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On ISS with respect to average value of disturbances: a time-delay approach

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Abstract—For input-affine nonlinear dynamical systems, an input-to-state stability analysis with respect to (weighted) average values of exogenous perturbations is proposed. The time-delay method is used to represent the system in a suitable form for investigation: a kind of neutral-type differential equation. The introduced approach allows the asymptotic gains with respect to zero-mean periodic signals to be evaluated for nonlinear systems (an analogue of Bode magnitude plot), as well as for integral input-to-state stable systems with periodic inputs. The results are illustrated on the class of homogeneous systems.

I. Introduction

Analysis of robustness of stability properties for dynamical systems with respect to external disturbances, measurement noises, variations of parameters, unmodeled dynamics, delays, etc., is one of the central issues in the control and estimation theory [1]. For linear systems all these notions have been well-studied through H_∞ norms or L_2 gains in 60s of the previous century. For nonlinear systems there are several approaches for robust stability analysis, and one of the most popular frameworks is built around the input-to-state stability (ISS) concept [2]. The advantage of this theory is that it has a list of properties describing different stability characteristics, as well as efficient necessary and sufficient conditions to check these properties in applications (existence of proper Lyapunov functions are usually verified). This framework allows one to make qualitative and quantitative analysis of behavior of the system trajectories with respect to the L_∞ norm of the input (or an integral of the norm).

In some cases, the analysis should be performed not with respect to the maximal amplitude of the input, but evaluating the bounds on trajectories as a function of the frequency of the disturbance, or its average value on an interval of time (see the respective extensions of the ISS framework [3], [4]). This is often the case in vibration analysis, or when the external inputs are (almost) periodic functions of time. In such a scenario, the time-averaging approach was found to be very efficient [5], and later its different extensions have been proposed [6], [7], [8], also in conjunction with ISS [9], [10] (where the stability property of the system is substantiated from the characteristics of the averaged dynamics). Recent works [11], [12], which are based on the time-delay approach, introduced the idea of a transformation of coordinates, which allowed quantitative estimates to be obtained on the averaging period.

The objective of this work is to use the latter results to propose a tool for studying ISS property of nonlinear systems with respect to an averaged value of the input, which can also be transformed in a frequency gain analysis (similar to the amplitude Bode plot for linear systems or its counterpart given for nonlinear convergent dynamics in [13]). Applying the transformation from [11], [12], it will be shown that a nonlinear affine in the inputs system can be transformed to the neutral-type delayed dynamics dependent on the average values of the inputs. The obtained theoretical findings are illustrated on a class of homogeneous systems.

For homogeneous dynamical systems, their local and global behaviors coincide, and for stability/instability analysis, Lyapunov or Chetaev function of a homogeneous system can also be chosen homogeneous [14]. In addition, if a system is homogeneous (considering disturbance as an auxiliary variable) and asymptotically stable without disturbances, then it is ISS [15]. These facts make homogeneous systems suitable for analysis and modeling of complex nonlinear compartments, disposing many efficient and helping design procedures. The averaging method has been already used with homogeneous systems in [16], [17], [18], [19], where stability of a system was related with the same property of its averaged dynamics.

The outline of this work is as follows. The preliminary definitions and the homogeneity framework are given in Section II. A method for analysis of robust stability with respect to an average value of perturbations is presented in Section III. Application of the developed theory to homogeneous dynamical systems is considered in Section IV.

II. Preliminaries

A. Notation

The real and integer numbers are denoted by $\mathbb{R}$ and $\mathbb{Z}$, respectively, $\mathbb{R}_+ = \{s \in \mathbb{R} : s \geq 0\}$ and $\mathbb{Z}_+ = \mathbb{R}_+ \cap \mathbb{Z}$. Euclidean norm for a vector $x \in \mathbb{R}^n$ is denoted by $\|x\|$ (for a matrix $A \in \mathbb{R}^{n\times m}$, $\|A\|$ corresponds to the induced norm), and the absolute value of $s \in \mathbb{R}$ by $|s|$. We denote by $C^n_{[a,b]}$, $-\infty < a < b < +\infty$ the Banach space of continuous functions $\phi : [a, b] \to \mathbb{R}^n$ with the uniform norm $|\phi|_C = \sup_{a\leq\varsigma\leq b} \|\phi(\varsigma)\|$ (similarly for $C^n_{[a,b)}$), for $x :$

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$[-T,+\infty)\to\mathbb{R}^n$ with $T>0$ we denote by $x_t\in C^n_{[-T,0]}$, $t\in\mathbb{R}_+$ its respective restriction $x_t(s)=x(t+s)$ for $s\in[-T,0]$. For a Lebesgue measurable function of time $d:[a,b)\to\mathbb{R}^m$ define the norm $\|d\|_{[a,b)}=\text{ess}\sup_{t\in[a,b)}\|d(t)\|$, then $\|d\|_\infty=\|d\|_{[0,+\infty)}$ and the space of d with $\|d\|_{[a,b)}<+\infty$ ($\|d\|_\infty<+\infty$) we further denote as $\mathcal{L}^m_{[a,b]}$ ($\mathcal{L}^m_\infty$).

For a continuously differentiable function $V:\mathbb{R}^n\to\mathbb{R}_+$, its directional derivative along a given vector $v\in\mathbb{R}^n$ at a point $x\in\mathbb{R}^n$ is defined as follows:

$$DV(x)v=\frac{\partial V(x)}{\partial x}v.$$

A continuous function $\sigma:\mathbb{R}_+\to\mathbb{R}_+$ belongs to class $\mathcal{K}$ if it is strictly increasing and $\sigma(0)=0$; it belongs to class $\mathcal{K}_\infty$ if it is also unbounded. A continuous function $\beta:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}_+$ belongs to class $\mathcal{KL}$ if $\beta(\cdot,r)\in\mathcal{K}$ and $\beta(r,\cdot)$ is a strictly decreasing to zero for any fixed $r>0$.

A sequence of integers, $1,2,\ldots,n$, is further denoted by $\overline{1,n}$.

B. Input-to-state stability

Consider a dynamical system:

$$\dot{x}(t)=F(x(t),u(t)),\ t\geq 0, \tag{1}$$

where $x(t)\in\mathbb{R}^n$ is the state and $u(t)\in\mathbb{R}^m$ is the input, $u\in\mathcal{L}^m_\infty$; $F:\mathbb{R}^{n+m}\to\mathbb{R}^n$ is a continuous function, and it ensures forward existence and uniqueness of solutions of the system at least locally in time (e.g., it is locally Lipschitz continuous out the origin), $F(0,0)=0$. For any $x_0\in\mathbb{R}^n$ and $u\in\mathcal{L}^m_\infty$ the respective solution is denoted by $x(t,x_0,u)$ with $x(0,x_0,u)=x_0$.

The definitions and results given in this subsection follow [2], [20].

Definition 1. The system (1) is called practically ISS, if there exist $\beta\in\mathcal{KL}$, $\gamma\in\mathcal{K}$ and $q\geq 0$ such that

$$\|x(t,x_0,u)\|\leq\max\{\beta(\|x_0\|,t),\gamma(\|u\|_{[0,t)}),q\},\quad\forall t\geq 0$$

for all $x_0\in\mathbb{R}^n$ and $u\in\mathcal{L}^m_\infty$.

