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Brief paper

Lie-brackets-based averaging of affine systems via a time-delay approach ☆

Jin Zhang [a,b,*], Emilia Fridman [b]

[a] School of Mechatronic Engineering and Automation, Shanghai University, Shanghai 200072, China
[b] School of Electrical Engineering, Tel Aviv University, Tel Aviv 69978, Israel

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ABSTRACT

In this paper, we study input-to-state stability (ISS) of affine systems with a small parameter $\varepsilon > 0$ and additive disturbances in the presence of state-delays. We present a time-delay approach to Lie-brackets-based averaging, where we transform the system to a time-delay (neutral type) one. The latter has a form of perturbed Lie brackets system. The ISS of the time-delay system guarantees the same for the original one. We present a direct Lyapunov–Krasovskii (L–K) method for the time-delay system and provide sufficient conditions for regional ISS. Further we apply the results to stabilization of linear uncertain systems under unknown control directions using the bounded extremum seeking controller with measurement delay. In contrast to the existing results that are all qualitative, we derive constructive linear matrix inequalities for finding quantitative upper bounds on ε and the time-delay that ensure regional ISS of the original system and on the resulting ultimate bound. Numerical examples illustrate the efficiency of our method.

1. Introduction

Extremum seeking (ES), as a real-time model-free optimization approach, has received much attention during the past decades, starting with the rigorous proof of local convergence in Krstić and Wang (2000) and extension to semi-global convergence in Tan, Nešić, and Mareels (2006). Various maps and dithers were introduced for ES systems in Dürr, Stanković, Ebenbauer, and Johansson (2013), Grushkovskaya, Zuyev, and Ebenbauer (2018), Guay and Dochain (2014), Oliveira, Feiling, Koga, and Krstić (2020), Scheinker and Krstić (2017) and Tan, Nešić, and Mareels (2008). Besides, ISS of nonlinear time-varying systems with application to ES was recently studied in Labar, Ebenbauer, and Marconi (2022). A majority of the aforementioned literature relies on the classical averaging method (Khalil, 2002) and Lie brackets approximation (Gurvits & Li, 1993; Sussmann & Liu, 1993). By exploiting the converging trajectories property of the original system and the averaged system, the stability of the original system is guaranteed provided that the parameter $\varepsilon > 0$ is small enough. In case of averaging-based stability analysis of time-varying systems, a direct Lyapunov method along solutions of an oscillatory system was suggested e.g. in Morin and Samson (1997) and Teel, Peuteman, and Aeyels (1999). Note that in Morin and Samson (1997) the first analytical upper bound on ε was suggested, but in the example the analytical upper bound was not calculated (being conservative) and appropriate values of ε were found from simulations. However, bounds on the small parameter found from simulations only are not reliable for the unknown systems. Some upper bounds on the small parameter for finite-time stabilization of linear systems under unknown control directions via classical ES method were presented in Mele, De Tommasi, and Pironti (2022). However, the latter bounds still employed approximations.

Recently, a constructive time-delay approach to periodic averaging was introduced in Fridman and Zhang (2020), where the system is transformed to a time-delay (neutral type) system whose nominal part is the stable averaged system. The direct Lyapunov-Krasovskii (L-K) method applied to the neutral system leads to linear matrix inequalities (LMIs) for finding an efficient upper bound on the small parameter ensuring the stability and ISS of the original system. In Zhang and Fridman (2022b), an improved time-delay approach to periodic averaging was presented with fewer terms to be compensated in the L–K analysis leading

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* Corresponding author at: School of Mechatronic Engineering and Automation, Shanghai University, Shanghai 200072, China

E-mail addresses: zhangjin1116@126.com (J. Zhang), emilia@tauex.tau.ac.il (E. Fridman).

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to less conservative and simpler LMIs. The time-delay approach to averaging was applied to power systems (Griñó, Ortega, Fridman, Zhang, & Mazenc, 2021), vibrational control (Zhang & Fridman, 2022a), and extended to ES (Zhu & Fridman, 2022; Zhu, Fridman, & Oliveira, 2023). Note that for the completely unknown models, the time-delay approach allows for new results (e.g. sampled-data and delays in Zhu et al., 2023) and more explicit bounds, whereas for partially unknown models (with applications of ES to control of vehicles in GPS-denied environment) it provides the quantitative bounds that are very important for reliable control.

As it is well known, input and output delays are unavoidable in practical applications. Classical ES subject to a large known constant delay was studied in Oliveira, Krstić, and Tsubakino (2016) by using backstepping-based predictors and in Malisoff and Krstic (2021) by using sequential predictors. Robustness of classical ES with constant sampling and small constant delays was presented in Zhu et al. (2023). We will consider, for the first time, affine systems in the presence of state time-varying delays (without any restriction on the delay derivative) that may include the sampling and constant delays as particular cases.

In this paper, motivated by Fridman and Zhang (2020), Zhang and Fridman (2022b), Zhu and Fridman (2022) and Zhu et al. (2023), we propose a time-delay approach to Lie-brackets-based averaging of affine systems with a small parameter $\varepsilon > 0$ and additive disturbances in the presence of state-delays. We transform the affine system to a time-delay system of neutral type without any approximations. The ISS of the resulting time-delay system guarantees the ISS of the original one. We provide L-K-based sufficient conditions for regional ISS. We further consider an application to stabilization of linear systems under unknown control directions (Scheinker & Krstić, 2013), where we employ a bounded ES controller (Scheinker & Krstić, 2014) in the presence of delayed measurements and derive LMIs. By verifying these LMIs, one can find quantitative bounds on ε and on the time-delay that ensure regional ISS along with the corresponding initial and attractive balls. A conference version of the results confined to the non-delayed case was presented in Zhang and Fridman (2022c).

Notation: The Lie bracket of two vector fields $f, g : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$ with $f(t,\cdot)$ and $g(t,\cdot)$ being continuously differentiable is defined by $[f,g](t,x) = \frac{\partial g(t,x)}{\partial x}f(t,x) - \frac{\partial f(t,x)}{\partial x}g(t,x)$. For $x, y \in \mathbb{R}^n$, we use $x \pm y$ to denote $x+y-y$ (not the set $\{x+y, x-y\}$).

We will employ extended Jensen's inequalities (Griñó et al., 2021):

Lemma 1. *Let $f : [a,b] \to \mathbb{R}$ and $\phi : [a,b] \to \mathbb{R}^n$, where $a \le b$, be functions such that the integration concerned is well-defined. Then for any $0 < R \in \mathbb{R}^{n\times n}$ the following extended Jensen's inequalities hold:*

$$\begin{aligned}&\int_a^b f(s)\phi^T(s)ds R \int_a^b f(s)\phi(s)ds \\ &\le \int_a^b |f(\theta)|d\theta \int_a^b |f(s)|\phi^T(s)R\phi(s)ds,\end{aligned} \tag{1}$$

$$\begin{aligned}&\int_a^b\int_s^b \phi^T(\theta)d\theta ds R \int_a^b\int_s^b \phi(\theta)d\theta ds \\ &\le \tfrac{(b-a)^2}{2}\int_a^b\int_s^b \phi^T(\theta)R\phi(\theta)d\theta ds,\end{aligned} \tag{2}$$

$$\begin{aligned}&\int_a^b\int_s^b\int_\theta^b \phi^T(\xi)d\xi d\theta ds R \int_a^b\int_s^b\int_\theta^b \phi(\xi)d\xi d\theta ds \\ &\le \tfrac{(b-a)^3}{6}\int_a^b\int_s^b\int_\theta^b \phi^T(\xi)R\phi(\xi)d\xi d\theta ds.\end{aligned} \tag{3}$$

2. A time-delay approach to Lie-brackets-based averaging

Consider the following input-affine system in the presence of state time-varying delay:

$$\begin{aligned}\dot{x}(t) &= f_0(t,x(t)) + v(t) \\ &+ \tfrac{1}{\sqrt{\varepsilon}}\sum_{i=1}^{\ell} u_i(\frac{t}{\varepsilon})f_i(t, x(t-\varepsilon\tau(t))), \quad t \ge t_0\end{aligned} \tag{4}$$

with the state $x(t) \in \mathbb{R}^n$, the fast-oscillating signals $u_i : [t_0,\infty) \to \mathbb{R}$ $(i = 1,\dots,\ell)$, small parameter $\varepsilon > 0$, map $f_0 : [t_0,\infty)\times\mathbb{R}^n \to \mathbb{R}^n$ that is continuous and locally Lipschitz continuous in the second argument, twice continuously differentiable maps $f_i : [t_0,\infty)\times\mathbb{R}^n \to \mathbb{R}^n$ $(i = 1,\dots,\ell)$ and the disturbance $v : [t_0,\infty) \to \mathbb{R}^n$. For simplicity we consider $t_0 = 0$. We assume the following:

A1 For $i, j = 1,\dots,\ell$, the following holds (Dürr et al., 2013; Grushkovskaya et al., 2018; Labar et al., 2022; Scheinker & Krstić, 2014):

(i) $u_i(\frac{t}{\varepsilon})$ is piecewise-continuous, bounded and ε-periodic in t, has zero mean value, i.e. $\int_0^\varepsilon u_i(\frac{s}{\varepsilon})ds = 0$;

(ii) for every compact set $\mathcal{C} \subseteq \mathbb{R}^n$, the functions $f_0(t,x)$, $f_i(t,x)$, $\frac{\partial f_i(t,x)}{\partial x}$, $\frac{\partial f_i(t,x)}{\partial t}$, $\frac{\partial}{\partial x}(\frac{\partial f_i(t,x)}{\partial x}f_j(t,x))$ and $\frac{\partial}{\partial t}(\frac{\partial f_i(t,x)}{\partial x}f_j(t,x))$ are uniformly bounded for all $x \in \mathcal{C}$ and $t \ge 0$.

A2 The disturbance $v(t)$ is assumed to be measurable and locally essentially bounded meaning that

$$\|v[0,t]\|_\infty = \operatorname{ess\,sup}_{\theta\in[0,t]}|v(\theta)| < \infty \quad \forall t \ge 0.$$

A3 The delay $\tau(t)$ is supposed to be bounded, i.e. $0 \le \tau(t) \le \tau_M$ and fast-varying (without any restriction on the delay derivative).

Assumption **A3** includes sawtooth delays that model network-based control. The initial condition of the delayed system (4) is given by $x(\theta) = \phi(\theta)$, $\theta \in [-\varepsilon\tau_M, 0]$ with $\phi \in C[-\varepsilon\tau_M, 0]$. The small delay consideration is consistent with the fast structure of (4), where $\dot{x}(t) = O(\varepsilon^{-0.5})$. The small delay $\varepsilon\tau(t)$ assumption is consistent with the same assumption (for the fast subsystem) for a singularly perturbed LTI system, which is necessary for delay-dependent stability of the system (Fridman, 2002).

Remark 1. As in Section 3, $v(t)$ in (4) may be dependent on x (i.e. $v(t) = v(t,x)$) provided that it is uniformly bounded in x and t as well as the solution of system (4) starting from the initial ball $|x(0)| \le \sigma_0$ is well-defined.

