P - 2\delta_1\rho\mathbb{E}\left[\sum_{n=N+1}^\infty z_n^2(t-\tau_y)\right]\\
&\overset{(2.37)}{\leq} -2\delta_1\mathbb{E}|X(t)-\nu_y(t)|^2_P - 2\delta_1\rho\|c\|_N^{-2}\mathbb{E}\zeta^2(t-\tau_y),
\end{aligned}
\tag{2.62}
$$

where $0<\delta_1<\delta_0$.

Remark 2.3. In (2.62), the bound (2.37) for $\zeta^2(t)$ is employed. For the case of boundary measurement, we have to use estimate (3.72) in [26] to compensate for $\zeta(t-\tau_y)$. In that scenario, the H^1 stability is desired (see [26]). However, the L^2 regularity in $H^1(0,1)$ (see the well-posedness in section 2.2) complicates studying the H^1 stability. That is why we consider the nonlocal measurement.

By Parseval's equality and (2.2), we have

$$
\begin{aligned}
|\Sigma_1(t)|^2 &= \sum_{n=1}^N \sigma^2_{1,n}(t) \leq \sum_{n=1}^\infty \sigma^2_{1,n}(t) = \int_0^1 |\sigma(x,z(x,t))|^2\mathrm{d}x\\
&\leq \bar{\sigma}^2\sum_{n=1}^\infty z_n^2(t) = \bar{\sigma}_1^2|\mathbb{I}X(t)|^2 + \bar{\sigma}_1^2\sum_{n=N+1}^\infty z_n^2(t).
\end{aligned}
\tag{2.63}
$$

Denote $\eta(t)=\mathrm{col}\{X(t),\zeta(t-\tau_y),\eta_1(t),\eta_2(t)\}$ with $\eta_1(t)$, $\eta_2(t)$ given by (2.59). By (2.51)–(2.54), (2.60)–(2.63), and using the S-Procedure (see proposition 3.2 in [8]), we obtain

$$
\begin{aligned}
&\mathbb{E}[\mathcal{L}V(t)+2\delta_0V(t)] - 2\delta_1\sup_{s_k\leq\theta\leq t}\mathbb{E}V(\theta)\\
&\quad + \beta\left[\bar{\sigma}_1^2\mathbb{E}|\mathbb{I}X(t)|^2 + \bar{\sigma}_1^2\mathbb{E}\sum_{n=N+1}^\infty z_n^2(t) - \mathbb{E}|\Sigma_1(t)|^2\right]\\
&\leq \mathbb{E}[\eta^{\mathrm{T}}(t)\Phi_1\eta(t)] + \mathbb{E}\left[\sum_{n=N+1}^\infty \mu_n z_n^2(t)\right] + \mathbb{E}[\Sigma_1^{\mathrm{T}}(t)\Phi_2\Sigma_1(t)] \leq 0,
\end{aligned}
\tag{2.64}
$$

provided $\mu_n := 2\rho(-\lambda_n+q+\delta_0) + \frac{2\rho^2}{\alpha}\lambda_n + (\beta+\rho)\bar{\sigma}_1^2 < 0$ for all $n>N$ and

$$
\begin{aligned}
\Phi_1 = &\left[\begin{array}{c|cc} \Omega_1 & \Theta_1 & \Theta_2 \\ \hline * & \multicolumn{2}{c}{\mathrm{diag}\{\Omega_2,\Omega_3\}} \end{array}\right] + \tau^2_{M,y}\Lambda_1^{\mathrm{T}}R_1\Lambda_1 + \tau_{M,y}\bar{\sigma}_2^2\Lambda_2^{\mathrm{T}}Q_{12}\Lambda_2\\
&+ \tau^2_{M,u}\Lambda_1^{\mathrm{T}}\mathbb{K}_0^{\mathrm{T}}R_2\mathbb{K}_0\Lambda_1 + \tau_{M,u}\bar{\sigma}_2^2\Lambda_2^{\mathrm{T}}\mathbb{K}_0^{\mathrm{T}}Q_{22}\mathbb{K}_0\Lambda_2 < 0,
\end{aligned}
\tag{2.65a}
$$

$$
\Phi_2 = P + \tau_{M,y}Q_{11} + \tau_{M,u}\mathbb{K}_0^{\mathrm{T}}Q_{21}\mathbb{K}_0 - (\rho+\beta)I < 0,
\tag{2.65b}
$$

where

$$
(2.66)\quad
\begin{aligned}
&\Omega_1 = \begin{bmatrix} \tilde{\Omega}_1^{(11)} + \Omega_1^{(11)} & P\mathbb{L}_0 + \bar{\sigma}_2^2 \mathbb{C}^{\mathrm{T}} \mathbb{L}_0^{\mathrm{T}} P \mathbb{L}_0 \\ * & -2\delta_1 \rho \|c\|_N^{-2} + \bar{\sigma}_2^2 \mathbb{L}_0^{\mathrm{T}} P \mathbb{L}_0 \end{bmatrix},\\
&\tilde{\Omega}_1^{(11)} = PF_0 + F_0^{\mathrm{T}} P + 2\delta P + \tfrac{2\alpha}{\pi^2 N}(\tfrac{1}{N}+1)\mathbb{K}_0^{\mathrm{T}}\mathbb{K}_0, \ \ \delta = \delta_0 - \delta_1,\\
&\Omega_1^{(11)} = \bar{\sigma}_2^2 \mathbb{C}^{\mathrm{T}} \mathbb{L}_0^{\mathrm{T}} P \mathbb{L}_0 \mathbb{C} + (1-\varepsilon_y) S_1 + (1-\varepsilon_u)\mathbb{K}_0^{\mathrm{T}} S_2 \mathbb{K}_0 + (\rho+\beta)\bar{\sigma}_1^2 \mathbb{I}^{\mathrm{T}}\mathbb{I},\\
&\Omega_2 = \left[\begin{array}{cc|c} \Omega_2^{(11)} & -\varepsilon_y(S_1+G_1) & \varepsilon_y[R_1, G_1, R_1, G_1] \\ * & -\varepsilon_y(S_1+R_1) & \varepsilon_y[G_1^{\mathrm{T}}, R_1, G_1^{\mathrm{T}}, R_1] \\ \hline * & * & -\varepsilon_y \Omega_2^{(33)} \end{array}\right],\\
&\Omega_2^{(11)} = \bar{\sigma}_2^2 \mathbb{C}^{\mathrm{T}} \mathbb{L}_0^{\mathrm{T}} P \mathbb{L}_0 \mathbb{C} - 2\delta_1 P - \varepsilon_y(S_1+R_1),\\
&\Omega_2^{(33)} = \mathrm{diag}\{R_1+Q_{11}, R_1+Q_{11}, R_1+Q_{12}, R_1+Q_{12}\},\\
&\Omega_3 = \left[\begin{array}{cc|c} \Omega_3^{(11)} & -\varepsilon_u(S_2+G_2) & \varepsilon_u[R_2, G_2, R_2, G_2] \\ * & -\varepsilon_u(S_2+R_2) & \varepsilon_u[G_2, R_2, G_2, R_2] \\ \hline * & * & -\varepsilon_u \Omega_3^{(33)} \end{array}\right],\\
&\Omega_3^{(11)} = \tfrac{2\alpha}{\pi^2 N}(\tfrac{1}{N}+1) - \varepsilon_u(S_2+R_2),\\
&\Omega_3^{(33)} = \mathrm{diag}\{R_2+Q_{21}, R_2+Q_{21}, R_2+Q_{22}, R_2+Q_{22}\},\\
&\Theta_1 = \begin{bmatrix} P(2\delta_1 I - F_1) - \bar{\sigma}_2^2 \mathbb{C}^{\mathrm{T}} \mathbb{L}_0^{\mathrm{T}} P \mathbb{L}_0 \mathbb{C} + \varepsilon_y S_1 & \varepsilon_y S_1 & 0_{2N\times 8N} \\ -\bar{\sigma}_2^2 \mathbb{L}_0^{\mathrm{T}} P \mathbb{L}_0 \mathbb{C} & 0_{1\times 2N} & 0_{1\times 8N} \end{bmatrix},\\
&\Theta_2 = \begin{bmatrix} -PF_2 + \varepsilon_u \mathbb{K}_0^{\mathrm{T}} S_2 & \varepsilon_u \mathbb{K}_0^{\mathrm{T}} S_2 & 0_{2N\times 4} \\ 0_{1\times 1} & 0_{1\times 1} & 0_{1\times 4} \end{bmatrix},\\
&\Lambda_1 = [F_0, \mathbb{L}_0, -F_1, 0_{2N\times 10N}, -F_2, 0_{2N\times 5}],\\
&\Lambda_2 = [\mathbb{L}_0\mathbb{C}, \mathbb{L}_0, -\mathbb{L}_0\mathbb{C}, 0_{2N\times(10N+6)}].
\end{aligned}
$$

From the monotonicity of λ_n and Schur complement, we find that $\mu_n < 0$ for all $n > N$ iff

$$
(2.67)\qquad \begin{bmatrix} 2\rho(-\lambda_{N+1}+q+\delta_0) + (\rho+\beta)\bar{\sigma}_1^2 & \rho \\ * & -\frac{\alpha}{2\lambda_{N+1}} \end{bmatrix} < 0.
$$

Applying Itô's formula for $V_P(t)$ along stochastic ODE (2.40a) (see [27, Theorem 4.18]), we obtain for all $t \geq 0$ and $\Delta > 0$

$$
(2.68)\qquad V_P(t+\Delta) - V_P(t) = \int_t^{t+\Delta} \mathcal{L}V_P(s)\mathrm{d}s + \sum_{i=1}^2 \int_t^{t+\Delta} \tfrac{\partial V_P}{\partial X}(s)\Sigma_i(s)\mathrm{d}\mathcal{W}_i(s),
$$

where $\frac{\partial V_P}{\partial X}(s) = 2X^{\mathrm{T}}(s)P$. Taking expectation on both sides of (2.68) and applying Fubini's theorem (see [27, Theorem 2.39]) we have

$$
\mathbb{E}V_P(t+\Delta) - \mathbb{E}V_P(t) = \mathbb{E}[\int_t^{t+\Delta} \mathcal{L}V_P(s)\mathrm{d}s] = \int_t^{t+\Delta} \mathbb{E}[\mathcal{L}V_P(s)]\mathrm{d}s,
$$

which implies that

$$
(2.69)\qquad
\begin{aligned}
D^+\mathbb{E}V_P(t) &= \limsup_{\Delta\searrow 0} \tfrac{\mathbb{E}V_P(t+\Delta) - \mathbb{E}V_P(t)}{\Delta}\\
&= \limsup_{\Delta\searrow 0} \tfrac{1}{\Delta}\int_t^{t+\Delta} \mathbb{E}[\mathcal{L}V_P(s)]\mathrm{d}s = \mathbb{E}[\mathcal{L}V_P(t)],
\end{aligned}
$$

where the limit exists because $\mathbb{E}[\mathcal{L}V_P(t)]$ is continuous (see Lemma 3.1 in [16]). Note that (2.69) also holds with V_P replaced by V_1, V_{S_i}, V_{R_i}, $V_{Q_{1i}}$ and $V_{Q_{2i}}$, $i = 1, 2$. Since $w(t)$ is a strong solution to (2.46), applying Itô's formula for $V_2(t, w(t))$ along with (2.46) (see [2, Theorem 7.2.1]), we have for all $t \geq 0$ and $\Delta > 0$

$$
(2.70)\qquad
\begin{aligned}
&V_2(t+\Delta, w(t+\Delta)) - V_2(t, w(t)) = \int_t^{t+\Delta} \mathcal{L}V_2(s, w(s))\mathrm{d}s\\
&\quad + 2\rho \int_t^{t+\Delta} \langle \tfrac{\partial V_2}{\partial w}(s, w(s)), \sigma_1(\cdot, w(s) + r\mathbb{K}_0 X(s-\tau_u(s)))\mathrm{d}\mathcal{W}_1(s)\rangle.
\end{aligned}
$$