If $q=0$ then (1) is called ISS.

The function γ from Definition 1 quantifies an asymptotic gain of (1):

$$\limsup_{t\to+\infty}\|x(t,x_0,u)\|\leq\max\{\gamma(\|u\|_{[0,t)}),q\}$$

satisfying for all $x_0\in\mathbb{R}^n$ and $u\in\mathcal{L}^m_\infty$. Another property, followed by (practical) ISS, is (practical) global stability: there exist $\vartheta_x,\vartheta_u\in\mathcal{K}_\infty$ such that

$$\|x(t,x_0,u)\|\leq\max\{\vartheta_x(\|x_0\|),\vartheta_u(\|u\|_{[0,t)}),q\},\quad\forall t\geq 0$$

for all $x_0\in\mathbb{R}^n$ and $u\in\mathcal{L}^m_\infty$ (obviously, a possible choice is $\vartheta_x(s)=\beta(s,0)$ and $\vartheta_u(s)=\gamma(s)$).

If $u\equiv 0$, then from ISS we recover the conventional global asymptotic stability at the origin.

Definition 2. A continuously differentiable function $V:\mathbb{R}^n\to\mathbb{R}_+$ is called practical ISS-LF for the system (1) if there exist $\alpha_1,\alpha_2\in\mathcal{K}_\infty$, $\eta,\varrho\in\mathcal{K}$ and $r\geq 0$ such that for all $x\in\mathbb{R}^n$ and all $u\in\mathbb{R}^m$:

$$\alpha_1(\|x\|)\leq V(x)\leq\alpha_2(\|x\|),$$
$$V(x)\geq\max\{\varrho(\|u\|),r\}\Longrightarrow DV(x)F(x,u)\leq-\eta(\|x\|).$$

If $r=0$, then such a function V is called ISS-LF.

Theorem 1. The system (1) is (practically) ISS if and only if it admits a (practical) ISS-LF.

C. Homogeneity

For any $r_i>0$, $i\in\overline{1,n}$ and $\lambda>0$, define the vector of weights $\mathbf{r}=[r_1,...,r_n]$ and the dilation matrix $\Lambda_r(\lambda)=\text{diag}\{\lambda^{r_i}\}_{i=1}^n$; $r_{\min}=\min_{i\in\overline{1,n}}r_i$ and $r_{\max}=\max_{i\in\overline{1,n}}r_i$.

Definition 3. [21], [14] The function $h:\mathbb{R}^n\to\mathbb{R}$ is called $\mathbf{r}$-homogeneous, if for any $x\in\mathbb{R}^n$ the relation

$$h(\Lambda_r(\lambda)x)=\lambda^\nu h(x)$$

holds for some $\nu\in\mathbb{R}$ and all $\lambda>0$.

The vector field $f:\mathbb{R}^n\to\mathbb{R}^n$ is called $\mathbf{r}$-homogeneous, if for any $x\in\mathbb{R}^n$ the relation

$$f(\Lambda_r(\lambda)x)=\lambda^\nu\Lambda_r(\lambda)f(x)$$

holds for some $\nu\geq-r_{\min}$ and all $\lambda>0$.

In both cases, the constant ν is called the degree of homogeneity.

A dynamical system

$$\dot{x}(t)=f(x(t)),\ t\geq 0 \tag{2}$$

is called $\mathbf{r}$-homogeneous of degree ν if this property is satisfied for f in the sense of Definition 3.

For any $x\in\mathbb{R}^n$ and $\varpi\geq r_{\max}$, a homogeneous norm can be defined as follows

$$\|x\|_r=\left(\sum_{i=1}^n|x_i|^{\varpi/r_i}\right)^{1/\varpi}.$$

For all $x\in\mathbb{R}^n$, its Euclidean norm $\|x\|$ is related with the homogeneous one:

$$\underline{\sigma}_r(\|x\|_r)\leq\|x\|\leq\bar{\sigma}_r(\|x\|_r) \tag{3}$$

for some $\underline{\sigma}_r,\bar{\sigma}_r\in\mathcal{K}_\infty$ [14]. In the following, due to this "equivalence", stability analysis with respect to the norm $\|x\|$ can be substituted with analysis for the norm $\|x\|_r$. The homogeneous norm has an important property: it is $\mathbf{r}$-homogeneous of degree 1, that is $\|\Lambda_r(\lambda)x\|_r=\lambda\|x\|_r$ for all $x\in\mathbb{R}^n$ and $\lambda>0$.

Homogeneous systems possess certain robustness with respect to external disturbances.

Theorem 2. [15] Let $F(\Lambda_r(\lambda)x,\Lambda_{\tilde{r}}(\lambda)u)=\lambda^\nu\Lambda_r(\lambda)F(x,u)$ for all $x\in\mathbb{R}^n$, $u\in\mathbb{R}^m$ and all $\lambda>0$ with the weights $\mathbf{r}=[r_1,\ldots,r_n]>0$, $\tilde{\mathbf{r}}=[\tilde{r}_1,\ldots,\tilde{r}_m]>0$ and a degree $\nu\geq-r_{\min}$. Assume that the system (1) is asymptotically stable for $u=0$, then (1) is ISS.

III. Main result

Consider an affine in the inputs nonlinear system:

$$\dot{x}(t) = f(x(t)) + g(x(t))u(t),\ t \geq 0, \tag{4}$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the input, $u \in \mathcal{L}_\infty^m$; $f : \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ are locally Lipschitz continuous functions for all $x \in \mathbb{R}^n \setminus \{0\}$, ensuring forward existence and uniqueness of the system solutions at least locally in time, $f(0) = 0$.

Our goal is to derive ISS-like estimates for (4) with respect to average or weighted average values of the input u, and for this purpose we define the signals of interest:

$$\delta_1(t) = \frac{1}{T}\int_{t-T}^{t}(s-t+T)u(s)ds,\ \delta_2(t) = \frac{1}{T}\int_{t-T}^{t}u(s)ds,$$
$$\tilde{\delta}_1(t) = \frac{1}{T}\int_{t-T}^{t}(s-t+T)\|u(s)\|ds,\ \tilde{\delta}_2(t) = \frac{1}{T}\int_{t-T}^{t}\|u(s)\|ds,$$

where $T > 0$ is the window of averaging (a parameter for our analysis). These quantities are well-defined for $t \geq T$, while for $t \in [0, T)$ in the sequel we will assume that $u(t-T) = 0$, then all these averaged variables are defined for all $t \geq 0$.