The Lie brackets averaging method (Gurvits & Li, 1993; Sussmann & Liu, 1993) has not been applied yet to the delayed system (4). To obtain the averaged system of (4), we first use the following:

$$\begin{aligned}&\tfrac{1}{\sqrt{\varepsilon}}u_i(\tfrac{t}{\varepsilon})f_i(t, x(t-\varepsilon\tau(t))) = \tfrac{1}{\sqrt{\varepsilon}}u_i(\tfrac{t}{\varepsilon}) \\ &\times [f_i(t,x(t)) - \int_{t-\varepsilon\tau(t)}^t \tfrac{\partial f_i(t,x)}{\partial x}|_{x=x(s)}\dot{x}(s)ds].\end{aligned} \tag{5}$$

By defining

$$Y_{\tau 1}(t) = -\tfrac{1}{\sqrt{\varepsilon}}\sum_{i=1}^{\ell} u_i(\frac{t}{\varepsilon})\int_{t-\varepsilon\tau(t)}^t \frac{\partial f_i(t,x)}{\partial x}|_{x=x(s)}\dot{x}(s)ds, \tag{6}$$

we rewrite the delayed system (4) as

$$\begin{aligned}\dot{x}(t) &= f_0(t,x(t)) + v(t) + Y_{\tau 1}(t) \\ &+ \tfrac{1}{\sqrt{\varepsilon}}\sum_{i=1}^{\ell} u_i(\frac{t}{\varepsilon})f_i(t,x(t)), \quad t \ge 0.\end{aligned} \tag{7}$$

Note that $Y_{\tau 1}(t)$ is of the order of $O(\tau_M)$ provided $\dot{x}(t)$ is of the order of $O(\frac{1}{\sqrt{\varepsilon}})$. Thus, the term $Y_{\tau 1}(t)$ will vanish as $\tau_M \to 0$. The resulting time-delay model (see (19)) will be a perturbation of the following "averaged" system

$$\begin{aligned}\dot{x}_{av}(t) &= f_0(t,x_{av}(t)) + \sum_{i=1}^{\ell}\sum_{j=i+1}^{\ell}\beta_{ij} \\ &\times [f_i,f_j](t,x_{av}(t)) + v(t) + Y_{\tau 1}(t)|_{x=x_{av}}\end{aligned} \tag{8}$$

with $x_{av}(t) \in \mathbb{R}^n$ and

$$\beta_{ij} = \tfrac{1}{\varepsilon^2}\int_0^\varepsilon\int_s^\varepsilon u_i(\tfrac{s}{\varepsilon})u_j(\tfrac{\theta}{\varepsilon})d\theta ds. \tag{9}$$

By changing the order of integration in (9) and adding the zero term $\frac{1}{\varepsilon^2}\int_0^\varepsilon\int_\varepsilon^0 u_i(\frac{s}{\varepsilon})u_j(\frac{\theta}{\varepsilon})dsd\theta$ (due to (i) of **A1**) to the right-hand side of (9), we obtain for $i,j=1,\dots,\ell$

$$\begin{aligned}\beta_{ij} &= \tfrac{1}{\varepsilon^2}\int_0^\varepsilon\int_0^\theta u_i(\tfrac{s}{\varepsilon})u_j(\tfrac{\theta}{\varepsilon})dsd\theta \\ &\quad+\tfrac{1}{\varepsilon^2}\int_0^\varepsilon\int_\varepsilon^0 u_i(\tfrac{s}{\varepsilon})u_j(\tfrac{\theta}{\varepsilon})dsd\theta \\ &= \tfrac{1}{\varepsilon^2}\int_0^\varepsilon\int_\varepsilon^\theta u_i(\tfrac{s}{\varepsilon})u_j(\tfrac{\theta}{\varepsilon})dsd\theta = -\beta_{ji}.\end{aligned} \tag{10}$$

The latter implies $\beta_{ii}=0$ $(i=1,\dots,\ell)$.

Differently from the Lie brackets system (8) approximating the behavior of the original system (4), we will directly transform system (4) to a time-delay system, which may be considered as a perturbation of (8). Namely, as in Fridman and Zhang (2020) and Zhang and Fridman (2022b) we integrate both sides of system (7) over $[t-\varepsilon,t]$ for $t\geq\varepsilon+\varepsilon\tau_M$, i.e.

$$\begin{aligned}&\tfrac{x(t)-x(t-\varepsilon)}{\varepsilon} = \tfrac{1}{\varepsilon}\int_{t-\varepsilon}^t[f_0(s,x(s))+Y_{\tau 1}(s) \\ &\quad+v(s)+\tfrac{1}{\sqrt{\varepsilon}}g(s)]ds,\quad t\geq\varepsilon+\varepsilon\tau_M\end{aligned} \tag{11}$$

with

$$g(t)=\sum_{i=1}^{\ell}u_i(\frac{t}{\varepsilon})f_i(t,x(t)). \tag{12}$$

Then we present the left-hand side of (11) as

$$\begin{aligned}&\tfrac{x(t)-x(t-\varepsilon)}{\varepsilon} = \tfrac{d}{dt}[x(t)+G(t)] \\ &\quad+\tfrac{1}{\varepsilon}\int_{t-\varepsilon}^t[f_0(s,x(s))+Y_{\tau 1}(s)+v(s)]ds \\ &\quad-f_0(t,x(t))-Y_{\tau 1}(t)-v(t),\quad t\geq\varepsilon+\varepsilon\tau_M,\end{aligned} \tag{13}$$

where

$$G(t)=-\tfrac{1}{\varepsilon\sqrt{\varepsilon}}\int_{t-\varepsilon}^t(s-t+\varepsilon)g(s)ds \tag{14}$$

with notation (12). Note that $G(t)$ depends on $g(t)$ only (that is the fast-varying term to be "averaged") and not on the whole $\dot{x}(t)$ as in Fridman and Zhang (2020). From (11) and (13) we obtain

$$\begin{aligned}&\tfrac{d}{dt}[x(t)+G(t)] = f_0(t,x(t))+Y_{\tau 1}(t)+v(t) \\ &\quad+\tfrac{1}{\varepsilon\sqrt{\varepsilon}}\int_{t-\varepsilon}^t g(s)ds,\quad t\geq\varepsilon+\varepsilon\tau_M.\end{aligned} \tag{15}$$

Denote for $t\geq\varepsilon+\varepsilon\tau_M$

$$\begin{aligned}Y_v(t) &= -\tfrac{1}{\varepsilon\sqrt{\varepsilon}}\sum_{i=1}^{\ell}\sum_{j=1}^{\ell}\int_{t-\varepsilon}^t\int_s^t u_i(\frac{s}{\varepsilon}) \\ &\qquad\times\tfrac{\partial f_i(\theta,x)}{\partial x}\Big|_{x=x(\theta)}v(\theta)d\theta ds, \\ Y_0(t) &= -\tfrac{1}{\varepsilon\sqrt{\varepsilon}}\sum_{i=1}^{\ell}\int_{t-\varepsilon}^t\int_s^t u_i(\frac{s}{\varepsilon})\frac{\partial f_i(\theta,x)}{\partial\theta}\Big|_{x=x(\theta)}d\theta ds, \\ Y_1(t) &= -\tfrac{1}{\varepsilon\sqrt{\varepsilon}}\sum_{i=1}^{\ell}\int_{t-\varepsilon}^t\int_s^t u_i(\frac{s}{\varepsilon})\frac{\partial f_i(\theta,x)}{\partial x}\Big|_{x=x(\theta)} \\ &\qquad\times f_0(\theta,x(\theta))d\theta ds, \\ Y_2(t) &= \tfrac{1}{\varepsilon^2}\sum_{i=1}^{\ell}\sum_{j=1}^{\ell}\int_{t-\varepsilon}^t\int_s^t\int_\theta^t u_i(\frac{s}{\varepsilon})u_j(\frac{\theta}{\varepsilon}) \\ &\qquad\times\tfrac{\partial}{\partial x}\Big(\tfrac{\partial f_i(\xi,x)}{\partial x}f_j(\xi,x)\Big)\Big|_{x=x(\xi)}\dot{x}(\xi)d\xi d\theta ds, \\ Y_3(t) &= \tfrac{1}{\varepsilon^2}\sum_{i=1}^{\ell}\sum_{j=1}^{\ell}\int_{t-\varepsilon}^t\int_s^t\int_\theta^t u_i(\frac{s}{\varepsilon})u_j(\frac{\theta}{\varepsilon}) \\ &\qquad\times\tfrac{\partial}{\partial\xi}\Big(\tfrac{\partial f_i(\xi,x)}{\partial x}f_j(\xi,x)\Big)\Big|_{x=x(\xi)}d\xi d\theta ds, \\ Y_{\tau 2}(t) &= \tfrac{1}{\varepsilon^2}\sum_{i=1}^{\ell}\sum_{j=1}^{\ell}\int_{t-\varepsilon}^t\int_s^t\int_{\theta-\varepsilon\tau(\theta)}^\theta u_i(\frac{s}{\varepsilon})u_j(\frac{\theta}{\varepsilon}) \\ &\qquad\times\tfrac{\partial f_i(\theta,x)}{\partial x}\Big|_{x=x(\theta)}\tfrac{\partial f_j(\theta,x)}{\partial x}\Big|_{x=x(\xi)}\dot{x}(\xi)d\xi d\theta ds.\end{aligned} \tag{16}$$

By subtracting the zero terms $\frac{1}{\varepsilon\sqrt{\varepsilon}}\int_{t-\varepsilon}^t u_i(\frac{s}{\varepsilon})dsf_i(t,x(t))$ (due to (i) of **A1**) for $i=1,\dots,\ell$, we present

$$\begin{aligned}&\tfrac{1}{\varepsilon\sqrt{\varepsilon}}\int_{t-\varepsilon}^t g(s)ds \overset{(12)}{=} \tfrac{1}{\varepsilon\sqrt{\varepsilon}}\sum_{i=1}^{\ell}\int_{t-\varepsilon}^t u_i(\frac{s}{\varepsilon}) \\ &\quad\times[f_i(s,x(s))-f_i(t,x(t))]ds \\ &= Y_0(t)-\tfrac{1}{\varepsilon\sqrt{\varepsilon}}\sum_{i=1}^{\ell}\int_{t-\varepsilon}^t\int_s^t u_i(\frac{s}{\varepsilon})\frac{\partial f_i(\theta,x)}{\partial x}\Big|_{x=x(\theta)}\dot{x}(\theta)d\theta ds.\end{aligned} \tag{17}$$

For the last term on the right-hand side of (17), we have

$$\begin{aligned}&-\tfrac{1}{\varepsilon\sqrt{\varepsilon}}\sum_{i=1}^{\ell}\int_{t-\varepsilon}^t\int_s^t u_i(\frac{s}{\varepsilon})\frac{\partial f_i(\theta,x)}{\partial x}\Big|_{x=x(\theta)}\dot{x}(\theta)d\theta ds \\ &\overset{(4)}{=} Y_1(t)+Y_v(t)-\tfrac{1}{\varepsilon^2}\sum_{i=1}^{\ell}\sum_{j=1}^{\ell}\int_{t-\varepsilon}^t\int_s^t u_i(\frac{s}{\varepsilon}) \\ &\quad\times u_j(\tfrac{\theta}{\varepsilon})[\tfrac{\partial f_i(\theta,x)}{\partial x}\Big|_{x=x(\theta)}(f_j(\theta,x(\theta-\varepsilon\tau(\theta))) \\ &\quad\pm f_j(\theta,x(\theta)))\pm\tfrac{\partial f_i(t,x)}{\partial x}\Big|_{x=x(t)}f_j(t,x(t))]d\theta ds \\ &= \sum_{i=1}^{\ell}\sum_{j=i+1}^{\ell}\beta_{ij}[f_i,f_j](t,x(t)) \\ &\quad+\sum_{i=1}^{3}Y_i(t)+Y_v(t)+Y_{\tau 2}(t),\end{aligned} \tag{18}$$

where in the first equality we added and subtracted

$$\begin{aligned}&\tfrac{1}{\varepsilon^2}\int_{t-\varepsilon}^t\int_s^t u_i(\tfrac{s}{\varepsilon})u_j(\tfrac{\theta}{\varepsilon})[\tfrac{\partial f_i(\theta,x)}{\partial x}\Big|_{x=x(\theta)}f_j(\theta,x(\theta)) \\ &+\tfrac{\partial f_i(t,x)}{\partial x}\Big|_{x=x(t)}f_j(t,x(t))]d\theta ds,\quad i,j=1,\dots,\ell,\end{aligned}$$

and in the second equality we used (9) and (10). Substituting (18) into (17) and further into (15), we transform system (4) to the following time-delay system:

$$\begin{aligned}&\tfrac{d}{dt}[x(t)+G(t)] = f_0(t,x(t))+\sum_{i=0}^{3}Y_i(t) \\ &\quad+\sum_{i=1}^{\ell}\sum_{j=i+1}^{\ell}\beta_{ij}[f_i,f_j](t,x(t)) \\ &\quad+Y_v(t)+\sum_{i=1}^{2}Y_{\tau i}(t)+v(t),\quad t\geq\varepsilon+\varepsilon\tau_M\end{aligned} \tag{19}$$

with $\dot{x}(t)$ satisfying (4), β_{ij} given by (9), $G(t)$ defined by (14) with notation (12), $Y_v(t)$, $Y_i(t)$ $(i=0,\dots,3)$ and $Y_{\tau 2}(t)$ given by (16), and $Y_{\tau 1}(t)$ given by (6).