Since the stochastic integral in (2.70) is a continuous L^2-martingale (see [5, subsection 3.4]), taking expectation on both sides of (2.70) and applying infinite-dimensional Fubini's theorem (see [5, Theorem 4.33]) we arrive at

$$\begin{aligned} D^+\mathbb{E}V_2(t,w(t)) &= \limsup_{\Delta\searrow 0} \tfrac{\mathbb{E}[V_2(t+\Delta,w(t+\Delta))-V_2(t,w(t))]}{\Delta} \\ &= \limsup_{\Delta\searrow 0} \tfrac{\mathbb{E}\int_t^{t+\Delta}\mathcal{L}V_2(w(s))\mathrm{d}s}{\Delta} = \limsup_{\Delta\searrow 0} \tfrac{\int_t^{t+\Delta}\mathbb{E}\mathcal{L}V_2(w(s))\mathrm{d}s}{\Delta} \\ &= \mathbb{E}[\mathcal{L}V_2(t,w(t))]. \end{aligned} \tag{2.71}$$

By the definition of $V(t)$ in (2.42), (2.43), and combining (2.69) and (2.71), we arrive at $D^+\mathbb{E}V(t) = \mathbb{E}[\mathcal{L}V(t)]$, which together with (2.64) gives

$$D^+\mathbb{E}V(t) + 2\delta_0\mathbb{E}V(t) - 2\delta_1 \sup_{s_k\le\theta\le t}\mathbb{E}V(\theta) \le 0,\ t\ge 0.$$

By (2.33) we have that $\mathbb{E}\|z(\cdot,t)\|^2_{L^2}$ and $\mathbb{E}|X(t)|^2$ are continuous. From the construction of $V(t)$ in (2.42), we see that $\mathbb{E}V(t)$ is continuous. Then employing Halanay's inequality (see [8, p. 138]) we arrive at

$$\mathbb{E}V(t) \le \mathbb{E}V(0)\mathrm{e}^{-2\delta_\tau t},\ t\ge 0, \tag{2.72}$$

where $\delta_\tau > 0$ is the unique solution of $\delta_\tau = \delta_0 - \delta_1\mathrm{e}^{2\delta_\tau\tau_{M,y}}$. From (2.42) we have

$$\begin{aligned} \mathbb{E}V(0) &= \mathbb{E}V_{\mathrm{nom}}(0) \le \max\{\lambda_{\max}(P),\rho\}\mathbb{E}\|z_0\|^2_{L^2}, \\ \mathbb{E}V(t) &\ge \mathbb{E}V_{\mathrm{nom}}(t) \ge \min\{\lambda_{\min}(P),\rho\}\mathbb{E}\|z(\cdot,t)\|^2_{L^2}, \end{aligned}$$

which together with (2.72) implies

$$\mathbb{E}[\|z(\cdot,t)\|^2_{L^2} + \|z(\cdot,t)-\hat{z}(\cdot,t)\|^2_{L^2}] \le M\mathrm{e}^{-2\delta_\tau t}\mathbb{E}\|z_0\|^2_{L^2}, \quad t\ge 0, \tag{2.73}$$

for some $M\ge 1$.

For the feasibility proof of inequalities (2.65) and (2.67) with large enough N and small enough $\tau_{M,y}$, $\tau_{M,u}$, $\bar{\sigma}_1$, and $\bar{\sigma}_2$, let $S_i=0$, $G_i=0$, $i=1,2$. First, choose $\alpha=2$, $\rho=1$. Monotonicity of $\{\lambda_n\}_{n=1}^\infty$ implies (2.67) for sufficiently large N. Next, to prove the feasibility of LMIs (2.65), taking $\bar{\sigma}_1,\bar{\sigma}_2\to 0^+$, $\tau_{M,y},\tau_{M,u}\to 0^+$, it is sufficient to show

$$P < (1+\beta)I,\ \left[\begin{array}{c|c} \tilde{\Omega}_1 & \tilde{\Theta}_1\ \ \tilde{\Theta}_2 \\ \hline * & \mathrm{diag}\{\tilde{\Omega}_2,\tilde{\Omega}_3\} \end{array}\right] < 0, \tag{2.74}$$

where

$$\begin{aligned} &\tilde{\Theta}_1 = \begin{bmatrix} P(2\delta_1 I - F_1) & 0_{2N\times 10N} \\ 0_{1\times 2N} & 0 \end{bmatrix},\ \tilde{\Theta}_2 = \begin{bmatrix} -PF_2 & 0_{2N\times 5} \\ 0_{1\times 1} & 0 \end{bmatrix}, \\ &\tilde{\Omega}_1 = \begin{bmatrix} \tilde{\Omega}_1^{(11)} & P\mathbb{L}_0 \\ * & -2\delta_1\|c\|_N^{-2} \end{bmatrix},\ \tilde{\Omega}_2 = \left[\begin{array}{cc|cccc} -2\delta_1 P - R_1 & 0 & R_1 & 0 & R_1 & 0 \\ * & -R_1 & 0 & R_1 & 0 & R_1 \\ \hline * & * & & -\Omega_2^{(33)} & & \end{array}\right], \\ &\tilde{\Omega}_3 = \left[\begin{array}{cc|cccc} \frac{4}{\pi^2 N}(\frac{1}{N}+1) - R_2 & 0 & R_2 & 0 & R_2 & 0 \\ * & -R_2 & 0 & R_2 & 0 & R_2 \\ \hline * & * & & -\Omega_3^{(33)} & & \end{array}\right]. \end{aligned}$$

Here $\tilde{\Omega}_1^{(11)}$, $\Omega_2^{(33)}$, and $\Omega_3^{(33)}$ are given in (2.66). Applying the Schur complement repeatedly and substituting $Q_{11}=Q_{12}=3R_1$ and $Q_{21}=Q_{22}=3R_2$ into (2.74), we find that (2.74) holds iff

$$\begin{aligned} &\tilde{\Omega}_1^{(11)} + \tfrac{\|c\|_N^2}{2\delta_1}P\mathbb{L}_0\mathbb{L}_0^{\mathrm{T}}P + \tfrac{PF_2F_2^{\mathrm{T}}P}{\frac{1}{2}R_2 - \frac{4}{\pi^2 N}(\frac{1}{N}+1)} \\ &\quad + P(2I - F_1)(2\delta_1 P + \tfrac{1}{2}R_1)^{-1}(2\delta_1 I - F_1)^{\mathrm{T}}P < 0, \quad P < (1+\beta)I. \end{aligned} \tag{2.75}$$

Inspired by the reduced-order LMI method in [25], we let P be of the form $P = \mathrm{diag}\{P_0, I_{N-N_0}, pI_{N-N_0}\}$ with $0 < P_0 \in \mathbb{R}^{2N_0\times 2N_0}$ and scalar $p > 0$. We rewrite F_0 defined in (2.39) as

$$\begin{aligned}
&F_0 = \left[\begin{array}{c|c} F_0^{(11)} & F_0^{(12)} \\ \hline F_0^{(21)} & F_0^{(22)} \end{array}\right],\ F_0^{(11)} = \begin{bmatrix} A_0 + B_0K_0 & L_0C_0 \\ 0 & A_0 - L_0C_0 \end{bmatrix},\\
&F_0^{(12)} = \begin{bmatrix} 0 & L_0C_1 \\ 0 & -L_0C_1 \end{bmatrix},\ F_0^{(21)} = \begin{bmatrix} B_1K_0 & 0 \\ 0 & 0 \end{bmatrix},\ F_0^{(22)} = \mathrm{diag}\{A_1, A_1\}.
\end{aligned}$$

Let $\hat{K}_0 = [K_0, 0_{1\times N_0}]$ and $\hat{L}_0 = \mathrm{col}\{L_0, -L_0\}$. We have

$$\begin{aligned}
&\tilde{\Omega}_1^{(11)} + \tfrac{\|c\|_N^2}{2\delta_1} P\mathbb{L}_0\mathbb{L}_0^{\mathrm{T}} P = \begin{bmatrix} \Xi_1 & P_0F_0^{(12)} + F_0^{(21)\mathrm{T}}\mathrm{diag}\{I_{N-N_0}, pI_{N-N_0}\} \\ * & \Xi_2 \end{bmatrix},\\
&\Xi_1 := P_0F_0^{(11)} + F_0^{(11)\mathrm{T}}P_0 + 2\delta P_0 + \tfrac{4}{\pi^2 N}(\tfrac{1}{N}+1)\hat{K}_0^{\mathrm{T}}\hat{K}_0 + \tfrac{\|c\|_N^2}{2\delta_1}P_0\hat{L}_0\hat{L}_0^{\mathrm{T}}P_0,\\
&\Xi_2 := 2\mathrm{diag}\{A_1 + \delta I,\ p(A_1 + \delta I)\}.
\end{aligned}$$

By (2.8), we have $\Xi_2 < 0$. Applying the Schur complement we find that

$$\tilde{\Omega}_1^{(11)} + \tfrac{\|c\|_N^2}{2\delta_1} P\mathbb{L}_0\mathbb{L}_0^{\mathrm{T}} P < 0 \tag{2.76}$$

iff

$$\begin{aligned}
&\Xi_3 := \Xi_1 + \tfrac{\bar{c}^N}{2p}P_0\hat{L}_0\hat{L}_0^{\mathrm{T}}P_0 + \tfrac{\bar{b}^N}{2}\hat{K}_0^{\mathrm{T}}\hat{K}_0 < 0,\\
&\bar{b}^N = \textstyle\sum_{n=N_0+1}^{N} \frac{b_n^2}{\lambda_n - q - \delta},\ \bar{c}^N = \sum_{n=N_0+1}^{N} \frac{c_n^2}{\lambda_n - q - \delta},
\end{aligned} \tag{2.77}$$

where $F_0^{(12)}\Xi_2^{-1}\mathrm{diag}\{I_{N-N_0}, pI_{N-N_0}\}F_0^{(21)} = 0$ is used. By using (1.2) and the integral convergence test, we have

$$\bar{b}^N \le 2\textstyle\sum_{n=N_0}^{\infty} \frac{1}{(n\pi)^2 - q - \delta} \le \bar{b},\ \bar{c}^N \le \sum_{n=N_0}^{\infty} \frac{\|c\|_{N_0}^2}{(n\pi)^2 - q - \delta} \le \bar{c}\ \ \forall N > N_0,$$

where $\bar{b}$ and $\bar{c}$ are positive constants which are independent of N. Let $0 < P_0 \in \mathbb{R}^{2N_0\times 2N_0}$ solve the Lyapunov equation

$$P_0\big(F_0^{(11)} + \delta I\big) + \big(F_0^{(11)} + \delta I\big)^{\mathrm{T}}P_0 = -\chi I, \tag{2.78}$$

where $\chi > 0$ is independent of N and large enough such that $-\chi I + \frac{\bar{b}}{2}\hat{K}_0^{\mathrm{T}}\hat{K}_0 < 0$. Clearly, P_0 is independent of N. Substituting (2.78) into Ξ_3 given in (2.77), we have

$$\Xi_3 = -\chi I + \tfrac{\bar{b}^N}{2}\hat{K}_0^{\mathrm{T}}\hat{K}_0 + \tfrac{\bar{c}^N}{2p}P_0\hat{L}_0\hat{L}_0^{\mathrm{T}}P_0 + \tfrac{4}{\pi^2 N}(\tfrac{1}{N}+1)\hat{K}_0^{\mathrm{T}}\hat{K}_0 + \tfrac{\|c\|_N^2}{2\delta_1}P_0\hat{L}_0^{\mathrm{T}}\hat{L}_0P_0.$$

By taking $\beta = N^\gamma$ with $0 < \gamma < 1$, $\delta_1 = 1$, and $p = \sqrt{N}$, we get $\Xi_3 < 0$ for large enough N, which implies that (2.76) holds for large enough N. Since $P = \mathrm{diag}\{P_0, I_{N-N_0}, \sqrt{N}I_{N-N_0}\} = O(\sqrt{N})$, $N \to \infty$, substituting these and $R_1 = N^2 I$, $R_2 = N^2$ into (2.75) we have that LMIs (2.75) hold for sufficiently large N. By arguments similar to [22, Theorem 3.1], by continuity, inequalities (2.65) and (2.67) hold for $\tau_{M,y} = \tau_{M,u} = \bar{\sigma}_1 = \bar{\sigma}_2 = N^{-2}$ and large enough N. Summarizing, we have the following.