Remark 1. Note that δ_1, δ_2, $\tilde{\delta}_1$ and $\tilde{\delta}_2$ are continuous functions of time $t \geq 0$, and the upper bounds of these averaged values are related to the respective norms of u and the period T:

$$\|\delta_1(t)\| \leq \|\tilde{\delta}_1(t)\| \leq \frac{T}{2}\|u\|_\infty,\ \|\delta_2(t)\| \leq \|\tilde{\delta}_2(t)\| \leq \|u\|_\infty.$$

A. Time-delay approach analysis

Similar to [11], [12], [22], let us define a transformation of state variables:

$$x(t) = \frac{1}{T}\int_{t-T}^{t}(s-t+T)g(x(s))u(s)ds + z(t), \tag{5}$$
$$x(s) = x(0),\ u(s) = 0,\ \forall s < 0,$$

where $z(t) \in \mathbb{R}^n$ is a new auxiliary residual variable defined for all $t \geq 0$. Differentiating (5) we obtain:

$$\begin{aligned}\dot{z}(t) &= f(x(t)) + \frac{1}{T}\int_{t-T}^{t}g(x(s))u(s)ds \\ &= f(z(t) + d_1(t)) + d_2(t),\end{aligned} \tag{6}$$

where

$$d_1(t) = -\frac{1}{T}\int_{t-T}^{t}(s-t+T)g(x(s))u(s)ds, \tag{7}$$
$$d_2(t) = \frac{1}{T}\int_{t-T}^{t}g(x(s))u(s)ds.$$

The right-hand side of (5) depends on the past values of $x(t)$ (i.e., $x(s)$ for $s \in [t-T, t)$). Thus, (5) can be written as $x(t) - G(x_t, u_t) = z(t)$ for $G(x_t, u_t) = \frac{1}{T}\int_{t-T}^{t}(s-t+T)g(x(s))u(s)ds$, where $x_t \in C^n_{[-T,0]}$ with $x_t(s) = x(t+s)$ and $u_t(s) = u(t+s)$ for $s \in [-T, 0]$. It can be considered as a difference equation with respect to $x(t)$ with disturbances u and z, which is ISS in $z(t)$ and $\tilde{\delta}_1(t)$ under introduced restrictions, as it is demonstrated later.

To this end note that for any $x(0) \in \mathbb{R}^n$ and $u \in \mathcal{L}_\infty^m$ in (5), $z(t)$ exists on maximal time interval $\mathcal{T} = [0, t')$ for some $t' > 0$ and $z(0) = x(0)$. By continuity of the solutions $x(t)$ of (4), $z(t)$ is a continuous function of time $t \in \mathcal{T}$ by construction.

The starting point for our analysis is ISS of (6) with respect to the auxiliary inputs d_1 and d_2:

Assumption 1. There exists a continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}_+$ such that

$$\begin{gathered}\alpha_1(\|z\|) \leq V(z) \leq \alpha_2(\|z\|), \\ V(z) \geq \max\{\sigma_1(\|d_1\|), \sigma_2(\|d_2\|)\} \Rightarrow \\ DV(z)\left(f(z+d_1) + d_2\right) \leq -\alpha(\|z\|),\end{gathered}$$

for all $z, d_1, d_2 \in \mathbb{R}^n$, with some $\alpha_1, \alpha_2, \alpha \in \mathcal{K}_\infty$ and $\sigma_1, \sigma_2 \in \mathcal{K}$.

These conditions imply that V is an ISS-LF for the system (6) with disturbances $d_1, d_2 \in \mathcal{L}_\infty^n$ and, hence, due to Theorem 1 this system is ISS with respect to these perturbations.

Since in (7) there is additionally a nonlinear gain g multiplying the input u, supplementary constraints have to be imposed. Note that by continuity of g, its norm can be bounded by a growing function of $\|x\|$, hence, there exist $\rho \in \mathcal{K}$ and $\rho_0 \geq 0$ such that

$$\|g(x)\| \leq \rho(\|x\|) + \rho_0,\ \forall x \in \mathbb{R}^n. \tag{8}$$

Assumption 2. There exist $\theta_\delta, \theta_z \in \mathcal{K}_\infty$ such that

$$\theta_z(s) + (\rho(s) + \rho_0)\theta_\delta(s) < s,\ \forall s > 0.$$

Assumption 3. There exist $\gamma_V \in \mathcal{K}$, $\gamma_1, \gamma_2 \in \mathcal{K}_\infty$ such that

$$\xi \geq \gamma_V(\zeta) \Rightarrow \sigma_i\left(\gamma_i(\xi)(\rho(\zeta) + \rho_0)\right) \leq \xi,\ i = 1, 2,\ \forall \xi, \zeta \in \mathbb{R}_+,$$

where $\sigma_1, \sigma_2 \in \mathcal{K}$ are given in Assumption 1.

In these assumptions, some functions from classes $\mathcal{K}$ and $\mathcal{K}_\infty$ are introduced, whose existence guarantees ISS of (4) with respect to averaged signals $\tilde{\delta}_1$ and $\tilde{\delta}_2$, as it will be shown below (assumptions 1 and 3 imply ISS of (4) with respect to the input u if $\gamma_V(s) \geq \alpha_1(s)$ for all $s \in \mathbb{R}_+$).

Lemma 1. Let Assumption 2 hold and given $x(0) \in \mathbb{R}^n$, $z \in \mathcal{L}_\infty^n \cap C^n_{[0,+\infty)}$, $u \in \mathcal{L}_\infty^m$ the relation (5) be satisfied for all $t \in \mathcal{T}$. Then $\mathcal{T} = \mathbb{R}_+$ and there exists $\beta_1 \in \mathcal{KL}$ such that

$$\begin{gathered}\|x(t)\| \leq \max\{\beta_1(\|x(0)\|, t), \theta_z^{-1}(\|z\|_{[0,t)}), \\ \theta_\delta^{-1}(\|\tilde{\delta}_1\|_{[0,t)})\}\end{gathered} \tag{9}$$

for all $t \geq 0$, all $x(0) \in \mathbb{R}^n$, $z \in \mathcal{L}_\infty^n \cap C^n_{[0,+\infty)}$ and $u \in \mathcal{L}_\infty^m$.

Proof. Using (8) an estimate follows from (5) for all $t \in \mathcal{T}$:

$$\begin{aligned}\|x(t)\| &\leq \frac{1}{T}\int_{t-T}^{t}(s-t+T)\|g(x(s))u(s)\|ds + \|z(t)\| \\ &\leq \left(\rho(|x_t|_C) + \rho_0\right)\tilde{\delta}_1(t) + \|z(t)\|.\end{aligned}$$

Under Assumption 2, this estimate leads to the following relation:

$$\|x(t)\| \geq \max\{\theta_z^{-1}(\|z(t)\|), \theta_\delta^{-1}(\tilde{\delta}_1(t))\} \Rightarrow$$
$$\|x(t)\| \leq \theta_z(|x_t|_C) + (\rho(|x_t|_C) + \rho_0)\,\theta_\delta(|x_t|_C) = \mu(|x_t|_C)$$

with $\mu \in \mathcal{K}_\infty$. The contraction property, $\mu(|x_t|_C) < |x_t|_C$ for all $|x_t|_C \neq 0$, is introduced in Assumption 2, and it implies convergence of $x(t)$ while the latter implication holds. Indeed, assume that $\mathcal{T} = \mathcal{T}_z \cup \mathcal{T}_x$, where

$$t \in \mathcal{T}_z : \|x(t)\| \leq \max\{\theta_z^{-1}(\|z\|_\infty), \theta_\delta^{-1}(\|\tilde{\delta}_1\|_\infty)\},$$
$$t \in \mathcal{T}_x : \|x(t)\| \geq \max\{\theta_z^{-1}(\|z\|_\infty), \theta_\delta^{-1}(\|\tilde{\delta}_1\|_\infty)\},$$