Differently from the time-delay approach to the classical averaging of linear systems (Fridman & Zhang, 2020; Zhang & Fridman, 2022b), we here propose a time-delay approach to the Lie-brackets-based averaging of nonlinear systems leading to a time-delay neutral type system. The latter system is a perturbation of the averaged system (8) given in terms of Lie brackets. Moreover, compared with the averaged system (8) via the Lie brackets averaging (Dürr et al., 2013; Grushkovskaya et al., 2018; Labar et al., 2022; Scheinker & Krstić, 2014), system (19) additionally includes perturbation terms $G(t)$, $Y_i(t)$ $(i=0,1,2)$, $Y_v(t)$ of the order of $O(\sqrt{\varepsilon})$, $Y_3(t)$ of the order of $O(\varepsilon)$ and $Y_{\tau 2}(t)$ of the order of $O(\sqrt{\varepsilon}\tau_M)$ provided $\dot{x}$ is of the order of $O(\frac{1}{\sqrt{\varepsilon}})$. Note that the perturbations as well as $Y_{\tau_1}(t)$ defined by (6) will vanish as $\varepsilon\to 0$ and $\tau_M\to 0$. If ε and τ_M increase, system (4) may become unstable. However, till recently bounds on the small parameter could be found from simulations only, which is not reliable for the unknown systems. Thus, differently from the qualitative analysis in Dürr et al. (2013), Grushkovskaya et al. (2018), Labar et al. (2022) and Scheinker and Krstić (2014), our objective is to find the first efficient quantitative upper bounds on ε and τ_M that ensure the stability.

Remark 2. Note that the term $Y_v(t)$ in system (19) may be treated in the Lyapunov analysis as a disturbance, which can be directly upper bounded (as considered in Zhu and Fridman (2022) for ES method). Alternatively, one can consider appropriate Lyapunov functional (see (46)) to compensate this term that leads to less conservative results than those via the upper bounding method.

Remark 3. From (11), (17) and (18) we obtain the following integral equation

$$\begin{aligned} x(t+\varepsilon) &= x(t) + \tfrac{1}{\sqrt{\varepsilon}}\int_t^{t+\varepsilon} g(s)ds \\ &\quad + \int_t^{t+\varepsilon}(f_0(s,x(s)) + Y_{\tau 1}(s) + v(s))ds \\ &= x(t) + \sum_{i=0}^{3}\varepsilon Y_i(t) + \varepsilon Y_v(t) + \varepsilon Y_{\tau 2}(t) \\ &\quad + \sum_{i=1}^{\ell}\sum_{j=i+1}^{\ell}\varepsilon\beta_{ij}[f_i,f_j](t,x(t)) \\ &\quad + \int_t^{t+\varepsilon}(f_0(s,x(s)) + Y_{\tau 1}(s) + v(s))ds, \quad t \geq 0, \end{aligned} \tag{20}$$

where $Y_v(t)$, $Y_i(t)$ $(i = 0, \dots, 3)$ and Y_{τ_2} are from (16) and Y_{τ_1} is from (6) with t and $t-\varepsilon$ changed by $t+\varepsilon$ and t, respectively. The latter is similar to the integral equations (A.2) in Grushkovskaya et al. (2018) and (A.23) in Labar et al. (2022). In the time-delay approach, instead of the integral Eq. (20), via (13) we arrive at the differential Eq. (19) with time-delays that allows to apply L–K method and to derive constructive and explicit LMIs for finding an efficient upper bound on ε that ensures the ISS along with the resulting ultimate bound (see Section 3).

We now present the relation between solutions of systems (4) and (19):

Proposition 1. *If $x(t)$ is a solution to system (4), then it satisfies the time-delay system (19) with notations (6), (12), (14) and (16), where $\dot{x}(t)$ is defined by (4).*

From Proposition 1 it follows that if solutions $x(t)$ of the time-delay system (19) for $t \geq \varepsilon + \varepsilon\tau_M$ satisfy some bound (e.g. ISS bound given by (22)), then the same bound holds for solutions of the affine system (4) for $t \geq \varepsilon + \varepsilon\tau_M$.

We will present next Lyapunov-based regional ISS conditions for system (4). We assume that given $0 < \sigma_0 < \sigma$, $\varepsilon > 0$ and $\tau_M \geq 0$ let there exists a constant $\delta > 0$ such that solutions of the Lie brackets system (8) starting from the initial ball $\|\phi\|_{C[-\varepsilon-\varepsilon\tau_M,0]} \leq \sigma_0$ are exponentially approaching an attractive ball of radius σ with a decay rate δ. Our conditions will be formulated in terms of Lyapunov functional $\tilde{V}(t) = V(t, x_t, \dot{x}_t, g_t, \varepsilon, \tau_M)$ for the time-delay system (19) (proved as Proposition 2 of Zhang and Fridman (2022c)):

Proposition 2 (*Regional ISS of System (4)*)**.** *Consider system (4) subject to* **A1**, **A2** *and* **A3**. *Given $\varepsilon^* > 0$ and $\tau_M \geq 0$, let there exists a locally Lipschitz in the first four arguments Lyapunov functional $V(\cdot,\cdot,\cdot,\cdot,\varepsilon,\tau_M) : [\varepsilon+\varepsilon\tau_M,\infty) \times C[-\varepsilon,0] \times L_2(-\varepsilon-\varepsilon\tau_M,0) \times L_2(-\varepsilon,0) \to \mathbb{R}_+$ with $\varepsilon \in (0,\varepsilon^*]$ such that $\tilde{V}(t) = V(t,x_t,\dot{x}_t,g_t,\varepsilon,\tau_M)$ is absolutely continuous along solutions of (19) for $t \geq \varepsilon+\varepsilon\tau_M$. Moreover, let there exist positive scalars σ_0, σ_i, ρ_i $(i = 1,2,3)$, δ, γ, γ_0 and σ such that $\sigma_0 < \sigma_1$ and the following conditions hold for all $\varepsilon \in (0,\varepsilon^*]$:*
(i) $\tilde{V}(\varepsilon+\varepsilon\tau_M) \leq \rho_1\|x_{\varepsilon+\varepsilon\tau_M}\|^2_{C[-\varepsilon,0]} + \rho_2\|g_{\varepsilon+\varepsilon\tau_M}\|^2_{L_2(-\varepsilon,0)} + \rho_3 \|\dot{x}_{\varepsilon+\varepsilon\tau_M}\|^2_{L_2(-\varepsilon-\varepsilon\tau_M,0)}$ and $\tilde{V}(\varepsilon+\varepsilon\tau_M) \geq |x(\varepsilon+\varepsilon\tau_M)|^2$;
(ii) $\dot{\tilde{V}}(t) + 2\delta\tilde{V}(t) - \gamma - \gamma_0|v(t)|^2 \leq 0$ $\forall t \geq \varepsilon+\varepsilon\tau_M$ for solutions of (19) subject to $|x(t)| \leq \sigma$ $\forall t \geq \varepsilon+\varepsilon\tau_M$;
(iii) $\|x_{\varepsilon+\varepsilon\tau_M}\|_{C[-\varepsilon,0]} \leq \sigma_1 < \sigma$, $\|g_{\varepsilon+\varepsilon\tau_M}\|_{L_2(-\varepsilon,0)} \leq \sigma_2$ and $\|\dot{x}_{\varepsilon+\varepsilon\tau_M}\|_{L_2(-\varepsilon-\varepsilon\tau_M,0)} \leq \sigma_3$ for solutions of (4) starting from $\|\phi\|_{C[-\varepsilon\tau_M,0]} \leq \sigma_0$.

If additionally the following holds:

$$\sum_{i=1}^{3}\rho_i\sigma_i^2 + \frac{\gamma+\gamma_0\cdot(v^*)^2}{2\delta} < \sigma^2, \tag{21}$$

then for all disturbances $v(t)$ subject to

$$\|v[0,t]\|_\infty \leq v^* \quad \forall t \geq 0, \quad \forall \varepsilon \in (0,\varepsilon^*]$$

and for all $\varepsilon \in (0,\varepsilon^*]$ the solution of (4) starting from $\|\phi\|_{C[-\varepsilon\tau_M,0]} \leq \sigma_0$ satisfies

$$\begin{aligned} &|x(t)|^2 \leq \sigma_1^2 < \sigma^2, \quad t \in [0,\varepsilon+\varepsilon\tau_M], \\ &|x(t)|^2 \leq e^{-2\delta(t-\varepsilon-\varepsilon\tau_M)}\sum_{i=1}^{3}\rho_i\sigma_i^2 \\ &\qquad + \tfrac{\gamma+\gamma_0\|v[0,t]\|^2_\infty}{2\delta} < \sigma^2, \quad t \geq \varepsilon+\varepsilon\tau_M. \end{aligned} \tag{22}$$

Moreover, the ball

$$\mathfrak{X} = \{x \in \mathbb{R}^n : |x| \leq \sqrt{\tfrac{\gamma+\gamma_0\cdot(v^*)^2}{2\delta}}\} \tag{23}$$

is exponentially attractive with a decay rate δ.

Remark 4. For system (4), we can consider also a state time-varying delay in the term $f_0(t,x(t))$, i.e. $f_0(t,x(t-\eta(t)))$ with fast-varying but bounded $\eta(t)$. In this case, we will arrive at system (19), where $f_0(t,x(t))$ is changed by $f_0(t,x(t-\eta(t)))$. The results of Propositions 1 and 2 can be correspondingly extended to this case.

3. Stabilization under unknown control directions

We consider the following linear system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \quad t \geq 0 \tag{24}$$

under a bounded ES controller with a time-varying delay that appears due to delayed measurement of the state

$$\begin{aligned} u(t) &= \sqrt{\alpha\omega}\cos(\omega t + k|x(t-\tfrac{2\pi}{\omega}\tau(t))|^2) \\ &= \sqrt{\alpha\omega}[\cos(\omega t)\cos(k|x(t-\tfrac{2\pi}{\omega}\tau(t))|^2) \\ &\quad - \sin(\omega t)\sin(k|x(t-\tfrac{2\pi}{\omega}\tau(t))|^2)]. \end{aligned} \tag{25}$$

Here $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}$ is the input, ω is the frequency of the dither signal whose magnitude is $\sqrt{\alpha\omega}$ with $\alpha > 0$, $k > 0$ is the controller gain, and the delay $\tau(t)$ satisfies **A3**. To implement $u(t)$ for $t \geq 0$, we assume that the measurements of $x(t)$ are available for $t \geq -\frac{2\pi}{\omega}\tau_M$. The time-varying coefficients $A(t)$ and $B(t)$ have the following form

$$A(t) = A_0 + \Delta A(t), \quad B(t) = B_0 + \frac{\sqrt{2\pi}}{\sqrt{\omega}}\Delta B(t), \tag{26}$$

where $A_0 \in \mathbb{R}^{n\times n}$ is a constant matrix, $B_0 \in \mathbb{R}^n$ is a known constant vector up to its sign, and $\Delta A(t) \in \mathbb{R}^{n\times n}$ and $\Delta B(t) \in \mathbb{R}^n$ denote the time-varying uncertainties that satisfy the following inequalities

$$\|\Delta A(t)\| \leq \Delta a, \quad |\Delta B(t)| \leq \Delta b \quad \forall t \geq 0 \tag{27}$$

with small constants $\Delta a \geq 0$ and $\Delta b \geq 0$. The latter implies

$$\|A(t)\| \leq a \quad \forall t \geq 0, \quad a = \|A_0\| + \Delta a. \tag{28}$$

Moreover, since the sign of B_0 entries is unknown, one cannot design for system (24) a classical PID type stabilizing controller.