THEOREM 2.4. *Consider* (2.1) *with nonlinear noise function* σ_1 *satisfying* (2.2), *control law* (2.13), *noisy measurement* (2.3) *with* σ_2 *satisfying* (2.4), $z_0 \in \mathcal{D}(\mathcal{A}_1)$ *almost surely, and* $z_0 \in L^2(\Omega; L^2(0,1))$. *Given* $\delta > 0$, *let* $N_0 \in \mathbb{N}$ *satisfy* (2.8) *and* $N \in \mathbb{N}$ *satisfy* $N \geq N_0$. *Let* L_0 *and* K_0 *be obtained from* (2.12) *and* (2.26), *respectively. Given* $\tau_{M,y}, \tau_{M,u}, \delta_0, \bar{\sigma}_1, \bar{\sigma}_2 > 0$, *let there exist* $0 < P, S_1, R_1, Q_{11}, Q_{12} \in \mathbb{R}^{2N\times 2N}$, *scalars* $S_2, R_2, Q_{21}, Q_{22}, \alpha, \beta, \rho > 0$, $G_1 \in \mathbb{R}^{2N\times 2N}$, *and* $G_2 \in \mathbb{R}$ *such that the following LMIs hold with* $\delta_1 = \delta_0 - \delta$: *LMIs* (2.56), (2.67), *and* $\Phi_i < 0$ *(*$i = 1,2$*) with* Φ_i *given in* (2.65)–(2.66). *Then the solution* $z(x,t)$ *to* (2.1) *subject to the control law* (2.10), (2.13) *and the corresponding observer* $\hat{z}(x,t)$ *given by* (2.9) *satisfy* (2.73) *for some* $M \geq 1$, *where* $\delta_\tau > 0$ *is the unique solution of* $\delta_\tau = \delta_0 - \delta_1 \mathrm{e}^{2\delta_\tau \tau_{M,y}}$. *Moreover, the above LMIs always hold for large enough* N *and small enough* $\tau_{M,y}, \tau_{M,u}, \bar{\sigma}_1, \bar{\sigma}_2$.

Remark 2.5. If noise functions σ_1, σ_2 are of the linear form

$$\sigma_1(x,z) \equiv \bar{\sigma}_1 z, \;\; \sigma_2(z) = \bar{\sigma}_2 z, \tag{2.79}$$

we have the closed-loop system (2.40) with

$$\begin{aligned} &\sigma_{1,n}(t) = \bar{\sigma}_1 z_n(t), \;\; \Sigma_2(t) = \bar{\sigma}_2 \mathbb{L}_0[\zeta(t-\tau_y) + \mathbb{C}X(t) - \mathbb{C}\nu_y(t)], \\ &\Sigma_1(t) = \bar{\sigma}_1 \mathcal{I}_1 X(t), \;\; \mathcal{I}_1 = \begin{bmatrix} 0_{N_0\times N_0} & 0 & 0 & 0 \\ I_{N_0} & I_{N_0} & 0 & 0 \\ 0_{(N-N_0)\times N_0} & 0 & 0 & 0 \\ 0 & 0 & I_{N-N_0} & I_{N-N_0} \end{bmatrix}. \end{aligned}$$

In this case, the constraint (2.65b) is not needed. By arguments similar to (2.42)–(2.73), we find that the mean-square L^2 exponential stability of the closed-loop system can be guaranteed if (2.65a) and (2.67) are feasible with $\beta = 0$ and $\Omega_1^{(11)}$ therein replaced by

$$\begin{aligned} \Omega_1^{(11)} = &\bar{\sigma}_2^2 \mathbb{C}^{\mathrm{T}} \mathbb{L}_0^{\mathrm{T}} P \mathbb{L}_0 \mathbb{C} + (1-\varepsilon_y)S_1 + (1-\varepsilon_u)\mathbb{K}_0^{\mathrm{T}} S_2 \mathbb{K}_0 \\ &+ \bar{\sigma}_1^2 \mathcal{I}_1^{\mathrm{T}}(P + \tau_{M,y} Q_{11} + \tau_{M,u} \mathbb{K}_0^{\mathrm{T}} Q_{21} \mathbb{K}_0)\mathcal{I}_1. \end{aligned} \tag{2.80}$$

3. Sampled-data control using a generalized hold device. In this section, inspired by [26], we employ dynamic extension and suggest a sample-data implementation of the controller via a generalized hold device. As seen from Table 1 and Table 3 in section 4, the dynamic-extension-based method allows for a larger bound on sampling intervals than the direct method.

Consider the stochastic 1D heat equation (2.1) with Neumann actuation where control signal $u(t)$ is generated by a generalized hold device and is of the following form (see, e.g., [26]):

$$\dot{u}(t) = v(t_j), \; t \in [t_j, t_{j+1}), \;\; u(0) = 0. \tag{3.1}$$

Here v is the new control input and the values $\{v(t_j)\}_{j=1}^{\infty}$ are to be determined, and $\{t_j\}_{j=1}^{\infty}$ are the controller hold times defined below (2.13). Given $v(t_j)$, $u(t)$ is calculated as

$$u(t) = u(t_j) + v(t_j)(t - t_j), \; t \in [t_j, t_{j+1}).$$

Consider the nonlocal noisy measurement (2.3). Following [19, 26, 34, 40], introduce the change of variables

$$w(x,t) = z(x,t) - r(x)u(t), \;\; r(x) = x \tag{3.2}$$

to obtain the equivalent system

(3.3a) $$\mathrm{d}u(t) = v(t_j)\mathrm{d}t,\ t \in [t_j, t_{j+1}),\ \ u(0) = 0,$$

(3.3b) $$\begin{aligned}\mathrm{d}w(x,t) = [\tfrac{\partial^2}{\partial x^2} w(x,t) + qw(x,t) + qr(x)u(t) - r(x)v(t_j)]\mathrm{d}t \\ + \sigma_1(x, w(x,t) + r(x)u(t))\mathrm{d}\mathcal{W}_1(t),\ \ t \in [t_j, t_{j+1}),\end{aligned}$$

(3.3c) $$w(0,t) = 0,\ w_x(1,t) = 0,\ \ w(x,0) = z_0(x), x \in [0,1],$$

with the noisy measurement output satisfying

(3.4) $$\begin{aligned}\mathrm{d}y(t) = \langle c, w(\cdot, s_k) + r(\cdot)u(s_k)\rangle \mathrm{d}t \\ + \sigma_2(\langle c, w(\cdot, s_k) + r(\cdot)u(s_k)\rangle)\mathrm{d}\mathcal{W}_2(t),\ t \in [s_k, s_{k+1}).\end{aligned}$$

By the well-posedness analysis in subsection 3.2, we can prove that for initial values $z_0 \in L^2(\Omega; L^2(0,1))$ and $z_0 \in \mathcal{D}(\mathcal{A}_1)$ almost surely, for (3.3) there exists a unique global solution which satisfies the regularity

(3.5) $$w \in L^2(\Omega; C([0,\infty); L^2(0,1))) \cap L^2(\Omega \times [0,\infty)\backslash \mathcal{J}; H^1(0,1)).$$

Therefore, we can present the solution to (3.3a) as

(3.6) $$w(x,t) = \textstyle\sum_{n=1}^{\infty} w_n(t)\phi_n(x),\ w_n(t) = \langle w(\cdot,t), \phi_n\rangle.$$

The series (3.6) convergence in $L^2(0,1)$ to w in mean square follows from (3.5). Differentiating w_n in (3.6) and integrating by parts, we obtain

(3.7) $$\begin{aligned}\mathrm{d}w_n(t) = [(-\lambda_n + q)w_n(t) + a_n u(t) - b_n v(t_j)]\mathrm{d}t \\ + \tilde{\sigma}_{1,n}(t)\mathrm{d}\mathcal{W}_1(t),\ t \in [t_j, t_{j+1}),\end{aligned}$$

where $\tilde{\sigma}_{1,n}(t) = \langle \sum_{j=1}^{\infty} w_j(t)\phi_j + ru(t), \phi_n\rangle$ and $b_n = \langle r, \phi_n\rangle$, $a_n = q\langle r, \phi_n\rangle$, which can be calculated as

(3.8) $$a_n = (-1)^{n+1}\tfrac{\sqrt{2}q}{\lambda_n},\ b_n = (-1)^{n+1}\tfrac{\sqrt{2}}{\lambda_n},\ \ n \geq 1.$$

By using (1.2), (3.8) and the integral convergence test, we have

(3.9) $$\|a\|_N^2 \leq q^2\chi_N,\ \|b\|_N^2 \leq \chi_N,\ \chi_N = \tfrac{2}{N^3\pi^4}(\tfrac{1}{N} + \tfrac{1}{3}).$$

Let $\delta > 0$ be a desired decay rate, and let $N_0 \in \mathbb{N}$ satisfy (2.8), where N_0 will be the number of modes used for the controller design. Let $N \in \mathbb{N}$, $N_0 \leq N$, where N will define the dimension of the observer.

Similar to section 2, we use the time-delay approach to sampled-data control and introduce the representations (2.14). Following [21, 37], we construct a finite-dimensional observer of the form

(3.10) $$\hat{w}(x,t) = \textstyle\sum_{n=1}^{N} \hat{w}_n(t)\phi_n(x),$$

where $\hat{w}_n(t)$ $(1 \leq n \leq N)$ satisfy the ODEs

(3.11) $$\begin{aligned}&\mathrm{d}\hat{w}_n(t) = [(-\lambda_n + q)\hat{w}_n(t) + a_n u(t) - b_n v(t - \tau_u)]\mathrm{d}t \\ &\quad -l_n[\langle c, \hat{w}(\cdot, t-\tau_y) + r(\cdot)u(t-\tau_y)\rangle \mathrm{d}t - \mathrm{d}y(t)],\ t \geq 0, \\ &\hat{w}_n(0) = 0,\ \ 1 \leq n \leq N,\end{aligned}$$

with $y(t)$ satisfying (3.4) and scalar observer gains $\{l_n\}_{n=1}^N$.