then due to continuity of $x(t)$ there exists a sequence of nonempty intervals such that $\mathcal{T}_x = \bigcup_{j\in\mathbb{Z}_+}[t_s^j, t_e^j]$ (all isolated points can be placed in $\mathcal{T}_z$). Denote $\tau_t = \min\{\arg\max_{s\in[-T,0]} \|x(t+s)\|\}$ (we use $\min\{\cdot\}$ to avoid non-uniqueness of $\arg\max$ operation), i.e., $|x_t|_C = \|x(\tau_t)\|$, and for some $j \in \mathbb{Z}_+$ consider $t \in [t_s^j, t_e^j]$, then

$$\|x(t)\| \leq \mu(\|x(\tau_t)\|) \leq \mu \circ \mu(|x_{\tau_t}|_C) \leq \mu \circ \mu(\|x(\tau_{\tau_t})\|) \leq$$
$$\cdots \leq \mu \overbrace{\circ \cdots \circ}^{\mathrm{ceil}\left(\frac{t-t_s^j}{T}\right)} (|x_{t_s^j}|_C),$$

where ceil$\,(s)$ is the function of truncation to the smallest integer bigger than $s \in \mathbb{R}$. Hence, there is $\beta_1 \in \mathcal{KL}$ such that

$$\|x(t)\| \leq \beta_1(|x_{t_s^j}|_C, t - t_s^j)$$

for all $t \in [t_s^j, t_e^j]$ and any $j \in \mathbb{Z}_+$. Note that $\beta_1(s, 0) = s$ by the construction, then for the value of $|x_{t_s^j}|_C$ two options are possible: either $t_s^j = 0$ and $|x_{t_s^j}|_C = |x_0|_C = \|x(0)\|$, or $t_s^j > 0$ and $|x_{t_s^j}|_C = \max\{\theta_z^{-1}(\|z\|_\infty), \theta_\delta^{-1}(\|\tilde{\delta}_1\|_\infty)\}$ by continuity. Therefore, the following estimate is satisfied for all $t \in \mathcal{T}$:

$$\|x(t)\| \leq \max\{\beta_1(\|x(0)\|, t), \theta_z^{-1}(\|z\|_\infty), \theta_\delta^{-1}(\|\tilde{\delta}_1\|_\infty)\},$$

implying that $\mathcal{T} = \mathbb{R}_+$, which finishes the proof. $\square$

In the framework of the trajectory based approach [23], a similar ISS result was obtained for $\mu(s) = \mu_0 s$ with $\mu_0 \in (0, 1)$ and the exponential convergence, then Lemma 1 can be seen as an extension of that method to the case of asymptotic convergence. Now let us investigate the behavior of $z(t)$ in (6). Under Assumption 1, for d_1 and d_2 defined by (7), considering $V(z)$ as an ISS-LF candidate for (6), the following relation is valid:

$$V(z(t)) \geq \max\{\sigma_1(\|d_1(t)\|), \sigma_2(\|d_2(t)\|)\} \Rightarrow$$
$$\dot{V}(t) = DV(z(t))\,(f(z(t) + d_1(t)) + d_2(t)) \leq -\alpha(\|z(t)\|),$$

Recalling (8),

$$\|d_1(t)\| \leq (\rho(|x_t|_C) + \rho_0)\,\tilde{\delta}_1(t),$$
$$\|d_2(t)\| \leq (\rho(|x_t|_C) + \rho_0)\,\tilde{\delta}_2(t).$$

In addition, under Assumption 3 the following implications are satisfied:

$$V(z(t)) \geq \max\{\gamma_V(|x_t|_C), \gamma_1^{-1}(\tilde{\delta}_1(t)),$$
$$\gamma_2^{-1}(\tilde{\delta}_2(t))\} \Rightarrow$$
$$V(z(t)) \geq \max\{\sigma_1\left((\rho(|x_t|_C) + \rho_0)\,\tilde{\delta}_1(t)\right),$$
$$\sigma_2\left((\rho(|x_t|_C) + \rho_0)\,\tilde{\delta}_2(t)\right)\} \Rightarrow$$
$$V(z(t)) \geq \max\{\sigma_1(\|d_1(t)\|), \sigma_2(\|d_2(t)\|)\},$$

which implies due to Definition 2 that V is an ISS-LF for (6) with respect to the inputs $|x_t|_C$, $\tilde{\delta}_1(t)$ and $\tilde{\delta}_2(t)$:

$$V(z) \geq \max\{\gamma_V(|x_t|_C), \gamma_1^{-1}(\tilde{\delta}_1), \gamma_2^{-1}(\tilde{\delta}_2)\} \Rightarrow \dot{V} \leq -\alpha(\|z\|)$$

for all $z \in \mathbb{R}^n$, $\tilde{\delta}_1, \tilde{\delta}_2 \in \mathbb{R}_+$ and $x_t \in C^n_{[-T,0]}$. Therefore, according to Theorem 1, the following estimate is valid:

$$\|z(t)\| \leq \alpha_1^{-1} \circ \max\{\beta_2(\|z(0)\|, t), \gamma_V(\|x\|_{[0,t)}), \quad (10)$$
$$\gamma_1^{-1}(\|\tilde{\delta}_1\|_{[0,t)}), \gamma_2^{-1}(\|\tilde{\delta}_2\|_{[0,t)})\}$$

for all $t \geq 0$ and some $\beta_2 \in \mathcal{KL}$ (note that again $\beta_2(s, 0) = s$).

We are in position to formulate the main result.

Theorem 3. Let Assumptions 1–3 be satisfied for (4), then there exists $\beta_3 \in \mathcal{KL}$ such that

$$\|x(t)\| \leq \max\{\beta_3(\|x(0)\|, t), \theta_\delta^{-1}(\|\tilde{\delta}_1\|_{[0,t)}),$$
$$\theta_z^{-1} \circ \alpha_1^{-1} \circ \gamma_1^{-1}(\|\tilde{\delta}_1\|_{[0,t)}), \theta_z^{-1} \circ \alpha_1^{-1} \circ \gamma_2^{-1}(\|\tilde{\delta}_2\|_{[0,t)})\}$$

for all $t \geq 0$, all $x(0) \in \mathbb{R}^n$ and $u \in \mathcal{L}_\infty^m$ provided that

$$\theta_z^{-1} \circ \alpha_1^{-1} \circ \gamma_V(s) < s, \ \forall s > 0,$$

meaning that (4) is ISS with respect to $\tilde{\delta}_1, \tilde{\delta}_2 \in \mathcal{L}_\infty^1$ that are averages of $|u|$.