For simplicity we here consider that $\Delta A(t)$ and $\Delta B(t)$ depend on t only. Both uncertainties can be dependent on t and x provided that they satisfy (27) for all t and x and the solution of system (24), (25) is well-defined.

By letting $\omega = \frac{2\pi}{\varepsilon}$, we rewrite system (24)–(26) in the following form

$$\begin{aligned}\dot{x}(t) &= [A_0 + \Delta A(t)]x(t) + \tfrac{\sqrt{2\pi\alpha}}{\sqrt{\varepsilon}} B_0 \Big[\cos(\tfrac{2\pi t}{\varepsilon}) \\ &\times \cos(k|x(t-\varepsilon\tau(t))|^2) - \sin(\tfrac{2\pi t}{\varepsilon}) \\ &\times \sin(k|x(t-\varepsilon\tau(t))|^2) \Big] + \sqrt{2\pi\alpha}\Delta B(t) \\ &\times \cos(\tfrac{2\pi t}{\varepsilon} + k|x(t-\varepsilon\tau(t))|^2), \quad t \ge 0,\end{aligned} \tag{29}$$

which can be presented as system (4) with

$$\begin{aligned}&\ell = 2, \quad f_0(t,x) = [A_0 + \Delta A(t)]x, \\ &f_1(t,x) = B_0 \cos(k|x|^2), \quad f_2(t,x) = -B_0 \sin(k|x|^2), \\ &u_1(\tfrac{t}{\varepsilon}) = \sqrt{2\pi\alpha}\cos(\tfrac{2\pi t}{\varepsilon}), \quad u_2(\tfrac{t}{\varepsilon}) = \sqrt{2\pi\alpha}\sin(\tfrac{2\pi t}{\varepsilon}), \\ &v(t) = \sqrt{2\pi\alpha}\Delta B(t)\cos(\tfrac{2\pi t}{\varepsilon} + k|x(t-\varepsilon\tau(t))|^2).\end{aligned} \tag{30}$$

The initial condition of system (29) is given by $x(\theta) = \phi(\theta)$, $\theta \in [-\varepsilon\tau_M, 0]$ with $\phi \in C[-\varepsilon\tau_M, 0]$.

Taking into account

$$\sum_{i=1}^{\ell}\sum_{j=i+1}^{\ell} \beta_{ij}[f_i, f_j](t,x) = -k\alpha B_0 B_0^T x,$$

the averaged system that corresponds to (29) is given by the following Lie brackets system:

$$\dot{x}_{av}(t) = [A_{av} + \Delta A(t)]x_{av}(t) + Y_{\tau 1}(t)|_{x=x_{av}} + v(t)|_{x=x_{av}}$$

with $x_{av}(t) \in \mathbb{R}^n$, $v(t)$ defined in (30) and

$$\begin{aligned}Y_{\tau 1}(t) &= \tfrac{2k\sqrt{2\pi\alpha}}{\sqrt{\varepsilon}} \int_{t-\varepsilon\tau(t)}^{t} \sin(\tfrac{2\pi t}{\varepsilon} + k|x(s)|^2) \\ &\times B_0 x^T(s)\dot{x}(s)ds, \quad A_{av} = A_0 - k\alpha B_0 B_0^T.\end{aligned} \tag{31}$$

We assume that there exist constants α and k leading to *Hurwitz* A_{av} in (31). This assumption guarantees the solvability of LMIs in Theorem 1 and Corollary 1 for small enough $\varepsilon^* > 0$, $\tau_M > 0$, $\Delta a > 0$ and $\Delta b > 0$. A sufficient condition for existence of a stabilizing αk is $B_0 B_0^T > 0$ (implying controllability of the corresponding system (4), where u_1 and u_2 are considered as control inputs). Note that in Example 2, where A_0 possesses purely imaginary eigenvalues, we managed to find a stabilizing αk though $B_0 B_0^T$ is singular.

We will further present explicitly the time-delay system (19) that corresponds to (29). Here $G(t)$ is defined by (14) with

$$g(t) = \sqrt{2\pi\alpha}B_0\cos(\tfrac{2\pi t}{\varepsilon} + k|x(t)|^2), \tag{32}$$

whereas notations (6) and (16) have the following form: $Y_0(t) = Y_3(t) = 0$ (due to $\dot{B}_0 = 0$), $Y_{\tau 1}(t)$ is defined in (31) and

$$\begin{aligned}Y_v(t) &= \tfrac{4\pi k\alpha}{\varepsilon\sqrt{\varepsilon}} \int_{t-\varepsilon}^{t}\int_{s}^{t} \sin(\tfrac{2\pi s}{\varepsilon} + k|x(\theta)|^2) \\ &\times \cos(\tfrac{2\pi\theta}{\varepsilon} + k|x(\theta)|^2) B_0 \Delta B^T(\theta)x(\theta)d\theta ds, \\ Y_1(t) &= \tfrac{2k\sqrt{2\pi\alpha}}{\varepsilon\sqrt{\varepsilon}} \int_{t-\varepsilon}^{t}\int_{s}^{t} \sin(\tfrac{2\pi s}{\varepsilon} + k|x(\theta)|^2) \\ &\times B_0 x^T(\theta)A(\theta)x(\theta)d\theta ds, \\ Y_2(t) &= -\tfrac{2\pi k\alpha}{\varepsilon^2} \int_{t-\varepsilon}^{t}\int_{s}^{t}\int_{\theta}^{t} B_0 B_0^T [\sin(\tfrac{2\pi}{\varepsilon}(s-\theta))I \\ &+ 4k\cos(\tfrac{2\pi}{\varepsilon}(s+\theta) + 2k|x(\xi)|^2)x(\xi)x^T(\xi) \\ &+ \sin(\tfrac{2\pi}{\varepsilon}(s+\theta) + 2k|x(\xi)|^2)I]\dot{x}(\xi)d\xi d\theta ds, \\ Y_{\tau 2}(t) &= \tfrac{8\pi k^2\alpha}{\varepsilon^2} \int_{t-\varepsilon}^{t}\int_{s}^{t}\int_{\theta-\varepsilon\tau(\theta)}^{\theta} \sin(\tfrac{2\pi s}{\varepsilon} + k|x(\theta)|^2) \\ &\times \sin(\tfrac{2\pi\theta}{\varepsilon} + k|x(\xi)|^2) B_0 B_0^T x(\theta)x^T(\xi)\dot{x}(\xi)d\xi d\theta ds.\end{aligned} \tag{33}$$

Thus, using (32) and (33) the time-delay system (19) is presented as follows:

$$\begin{aligned}&\tfrac{d}{dt}[x(t) + G(t)] = [A_{av} + \Delta A(t)]x(t) + Y_v(t) \\ &+ \sum_{i=1}^{2}(Y_i(t) + Y_{\tau i}(t)) + v(t), \quad t \ge \varepsilon + \varepsilon\tau_M,\end{aligned} \tag{34}$$

where $\dot{x}(t)$ satisfies (29).

Remark 5. Note that system (29) can be also presented as system (4) with $v(t) = 0$ and

$$f_1(t,x) = B(t)\cos(k|x|^2), \quad f_2(t,x) = -B(t)\sin(k|x|^2),$$

where the other terms coincide with (30). The latter leads to zero term $Y_v(t)$ and non-zero terms $Y_0(t)$, $Y_3(t)$ (to be compared with zero terms $Y_0(t)$, $Y_3(t)$ and non-zero term $Y_v(t)$ under the expression (30)), which complicate the stability analysis. Moreover, one needs to impose an additional assumption on the derivative of $B(t)$ (which should be small) such that non-zero terms $Y_0(t)$ and $Y_3(t)$ are small perturbations. It is clear that expression (30) provides a simpler stability analysis and removes the assumption on the derivative of $B(t)$.

Theorem 1. *Consider system (29) subject to (27) under $\|\phi\|_{C[-\varepsilon\tau_M, 0]} \le \sigma_0$. Let positive α and k lead to Hurwitz A_{av} in (31). Given $\Delta a \ge 0$, $\Delta b \ge 0$ and $\tau_M \ge 0$, positive scalars δ, ε^*, $\sigma_0 < \sigma$ and a tuning parameter $q > 0$, let there exist $n \times n$ symmetric positive definite matrices P, R, Q_v, Q_i, $Q_{\tau i}$ $(i = 1, 2)$ and positive scalars λ_P, λ_R, λ_{Q_v}, λ_{Q_i}, $\lambda_{Q_{\tau i}}$ $(i = 1, 2)$, λ, γ, γ_0 that satisfy the following inequalities for all $\varepsilon \in (0, \varepsilon^*]$: (21) and*

$$\Theta = \begin{bmatrix} P - I & P \\ * & P + Re^{-2\delta\varepsilon^*} \end{bmatrix} > 0, \quad \Phi \le 0, \tag{35}$$

$$\gamma_M = \gamma - 2\lambda_R \pi\alpha|B_0|^2 - \sqrt{\varepsilon}\lambda_{Q_2}(\bar{\mu}|B_0|)^2 \ge 0, \tag{36}$$

$$\begin{aligned}&P \le \lambda_P I, \quad R \le \lambda_R I, \quad Q_v \le \lambda_{Q_v} I, \\ &Q_i \le \lambda_{Q_i} I, \quad Q_{\tau i} \le \lambda_{Q_{\tau i}} I, \quad i = 1, 2,\end{aligned} \tag{37}$$

$$\begin{aligned}\sigma_1 &= e^{a\varepsilon^*(1+\tau_M)}(\sigma_0 + \sqrt{2\varepsilon^*\pi\alpha}(|B_0| + \sqrt{\varepsilon^*}\Delta b) \\ &\times (1 + \tau_M)) < \sigma,\end{aligned} \tag{38}$$