Recall the notation of (2.11). Assume that $c_n \neq 0$, $1 \leq n \leq N_0$. Choose $l_1, \ldots, l_{N_0}$ such that $L_0 = [l_1, \ldots, l_{N_0}]^{\mathrm{T}}$ satisfies (2.12). Define

(3.12) $$\mathbf{a}_0 = [a_1, \ldots, a_{N_0}]^{\mathrm{T}},\ \ \tilde{B}_0 = \mathrm{col}\{1, -B_0\},\ \tilde{A}_0 = \begin{bmatrix} 0 & 0 \\ \mathbf{a}_0 & A_0 \end{bmatrix}.$$

Since $b_n \neq 0$ (see (3.8)), the pair $(\tilde{A}_0, \tilde{B}_0)$ is controllable. We propose an $N_0 + 1$-dimensional controller of the form

(3.13) $$v(t_j) = \tilde{K}_0 \hat{w}^{N_0}(t_j),\ \ \hat{w}^{N_0}(t) = [u(t), \hat{w}_1(t), \ldots, \hat{w}_{N_0}(t)]^{\mathrm{T}},$$

where $\tilde{K}_0 \in \mathbb{R}^{1\times(N_0+1)}$ will be obtained by the state-feedback controller design.

3.1. Design of gain $\tilde{K}_0$. Consider (3.3) with $\sigma_1 \equiv 0$. By presenting the solution as (3.6) and differentiating z_n in (3.6), we obtain (3.7) with $\tilde{\sigma}_{1,n} \equiv 0$. Let $\delta > 0$ be a desired decay rate, and let $N_0 \in \mathbb{N}$ satisfy $-\lambda_n + q + \delta < 0$, $n > N_0$.

Since $b_n \neq 0$ (see (3.8)), the pair $(\tilde{A}_0, \tilde{B}_0)$ is controllable ($\tilde{A}_0, \tilde{B}_0$ are defined as in (3.12)). We propose a (N_0+1)-dimensional controller of the form

$$v(t_j) = \tilde{K}_0 w^{N_0}(t_j), \;\; w^{N_0}(t) = [u(t), w_1(t), \dots, w_{N_0}(t)]^{\mathrm{T}}, \tag{3.14}$$

where $\tilde{K}_0 \in \mathbb{R}^{1\times(N_0+1)}$ is obtained from the LMIs below. We have the closed-loop system

$$\begin{aligned}&\dot{w}^{N_0}(t) = \tilde{A}_0 w^{N_0}(t) + \tilde{B}_0 \tilde{K}_0 w^{N_0}(t-\tau_u), \quad t \geq 0,\\ &\dot{w}_n(t) = (-\lambda_n + q) w_n(t) + a_n \mathbb{1} w^{N_0}(t) + b_n \tilde{K}_0 w^{N_0}(t-\tau_u), t \geq 0, n > N_0,\end{aligned}$$

where τ_u is defined as in (2.14). Consider the Lyapunov–Krasovskii functional:

$$\begin{aligned}V(t) = &\textstyle\sum_{n=N_0+1}^{\infty} \rho w_n^2(t) + |w^{N_0}(t)|_P^2 + \int_{t-\tau_M}^{t} \mathrm{e}^{-2\delta(t-s)} |w^{N_0}(s)|_S^2 \mathrm{d}s\\ &+ \tau_M \textstyle\int_{-\tau_M}^{0} \int_{t+\theta}^{t} \mathrm{e}^{-2\delta(t-s)} |\dot{w}^{N_0}(s)|_R \mathrm{d}s\mathrm{d}\theta,\end{aligned}$$

where $0 < P, S, R \in \mathbb{R}^{(N_0+1)\times(N_0+1)}$, and $\rho > 0$. Let $G \in \mathbb{R}^{(N_0+1)\times(N_0+1)}$ satisfy (2.17). Choose $P_3 = \varepsilon P_2$ with a tuning scalar $\varepsilon > 0$ and denote

$$\bar{P}_2 = P_2^{-1}, \, Y = K\bar{P}_2, \;\; [\bar{P}, \bar{R}, \bar{S}, \bar{G}] = \bar{P}_2^{\mathrm{T}} [P, R, S, G] \bar{P}_2, \;\; \bar{\rho} = 1/\rho.$$

By arguments similar to (2.18)–(2.24) and using following Young's inequalities for $\alpha_1, \alpha_2 > 0$:

$$\begin{aligned}&\rho \textstyle\sum_{n=N_0+1}^{\infty} 2 w_n(t) a_n \mathbb{1} w^{N_0}(t) \leq \sum_{n=N_0+1}^{\infty} \rho^2 \alpha_1 w_n^2(t) + \frac{\|a\|_{N_0}^2}{\alpha_1} |\mathbb{1} w^{N_0}(t)|^2,\\ &\rho \textstyle\sum_{n=N_0+1}^{\infty} 2 w_n(t) b_n \tilde{K}_0 w^{N_0}(t-\tau_u) \leq \sum_{n=N_0+1}^{\infty} \rho^2 \alpha_2 w_n^2(t) + \frac{\|b\|_{N_0}^2}{\alpha_2} |\tilde{K}_0 w^{N_0}(t-\tau_u)|^2,\end{aligned}$$

we arrive at

$$\dot{V}(t) + 2\delta V(t) \leq 0, \tag{3.15}$$

provided (2.24) and

$$\begin{aligned}&2\bar{\rho}(-\lambda_{N_0+1} + q + \delta) + \alpha_1 + \alpha_2 < 0,\\ &\begin{bmatrix} \bar{\Phi}_{11} & \bar{\Phi}_{12} & \varepsilon_u \bar{G} & \varepsilon_u(\bar{R} - \bar{G}) + \tilde{B}_0 Y & 0 & \bar{P}_2^{\mathrm{T}} \mathbb{1}^{\mathrm{T}} \\ * & \bar{\Phi}_{22} & 0 & \varepsilon \tilde{B}_0 Y & 0 & 0 \\ * & * & -\varepsilon_u(\bar{S} + \bar{R}) & \varepsilon_u(\bar{R} - \bar{G}^{\mathrm{T}}) & 0 & 0 \\ * & * & * & \varepsilon_u(-2\bar{R} + \bar{G} + \bar{G}^{\mathrm{T}}) & Y^{\mathrm{T}} & 0 \\ * & * & * & * & -\frac{\alpha_2}{\|b\|_{N_0}^2} & 0 \\ * & * & * & * & * & -\frac{\alpha_1}{\|a\|_{N_0}^2} \end{bmatrix} < 0,\\ &\bar{\Phi}_{11} = \bar{P}_2^{\mathrm{T}} \tilde{A}_0^{\mathrm{T}} + \tilde{A}_0 \bar{P}_2 + \bar{S} - \varepsilon_u \bar{R}, \;\; \varepsilon_u = \mathrm{e}^{-2\delta \tau_{M,u}},\\ &\bar{\Phi}_{12} = \bar{P} - \bar{P}_2 + \varepsilon \bar{P}_2^{\mathrm{T}} \tilde{A}_0^{\mathrm{T}}, \;\; \bar{\Phi}_{22} = -\varepsilon(\bar{P}_2 + \bar{P}_2^{\mathrm{T}}) + \tau_{M,u}^2 \bar{R}.\end{aligned} \tag{3.16}$$

The feasibility of (3.15) guarantees the exponential stability of closed-loop system (3.3) with $\sigma_1 \equiv 0$ under state-feedback controller (3.14). In addition, if LMIs (2.24) and (3.16) hold, the controller gain is obtained by $\tilde{K}_0 = Y \bar{P}_2^{-1}$.

Remark 3.1. In [25, 26], the controller gain is obtained from

$$P_c(\tilde{A}_0 + \tilde{B}_0 \tilde{K}_0) + (\tilde{A}_0 + \tilde{B}_0 \tilde{K}_0)^{\mathrm{T}} P_c \leq -2\delta P_c, \tag{3.17}$$

where $0 < P_c \in \mathbb{R}^{(N_0+1)\times(N_0+1)}$. Let $\bar{P}_c = P_c^{-1}$ and $Y = K_0\bar{P}_c$. By arguments similar to those in Remark 2.2, we have that (3.17) holds iff

$$\tilde{A}_0\bar{P}_c + \bar{P}_c\tilde{A}_0^{\mathrm{T}} + \tilde{B}_0Y + Y^{\mathrm{T}}\tilde{B}_0^{\mathrm{T}} + 2\delta\bar{P}_c \le 0. \tag{3.18}$$

If LMI (3.18) is feasible, the controller gain is given by $K_0 = Y\bar{P}_c^{-1}$. As seen from Table 3 in section 4, the state-feedback controller design based on LMIs (2.24) are (3.16) allows larger $\tau_{M,u}$ than the gain obtained from (3.17).

3.2. Well-posedness of the closed-loop system. For well-posedness of the closed-loop system (3.3) and (3.11) with control input (3.13), we recall (2.30) and the notation of (2.31), and we further define

$$\begin{aligned}&\tilde{B} = \mathrm{col}\{1, -B_0, -B_1\},\ \mathbb{1}_0 = [1, 0_{1\times N_0}],\ \mathbb{1}_1 = [1, 0_{1\times N}],\\ &\tilde{C} = [0_{1\times 1}, c_1, \ldots, c_N],\ \tilde{L} = \mathrm{col}\{0_{1\times 1}, L_0, 0_{(N-N_0)\times 1}\},\\ &\mathbf{a}_1 = [a_{N_0+1}, \ldots, a_N]^{\mathrm{T}}, \tilde{K}_1 = [\tilde{K}_0, 0_{1\times(N-N_0)}],\ r_c = \langle r, c\rangle.\end{aligned} \tag{3.19}$$

Consider $\mathcal{H}$, $\mathcal{V}$, and $\mathcal{V}'$ as in section 2.2 with $\mathbb{R}^N$ therein replaced by $\mathbb{R}^{N+1}$. Define the state $\xi(t)$ as $\xi(t) = \mathrm{col}\{w(\cdot,t), \hat{w}^N(t)\}$, $\hat{w}^N(t) = [u(t), \hat{w}_1(t), \ldots, \hat{w}_N(t)]^{\mathrm{T}}$. We can present the closed-loop system as (2.32) with $\mathcal{W}(t) = [\mathcal{W}_1(t), \mathcal{W}_2(t)]^{\mathrm{T}}$ and

$$\begin{aligned}&\mathcal{A} = \mathrm{diag}\{\mathcal{A}_1, \mathcal{A}_2\},\ \mathcal{A}_2 = \begin{bmatrix}\tilde{A}_0 & 0\\ \mathbf{a}_1\mathbb{1}_0 & A_1\end{bmatrix},\\ &f(t,\xi(t)) = \begin{bmatrix} qw(\cdot,t) + qr(\cdot)\mathbb{1}_1\hat{w}^N(t) - r(\cdot)\tilde{K}_1\hat{w}^N(t-\tau_u)\\ \tilde{B}\tilde{K}_1\hat{w}^N(t-\tau_u) - \tilde{L}\tilde{C}\hat{w}^N(t-\tau_y) + \tilde{L}\langle c, w(\cdot, t-\tau_y)\rangle\end{bmatrix},\\ &g(t,\xi(t)) = \mathrm{diag}\{\sigma_1(\cdot, w(\cdot,t) + r(\cdot)\mathbb{1}_1\hat{w}^N(t)),\ \tilde{L}\sigma_2(\langle c, w(\cdot,t-\tau_y)\rangle + r_c\mathbb{1}_1\hat{w}^N(t-\tau_y))\}.\end{aligned}$$

By arguments similar to those in section 2.2, we obtain, for $z_0 \in \mathcal{D}(\mathcal{A}_1)$ almost surely and $z_0 \in L^2(\Omega; L^2(0,1))$, existence of a unique solution $\xi \in L^2(\Omega; C([0,\infty); \mathcal{H})) \cap L^2([0,\infty)\backslash\mathcal{J}\times\Omega; \mathcal{V})$, where $\mathcal{J} = \{s_k\}_{k=0}^{\infty} \cup \{t_j\}_{j=0}^{\infty}$, such that $\xi(t) \in \mathcal{D}(\mathcal{A})$, $t \ge 0$, almost surely.