Proof. Since all conditions of Lemma 1 are verified, the estimate (9) is valid for $t \in \mathcal{T} \subseteq \mathbb{R}_+$. Above, it has been shown that the estimate (10) is also true for all $t \in \mathcal{T}$. Using the standard small-gain arguments [24], [25], first, the global stability of the feedback interconnection (5), (6) can be proven, i.e., there exist functions $\psi_i \in \mathcal{K}_\infty$ for $i = 0, 1, 2$ such that

$$\max\{\|x(t)\|, \|z(t)\|\} \leq \max\{\psi_0 \circ \max\{\|x(0)\|, \|z(0)\|\},$$
$$\psi_1(\|\tilde{\delta}_1\|_\infty), \psi_2(\|\tilde{\delta}_2\|_\infty)\}, \ \forall t \geq 0,$$

for all $x(0), z(0) \in \mathbb{R}^n$ and $u \in \mathcal{L}_\infty^m$, implying also that $\mathcal{T} = \mathbb{R}_+$. Under the observation that $\|z(0)\| \leq (\rho(\|x(0)\|) + \rho_0)\,\|\tilde{\delta}_1\|_\infty + \|x(0)\|$, the latter estimate can be rewritten as

$$\max\{\|x(t)\|, \|z(t)\|\} \leq \max\{\tilde{\psi}_0(\|x(0)\|),$$
$$\tilde{\psi}_1(\|\tilde{\delta}_1\|_\infty), \psi_2(\|\tilde{\delta}_2\|_\infty)\}$$

for all $t \geq 0$, $x(0) \in \mathbb{R}^n$ and $u \in \mathcal{L}_\infty^m$, for suitably modified $\tilde{\psi}_0, \tilde{\psi}_1 \in \mathcal{K}_\infty$. Second, the asymptotic gain function can be derived (we are further interested only in the behavior of $x(t)$ due to its physical meaning in (4)):

$$\limsup_{t\to+\infty} \|x(t)\| \leq \max\{\theta_\delta^{-1}(\|\tilde{\delta}_1\|_\infty),$$
$$\theta_z^{-1} \circ \alpha_1^{-1} \circ \max\{\gamma_1^{-1}(\|\tilde{\delta}_1\|_\infty), \gamma_2^{-1}(\|\tilde{\delta}_2\|_\infty)\}\}.$$

If the system (4) admits both properties, global stability and asymptotic gain, then it is ISS [26, Lemma 2.7]. □

Remark 2. All assumptions, 1–3, as well as the small-gain condition in Theorem 3, are formulated globally (i.e., for all $x \in \mathbb{R}^n$ and $s > 0$). Restricting validity of the introduced inequalities it is possible to get the respective local stability results.

In the conditions of Theorem 3, the system (4) is not obligatory ISS in u, while the power ISS or the moving average ISS properties from [3] are equivalent to ISS under Assumption 1. In [4], ISS of (4) with respect to $\tilde{\delta}_2$ was established starting from ISS property in u.

Note that rewriting (6) in the coordinates $x(t)$,

$$\frac{d}{dt}\left(x(t) - \frac{1}{T}\int_{t-T}^{t}(s-t+T)g(x(s))u(s)ds\right) = f(x(t)) + \frac{1}{T}\int_{t-T}^{t} g(x(s))u(s)ds,$$

we get a special neutral-type time-delay system. Then the main result of this work can be interpreted as an extension of Lyapunov-Razumikhin ISS result of [27] to this neutral-type system.

The results of Theorem 3 can be easily applied if the function $g(x)$ is just bounded, then $\rho \in \mathcal{K}$ from (8) admits a restriction:

$$\rho(s) \le \rho_1 < +\infty, \ \forall s > 0,$$

for some $\rho_1 > 0$, and we can take $\theta_\delta(s) = \frac{\theta_z(s)}{\rho_1+\rho_0}$ with $\theta_z(s) < \frac{s}{2}$, $\gamma_i(s) = \frac{\sigma_i^{-1}(s)}{\rho_1+\rho_0}$ with $i = 1, 2$ and $\gamma_V(s) < \alpha_1 \circ \theta_z(s)$. In addition, if g is a constant function, then a more constructive result can be derived given below.

B. A static input gain

A constant input gain, $g(x) = B \in \mathbb{R}^{n\times m}$, represents an interesting practical case, where rather technical Assumptions 2, 3 and the small-gain condition can be skipped (they characterize the properties of g as a function of x):

Theorem 4. Let Assumption 1 be satisfied for (4) with $g(x) \equiv B$, then there exists $\beta_4 \in \mathcal{KL}$ such that

$$\|x(t)\| \le \max\{\beta_4(\|x(0)\|, t), \alpha_1^{-1} \circ \sigma_1(\|B\|\|\delta_1\|_{[0,t)}), \alpha_1^{-1} \circ \sigma_2(\|B\|\|\delta_2\|_{[0,t)})\} + \|B\|\|\delta_1\|_{[0,t)}$$

for all $t \ge 0$, all $x(0) \in \mathbb{R}^n$ and $u \in \mathcal{L}_\infty^m$, meaning that (4) is ISS with respect to $\delta_1, \delta_2 \in \mathcal{L}_\infty^m \cap C_{[0,+\infty)}^m$ that are averages of u.

Proof. In such a case the equations (5) and (6) take the form:

$$x(t) = B\delta_1(t) + z(t)$$

and

$$\dot{z}(t) = f(z(t) - B\delta_1(t)) + B\delta_2(t),$$

respectively. Hence, from the first equation, which is static here, we immediately get:

$$\|x(t)\| \le \|B\|\|\delta_1(t)\| + \|z(t)\|$$

for all $t \ge 0$, while applying Assumption 1 to the dynamics of z, its ISS property can be substantiated, i.e., there exists $\tilde{\beta}_4 \in \mathbb{R}^n$ such that

$$\|z(t)\| \le \alpha_1^{-1} \circ \max\{\tilde{\beta}_4(\|z(0)\|, t), \sigma_1(\|B\|\|\delta_1\|_{[0,t)}), \sigma_2(\|B\|\|\delta_2\|_{[0,t)})\}, \ \forall t \ge 0,$$

for all $z(0) \in \mathbb{R}^n$ and $\delta_1, \delta_2 \in \mathcal{L}_\infty^m$. Combining both inequalities, due to $\|z(0)\| = \|x(0)\|$ we obtain:

$$\|x(t)\| \le \|B\|\|\delta_1\|_{[0,t)} + \alpha_1^{-1} \circ \max\{\tilde{\beta}_4(\|x(0)\|, t), \sigma_1(\|B\|\|\delta_1\|_{[0,t)}), \sigma_2(\|B\|\|\delta_2\|_{[0,t)})\}, \ \forall t \ge 0,$$

for all $x(0) \in \mathbb{R}^n$ and $\delta_1, \delta_2 \in \mathcal{L}_\infty^m$, leading to the required estimate. □

Remark 3. Note that we may have $\delta_2 \in \mathcal{L}_\infty^m$ while $u \notin \mathcal{L}_\infty^m$.

For a T-periodic time signal u with zero mean value (in such a case $\delta_2(t) = 0$ for all $t \ge 0$), the following corollary is a direct consequence of Theorem 4 and Remark 1:

Corollary 1. Let Assumption 1 be satisfied for $d_2 = 0$ and $g(x) \equiv B$ in (4). Then for all $x(0) \in \mathbb{R}^n$ and all T-periodic signals $u \in \mathcal{L}_\infty^m$ with zero mean value the following holds:

$$\limsup_{t\to+\infty}\|x(t)\| \le \alpha_1^{-1} \circ \sigma_1(\|B\|\frac{T}{2}\|u\|_\infty) + \|B\|\frac{T}{2}\|u\|_\infty.$$

Remark 4. If the signal u is T-periodic with non-zero average, then practical asymptotic gain can be obtained. Local versions of the above results follow the respective restrictions in Assumption 1 (e.g., if V is a local ISS-LF).