where Φ is the symmetric matrix composed of

$$\begin{aligned}\Phi_{11} &= PA_{av} + A_{av}^T P + 2\delta P + \lambda(\Delta a)^2 I + [\sqrt{\varepsilon^*}\lambda_{Q_v} \\ &\times (\Delta b)^2 + \sqrt{\varepsilon^*}\lambda_{Q_1}(ka\sigma)^2 + 4\sqrt{\varepsilon^*}\lambda_{Q_2}k\bar{\mu}^2(1 + k\sigma^2) \\ &+ \tau_M\lambda_{Q_{\tau 1}}(k\bar{\mu})^2 + \sqrt{\varepsilon^*}\tau_M\lambda_{Q_{\tau 2}}(e^{\delta\varepsilon^*\tau_M}\sigma k\bar{\mu})^2]|B_0|^2 I, \\ \Phi_{12} &= A_{av}^T P + 2\delta P, \quad \Phi_{ij} = P, \ i = 1, 2, \ j = 3, \ldots, 9, \\ \Phi_{22} &= -\tfrac{4R}{\varepsilon^*}e^{-2\delta\varepsilon^*} + 2\delta P, \ \Phi_{33} = -\tfrac{1}{4\sqrt{\varepsilon^*}(\pi k\alpha)^2}e^{-2\delta\varepsilon^*}Q_v, \\ \Phi_{44} &= -\tfrac{1}{2\sqrt{\varepsilon^*}\pi\alpha}e^{-2\delta\varepsilon^*}Q_1, \ \Phi_{66} = -\tfrac{1}{8\tau_M\pi\alpha}e^{-2\delta\varepsilon^*\tau_M}Q_{\tau 1}, \\ \Phi_{55} &= -\tfrac{9}{4\sqrt{\varepsilon^*}(\pi k\alpha|B_0|)^2}e^{-2\delta\varepsilon^*}Q_2, \quad \Phi_{88} = -\lambda I, \\ \Phi_{77} &= -\tfrac{1}{16\sqrt{\varepsilon^*}\tau_M(\pi k\alpha|B_0|)^2}e^{-2\delta\varepsilon^*}Q_{\tau 2}, \quad \Phi_{99} = -\gamma_0 I\end{aligned} \tag{39}$$

with other blocks being zero, and

$$\begin{aligned}\sigma_2 &= \sqrt{2\varepsilon^*\pi\alpha}|B_0|, \quad v^* = \sqrt{2\pi\alpha}\Delta b, \\ \sigma_3 &= \sqrt{1+\tau_M}[ae^{a\varepsilon^*(1+\tau_M)}(\sqrt{\varepsilon^*}\sigma_0 + \sqrt{2\pi\alpha}\varepsilon^*(|B_0| \\ &+ \sqrt{\varepsilon^*}\Delta b)(1+\tau_M)) + \sqrt{2\pi\alpha}(|B_0| + \sqrt{\varepsilon^*}\Delta b)], \\ \rho_1 &= (1 + \tfrac{1}{q})\lambda_P + \varepsilon^*\sqrt{\varepsilon^*}|B_0|^2[\lambda_{Q_v}(\Delta b)^2 + \lambda_{Q_1}(ka\sigma)^2], \\ \rho_2 &= (1+q)\lambda_P + \lambda_R, \\ \rho_3 &= \varepsilon^*|B_0|^2[\sqrt{\varepsilon^*}\lambda_{Q_2}(1 + 2k\sigma^2)^2 + \tau_M\lambda_{Q_{\tau 1}}(k\sigma)^2 \\ &+ \sqrt{\varepsilon^*}\lambda_{Q_{\tau 2}}(k\sigma^2)^2(\tfrac{1}{3} + \tau_M e^{2\delta\varepsilon^*\tau_M})].\end{aligned} \tag{40}$$

Here a is defined in (28) and $\bar{\mu} = \sqrt{\varepsilon^*}a\sigma + \sqrt{2\pi\alpha}(|B_0| + \sqrt{\varepsilon^*}\Delta b)$. *Then for all* $\varepsilon \in (0, \varepsilon^*]$ *the solution of system (29) with the initial condition* $\|\phi\|_{C[-\varepsilon\tau_M,0]} \le \sigma_0$ *satisfies (22). For all the initial conditions* $\|\phi\|_{C[-\varepsilon\tau_M,0]} \le \sigma_0$ *the ball (23), where* v^* *is defined in (40), is exponentially attractive with a decay rate* δ. *Moreover, the inequalities (21) and (35)–(38) are always feasible for small enough* $\varepsilon^* > 0$, $\tau_M > 0$, $\Delta a > 0$ *and* $\Delta b > 0$.

Proof. As in Zhang and Fridman (2022c) and Zhu and Fridman (2022), we assume that

$$|x(t)| < \sigma \quad \forall t \ge 0. \tag{41}$$

holds for solutions of system (29). Note that (41) can proven by following the contradiction-based arguments in Zhang and Fridman (2022c) and Zhu and Fridman (2022). For the analysis of system (29), we employ Proposition 2 and choose

$$V_P(t) = |x(t) + G(t)|_P^2, \quad 0 < P \in \mathbb{R}^{n\times n}, \tag{42}$$

where $G(t)$ is defined by (14) with (32). Differentiating $V_P(t)$ along (34) we obtain

$$\begin{aligned}\dot{V}_P(t) &= 2[x(t) + G(t)]^T P[(A_{av} + \Delta A(t))x(t)\\ &\quad + Y_v(t) + \sum_{i=1}^{2}(Y_i(t) + Y_{\tau i}(t)) + v(t)].\end{aligned} \tag{43}$$

To compensate the term $G(t)$ in (43), we employ (Fridman & Shaikhet, 2016)

$$V_R(t) = \tfrac{1}{\varepsilon^2}\int_{t-\varepsilon}^{t} e^{-2\delta(t-s)}(s-t+\varepsilon)^2|g(s)|_R^2 ds, \tag{44}$$

where $0 < R \in \mathbb{R}^{n\times n}$. We have

$$\begin{aligned}&\dot{V}_R(t) + 2\delta V_R(t) = |g(t)|_R^2\\ &\quad - \tfrac{2}{\varepsilon^2}\int_{t-\varepsilon}^{t} e^{-2\delta(t-s)}(s-t+\varepsilon)|g(s)|_R^2 ds\\ &\overset{(37)}{\le} 2\lambda_R\pi\alpha|B_0|^2 - \tfrac{4}{\varepsilon}e^{-2\delta\varepsilon}|G(t)|_R^2,\end{aligned} \tag{45}$$

where we used the extended Jensen's inequality (1)

$$|G(t)|_R^2 \le \tfrac{1}{2\varepsilon}\int_{t-\varepsilon}^{t}(s-t+\varepsilon)|g(s)|_R^2 ds.$$

To compensate the term $Y_v(t)$ in (43), we suggest

$$\begin{aligned}V_{Q_v(t)} &= \tfrac{2}{\varepsilon\sqrt{\varepsilon}}\int_{t-\varepsilon}^{t}\int_s^t e^{-2\delta(t-\theta)}(s-t+\varepsilon)\\ &\quad \times|B_0\Delta B^T(\theta)x(\theta)|_{Q_v}^2 d\theta ds, \quad 0 < Q_v \in \mathbb{R}^{n\times n}.\end{aligned} \tag{46}$$

Via (26) and (37), we have

$$\begin{aligned}&\dot{V}_{Q_v(t)} + 2\delta V_{Q_v}(t) = \sqrt{\varepsilon}|B_0\Delta B^T(t)x(t)|_{Q_v}^2\\ &\quad - \tfrac{2}{\varepsilon\sqrt{\varepsilon}}\int_{t-\varepsilon}^{t}\int_s^t e^{-2\delta(t-\theta)}|B_0\Delta B^T(\theta)x(\theta)|_{Q_v}^2 d\theta ds\\ &\le \sqrt{\varepsilon}\lambda_{Q_v}(\Delta b|B_0|)^2|x(t)|^2 - \tfrac{1}{4\sqrt{\varepsilon}(\pi k\alpha)^2}e^{-2\delta\varepsilon}|Y_v(t)|_{Q_v}^2,\end{aligned} \tag{47}$$

where we used the extended Jensen's inequality (2)

$$|Y_v(t)|_{Q_v}^2 \le \tfrac{8(\pi k\alpha)^2}{\varepsilon}\int_{t-\varepsilon}^{t}\int_s^t |B_0\Delta B^T(\theta)x(\theta)|_{Q_v}^2 d\theta ds.$$

To compensate the term $Y_1(t)$ in (43), we suggest

$$\begin{aligned}V_{Q_1(t)} &= \tfrac{2k^2}{\varepsilon\sqrt{\varepsilon}}\int_{t-\varepsilon}^{t}\int_s^t e^{-2\delta(t-\theta)}(s-t+\varepsilon)\\ &\quad \times|B_0x^T(\theta)A(\theta)x(\theta)|_{Q_1}^2 d\theta ds, \quad 0 < Q_1 \in \mathbb{R}^{n\times n}.\end{aligned} \tag{48}$$

From (28), (37) and (41) we obtain

$$\begin{aligned}&\dot{V}_{Q_1}(t) + 2\delta V_{Q_1}(t) = \sqrt{\varepsilon}k^2|B_0x^T(t)A(t)x(t)|_{Q_1}^2\\ &\quad - \tfrac{2k^2}{\varepsilon\sqrt{\varepsilon}}\int_{t-\varepsilon}^{t}\int_s^t e^{-2\delta(t-\theta)}|B_0x^T(\theta)A(\theta)x(\theta)|_{Q_1}^2 d\theta ds\\ &\le \sqrt{\varepsilon}\lambda_{Q_1}(ka\sigma|B_0|)^2|x(t)|^2 - \tfrac{1}{2\sqrt{\varepsilon}\pi\alpha}e^{-2\delta\varepsilon}|Y_1(t)|_{Q_1}^2,\end{aligned} \tag{49}$$

where we used the extended Jensen's inequality (2)

$$|Y_1(t)|_{Q_1}^2 \le \tfrac{4\pi k^2\alpha}{\varepsilon}\int_{t-\varepsilon}^{t}\int_s^t |B_0x^T(\theta)Ax(\theta)|_{Q_1}^2 d\theta ds.$$

To compensate $Y_2(t)$ in (43), we consider

$$\begin{aligned}V_{Q_2}(t) &= \tfrac{6}{\varepsilon\sqrt{\varepsilon}|B_0|^2}\int_{t-\varepsilon}^{t}\int_s^t\int_\theta^t e^{-2\delta(t-\xi)}(s-t+\varepsilon)|B_0B_0^T\\ &\quad \times(I + 2kx(\xi)x^T(\xi))\dot{x}(\xi)|_{Q_2}^2 d\xi d\theta ds, \quad 0 < Q_2 \in \mathbb{R}^{n\times n}.\end{aligned} \tag{50}$$

From (27)–(29), (37) and (41) we obtain

$$\begin{aligned}&\dot{V}_{Q_2}(t) + 2\delta V_{Q_2}(t) = \tfrac{\varepsilon\sqrt{\varepsilon}}{|B_0|^2}|B_0B_0^T(I + 2kx(t)\\ &\quad \times x^T(t))\dot{x}(t)|_{Q_2}^2 - \tfrac{6}{\varepsilon\sqrt{\varepsilon}|B_0|^2}\int_{t-\varepsilon}^{t}\int_s^t\int_\theta^t e^{-2\delta(t-\xi)}\\ &\quad \times|B_0B_0^T(I + 2kx(\xi)x^T(\xi))\dot{x}(\xi)|_{Q_2}^2 d\xi d\theta ds\\ &\le \sqrt{\varepsilon}\lambda_{Q_2}(\mu|B_0|)^2(1 + 4k(1 + k\sigma^2)|x(t)|^2)\\ &\quad - \tfrac{9}{4\sqrt{\varepsilon}(\pi k\alpha|B_0|)^2}e^{-2\delta\varepsilon}|Y_2(t)|_{Q_2}^2,\end{aligned} \tag{51}$$

where we used the extended Jensen's inequality (3)

$$\begin{aligned}|Y_2(t)|_{Q_2}^2 &\le \tfrac{8(\pi k\alpha)^2}{3\varepsilon}\int_{t-\varepsilon}^{t}\int_s^t\int_\theta^t |B_0B_0^T\\ &\quad \times(I + 2kx(\xi)x^T(\xi))\dot{x}(\xi)|_{Q_2}^2 d\xi d\theta ds,\end{aligned}$$

and

$$\sqrt{\varepsilon}|x(t)| < \sqrt{\varepsilon}a\sigma + \sqrt{2\pi\alpha}(|B_0| + \sqrt{\varepsilon}\Delta b) \triangleq \mu. \tag{52}$$