3.3. Mean-square L^2 stability analysis. Let

$$e_n(t) = w_n(t) - \hat{w}_n(t). \tag{3.20}$$

By using (3.4) and (3.10), we write the last term on the right-hand side of (3.11) as

$$\begin{aligned}&\langle c, \hat{w}(\cdot, t-\tau_y) + r(\cdot)u(t-\tau_y)\rangle \mathrm{d}t - \mathrm{d}y(t)\\ &\quad = [-\textstyle\sum_{n=1}^N c_n e_n(t-\tau_y) - \zeta(t-\tau_y)]\mathrm{d}t - \sigma_2(\hat{\zeta}(t-\tau_y))\mathrm{d}\mathcal{W}_2(t),\\ &\quad \zeta(t) = \textstyle\sum_{n=N+1}^{\infty} c_n w_n(t),\ \hat{\zeta}(t) = \zeta(t) + \sum_{n=1}^N c_n[\hat{w}_n(t) + e_n(t)] + r_c u(t).\end{aligned}$$

By the Cauchy–Schwarz inequality, we have

$$\zeta^2(t) \le \|c\|_N^2 \textstyle\sum_{n=N+1}^{\infty} w_n^2(t). \tag{3.21}$$

Then the error equations have the form

$$\begin{aligned}\mathrm{d}e_n(t) = \{&(-\lambda_n + q)e_n(t) - l_n[\textstyle\sum_{n=1}^N c_n e_n(t-\tau_y) + \zeta(t-\tau_y)]\}\mathrm{d}t\\ &+ \sigma_{1,n}(t)\mathrm{d}\mathcal{W}_1(t) - l_n\sigma_2(\hat{\xi}(t-\tau_y))\mathrm{d}\mathcal{W}_2(t), \quad t \ge 0.\end{aligned} \tag{3.22}$$

Recall the notation in (3.12), (3.13), (3.19) and $e^{N_0}(t)$, $e^{N-N_0}(t)$, C_1 given in (2.39). Let

(3.23)
$$\begin{aligned}
&\hat{w}^{N-N_0}(t)=[\hat{w}_{N_0+1}(t),\dots,\hat{w}_N(t)]^{\mathrm{T}}, \quad \mathbb{K}_0=[\tilde{K}_0,0_{1\times(2N-N_0)}], \quad \mathbb{1}=[1,0_{1\times 2N}],\\
&\tilde{L}_0=\mathrm{col}\{0_{1\times 1},L_0\}, \quad \mathbb{L}_0=\mathrm{col}\{\tilde{L}_0,-L_0,0_{2(N-N_0)\times 1}\},\ \mathbb{C}=[r_c,C_0,C_0,C_1,C_1],\\
&X(t)=\mathrm{col}\{\hat{w}^{N_0}(t),e^{N_0}(t),\hat{w}^{N-N_0}(t),e^{N-N_0}(t)\}, \quad F_1=\mathbb{L}_0\cdot[0,C_0,0,C_1],\\
&F_0=\begin{bmatrix} \tilde{A}_0+\tilde{B}_0\tilde{K}_0 & \tilde{L}_0C_0 & 0 & \tilde{L}_0C_1\\ 0 & A_0-L_0C_0 & 0 & -L_0C_1\\ \mathbf{a}_1\mathbb{1}_0-B_1\tilde{K}_0 & 0 & A_1 & 0\\ 0 & 0 & 0 & A_1\end{bmatrix}, \quad F_2=\mathrm{col}\{\tilde{B}_0,0,-B_1,0\}.
\end{aligned}$$

In the following analysis, the definition for notations ν_y and ν_u are the same as that in (2.39). By (3.1), (3.11) and (3.22), we have the closed-loop system

(3.24a)
$$\mathrm{d}X(t)=F(t)\mathrm{d}t+\Sigma_1(t)\mathrm{d}\mathcal{W}_1(t)+\Sigma_2(t)\mathrm{d}\mathcal{W}_2(t),\ t\geq 0,$$

(3.24b)
$$\begin{aligned}\mathrm{d}w_n(t)=&[(-\lambda_n+q)w_n(t)+(a_n\mathbb{1}+b_n\mathbb{K}_0)X(t)\\ &-b_n\mathbb{K}_0\nu_u(t)]\mathrm{d}t+\tilde{\sigma}_{1,n}(t)\mathrm{d}\mathcal{W}_1(t),\ t\geq 0,\ n>N,\end{aligned}$$

where

(3.25)
$$\begin{aligned}
&F(t)=F_0X(t)-F_1\nu_y(t)-F_2\mathbb{K}_0\nu_u(t)+\mathbb{L}_0\zeta(t-\tau_y),\\
&\Sigma_1(t)=\mathrm{col}\{0_{(N_0+1)\times 1},\tilde{\sigma}^{N_0}(t),0_{(N-N_0)\times 1},\tilde{\sigma}^{N-N_0}(t)\},\\
&\tilde{\sigma}^{N_0}(t)=\mathrm{col}\{\tilde{\sigma}_{1,n}(t)\}_{n=1}^{N_0},\ \tilde{\sigma}^{N-N_0}(t)=\mathrm{col}\{\tilde{\sigma}_{1,n}(t)\}_{n=N_0+1}^{N},\\
&\Sigma_2(t)=\mathbb{L}_0\sigma_2(\zeta(t-\tau_y)+\mathbb{C}X(t)-\mathbb{C}\nu_y(t)).
\end{aligned}$$

For mean-square L^2 exponential stability of the closed-loop system (2.40), we define the Lyapunov functional as (2.42) with $0<P,S_1,R_1,Q_{11},Q_{12}\in\mathbb{R}^{(2N+1)\times(2N+1)}$, positive scalars $S_2,R_2,Q_{21},Q_{22},\rho$, and $\sum_{n=N+1}^{\infty}z_n^2(t)$ therein replaced by $\sum_{n=N+1}^{\infty}w_n^2(t)$. By Parseval's equality we present $V_{\mathrm{nom}}(t)$ in (2.42) as

(3.26)
$$\begin{aligned}
&V_{\mathrm{nom}}(t)=V_P(t)-V_1(t)+V_2(t),\\
&V_1(t)=\rho|\mathbb{I}_0X(t)|^2, \quad V_2(t)=\rho\|w(\cdot,t)\|_{L^2}^2,\\
&\mathbb{I}_0=\begin{bmatrix} 0_{N_0\times 1} & I_{N_0} & I_{N_0} & 0 & 0\\ 0_{(N-N_0)\times 1} & 0 & 0 & I_{N-N_0} & I_{N-N_0}\end{bmatrix}.
\end{aligned}$$

Recalling the operator $\mathcal{A}_1$ in (2.30), we can rewrite the stochastic PDE in (3.3) as

(3.27)
$$\begin{aligned}\mathrm{d}w(t)=&[\mathcal{A}_1w(t)+qw(t)+qr\mathbb{1}X(t)-r\mathbb{K}_0X(t-\tau_u)]\mathrm{d}t\\ &+\sigma_1(\cdot,w(t)+r\mathbb{1}X(t))\mathrm{d}\mathcal{W}_1(t),\ t\geq 0,\end{aligned}$$

where $w(t)=w(\cdot,t)$, $r=r(\cdot)$. Since $w(t)$ is a strong solution to (3.27), for function $V_2(t)$ in (3.26), we estimate the generator $\mathcal{L}$ of (3.27) as follows (see [2, p. 228]):
(3.28)
$$\begin{aligned}\mathcal{L}V_2(t)\overset{(2.2)}{\leq}\ &2\rho\int_0^1 w(x,t)[\tfrac{\partial^2 w(x,t)}{\partial x^2}+qw(x,t)]\mathrm{d}x+\bar{\sigma}_1^2\rho\int_0^1[w(x,t)+r(x)\mathbb{1}X(t)]^2\mathrm{d}x\\ &+2\rho\int_0^1 w(x,t)[qr(x)\mathbb{1}X(t)-r(x)\mathbb{K}_0X(t-\tau_u)]\mathrm{d}x.\end{aligned}$$

By Parseval's equality (see [33, Proposition 10.29]), we have

(3.29)
$$\begin{aligned}
&\textstyle\int_0^1 w(x,t)r(x)\mathrm{d}x=\sum_{n=1}^{\infty}\langle w(\cdot,t),\phi_n\rangle\langle r,\phi_n\rangle=\sum_{n=1}^{\infty}b_nw_n(t),\\
&\textstyle\int_0^1 w(x,t)qr(x)\mathrm{d}x=\sum_{n=1}^{\infty}a_nw_n(t),\\
&\textstyle\int_0^1[w(x,t)+r(x)\mathbb{1}X(t)]^2\mathrm{d}x=\sum_{n=1}^{\infty}[w_n(t)+b_n\mathbb{1}X(t)]^2.
\end{aligned}$$

Integrating by parts and substituting (3.29) into (3.28), we get

$$
\begin{aligned}
&\mathcal{L}V_2(t)+2\delta_0V_2(t)\le 2\rho\textstyle\sum_{n=1}^{\infty}(-\lambda_n+q+\delta_0)w_n^2(t)\\
&\quad+2\rho\textstyle\sum_{n=1}^{\infty}w_n(t)a_n\mathbb{1}X(t)-2\rho\sum_{n=1}^{\infty}w_n(t)b_n\mathbb{K}_0[X(t)-\nu_u(t)]\\
&\quad+\rho\bar\sigma_1^2\textstyle\sum_{n=N+1}^{\infty}[w_n(t)+b_n\mathbb{1}X(t)]^2+\bar\sigma_1^2\rho X^{\mathrm T}(t)\mathbb{B}^{\mathrm T}\mathbb{B}X(t),\\
&\mathbb{B}=\begin{bmatrix} B_0 & I_{N_0} & I_{N_0} & 0 & 0\\ B_1 & 0 & 0 & I_{N-N_0} & I_{N-N_0}\end{bmatrix}.
\end{aligned}\tag{3.30}
$$