Defining the frequency $\omega = \frac{2\pi}{T}$, this result allows us to introduce an analogue of Bode magnitude plot for the nonlinear system (4) with $g(x) \equiv B$:

$$H(\omega) = 20\log\left(\alpha_1^{-1} \circ \sigma_1(\|B\|\frac{\pi}{\omega}\|u\|_\infty) + \|B\|\frac{\pi}{\omega}\|u\|_\infty\right),$$

which indicates that for a constant input amplitude, increasing the frequency improves the filtering abilities for the class of nonlinear systems satisfying the conditions of Corollary 1. Comparing with [13] (see also [28]), where a frequency response of convergent systems is studied, and to find the response it is suggested either to solve a PDE or to make simulations, in the present work an upper estimate $H(\omega)$ of the response is obtained dependent on the functions α_1 and σ_1 defining an ISS-LF of (6), which are easier to derive.

Example 1. In [13], the frequency response of

$$\dot{x}_1(t) = -x_1(t) + x_2^2(t),$$
$$\dot{x}_2(t) = -x_2(t) + u(t),$$

was analytically derived, $x(t) = [x_1(t)\ x_2(t)]^\top \in \mathbb{R}^2$. To apply Corollary 1, consider in Assumption 1 a Lyapunov function $V(x) = \frac{1}{2}x_1^2 + \frac{\varkappa}{2}x_2^2 + \frac{\ell}{4}x_2^4$ for some $\varkappa > 0$ and $\ell > 0$, then an estimate $DV(x)f(x+d_1) \le -(1-1.5\epsilon)x_1^2 - \frac{\varkappa}{2}x_2^2 - \left(\frac{\ell}{4} - \epsilon^{-1}\right)x_2^4 + 0.5\epsilon^{-1}(2d_{12}^2 + |d_{11}|)^2 + \frac{\varkappa}{2}d_{12}^2 + \frac{\ell}{4}d_{12}^4$ for

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$d_1 = [d_{11}\ d_{12}]^\top \in \mathbb{R}^2$ and some $\epsilon \in (0, \frac{2}{3})$ can be obtained, which for $\ell > 4\epsilon^{-1}$ gives

$$\alpha_1(s) = \frac{\min\{1, \varkappa\}}{2} s^2,$$
$$\sigma_1(s) = 2\frac{\left(\frac{\ell}{4} + \frac{4}{\epsilon}\right) s^4 + \max\{\epsilon^{-1}, \frac{\varkappa}{2}\} s^2}{\min\{1, 2 - 3\epsilon, 1 - \frac{4}{\ell\epsilon}\}}.$$

An interesting feature of the obtained result is that in (4), in order to have a well-defined asymptotic gain with respect to an average value of the input u, according to Assumption 1, the system should be ISS with respect to a "measurement noise" signal d_1. However, the noise sensitivity of a system usually has other origins and mechanisms than the robustness in the exogenous state disturbances leading to insightful results, which is demonstrated in the next example.

Example 2. Consider a system

$$\dot{x}(t) = -k|x(t)|^\kappa \mathrm{sign}(x(t)) + \sin(\omega t),$$

where $x(t) \in \mathbb{R}$ and $k, \kappa > 0$ are parameters. Assumption 1 is satisfied for $V(x) = x^2$ with

$$\alpha_1(s) = \alpha_2(s) = s^2,\ \sigma_1(s) = \epsilon_1^{-2} s^2,\ \sigma_2(s) = \epsilon_2^{-\frac{2}{\kappa}} s^{\frac{2}{\kappa}},$$
$$\alpha(s) = 2\epsilon_3 s^{1+\kappa},$$

where $\epsilon_1 \in (0, 1)$, $\epsilon_2, \epsilon_3 > 0$ and $k(1 - \epsilon_1)^\kappa \geq \epsilon_2 + \epsilon_3$. Therefore,

$$H(\omega) = 20 \log \left((1 + \epsilon_1^{-1}) \frac{\pi}{\omega} \right)$$

and optimizing the value of ϵ_1 we can get a lower bound $\underline{H}(\omega) = 20 \log \left(2\frac{\pi}{\omega}\right)$ independent in k and κ, which may indicate that the obtained bound is qualitative representing the worst-case estimate. To evaluate its conservativeness, we simulate this system on the time interval $[0, 500]$ using the explicit Euler method with the time step $h = 0.005$, for different values of ω, κ, k and calculating the map

$$a_k(\omega, \kappa) = \log \left[\max_{400 \leq t \leq 500} |x(t)| \right]$$

we obtain the results given in Fig. 1 for $k = 0.01, 1, 100$. These results surprisingly demonstrate indeed a weak dependence on κ, and also a limited sensitivity for the decay in k, while increasing ω leads to lower values of a_k confirming the theoretical findings.

C. Integral ISS

Another interesting consequence of Corollary 1 is that if the system (4) is integral ISS (iISS) [2], [20], then it has an asymptotic gain with respect to $\eta\left(\int_0^t \gamma(\|u(s)\|) ds\right)$ for some $\eta, \gamma \in \mathcal{K}_\infty$ (and not with respect to $\gamma(\|u\|_\infty)$ as in the ISS case), and it is easy to observe that $\int_0^t \gamma(\|u(s)\|) ds$ for any $\gamma \in \mathcal{K}_\infty$ and a bounded non-zero periodic u is an unbounded signal in t. Therefore, iISS property, formally, does not provide a reliable conclusion about the boundedness of solutions of (4) in such a setting. However, if the conditions of Corollary 1 are verified, it could be the case that the system is still ISS with respect to d_1

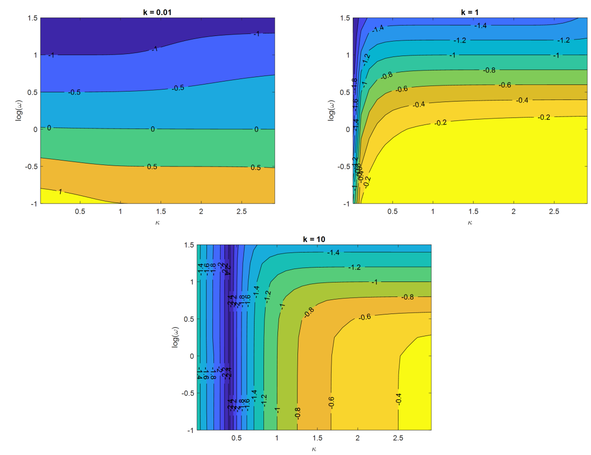

Figure 1. Plot of a_k for different value of k

(Assumption 1 is satisfied for $d_2 = 0$), then the theory proposed in this work allows to guarantee the boundedness with respect to periodic signals for iISS systems.