In order to compensate the term $Y_{\tau 1}(t)$ in (43), we suggest

$$\begin{aligned}V_{Q_{\tau 1}}(t) &= k^2\int_{t-\varepsilon\tau_M}^{t}\int_\theta^t e^{-2\delta(t-s)}|B_0x^T(s)\\ &\quad \times\dot{x}(s)|_{Q_{\tau 1}}^2 ds d\theta, \quad 0 < Q_{\tau 1} \in \mathbb{R}^{n\times n}.\end{aligned} \tag{53}$$

From (27)–(29), (37), (41) and (52), we obtain

$$\begin{aligned}&\dot{V}_{Q_{\tau 1}}(t) + 2\delta V_{Q_{\tau 1}}(t) = \varepsilon\tau_M k^2|B_0x^T(t)\dot{x}(t)|_{Q_{\tau 1}}^2\\ &\quad - k^2\int_{t-\varepsilon\tau_M}^{t} e^{-2\delta(t-s)}|B_0x^T(s)\dot{x}(s)|_{Q_{\tau 1}}^2 ds\\ &\le \tau_M\lambda_{Q_{\tau 1}}(k\mu|B_0|)^2|x(t)|^2 - \tfrac{1}{8\tau_M\pi\alpha}e^{-2\delta\tau_M}|Y_{\tau 1}(t)|_{Q_{\tau 1}}^2,\end{aligned} \tag{54}$$

where we employed Jensen's inequality (3.87) in Fridman (2014)

$$|Y_{\tau 1}(t)|_{Q_{\tau 1}}^2 \le 8\tau_M\pi k^2\alpha\int_{t-\varepsilon\tau_M}^{t}|B_0x^T(s)\dot{x}(s)|_{Q_{\tau 1}} ds.$$

To compensate the term $Y_{\tau 2}(t)$ in (43), we suggest

$$\begin{aligned}V_{Q_{\tau 2}}(t) &= \tfrac{2k^2}{\varepsilon\sqrt{\varepsilon}|B_0|^2}\int_{t-\varepsilon}^{t}\int_s^t\int_{\theta-\varepsilon\tau_M}^{\theta}(s-t+\varepsilon)\\ &\quad \times e^{-2\delta(t-\theta)}|B_0B_0^Tx(\theta)x^T(\xi)\dot{x}(\xi)|_{Q_{\tau 2}}^2 d\xi d\theta ds,\end{aligned} \tag{55}$$

where $0 < Q_{\tau 2} \in \mathbb{R}^{n\times n}$. We obtain

$$\begin{aligned}&\dot{V}_{Q_{\tau 2}}(t) + 2\delta V_{Q_{\tau 2}}(t) = \tfrac{\sqrt{\varepsilon}k^2}{|B_0|^2}\int_{t-\varepsilon\tau_M}^{t}|B_0B_0^Tx(t)\\ &\quad \times x^T(\xi)\dot{x}(\xi)|_{Q_{\tau 2}}^2 d\xi - \tfrac{2k^2}{\varepsilon\sqrt{\varepsilon}|B_0|^2}\int_{t-\varepsilon}^{t}\int_s^t\int_{\theta-\varepsilon\tau_M}^{\theta}\\ &\quad \times e^{-2\delta(t-\theta)}|B_0B_0^Tx(\theta)x^T(\xi)\dot{x}(\xi)|_{Q_{\tau 2}}^2 d\xi d\theta ds\\ &\le \sqrt{\varepsilon}\lambda_{Q_{\tau 2}}(k|B_0|)^2\int_{t-\varepsilon\tau_M}^{t}|x(t)x^T(\xi)\dot{x}(\xi)|^2 d\xi\\ &\quad - \tfrac{1}{16\sqrt{\varepsilon}\tau_M(\pi k\alpha|B_0|)^2}e^{-2\delta\varepsilon}|Y_{\tau 2}(t)|_{Q_{\tau 2}}^2,\end{aligned} \tag{56}$$

where we employed the extended Jensen's inequality

$$\begin{aligned}|Y_{\tau 2}(t)|_{Q_{\tau 2}}^2 &\le \tfrac{32\tau_M(\pi k^2\alpha)^2}{\varepsilon}\int_{t-\varepsilon}^{t}\int_s^t\int_{\theta-\varepsilon\tau_M}^{\theta}\\ &\quad \times|B_0B_0^Tx(\theta)x^T(\xi)\dot{x}(\xi)|_{Q_{\tau 2}}^2 d\xi d\theta ds.\end{aligned}$$

To cancel the positive term on the right-hand side of (56), we additionally consider

$$\begin{aligned}\tilde{V}_{Q_{\tau 2}}(t) &= \sqrt{\varepsilon}\lambda_{Q_{\tau 2}}(k|B_0|)^2\int_{t-\varepsilon\tau_M}^{t}\int_s^t\\ &\quad \times e^{-2\delta(t-\xi-\varepsilon\tau_M)}|x(t)x^T(\xi)\dot{x}(\xi)|^2 d\xi ds.\end{aligned} \tag{57}$$

From (27)–(29), (37), (41) and (52), we obtain

$$\begin{aligned}&\dot{\tilde{V}}_{Q_{\tau2}}(t)+2\delta\tilde{V}_{Q_{\tau2}}(t)=\sqrt{\varepsilon}\varepsilon\tau_M\lambda_{Q_{\tau2}}(e^{\delta\varepsilon\tau_M}k|B_0|)^2\\&\quad\times|x(t)x^T(t)\dot{x}(t)|^2-\sqrt{\varepsilon}\lambda_{Q_{\tau2}}(k|B_0|)^2\int_{t-\varepsilon\tau_M}^{t}\\&\quad\times e^{-2\delta(t-\xi-\varepsilon\tau_M)}|x(t)x^T(\xi)\dot{x}(\xi)|^2d\xi\\&\leq\sqrt{\varepsilon}\tau_M\lambda_{Q_{\tau2}}(e^{\delta\varepsilon\tau_M}\sigma k\mu|B_0|)^2|x(t)|^2\\&\quad-\sqrt{\varepsilon}\lambda_{Q_{\tau2}}(k|B_0|)^2\int_{t-\varepsilon\tau_M}^{t}|x(t)x^T(\xi)\dot{x}(\xi)|^2d\xi.\end{aligned}\tag{58}$$

From (56) and (58), it follows that

$$\begin{aligned}&\dot{V}_{Q_{\tau2}}(t)+2\delta V_{Q_{\tau2}}(t)+\dot{\tilde{V}}_{Q_{\tau2}}(t)+2\delta\tilde{V}_{Q_{\tau2}}(t)\\&\leq\sqrt{\varepsilon}\tau_M\lambda_{Q_{\tau2}}(e^{\delta\varepsilon\tau_M}\sigma k\mu|B_0|)^2|x(t)|^2\\&\quad-\tfrac{1}{16\sqrt{\varepsilon}\tau_M(\pi k\alpha|B_0|)^2}e^{-2\delta\varepsilon}|Y_{\tau2}(t)|^2_{Q_{\tau2}}.\end{aligned}\tag{59}$$

Define a Lyapunov functional as

$$\begin{aligned}\bar{V}(t)&=V(t,x_t,\dot{x}_t,g_t,\varepsilon,\tau_M)\\&=V_P(t)+V_R(t)+V_{Q_v(t)}+\tilde{V}_{Q_{\tau2}}(t)\\&\quad+\sum_{i=1}^{2}(V_{Q_i}(t)+V_{Q_{\tau i}}(t)),\quad t\geq\varepsilon+\varepsilon\tau_M,\end{aligned}\tag{60}$$

where $V_P(t)$, $V_R(t)$, $V_{Q_v}(t)$, $V_{Q_1}(t)$, $V_{Q_2}(t)$, $V_{Q_{\tau1}}(t)$, $V_{Q_{\tau2}}(t)$ and $\tilde{V}_{Q_{\tau2}}(t)$ are given by (42), (44), (46), (48), (50), (53), (55) and (57), respectively. Note that from (27) we obtain $|\Delta A(t)x(t)|\leq\Delta a|x(t)|$. To compensate $\Delta A(t)x(t)$ in (43), we apply S-procedure: we add to $\dot{\bar{V}}(t)+2\delta\bar{V}(t)-\gamma-\gamma_0|v(t)|^2$ the left-hand part of $\lambda((\Delta a)^2|x(t)|^2-|\Delta A(t)x(t)|^2)\geq0$ with some $\lambda>0$. In view of (43), (45), (47), (49), (51), (54) and (59), we obtain for all $\varepsilon\in(0,\varepsilon^*]$ and $t\geq\varepsilon+\varepsilon\tau_M$

$$\dot{\bar{V}}(t)+2\delta\bar{V}(t)-\gamma-\gamma_0|v(t)|^2\leq\zeta^T(t)\Phi\zeta(t)-\gamma_M,$$

where γ_M is defined in (36), Φ is composed of (39), and

$$\begin{aligned}\zeta(t)=\mathrm{col}\{&x(t),G(t),Y_v(t),Y_1(t),Y_2(t),Y_{\tau1}(t),\\&Y_{\tau2}(t),\Delta A(t)x(t),v(t)\}.\end{aligned}$$

Thus, (ii) of Proposition 2 holds since $\Phi\leq0$ in (35) and $\gamma_M\geq0$ in (37).

We now prove (i) of Proposition 2. By Jensen's inequality (3.87) in Fridman (2014)

$$V_R(t)\geq\tfrac{1}{\varepsilon^3}e^{-2\delta\varepsilon}|\int_{t-\varepsilon}^{t}(s-t+\varepsilon)g(s)ds|^2_R=e^{-2\delta\varepsilon}|G(t)|^2_R,$$

we find for all $\varepsilon\in(0,\varepsilon^*]$

$$\begin{aligned}\bar{V}(t)&\geq V_P(t)+V_R(t)\\&\geq\begin{bmatrix}x(t)\\G(t)\end{bmatrix}^T\Theta\begin{bmatrix}x(t)\\G(t)\end{bmatrix}+|x(t)|^2\geq|x(t)|^2,\end{aligned}\tag{61}$$

where the last inequality holds due to $\Theta>0$ in (35). By Young's inequality with tuning parameter $q>0$ and Jensen's inequality (3.87) in Fridman (2014) to $V_P(t)$-term we obtain

$$\begin{aligned}V_P(t)&\leq(1+\tfrac{1}{q})|x(t)|^2_P+(1+q)|G(t)|^2_P\\&\leq(1+\tfrac{1}{q})|x(t)|^2_P+\tfrac{1+q}{\varepsilon^2}\int_{t-\varepsilon}^{t}(s-t+\varepsilon)^2|g(s)|^2_Pds.\end{aligned}$$

Thus, from (37) we arrive at

$$\begin{aligned}&V_P(t)+V_R(t)\leq(1+\tfrac{1}{q})\lambda_P|x(t)|^2\\&\quad+((1+q)\lambda_P+\lambda_R)\int_{t-\varepsilon}^{t}|g(s)|^2ds.\end{aligned}\tag{62}$$

From (41), (27), (28) and (37), we have

$$\begin{aligned}&V_{Q_v}(t)+V_{Q_1}(t)\leq\sqrt{\varepsilon}|B_0|^2[\lambda_{Q_v}(\Delta b)^2\\&\quad+\lambda_{Q_1}(ka\sigma)^2]\int_{t-\varepsilon}^{t}|x(s)|^2ds,\\&V_{Q_2}(t)+V_{Q_{\tau1}}(t)+V_{Q_{\tau2}}(t)+\tilde{V}_{Q_{\tau2}}(t)\leq\varepsilon|B_0|^2[\sqrt{\varepsilon}\\&\quad\times\lambda_{Q_2}(1+2k\sigma^2)^2+\tau_M\lambda_{Q_{\tau1}}(k\sigma)^2+\sqrt{\varepsilon}\lambda_{Q_{\tau2}}(k\sigma^2)^2\\&\quad\times(\tfrac{1}{3}+\tau_Me^{2\delta\varepsilon\tau_M})]\int_{t-\varepsilon-\varepsilon\tau_M}^{t}|\dot{x}(s)|^2ds.\end{aligned}\tag{63}$$

Thus, from (40) and (61)–(63), it follows that $\bar{V}(t)$ in (60) satisfies (i) of Proposition 2 for all $\varepsilon\in(0,\varepsilon^*]$.