We calculate $\mathcal{L}V_P(t)$ and $\mathcal{L}V_1(t)$ along with (3.24a) (similar to (2.44), (2.45)) and combine (3.30) as well as the following estimates by Young's inequality:

$$
\begin{aligned}
&\rho\textstyle\sum_{n=N+1}^{\infty}2w_n(t)a_n\mathbb{1}X(t)\le\frac{\rho^2}{\alpha_1}\sum_{n=N+1}^{\infty}w_n^2(t)+\alpha_1\|a\|_N^2|\mathbb{1}X(t)|^2,\\
&-\rho\textstyle\sum_{n=N+1}^{\infty}2w_n(t)b_n\mathbb{K}_0[X(t)-\nu_u(t)]\le\frac{2\rho^2}{\alpha_2}\sum_{n=N+1}^{\infty}w_n^2(t)\\
&\qquad\qquad+\alpha_2\|b\|_N^2\left[|\mathbb{K}_0X(t)|^2+|\mathbb{K}_0\nu_u(t)|^2\right],\\
&\rho\textstyle\sum_{n=N+1}^{\infty}[w_n(t)+b_n\mathbb{1}X(t)]^2\le\sum_{n=N+1}^{\infty}(\rho+\frac{\rho^2}{\alpha_3})w_n^2(t)+(\rho+\alpha_3)\|b\|_N^2|\mathbb{1}X(t)|^2,
\end{aligned}\tag{3.31}
$$

where $\alpha_1,\alpha_2,\alpha_3>0$. We arrive at

$$
\begin{aligned}
&\mathcal{L}V_{\mathrm{nom}}(t)+2\delta_0V_{\mathrm{nom}}(t)\le X^{\mathrm T}(t)\big[PF_0+F_0^{\mathrm T}P+2\delta_0P+\rho\bar\sigma_1^2\mathbb{B}^{\mathrm T}\mathbb{B}\\
&\quad+\big[\alpha_1\|a\|_N^2+\bar\sigma_1^2(\rho+\alpha_3)\|b\|_N^2\big]\mathbb{1}^{\mathrm T}\mathbb{1}+\alpha_2\|b\|_N^2\mathbb{K}_0^{\mathrm T}\mathbb{K}_0\big]X(t)\\
&\quad-2X^{\mathrm T}(t)PF_2\mathbb{K}_0\nu_u(t)-2X^{\mathrm T}(t)PF_1\nu_y(t)+\Sigma_1^{\mathrm T}(t)(P-\rho I)\Sigma_1(t)\\
&\quad+\bar\sigma_2^2\mathbb{L}_0^{\mathrm T}P\mathbb{L}_0[\zeta(t-\tau_y)+\mathbb{C}X(t)-\mathbb{C}\nu_y(t)]^2+2X^{\mathrm T}(t)P\mathbb{L}_0\zeta(t-\tau_y)\\
&\quad+\textstyle\sum_{n=N+1}^{\infty}\left[2\rho(-\lambda_n+q+\delta_0+\frac{\bar\sigma_1^2}{2})+\frac{\rho^2}{\alpha_1}+\frac{2\rho^2}{\alpha_2}+\frac{\bar\sigma_1^2\rho^2}{\alpha_3}\right]w_n^2(t).
\end{aligned}\tag{3.32}
$$

Consider the presentations (2.53). By arguments similar to (2.54)–(2.61) and using (3.32) and the bound

$$
\begin{aligned}
|\Sigma_1(t)|^2&\le\textstyle\sum_{n=1}^{\infty}\tilde\sigma_{1,n}^2(t)=\int_0^1|\sigma(x,w(x,t)+r(x)\mathbb{1}X(t))|^2\mathrm{d}x\\
&\le\bar\sigma_1^2\textstyle\int_0^1[w(x,t)+r(x)\mathbb{1}X(t)]^2\mathrm{d}x=\bar\sigma_1^2\sum_{n=1}^{\infty}[w_n(t)+b_n\mathbb{1}X(t)]^2\\
&\le\bar\sigma_1^2\textstyle\sum_{n=N+1}^{\infty}2w_n^2(t)+\bar\sigma_1^2X^{\mathrm T}(t)[2\|b\|_N^2\mathbb{1}^{\mathrm T}\mathbb{1}+\mathbb{B}^{\mathrm T}\mathbb{B}]X(t),\\
&-2\delta_1\sup_{s_k\le\theta\le t}\mathbb{E}V(\theta)\overset{(2.14),(2.42)}{\le}-2\delta_1\mathbb{E}V_{\mathrm{nom}}(t-\tau_y)\\
&=-2\delta_1\mathbb{E}|X(t)-\nu_y(t)|_P^2-2\delta_1\rho\mathbb{E}[\textstyle\sum_{n=N+1}^{\infty}w_n^2(t-\tau_y)]\\
&\overset{(2.37)}{\le}-2\delta_1\mathbb{E}|X(t)-\nu_y(t)|_P^2-2\delta_1\rho\|c\|_N^{-2}\mathbb{E}\zeta^2(t-\tau_y),
\end{aligned}
$$

where $0<\delta_1<\delta_0$, we arrive at

$$
\begin{aligned}
&\mathbb{E}[\mathcal{L}V(t)+2\delta_0V(t)]-2\delta_1\sup_{s_k\le\theta\le t}\mathbb{E}V(\theta)\\
&\quad+\beta[\bar\sigma_1^2\textstyle\sum_{n=N+1}^{\infty}2w_n^2(t)+\bar\sigma_1^2X^{\mathrm T}(t)[2\|b\|_N^2\mathbb{1}^{\mathrm T}\mathbb{1}+\mathbb{B}^{\mathrm T}\mathbb{B}]X(t)-|\Sigma_1(t)|^2]\\
&\le\mathbb{E}[\eta^{\mathrm T}(t)\Phi_1\eta(t)]+\mathbb{E}[\textstyle\sum_{n=N+1}^{\infty}\mu_nw_n^2(t)]+\mathbb{E}[\Sigma_1^{\mathrm T}(t)\Phi_2\Sigma_1(t)]\le 0,
\end{aligned}\tag{3.33}
$$

where $\eta(t)$ is defined above (2.64), provided $\mu_n:=2\rho(-\lambda_n+q+\delta_0+\frac{\bar\sigma_1^2}{2})+\frac{\rho^2}{\alpha_1}+\frac{2\rho^2}{\alpha_2}+\frac{\bar\sigma_1^2\rho^2}{\alpha_3}+2\beta\bar\sigma_1^2$ for all $n\ge N+1$ and

$$
\begin{aligned}
\Phi_1=&\left[\begin{array}{c|c}\Omega_1 & \Theta_1\quad\Theta_2\\ \hline * & \mathrm{diag}\{\Omega_2,\Omega_3\}\end{array}\right]+\tau_{M,y}^2\Lambda_1^{\mathrm T}R_1\Lambda_1+\tau_{M,y}\bar\sigma_2^2\Lambda_2^{\mathrm T}Q_{12}\Lambda_2\\
&+\tau_{M,u}^2\Lambda_1^{\mathrm T}\mathbb{K}_0^{\mathrm T}R_2\mathbb{K}_0\Lambda_1+\tau_{M,u}\bar\sigma_2^2\Lambda_2^{\mathrm T}\mathbb{K}_0^{\mathrm T}Q_{22}\mathbb{K}_0\Lambda_2<0,
\end{aligned}\tag{3.34a}
$$

$$
\Phi_2=P+\tau_{M,y}Q_{11}+\tau_{M,u}\mathbb{K}_0^{\mathrm T}Q_{21}\mathbb{K}_0-(\rho+\beta)I<0\tag{3.34b}
$$

hold, where

$$
\begin{aligned}
&\Omega_1 = \begin{bmatrix} \tilde{\Omega}_1^{(11)} + \Omega_1^{(11)} & P\mathbb{L}_0 + \bar{\sigma}_2^2 \mathbb{C}^{\mathrm{T}} \mathbb{L}_0^{\mathrm{T}} P \mathbb{L}_0 \\ * & -2\delta_1 \rho \|c\|_N^{-2} + \bar{\sigma}_2^2 \mathbb{L}_0^{\mathrm{T}} P \mathbb{L}_0 \end{bmatrix}, \quad \delta = \delta_0 - \delta_1, \\
&\tilde{\Omega}_1^{(11)} = PF_0 + F_0^{\mathrm{T}} P + 2\delta P + \alpha_1 \|a\|_N^2 \mathbb{1}^{\mathrm{T}} \mathbb{1} + \alpha_2 \|b\|_N^2 \mathbb{K}_0^{\mathrm{T}} \mathbb{K}_0, \\
&\Omega_1^{(11)} = \bar{\sigma}_1^2 [\rho + \alpha_3 + 2\beta] \|b\|_N^2 \mathbb{1}^{\mathrm{T}} \mathbb{1} + \bar{\sigma}_2^2 \mathbb{C}^{\mathrm{T}} \mathbb{L}_0^{\mathrm{T}} P \mathbb{L}_0 \mathbb{C} \\
&\qquad + (1-\varepsilon_y) S_1 + (1-\varepsilon_u) \mathbb{K}_0^{\mathrm{T}} S_2 \mathbb{K}_0 + \bar{\sigma}_1^2 (\rho + \beta) \mathbb{B}^{\mathrm{T}} \mathbb{B}, \\
&\Omega_2 = \left[\begin{array}{cc|c} \Omega_2^{(11)} & -\varepsilon_y (S_1 + G_1) & \varepsilon_y [R_1, G_1, R_1, G_1] \\ * & -\varepsilon_y (S_1 + R_1) & \varepsilon_y [G_1^{\mathrm{T}}, R_1, G_1^{\mathrm{T}}, R_1] \\ \hline * & * & -\varepsilon_y \Omega_2^{(33)} \end{array}\right], \\
&\Omega_2^{(11)} = \bar{\sigma}_2^2 \mathbb{C}^{\mathrm{T}} \mathbb{L}_0^{\mathrm{T}} P \mathbb{L}_0 \mathbb{C} - 2\delta_1 P - \varepsilon_y (S_1 + R_1), \\
&\Omega_2^{(33)} = \mathrm{diag}\{R_1 + Q_{11}, R_1 + Q_{11}, R_1 + Q_{12}, R_1 + Q_{12}\}, \\
&\Omega_3 = \left[\begin{array}{cc|c} \alpha_2 \|b\|_N^2 - \varepsilon_u (S_2 + R_2) & -\varepsilon_u (S_2 + G_2) & \varepsilon_u [R_2, G_2, R_2, G_2] \\ * & -\varepsilon_u (S_2 + R_2) & \varepsilon_u [G_2, R_2, G_2, R_2] \\ \hline * & * & -\varepsilon_u \Omega_3^{(33)} \end{array}\right], \\
&\Omega_3^{(33)} = \mathrm{diag}\{R_2 + Q_{21}, R_2 + Q_{21}, R_2 + Q_{22}, R_2 + Q_{22}\}, \\
&\Theta_1 = \begin{bmatrix} \Theta_1^{(11)} & \varepsilon_y S_1 & 0_{(2N+1)\times(8N+4)} \\ -\bar{\sigma}_2^2 \mathbb{L}_0^{\mathrm{T}} P \mathbb{L}_0 \mathbb{C} & 0_{1\times(2N+1)} & 0_{1\times(8N+4)} \end{bmatrix}, \\
&\Theta_1^{(11)} = P(2\delta_1 I - F_1) - \bar{\sigma}_2^2 \mathbb{C}^{\mathrm{T}} \mathbb{L}_0^{\mathrm{T}} P \mathbb{L}_0 \mathbb{C} + \varepsilon_y S_1, \\
&\Theta_2 = \begin{bmatrix} -PF_2 + \varepsilon_u \mathbb{K}_0^{\mathrm{T}} S_2 & \varepsilon_u \mathbb{K}_0^{\mathrm{T}} S_2 & 0_{(2N+1)\times 4} \\ 0_{1\times1} & 0_{1\times1} & 0_{1\times4} \end{bmatrix}, \\
&\Lambda_1 = [F_0, \mathbb{L}_0, -F_1, 0_{(2N+1)\times(10N+5)}, -F_2, 0_{(2N+1)\times5}], \\
&\Lambda_2 = [\mathbb{L}_0 \mathbb{C}, \mathbb{L}_0, -\mathbb{L}_0 \mathbb{C}, 0_{(2N+1)\times(10N+11)}].
\end{aligned} \tag{3.35}
$$