Example 3. For an illustration consider a scalar system

$$\dot{x}(t) = -k\varphi(x(t)) + \sin(\omega t),$$

where $x(t) \in \mathbb{R}$, $k > 0$ is a parameter, and $\varphi : \mathbb{R} \to \mathbb{R}$ is a bounded continuous function such that $\varphi(s)s > 0$ for all $s \in \mathbb{R} \setminus \{0\}$. Due to boundedness of φ this system is only iISS with respect to an additive input. However, Assumption 1 is satisfied for $d_2 = 0$ with $V(x) = x^2$ and

$$\alpha_1(s) = \alpha_2(s) = s^2,\ \sigma_1(s) = \epsilon_1^{-2} s^2,$$
$$\alpha(s) = 2ks\varphi_{\epsilon_1}(s),$$

where $\epsilon_1 \in (0, 1)$ is a parameter, and

$$\varphi_{\epsilon_1}(x) = \begin{cases} \min_{s \in [(1-\epsilon_1)x, (1+\epsilon_1)x]} f(s) & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ \max_{s \in [(1+\epsilon_1)x, (1-\epsilon_1)x]} f(s) & \text{if } x < 0 \end{cases}$$

is a continuous function verifying the passivity condition $\varphi_{\epsilon_1}(s)s > 0$ for all $s \in \mathbb{R} \setminus \{0\}$. Therefore, again

$$H(\omega) = 20 \log \left((1 + \epsilon_1^{-1}) \frac{\pi}{\omega} \right)$$

and optimizing the value of ϵ_1 we can get a lower bound $\underline{H}(\omega) = 20 \log \left(2\frac{\pi}{\omega}\right)$.

IV. Application to homogeneous systems

Let us consider a subclass of (4) with homogeneous functions f and g, which allows us to replace Assumptions 1–3 by the following one (denote $g(x) = [g_1(x) \ldots g_m(x)]$, where $g_i(x)$ is a column of g for $i \in \overline{1, m}$, then $g_{i,j}(x)$ is the element of g_i for $j \in \overline{1, n}$):

Assumption 4. The vector field f and functions $g_{i,j}$ for $i \in \overline{1, m}$, $j \in \overline{1, n}$ are $\mathbf{r}$-homogeneous of degrees $\nu \geq -r_{\min}$ and $\varsigma_i > 0$, respectively, and the system (4) is asymptotically stable for $u \equiv 0$.

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Following [21], [29], under Assumption 4, for the system (4) there is a twice continuously differentiable and $\mathbf{r}$-homogeneous of degree $\mu > r_{\max}$ Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}_+$, which implies fulfillment of the following inequalities:

$$c_1\|x\|_r^\mu \le V(x) \le c_2\|x\|_r^\mu, DV(x)f(x) \le -c_3\|x\|_r^{\mu+\nu},$$
$$\sup_{\|x\|_r=1}\left\|\frac{\partial V(x)}{\partial x}\right\| \le c_4$$

for all $x \in \mathbb{R}^n$ and some $c_i > 0$, $i = 1, \dots, 4$. Moreover, following [15] and using (3) there exist $\omega_1 > 0$ and $\omega_2 > 0$ (dependent only on the properties of f and c_3, c_4) such that

$$\|x\|_r \ge \max\{\omega_1\|d_1\|_r, \omega_2\|d_2\|_{\tilde r}\} \Rightarrow$$
$$DV(x)(f(x + d_1) + d_2) \le -\frac{c_3}{2}\|x\|_r^{\mu+\nu}$$

for all $x, d_1, d_2 \in \mathbb{R}^n$ (see Theorem 2), $\tilde{\mathbf{r}} = \mathbf{r} + \nu$. Hence, Assumption 1 is verified for this V with $\alpha(s) = \frac{c_3}{2}s^{\mu+\nu}$, $\sigma_i(s) = c_2\omega_i^\mu s^\mu$ for $i = 1, 2$, where the homogeneous norm is used instead of the standard one (recall that these norms are equivalent due to (3)), then $|x_t|_C = \sup_{-T\le\varsigma\le 0}\|x(t+\varsigma)\|_r$ in this section. As a consequence, to verify Assumptions 2–3 we will use also the homogeneous norm, then to simplify the computations let $m = 1$ and $\varsigma = \varsigma_1$. From Assumption 4 we get that $\|g(x)\| \le c_5\|x\|_r^\varsigma$ for all $x \in \mathbb{R}^n$ and some $c_5 > 0$, therefore, $\rho(s) = c_5\underline{\sigma}_r^{-\varsigma}(s)$ and $\rho_0 = 0$ in (8). Next, let us repeat the steps made in the previous section in order to replace the conditions in Assumptions 2 and 3.

An estimate follows from (5) for all $t \in \mathcal{T}$:

$$\|x(t)\| \le \frac{1}{T}\int_{t-T}^t (s - t + T)\|g(x(s))u(s)\|ds + \|z(t)\|$$
$$\le \frac{1}{T}\int_{t-T}^t (s - t + T)\|u(s)\|\|g(x(s))\|ds + \|z(t)\|$$
$$\le c_5|x_t|_C^\varsigma\tilde\delta_1(t) + \|z(t)\|$$

for $x(0) \in \mathbb{R}^n$, $z \in \mathcal{L}_\infty^n$ and $u \in \mathcal{L}_\infty^m$. This estimate leads to the following relation:

$$\|x(t)\|_r \ge \max\{\theta_z^{-1}(\|z(t)\|), \theta_\delta^{-1}(\tilde\delta_1(t))\} \Rightarrow$$
$$\|x(t)\| \le \theta_z(|x_t|_C) + c_5|x_t|_C^\varsigma\theta_\delta(|x_t|_C) \Rightarrow$$
$$\|x(t)\|_r \le \underline{\sigma}_r^{-1}\left(\theta_z(|x_t|_C) + c_5|x_t|_C^\varsigma\theta_\delta(|x_t|_C)\right)$$
$$< |x_t|_C,$$

which is valid provided that

$$\underline{\sigma}_r^{-1}(\theta_z(s) + c_5 s^\varsigma\theta_\delta(s)) < s, \ \forall s > 0. \tag{11}$$

Repeating the same computations as before, the following inequality is satisfied for all $t \ge 0$, all $x(0) \in \mathbb{R}^n$, $z \in \mathcal{L}_\infty^n$ and $u \in \mathcal{L}_\infty^m$:

$$\|x(t)\|_r \le \max\{\beta_1(\|x(0)\|_r, t), \theta_z^{-1}(\|z\|_\infty), \theta_\delta^{-1}(\|\tilde\delta_1\|_\infty)\}$$

for some $\beta_1 \in \mathcal{KL}$. Recalling the definition of d_1 and d_2 in (7), we obtain

$$\underline{\sigma}_r(\|d_1(t)\|_r) \le c_5|x_t|_C^\varsigma\tilde\delta_1(t),$$
$$\underline{\sigma}_{\tilde r}(\|d_2(t)\|_{\tilde r}) \le c_5|x_t|_C^\varsigma\tilde\delta_2(t).$$