To prove (iii) of Proposition 2, we denote $x_t(\theta)=x(t+\theta)$, $\theta\in[-\varepsilon\tau_M,0]$. From (29), it follows that

$$x_t(\theta)=\begin{cases}\phi(t+\theta), & t+\theta<0,\\ \phi(0)+\int_0^{t+\theta}[A(s)x(s)+\frac{\sqrt{2\pi\alpha}}{\sqrt{\varepsilon}}B(s) & \\ \quad\times\cos(\frac{2\pi s}{\varepsilon}+k|x(s-\varepsilon\tau(s))|^2)]ds, & t+\theta\geq0.\end{cases}$$

The latter together with (27) and (28) implies

$$\begin{aligned}&\|x_t\|_{C[-\varepsilon\tau_M,0]}\leq\|\phi\|_{C[-\varepsilon\tau_M,0]}+a\int_0^t|x(s)|ds\\&\quad+\tfrac{\sqrt{2\pi\alpha}}{\sqrt{\varepsilon}}(|B_0|+\sqrt{\varepsilon}\Delta b)t\\&\leq\|\phi\|_{C[-\varepsilon\tau_M,0]}+\sqrt{2\varepsilon\pi\alpha}(|B_0|+\sqrt{\varepsilon}\Delta b)(1+\tau_M)\\&\quad+a\int_0^t\|x_s\|_{C[-\varepsilon\tau_M,0]}ds,\quad t\in[0,\varepsilon+\varepsilon\tau_M].\end{aligned}$$

By applying the Gronwall inequality, under the initial condition $\|\phi\|_{C[-\varepsilon\tau_M,0]}\leq\sigma_0$ we arrive at

$$\begin{aligned}&\|x_t\|_{C[-\varepsilon\tau_M,0]}\leq e^{at}(\sigma_0+\sqrt{2\varepsilon\pi\alpha}(|B_0|\\&\quad+\sqrt{\varepsilon}\Delta b)(1+\tau_M)),\quad t\in[0,\varepsilon+\varepsilon\tau_M].\end{aligned}\tag{64}$$

Using (38) we obtain $\|x_{\varepsilon+\varepsilon\tau_M}\|_{C[-\varepsilon,0]}\leq\sigma_1$ for all $\varepsilon\in(0,\varepsilon^*]$. From (32) and (40) we obtain $\|g_{\varepsilon+\varepsilon\tau_M}\|_{L_2(-\varepsilon,0)}\leq\sqrt{2\varepsilon\pi\alpha}|B_0|\leq\sigma_2$ for all $\varepsilon\in(0,\varepsilon^*]$. Moreover, from (28), (29) and (64) we find

$$\begin{aligned}|\dot{x}(t)|&\leq a|x(t)|+\tfrac{\sqrt{2\pi\alpha}}{\sqrt{\varepsilon}}(|B_0|+\sqrt{\varepsilon}\Delta b)\\&\leq ae^{at}(\sigma_0+\sqrt{2\varepsilon\pi\alpha}(|B_0|+\sqrt{\varepsilon}\Delta b)(1+\tau_M))\\&\quad+\tfrac{\sqrt{2\pi\alpha}}{\sqrt{\varepsilon}}(|B_0|+\sqrt{\varepsilon}\Delta b),\quad t\in[0,\varepsilon+\varepsilon\tau_M].\end{aligned}$$

Then, using (40) we have $\|\dot{x}_{\varepsilon+\varepsilon\tau_M}\|_{L_2(-\varepsilon-\varepsilon\tau_M,0)}\leq\sigma_3$ for all $\varepsilon\in(0,\varepsilon^*]$. Thus, we arrive at (iii) of Proposition 2. Then Theorem 1 follows from Proposition 2.

Finally, we show the feasibility of inequalities (21) and (35)–(38). Choose

$$\begin{aligned}&\sigma=(\varepsilon^*)^{-\frac{1}{5}},\quad Q_v=Q_i=Q_{\tau i}=I,\\&\lambda_{Q_v}=\lambda_{Q_i}=\lambda_{Q_{\tau i}}=q=1,\quad i=1,2,\\&\gamma=2.1\lambda_R\pi\alpha|B_0|^2,\quad\lambda=(\Delta a)^{-1},\quad\gamma_0=(\Delta b)^{-1},\end{aligned}$$

where λ_R satisfies $\lambda_RI\geq R$ with R found from $\Theta>0$ defined in (35) with $\varepsilon^*=0$. The feasibility of (21), (36)–(38) and $\Theta>0$ in (35) is self-evident. We now check the feasibility of $\Phi\leq0$ in (35). Since A_{av} in (31) is Hurwitz, there exists a $n\times n$ matrix $P>0$ such that for any $\delta\in(0,\bar{\delta})$ where $-\bar{\delta}$ equals to the largest real part of the eigenvalues of matrix A_{av}, the following inequality holds: $PA_{av}+A_{av}^TP+2\delta P\leq0$. By using Schur complements to $\Phi\leq0$ in (35), we find that $\Phi\leq0$ is equivalent to

$$\begin{aligned}&PA_{av}+A_{av}^TP+2\delta P+[O(\varepsilon^{\frac{1}{10}})\\&\quad+O(\tau_M)+O(\Delta a)+O(\Delta b)]I\leq0.\end{aligned}\tag{65}$$

As $\varepsilon>0$, $\tau_M>0$, $\Delta a>0$ and $\Delta b>0$ go to 0, the inequality (65) (thus, $\Phi\leq0$ in (35)) is always feasible provided A_{av} in (31) is Hurwitz. This completes the proof. □

Remark 6. It should be pointed out that in the transformation of the delayed system (29), we used the same term $G(t)$ defined by (14) and (32) as in the non-delay case (Zhang & Fridman, 2022c). The latter leads to the time-delay system (34). If one uses $G(t)$ defined by (14) with $g(t)=\sqrt{2\pi\alpha}B_0\cos(\frac{2\pi t}{\varepsilon}+k|x(t-\varepsilon\tau(t))|^2)$, system (34) will take the following form:

$$\begin{aligned}&\tfrac{d}{dt}[x(t)+G(t)]=[A_{av}+\Delta A(t)]x(t)+Y_v(t)+\sum_{i=1}^{2}Y_i(t)\\&\quad+\tfrac{1}{\varepsilon}\int_{t-\varepsilon}^{t}Y_{\tau1}(s)ds+Y_{\tau2}(t)+v(t),\ t\geq\varepsilon+\varepsilon\tau_M.\end{aligned}$$

In order to compensate the new term $\frac{1}{\varepsilon}\int_{t-\varepsilon}^{t}Y_{\tau 1}(s)ds$, we need to add the term $\frac{1}{\varepsilon}\int_{t-\varepsilon}^{t}\int_{s-\varepsilon\tau_M}^{s}e^{-2\delta(t-s+\varepsilon\tau_M)}(s-t+\varepsilon)|B_0x^T(\theta)\dot{x}(\theta)|^2_{Q_{\tau 1}}d\theta ds$ to $V_{Q_{\tau 1}}(t)$ given by (53). The latter leads to the following more conservative upper bounding on $\dot{V}_{Q_{\tau 1}}(t)+2\delta V_{Q_{\tau 1}}(t)$ (to be compared with (54)):

$$\dot{V}_{Q_{\tau 1}}(t)+2\delta V_{Q_{\tau 1}}(t)\le \tau_M\lambda_{Q_{\tau 1}}(k\mu|B_0|)^2|x(t)|^2 -\frac{1}{8\tau_M\pi\alpha}e^{-2\delta\varepsilon(1+\tau_M)}\left|\frac{1}{\varepsilon}\int_{t-\varepsilon}^{t}Y_{\tau 1}(s)ds\right|^2_{Q_{\tau 1}},$$

and gives a larger upper bounding on $V_{Q_{\tau 1}}(t)$ (to be compared with that in (63)), which result in more conservative results.

When the time-varying delay $\tau(t)$ are absent in (29), i.e.

$$\begin{aligned}\dot{x}(t)&=[A_0+\Delta A(t)]x(t)+\tfrac{\sqrt{2\pi\alpha}}{\sqrt{\varepsilon}}B_0\Big[\cos(\tfrac{2\pi t}{\varepsilon})\\&\quad\times\cos(k|x(t)|^2)-\sin(\tfrac{2\pi t}{\varepsilon})\sin(k|x(t)|^2)\Big]\\&\quad+\sqrt{2\pi\alpha}\Delta B(t)\cos(\tfrac{2\pi t}{\varepsilon}+k|x(t)|^2),\quad t\ge 0,\end{aligned}\tag{66}$$

then the time-delay system that corresponds to (66) is given by (34) with $Y_{\tau i}(t)=0$ $(i=1,2)$ (see also the conference version of this paper Zhang & Fridman, 2022c for the direct derivation), where $\dot{x}(t)$ satisfies (66). Based on Theorem 1, we have the following corollary (that coincides with Theorem 1 of Zhang and Fridman (2022c)).

Corollary 1. *Consider system (66) subject to (27) under $|x(0)|\le\sigma_0$. Let positive α and k lead to Hurwitz A_{av} in (31). Given $\Delta a\ge 0$ and $\Delta b\ge 0$, positive scalars δ, ε^*, $\sigma_0<\sigma$ and a tuning parameter $q>0$, let there exist $n\times n$ symmetric positive definite matrices P, R, Q_v, Q_i $(i=1,2)$ and positive scalars λ_P, λ_R, λ_{Q_v}, λ_{Q_i} $(i=1,2)$, λ, γ, γ_0 that satisfy the following inequalities for all $\varepsilon\in(0,\varepsilon^*]$: (21) and (35)–(38), where we set $\tau_M=\lambda_{Q_{\tau i}}=0$ and $Q_{\tau i}=0$ $(i=1,2)$, and take away the 6th and 7th block-columns and block-rows of Φ. Then for all $\varepsilon\in(0,\varepsilon^*]$ the solution of system (66) with the initial condition $|x(0)|\le\sigma_0$ satisfies (22) with $\tau_M=0$. For all the initial conditions $|x(0)|\le\sigma_0$ the ball (23), where v^* is defined in (40), is exponentially attractive with a decay rate δ. Moreover, the derived inequalities are always feasible for small enough $\varepsilon^*>0$, $\tau_M>0$, $\Delta a>0$ and $\Delta b>0$.*

Remark 7. Note that for $A(t)=0$, $B(t)\in\{-1,1\}$ and $n=1$, system (66) coincides with the ES system in Zhu and Fridman (2022) with $f''=2$. For this case our time-delay system has the same from as (64) in Zhu and Fridman (2022):

$$\tfrac{d}{dt}[x(t)+G(t)]=-k\alpha x(t)+Y_2(t)$$

with $Y_2(t)$ given by (33). Differently from Zhu and Fridman (2022), where the term $Y_2(t)$ was treated as a disturbance in the Lyapunov analysis, we employ $V_{Q_2}(t)$ in (50) to compensate this term, which is consistent with Lyapunov functional construction in Zhu et al. (2023). As a result, Corollary 1 leads to a larger upper bound ε and a smaller ultimate bound than those via (Zhu & Fridman, 2022) in Example 1.