From the monotonicity of λ_n we find that $\mu_n < 0$ for all $n \geq N+1$ iff

$$
\left[\begin{array}{c|ccc} 2\rho(-\lambda_{N+1} + q + \delta_0) + \bar{\sigma}_1^2(\rho + 2\beta) & \rho & \rho & \bar{\sigma}\rho \\ \hline * & \multicolumn{3}{c}{\mathrm{diag}\{-\alpha_1, -\frac{\alpha_2}{2}, -\alpha_3\}} \end{array}\right] < 0. \tag{3.36}
$$

Applying Itô's formula and Halanay's inequality (similar to the arguments (2.68)–(2.72)), we arrive at

$$
\mathbb{E}V(t) \leq \mathbb{E}V(0)\mathrm{e}^{-2\delta_\tau t}, \quad t \geq 0, \tag{3.37}
$$

where $\delta_\tau > 0$ is the unique solution of $\delta_\tau = \delta_0 - \delta_1 \mathrm{e}^{2\delta_\tau \tau_{M,y}}$. Since $u(0) = 0$ and $\hat{w}_n(0) = 0$, $1 \leq n \leq N$, we have

$$
\mathbb{E}V(0) = \mathbb{E}V_{\mathrm{nom}}(0) \leq \max\{\lambda_{\max}(P), \rho\} \mathbb{E}\|w(\cdot,0)\|_{L^2}^2. \tag{3.38}
$$

Note that $\hat{w}_n^2 + e_n^2 = (w_n - e_n)^2 + e_n^2 \geq 0.5 w_n^2$. Then by Parseval's equality, we have for some $M > 0$

$$
\begin{aligned}
\mathbb{E}V(t) &\geq \mathbb{E}V_{\mathrm{nom}}(t) \geq \lambda_{\min}(P)\mathbb{E}[u^2(t) + \textstyle\sum_{n=1}^N (\hat{w}_n^2(t) + e_n^2(t))] \\
&\quad + \rho\mathbb{E}[\textstyle\sum_{n=N+1}^\infty w_n^2(t)] \geq \min\{\frac{\lambda_{\min}(P)}{2}, \rho\}\mathbb{E}[u^2(t) + \|w(\cdot,t)\|_{L^2}^2],\ t \geq 0.
\end{aligned} \tag{3.39}
$$

Finally, (3.37), (3.38), and (3.39) imply

$$
\mathbb{E}[u^2(t) + \|w(\cdot,t)\|_{L^2}^2 + \|w(\cdot,t) - \hat{w}(\cdot,t)\|_{L^2}^2] \leq M\mathrm{e}^{-2\delta_\tau t}\mathbb{E}\|w(\cdot,0)\|_{L^2}^2, \quad t \geq 0, \tag{3.40}
$$

for some $M \geq 1$.

For the feasibility proof of inequalities (3.34) and (3.36) with large enough N and small enough $\tau_{M,y}$, $\tau_{M,u}$, $\bar\sigma_1$, and $\bar\sigma_2$, let $S_i=0$, $G_i=0$, $i=1,2$. Taking $\bar\sigma_1,\bar\sigma_2,\tau_{M,y}$, $\tau_{M,u}\to 0^+$, it is sufficient to show

$$P<(\rho+\beta)I,\quad \left[\begin{array}{c|c}\tilde\Omega_1 & \tilde\Theta_1\quad\tilde\Theta_2\\ \hline * & \operatorname{diag}\{\tilde\Omega_2,\tilde\Omega_3\}\end{array}\right]<0, \tag{3.41}$$

where

$$\tilde\Theta_1=\begin{bmatrix}P(2\delta_1 I-F_1) & 0\\ 0_{1\times(2N+1)} & 0\end{bmatrix},\ \tilde\Theta_2=\begin{bmatrix}-PF_2 & 0_{(2N+1)\times 5}\\ 0_{1\times 1} & 0\end{bmatrix},\ \tilde\Omega_1=\begin{bmatrix}\tilde\Omega_1^{(11)} & P\mathbb{L}_0\\ * & -2\delta_1\rho\|c\|_N^{-2}\end{bmatrix},$$
$$\tilde\Omega_2=\left[\begin{array}{cc|c}-2\delta_1 P-R_1 & 0 & R_1\ 0\ R_1\ 0\\ * & -R_1 & 0\ R_1\ 0\ R_1\\ \hline * & * & -\Omega_2^{(33)}\end{array}\right],\ \tilde\Omega_3=\left[\begin{array}{cc|c}\alpha_2\|b\|_N^2-R_2 & 0 & R_2\ 0\ R_2\ 0\\ * & -R_2 & 0\ R_2\ 0\ R_2\\ \hline * & * & -\Omega_3^{(33)}\end{array}\right].$$

Here $\tilde\Omega_1^{(11)}$, $\Omega_2^{(33)}$, $\Omega_3^{(33)}$ are defined as in (3.35). Let $P\in\mathbb{R}^{(2N+1)\times(2N+1)}$ solve the Lyapunov equation $P(F_0+\delta I)+(F_0+\delta I)^{\mathrm T}P=-\frac{1}{N}I$. Theorem 3.3 in [21] implies $\|P\|=O(1)$, uniformly in N. Applying the Schur complement repeatedly and substituting P, $\alpha_1=\delta_1=1$, $\alpha_2=2$, $\alpha_3=1$, $\rho=1$, $\beta=N$, $R_1=N^2I$, $Q_{11}=Q_{12}=3N^2I$, $R_2=N^2$, and $Q_{21}=Q_{22}=3N^2$ into (3.41), we find that (3.41) hold iff

$$\begin{aligned}&-\tfrac{1}{N}I+\|a\|_N^2\mathbb{1}^{\mathrm T}\mathbb{1}+2\|b\|_N^2\mathbb{K}_0^{\mathrm T}\mathbb{K}_0+\tfrac{2PF_2F_2^{\mathrm T}P}{N^2-\|b\|_N^2}+\tfrac{\|c\|_N^2}{2}P\mathbb{L}_0\mathbb{L}_0^{\mathrm T}P\\ &\quad+P(2I-F_1)(2P+\tfrac{1}{2}N^2I)^{-1}(2I-F_1)^{\mathrm T}P<0,\quad P<(1+N)I.\end{aligned}\tag{3.42}$$

Since $\|a\|_N^2$, $\|b\|_N^2$ satisfy (3.9), $c\in L^2(0,1)$, $\|\mathbb{1}^{\mathrm T}\mathbb{1}\|=1$, $\|P\|=O(1)$, $\|\mathbb{K}_0^{\mathrm T}\mathbb{K}_0\|=O(1)$, $N\to\infty$, we find that (3.42) hold for large enough N. By arguments similar to [22, Theorem 3.1], by continuity, inequalities (3.34) and (3.36) hold for $\tau_{M,y}=\tau_{M,u}=\bar\sigma_1=\bar\sigma_2=N^{-2}$ and large enough N. Summarizing, we arrive at the following.

THEOREM 3.2. *Consider* (3.3) *with nonlinear noise function* σ_1 *satisfying* (2.2), *control law* (3.13), *noise measurement* (3.4) *with* σ_2 *satisfying* (2.4), *and* $w(\cdot,0)\in\mathcal{D}(\mathcal{A}_1)\cap L^2(\Omega;L^2(0,1))$. *Given* $\delta>0$, *let* $N_0\in\mathbb{N}$ *satisfy* (2.8) *and* $N\in\mathbb{N}$ *satisfy* $N\ge N_0$. *Let* L_0 *and* $\tilde K_0$ *be obtained from* (2.12) *and* (3.17), *respectively. Given* $\tau_{M,y},\tau_{M,u},\delta_1,\bar\sigma_1,\bar\sigma_2>0$ *and* $\delta_0=\delta_1+\delta$, *let there exist* $0<P,S_1,R_1,Q_{11},Q_{12}\in\mathbb{R}^{(2N+1)\times(2N+1)}$, *scalars* $S_2,R_2,Q_{21},Q_{22},\alpha_1,\alpha_2,\alpha_3,\beta,\rho>0$, $G_1\in\mathbb{R}^{(2N+1)\times(2N+1)}$, *and* $G_2\in\mathbb{R}$ *such that the following LMIs hold with* $\delta_1=\delta_0-\delta$: *LMIs* (2.56), (3.36), *and* $\Phi_i<0$ ($i=1,2$) *with* Φ_i *defined as in* (3.34)–(3.35). *Then the solution* $u(t)$, $w(x,t)$ *to* (3.3) *subject to the control law* (3.11), (3.13) *and the corresponding observer* $\hat w(x,t)$ *given by* (3.10) *satisfies* (3.40) *for some* $M\ge 1$, *where* $\delta_\tau>0$ *is the unique solution of* $\delta_\tau=\delta_0-\delta_1 e^{2\delta_\tau\tau_{M,y}}$. *Moreover, the above LMIs always hold for large enough* N *and small enough* $\tau_{M,y},\tau_{M,u},\bar\sigma_1,\bar\sigma_2$.

Remark 3.3. If noise functions σ_1, σ_2 are of the linear form (2.79), we have the closed-loop system (3.24) with

$$\tilde\sigma_{1,n}(t)=\bar\sigma_1[w_n(t)+b_n\mathbb{1}X(t)],\ \Sigma_2(t)=\bar\sigma_2\mathbb{L}_0[\zeta(t-\tau_y)+\mathbb{C}X(t)-\mathbb{C}\nu_y(t)],$$
$$\Sigma_1(t)=\bar\sigma_1\mathcal{I}_2X(t),\ \mathcal{I}_2=\begin{bmatrix}0_{(N_0+1)\times 1} & 0 & 0 & 0 & 0\\ B_0 & I_{N_0} & I_{N_0} & 0 & 0\\ 0_{(N-N_0)\times 1} & 0 & 0 & 0 & 0\\ B_1 & 0 & 0 & I_{N-N_0} & I_{N-N_0}\end{bmatrix}.$$

By arguments similar to (3.26)–(3.40), we find that if (3.34a) and (3.36) hold with $\beta=0$ and $\Omega_1^{(11)}$ defined as in (3.35) replaced by

$$\Omega_1^{(11)} = \bar{\sigma}_1^2(\rho+\alpha_3)\|b\|_N^2 \mathbb{1}^{\mathrm{T}}\mathbb{1} + \bar{\sigma}_2^2 \mathbb{C}^{\mathrm{T}}\mathbb{L}_0^{\mathrm{T}} P \mathbb{L}_0 \mathbb{C} + (1-\varepsilon_y)S_1 + (1-\varepsilon_u)\mathbb{K}_0^{\mathrm{T}} S_2 \mathbb{K}_0 + \bar{\sigma}_1^2 \mathcal{I}_2^{\mathrm{T}}(P + \tau_{M,y}Q_{11} + \tau_{M,u}\mathbb{K}_0^{\mathrm{T}} Q_{21}\mathbb{K}_0)\mathcal{I}_2, \tag{3.43}$$

the mean-square L^2 exponential stability of the closed-loop system can be guaranteed. Here the constraint (3.34b) for $\Sigma_1(t)$ is not needed.