Next, the following implications should be satisfied:

$$V(z(t)) \ge \max\{\gamma_V(|x_t|_C), \gamma_1^{-1}(\tilde\delta_1(t)), \gamma_2^{-1}(\tilde\delta_2(t))\} \Rightarrow$$
$$V(z(t)) \ge \max\{c_2\omega_1^\mu\underline{\sigma}_r^{-\mu}\left(c_5|x_t|_C^\varsigma\tilde\delta_1(t)\right),$$
$$c_2\omega_2^\mu\underline{\sigma}_{\tilde r}^{-\mu}\left(c_5|x_t|_C^\varsigma\tilde\delta_2(t)\right)\} \Rightarrow$$
$$V(z(t)) \ge \max\{c_2\omega_1^\mu\|d_1(t)\|_r^\mu, c_2\omega_2^\mu\|d_2(t)\|_{\tilde r}^\mu\},$$

leading to the property:

$$\xi \ge \gamma_V(\zeta) \Rightarrow c_2\omega_1^\mu\underline{\sigma}_r^{-\mu}\left(c_5\zeta^\varsigma\gamma_1(\xi)\right) \le \xi, \tag{12}$$
$$c_2\omega_2^\mu\underline{\sigma}_{\tilde r}^{-\mu}\left(c_5\zeta^\varsigma\gamma_2(\xi)\right) \le \xi, \ \forall\xi, \zeta \in \mathbb{R}_+.$$

Therefore, according to Theorem 1, the following estimate is valid for all $t \ge 0$:

$$\|z(t)\|_r \le c_1^{-1/\mu}\max\{\beta_2(\|z(0)\|_r, t), \gamma_V(\sup_{s\in[0,t)}\|x(s)\|_r),$$
$$\gamma_1^{-1}(\|\tilde\delta_1\|_{[0,t)}), \gamma_2^{-1}(\|\tilde\delta_2\|_{[0,t)})\}^{1/\mu},$$

for all $z(0) \in \mathbb{R}^n$, $x \in \mathcal{L}_\infty^n$ and $u \in \mathcal{L}_\infty^m$, and a suitably defined $\beta_2 \in \mathcal{KL}$. Finally, the small-gain condition of Theorem 3 takes the form:

$$\theta_z^{-1} \circ \bar\sigma_r\left(c_1^{-1/\mu}\gamma_V^{1/\mu}(s)\right) < s, \ \forall s > 0. \tag{13}$$

Recall from [30] that for $r_{\max} \le 1$ and $\varpi = 1$ (this property can be always provided by scaling the weights in $\mathbf{r}$) the following choice is possible

$$\underline{\sigma}_r^{-1}(s) = n\begin{cases} s^{\frac{1}{r_{\max}}} & s \le 1 \\ s^{\frac{1}{r_{\min}}} & s > 1\end{cases}, \ \bar\sigma_r(s) = \sqrt{n}\begin{cases} s^{r_{\min}} & s \le 1 \\ s^{r_{\max}} & s > 1\end{cases}.$$

Let us perform verification of the conditions (11)–(13) (Assumptions 2–3 and the small-gain condition) for $s > 1$. After straightforward computations (applying Jensen's inequality) we get that (11) is satisfied for $\theta_z(s) = c_6 s^{r_{\min}}$ and $\theta_\delta(s) = c_6 s^{r_{\min}-\varsigma}$ with some sufficiently small $c_6 > 0$ under constraint $\varsigma < r_{\min}$; the condition (12) is valid for $\gamma_V(s) = c_7 s^{\frac{2\varsigma\mu}{r_{\min}}}$ and $\gamma_1(s) = c_8 s^{\frac{r_{\min}}{2\mu}}$, $\gamma_2(s) = c_8 s^{\frac{r_{\min}+2\nu}{2\mu}}$ for some $c_7 > 0$ and $c_8 > 0$ if $\nu \ge 0$, and $\gamma_V(s) = c_7 s^{\frac{2\varsigma\mu}{r_{\min}+\nu}}$, $\gamma_1(s) = c_8 s^{\frac{r_{\min}}{2\mu}}$, $\gamma_2(s) = c_8 s^{\frac{r_{\min}+\nu}{2\mu}}$ otherwise; finally the small-gain relation (13) can be checked for $s > 1$ if $\varsigma \le \frac{r_{\min}}{2r_{\max}}\min\{r_{\min}, r_{\min} + \nu\}$. Since $r_{\max} = 1$ is a possible/maximal choice, the restriction on the degree of g is $\varsigma \le \frac{r_{\min}}{2}\min\{r_{\min}, r_{\min} + \nu\}$, and the following result can be deduced from Theorem 3:

Corollary 2. For the case $m = 1$, the system (4) under Assumption 4 with $\varsigma \le \frac{r_{\min}}{2}\min\{r_{\min}, r_{\min}+\nu\}$ is practically ISS with respect to the inputs $\tilde\delta_1, \tilde\delta_2 \in \mathcal{L}_\infty^1$.

Example 4. Consider a system

$$\dot x(t) = -k|x(t)|^\kappa\text{sign}(x(t)) + |x(t)|^\varsigma\text{sign}(x(t))u(t),$$

where $x(t) \in \mathbb{R}$ and $k, \kappa, \varsigma > 0$ are parameters. Assumption 4 is satisfied for $\mathbf{r} = 1$ and $\nu = \kappa - 1$, and according to the established result, $\varsigma \le \frac{\kappa}{2}$. Obviously, if $\kappa > \varsigma$ then the system is ISS with respect to $u \in \mathcal{L}_\infty^1$ and for $u(t) = A\sin(\frac{2\pi}{T}t)$, for some $A > 0$ and $T > 0$, this system is practically ISS with respect to $\tilde\delta_1(t) \le \frac{AT}{2}$ and $\tilde\delta_2(t) = \frac{2A}{\pi}$.

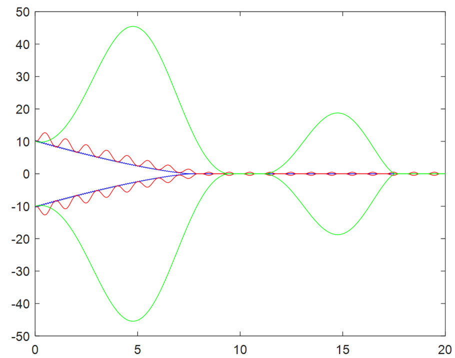

Figure 2. An illustration for Corollary 2

The results of simulation for $k = 1$, $\kappa = 0.25$, $\varsigma = 1/3$ and $A = 5$ are presented in Fig. 2, where blue, red and green curves correspond to $T = 0.1, 1, 10$, respectively. A kind of finite-time convergence to the origin is observed since the term $g(x(t))u(t)$ acts as a stabilizer for the instants of time when $u(t)$ is negative.

V. Conclusions

Using the time-delay approach, the conditions of ISS are investigated for nonlinear affine in the input systems with respect to an average value of the disturbances. For periodic signals and static input gains, a kind of Bode amplitude plot is derived for this class of nonlinear dynamics. For integral ISS systems, the gains are obtained for periodic inputs implying boundedness of solutions. For the case of homogeneous systems, it is shown that stability can be preserved with respect to an average value of the input even in the case when degree of the perturbation gain is higher than the degree of the nominal dynamics. The results are illustrated by simple examples. Relaxing conservatism of the imposed assumptions can be considered as a direction for further research.

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