4. Examples

Example 1. Consider the scalar system (24) with

$$|A(t)|=|\Delta A(t)|\le\Delta a,\quad B_0\in\{-1,1\},\quad \Delta B(t)=0\tag{67}$$

under the bounded ES controller (25) with $\tau(t)\equiv 0$, $\alpha=0.1$ and $k=9$. It is clear that $a=\Delta a$, $\Delta b=0$ and $b=|B_0|=1$. We have $A_{av}=-0.9$. Let the desired decay rate be $\delta=0.5$.

For different values of σ_0,σ and Δa, we verify LMIs of Theorem 3 in Zhu and Fridman (2022) and of Corollary 1 with $q=10$ that lead to the upper bounds ε^* (that preserve the ISS for all $\varepsilon\in(0,\varepsilon^*]$) and the resulting ultimate bound (see Table 1). As expected, Corollary 1 provides an essentially larger ε and a smaller ultimate bound than those via Theorem 3 in Zhu and Fridman (2022). Note that Zhu and Fridman (2022) has not considered the case of $A(t)\neq 0$.

Table 1
Solutions by Zhu and Fridman (2022), Corollary 1 and Theorem 1 (Example 1: $\delta=0.5$).

Method	σ_0, σ	Δa	ε^*	τ_M	UB
Zhu and Fridman (2022)	0.01, 1	–	$0.42\cdot 10^{-4}$	–	0.9986
	1, 2	–	$0.16\cdot 10^{-4}$	–	1.4103
Corollary 1	0.01, 1	0	$1.99\cdot 10^{-4}$	–	0.9541
	0.01, 1	0.1	$1.06\cdot 10^{-4}$	–	0.9203
	1, 2	0	$0.53\cdot 10^{-4}$	–	1.2572
	1, 2	0.1	$0.28\cdot 10^{-4}$	–	1.2589
Theorem 1	0.01, 1	0	$0.9\cdot 10^{-4}$	0.0108	0.9118
	0.01, 1	0.1	$0.9\cdot 10^{-4}$	0.0019	0.9073
	1, 2	0	$0.2\cdot 10^{-4}$	0.0129	1.2448
	1, 2	0.1	$0.2\cdot 10^{-4}$	0.0041	1.2449

For the numerical simulations, under the initial condition $x(0)=10$, $\varepsilon=0.02$ (that is essentially larger than that from Table 1) and $A(t)=0.39\sin(t)$, the state response of system (24), (67) under the bounded ES controller (25) with $\tau(t)\equiv 0$, $\alpha=0.1$, $k=9$ and its norm on a logarithmic scale (from which we obtain the practical decay rate as 0.8421 that is larger than the theoretical decay rate $\delta=0.5$) are shown in Fig. 1, which confirms our theoretical results illustrating the conservatism.

Moreover, by verifying LMIs of Theorem 1 with $q=10$ and different values of σ_0, σ, Δa and ε^*, we find the quantitative upper bounds τ_M (that preserve the ISS for all $\varepsilon\in(0,\varepsilon^*]$) and the resulting ultimate bound, see Table 1.

Example 2. Consider system (24), where

$$\begin{aligned}&A_0=\begin{bmatrix}0&0.1\\0.1&0\end{bmatrix},\quad B_0\in\left\{\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}-1\\-1\end{bmatrix}\right\},\\&\|\Delta A(t)\|\le\Delta a,\quad \Delta B(t)=[\Delta b_1(t),\Delta b_2(t)]^T,\\&\sum_{i=1}^{2}|\Delta b_i(t)|^2\le(\Delta b)^2,\quad t\ge 0\end{aligned}\tag{68}$$

under the delayed bounded ES controller (25) with $\alpha=0.01$ and $k=10$. We obtain $a=\Delta a+0.1$, $|B(t)|\le\sqrt{2}+\sqrt{\varepsilon}\Delta b$, and $A_{av}=-0.1I$. Let the desired decay rate be $\delta=0.067$. By verifying LMIs of Theorem 1 with $q=5$ and different values of σ_0, σ, Δa, Δb and ε^*, we find the quantitative upper bounds τ_M (that preserve the ISS for all $\varepsilon\in(0,\varepsilon^*]$) and the resulting ultimate bound, see Table 2.

For the numerical simulations, under the initial condition $\phi(t)=[5,-5]^T$ for $t\le 0$, $\varepsilon=0.05$ (that is essentially larger than that from Table 2), $\Delta A(t)=0.032\sin(t)$ and $\Delta B(t)=0.25[\sin(t),\cos(t)]^T$, the state responses of system (24), (68) under the delayed bounded ES controller (25) with $\alpha=0.01$, $k=10$ and $\tau(t)\equiv 0.16$ (that is essentially larger than that from Table 2) and their norm on a logarithmic scale (from which we obtain the practical decay rate as 0.092 that is larger than the theoretical decay rate $\delta=0.067$) are shown in Fig. 2, which confirms our theoretical results illustrating the conservatism.

Remark 8. To enlarge ε^* and τ_M, one can decrease α to $\frac{\alpha}{N}$ with some large $N>0$, where accordingly the parameters $A(t)$, $\Delta B(t)$ and the decay rate δ are changed by $\frac{A(t)}{N}$, $\frac{\Delta B(t)}{N}$ and $\frac{\delta}{N}$ (that is N times smaller), respectively. Then by verifying the LMIs with the same B_0, k, σ_0, σ and q, one can find the corresponding upper

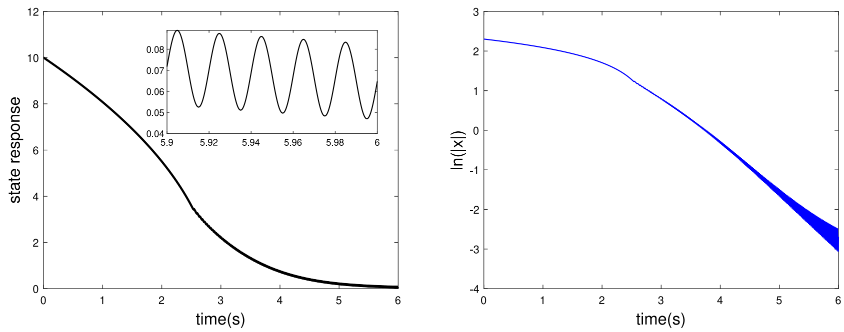

Fig. 1. State response and its norm on a logarithmic scale (Example 1).

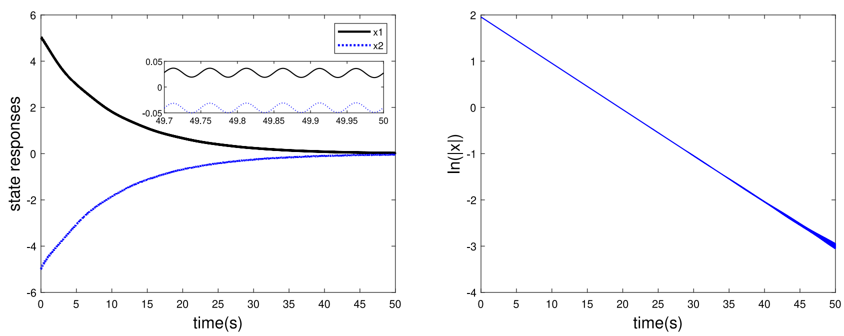

Fig. 2. State responses and their norms on a logarithmic scale (Example 2).

Table 2
Solutions by Theorem 1 (Example 2: $\delta = 0.067$).

σ_0, σ	Δa, Δb	ε^*	τ_M	UB
0.01, 1	0, 0	$0.3{\cdot}10^{-4}$	0.0054	0.9914
0.01, 1	0.002, 0.002	$0.3{\cdot}10^{-4}$	0.0045	0.9963
0.5, 2	0, 0	$0.1{\cdot}10^{-4}$	0.0045	1.3176
0.5, 2	0.002, 0.002	$0.1{\cdot}10^{-4}$	0.0037	1.3523

Table 3
Solutions by Theorem 1 (Example 1: $\delta = 0.05/N$ and Example 2: $\delta = 0.067/N$ after the scaling in Remark 8 with $N = 100$).

σ_0, σ	Δa, Δb	ε^*	τ_M	UB
0.01, 1	0, 0	0.0045	0.0177	0.9427
0.01, 1	$0.1{\cdot}10^{-4}$, 0	0.0045	0.0088	0.9256
1, 2	0, 0	0.001	0.0192	1.2786
1, 2	$0.1{\cdot}10^{-4}$, 0	0.001	0.0103	1.2611
0.01, 1	0, 0	0.0015	0.0076	0.9923
0.01, 1	$0.2{\cdot}10^{-4}$, $0.2{\cdot}10^{-4}$	0.0015	0.0068	0.9923
0.5, 2	0, 0	0.0005	0.0070	1.2999
0.5, 2	$0.2{\cdot}10^{-4}$, $0.2{\cdot}10^{-4}$	0.0005	0.0062	1.3202

bounds on ε^* and τ_M (see Table 3, where lines 2–5 and 6–9 are, respectively, for Examples 1 and 2 after the scaling with $N = 100$) that are larger than those in Tables 1 and 2. Thus, there is a tradeoff between the decay rate δ and the values of ε^* and τ_M.

5. Conclusions

We have presented a constructive time-delay approach to Lie-brackets-based averaging that transforms the affine system with state-delays and additive disturbances to a time-delay system without any approximations. The latter allows to provide sufficient L-K-based conditions for regional ISS of the original system. We have further applied the results to stabilization of linear uncertain systems under unknown control directions via a bounded ES controller with a measurement time-varying delay and have derived constructive LMIs for finding quantitative upper bounds on ε, the delay (that ensure the ISS) and on the resulting ultimate bound. Alternative constructive results can be derived by using the classical Lie brackets averaging methods if the remainder terms are taken into account. This may be a topic for future research.

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Jin Zhang received the Ph.D. degree in control theory and control engineering from Shanghai University, Shanghai, China, in 2018. From Oct. 2018 to May 2022, he was a Post-Doctoral Fellow with the School of Electrical Engineering, Tel Aviv University, Tel Aviv, Israel. Currently, he is a faculty with the School of Mechatronic Engineering and Automation, Shanghai University, Shanghai, China. His research interests include time-delay systems, networked control systems, and event-triggered control.

Emilia Fridman received the M.Sc. degree from Kuibyshev State University, USSR, in 1981 and the Ph.D. degree from Voronezh State University, USSR, in 1986, all in mathematics. From 1986 to 1992 she was an Assistant and Associate Professor in the Department of Mathematics at Kuibyshev Institute of Railway Engineers, USSR. Since 1993 she has been at Tel Aviv University, where she is currently Professor of Electrical Engineering-Systems. She has held visiting positions at the Weierstrass Institute for Applied Analysis and Stochastics in Berlin (Germany), INRIA in Rocquencourt (France), Ecole Centrale de Lille (France), Valenciennes University (France), Leicester University (UK), Kent University (UK), CINVESTAV (Mexico), Zhejiang University (China), St. Petersburg IPM (Russia), Melbourne University (Australia), Supelec (France), KTH (Sweden).

Her research interests include time-delay systems, networked control systems, distributed parameter systems, robust control, singular perturbations and nonlinear control. She has published 200 articles in international scientific journals and the author of two monographs. She serves/served as Associate Editor in Automatica, SIAM Journal on Control and Optimization and IMA Journal of Mathematical Control and Information. In 2014 she was Nominated as a Highly Cited Researcher by Thomson ISI. Since 2018, she has been the incumbent for Chana and Heinrich Manderman Chair on System Control at Tel Aviv University. She is IEEE Fellow since 2019. She is currently a member of the IFAC Council. In 2021 she was recipient of IFAC Delay Systems Life Time Achievement Award and of Kadar Award for outstanding research at Tel Aviv University.