Remark 3.4. Compared to the controller based on the direct method in section 2, the dynamic-extension-based one is more difficult for implementation because of the generalized hold device, but it allows an essentially larger bound on the sampling intervals $\tau_{M,u}$.

4. A numerical example. Consider (2.1) with $q = 6$, which results in an unstable open-loop system.

First, we consider the direct method presented in section 2. Let $N_0 = 2$ and $c(x) = \sqrt{2}\chi_{[0.24,0.75]}(x)$ (i.e., the indicator function of $[0.24, 0.75]$). The observer and controller gains L_0 and K_0 are found from (2.12) and (2.23)–(2.24) (with tuning parameter $\varepsilon = 0.5$) and are given by

$$\delta = 5: \ L_0 = [13.3782, -18.8783]^{\mathrm{T}}, \tag{4.1a}$$

$$\delta = 1, \tau_{M,u} = 0.02: \ K_0 = [-5.4751, -0.6505]. \tag{4.1b}$$

Take $\delta_0 = 0.55$ and $\delta = 0.01$. Choose $\bar{\sigma}_1 \in \{0.2, 0.4\}$, $\bar{\sigma}_2 = 0.01$, $N \in \{6, 8, 10, 12\}$, $\tau_{M,y} \in \{0.01, 0.02\}$. The LMIs of Theorem 2.4 with gains (4.1) were verified to obtain $\tau_{M,u}^{\max}$ (the maximal value of $\tau_{M,u}$) which preserves the feasibility. The results are given in Table 1. On the other hand, we find the controller gain from LMI (2.27):

$$\delta = 1: \ K_0 = [-5.2323, -11.7298]. \tag{4.2}$$

The LMIs of Theorem 2.4 with gains (4.1a) and (4.2) are not feasible even for $\tau_{M,u} = 0$.

For the case of linear noise functions (2.79), inequalities (2.67) and (2.65a) with $\Omega_1^{(11)}$ therein replaced by (2.80) and gains (4.1) were verified to obtain $\tau_{M,u}^{\max}$, which preserves the feasibility. The results are given in Table 2. As expected, the values of $\tau_{M,u}^{max}$ under linear noise are essentially larger than those under the nonlinear one.

For simulations of the closed-loop system (2.1) subject to the control law (2.10), (2.13), we consider nonlinear noise functions

TABLE 1
Direct method: $\tau_{M,u}^{\max}$ for nonlinear noise with K_0 in (4.1b).

N	6		8		10		12	
$\bar{\sigma}_1 \backslash \tau_{M,y}$	0.01	0.02	0.01	0.02	0.01	0.02	0.01	0.02
0.2	0.024	0.002	0.040	0.025	0.054	0.040	0.060	0.047
0.4	0.017	–	0.032	0.015	0.044	0.029	0.051	0.036

TABLE 2
Direct method: $\tau_{M,u}^{\max}$ for linear noise with K_0 in (4.1b).

N	6		8		10		12	
$\bar{\sigma}_1 \backslash \tau_{M,y}$	0.01	0.02	0.01	0.02	0.01	0.02	0.01	0.02
0.2	0.026	0.005	0.043	0.027	0.057	0.043	0.064	0.050
0.4	0.022	–	0.038	0.020	0.051	0.034	0.058	0.041

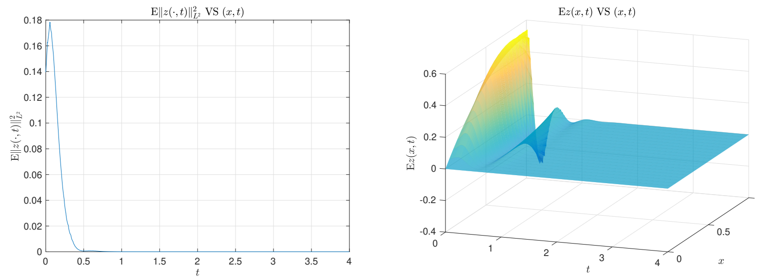

FIG. 1. $\mathbb{E}\|z(\cdot,t)\|^2_{L^2}$ *vs.* t *and* $\mathbb{E}z(x,t)$ *vs.* (x,t) *(*$\mathbb{E}$ *means taking the average over* 500 *sample trajectories).*

TABLE 3
Dynamic-extension-based method: $\tau^{\max}_{M,u}$ *for nonlinear noise,* $\tilde{K}_0$ *in* (4.5) *vs.* $\tilde{K}_0$ *in* (4.6).

	N	6		8		10		12	
$\tilde{K}_0$	$\bar{\sigma}_1\backslash\tau_{M,y}$	0.01	0.02	0.01	0.02	0.01	0.02	0.01	0.02
(4.5)	0.2	0.0633	0.0617	0.0636	0.0621	0.0639	0.0624	0.0640	0.0625
(4.5)	0.4	0.0496	0.0418	0.0512	0.0441	0.0525	0.0462	0.0528	0.0470
(4.6)	0.2	0.0458	0.0421	0.0468	0.0433	0.0475	0.0443	0.0478	0.0446
(4.6)	0.4	0.0325	0.0237	0.0344	0.0263	0.0363	0.0288	0.0368	0.0295

$$\sigma_1(x,z)=\bar{\sigma}_1\sin z,\ \ \sigma_2(z)=\bar{\sigma}_2\sin z \tag{4.3}$$

that satisfy (2.2) and (2.4). The variable sampling instances and variable controller hold times were generated by

$$s_{k+1}=s_k+0.5(1+U_k)\tau_{M,y},\ \ t_{j+1}=t_j+0.5(1+U_j)\tau_{M,u}, \tag{4.4}$$

respectively, where $U_k\sim Unif(0,1)$, $U_j\sim Unif(0,1)$. Fix $N=10$, $\bar{\sigma}_1=0.2$, $\bar{\sigma}_2=\tau_{M,y}=0.01$. From Table 1, we can choose $\tau_{M,u}=0.054$. Take the initial condition $z_0(x)=x-0.5x^2$. The simulation was carried out by using the FTCS (Forward Time Centered Space) method and the Euler–Maruyama method (see [14]) with time step 0.001 and space step 0.05. The simulation results are presented in Figure 1 and confirm the theoretical analysis. Moreover, in simulations, stability of the closed-loop system with the same gains is preserved up to $\tau_{M,u}=0.29$ (compared with the theoretical value 0.054) for $\bar{\sigma}_1=0.2$, $\bar{\sigma}_2=\tau_{M,y}=0.01$, which may illustrate some conservatism of our method.

We next consider the method via dynamic extension presented in section 3. Take $N_0=2$ and $c(x)=\sqrt{2}\chi_{[0.24,0.75]}(x)$. The observer gain L_0 is given by (4.1a) and the controller gain $\tilde{K}_0$ is found from (2.24), (3.16) (with $\varepsilon=0.5$) and are given by

$$\delta=1,\tau_{M,u}=0.02:\ \ \tilde{K}_0=[-50.1336,-61.6398,0.0106]. \tag{4.5}$$

Note that, compared with (4.1b), the large value of controller gains (4.5) is caused by the smaller value of b_1 and b_2. On the other hand, we also obtain controller gain $\tilde{K}_0$ from LMI (3.18):

$$\delta=1:\ \ \tilde{K}_0=[-50.3816,-68.9515,22.8149]. \tag{4.6}$$

Choose $\delta_0=0.55$ and $\delta=0.01$. Take $\bar{\sigma}_1\in\{0.2,0.4\}$, $\bar{\sigma}_2=0.01$, $N\in\{6,8,10,12,14\}$, $\tau_{M,y}\in\{0.01,0.02\}$. The LMIs of Theorem 3.2 were verified to obtain $\tau^{\max}_{M,u}$, which preserves the feasibility. The results are given in Table 3. From Table 3, it is clear that the state-feedback design based on LMIs (2.24) and (3.16) allows for larger $\tau_{M,u}$ than the one obtained by (3.17) for all values of N.

TABLE 4
Dynamic-extension-based method: $\tau_{M,u}^{\max}$ for linear noise with $\tilde{K}_0$ in (4.5).

N	6		8		10		12	
$\bar{\sigma}_1 \backslash \tau_{M,y}$	0.01	0.02	0.01	0.02	0.01	0.02	0.01	0.02
0.2	0.0673	0.0660	0.0676	0.0664	0.0677	0.0665	0.0677	0.0666
0.4	0.0634	0.0609	0.0636	0.0612	0.0637	0.0614	0.0637	0.0614

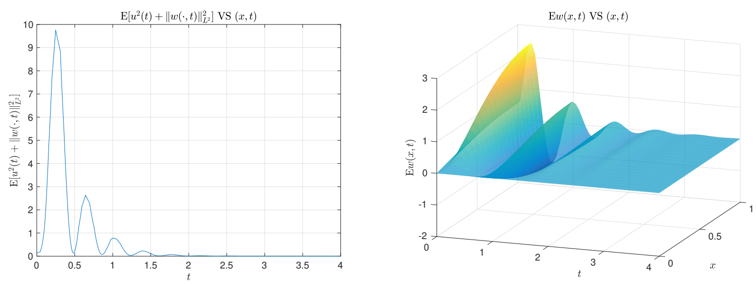

FIG. 2. $\mathbb{E}[u^2(t) + \|w(\cdot,t)\|_{L^2}^2]$ *vs.* t *and* $\mathbb{E}w(x,t)$ *vs.* (x,t) *(*$\mathbb{E}$ *means taking the average over* 500 *sample trajectories).*

For the case of linear noise functions (2.79), inequalities (3.36) and (3.34a) with $\Omega_1^{(11)}$ therein replaced by (3.43) and gains (4.5) were verified to obtain $\tau_{M,u}^{\max}$, which preserves the feasibility. The results are given in Table 4. As expected, the values of $\tau_{M,u}^{max}$ under linear noise are essentially larger than those under the nonlinear one. Moreover, from Tables 1 and 3 for nonlinear noise as well as Tables 2 and 4 for linear noise, we see that the dynamic-extension-based method always allows for larger $\tau_{M,u}$ than the direct one for $N \leq 12$.

For simulations of the closed-loop system (3.3) subject to the control law (3.11), (3.13), we take nonlinear noise functions (4.3). The variable sampling instances and the variable controller hold times were generated by (4.4), respectively. Fix $N = 10$, $\tau_{M,y} = 0.01$, $\bar{\sigma}_1 = 0.2$, and $\bar{\sigma}_2 = 0.01$. From Table 3, we take $\tau_{M,u} = 0.0677$. The simulation results are presented in Figure 2 and confirm the theoretical analysis. Moreover, in simulations, stability of the closed-loop system with the same gain is preserved up to $\tau_{M,u} = 0.18$ (compared with the theoretical value 0.0677) for $\bar{\sigma}_1 = 0.2$, $\bar{\sigma}_2 = \tau_{M,y} = 0.01$, which may illustrate some conservatism of our method.

5. Conclusions. In this paper, we considered a sampled-data implementation of a finite-dimensional observer-based boundary controller for 1D stochastic parabolic PDEs under discrete-time nonlocal measurement. We presented two methods: a direct one with sampled-data controller implemented via zero-order hold device, and a dynamic-extension-based one with sampled-data controller implemented via a generalized hold device. For both methods, we provided mean-square L^2 exponential stability analysis of the full-order closed-loop system. Improvements and extension of sampled-data control to various stochastic PDEs may be topics for future research.

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