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Stabilization under unknown control directions and disturbed measurements via a time-delay approach to extremum seeking

Jin Zhang[1] | **Emilia Fridman**[2]

[1]School of Mechatronic Engineering and Automation, Shanghai University, Shanghai, China

[2]School of Electrical Engineering, Tel Aviv University, Tel Aviv, Israel

Correspondence
Jin Zhang, School of Mechatronic Engineering and Automation, Shanghai University, Shanghai 200072, China.
Email: zhangjin1116@126.com

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Abstract

We study stabilization of linear uncertain systems under unknown control directions using a bounded extremum seeking controller in the presence of a small time-varying measurement delay. We assume that the measurements are subject to discontinuous disturbances. The main novelty is that these disturbances possess not only the constant part as in the existing results, but also small discontinuous part that may appear due to quantization. We consider two types of measurements: the state measurements and the state quadratic norm ones. In the latter case the constant part of the disturbances may be arbitrary large. By using the recently proposed time-delay approach to Lie-brackets-based averaging, we transform the closed-loop system to a time-delay (neutral type) one with no terms depending on the disturbance derivative, which has a form of perturbed Lie brackets system. The input-to-state stability (ISS) of the time-delay system guarantees the same for the original one. We further transform the neutral system to an ordinary differential equation (ODE) with delayed perturbations and employ variation of constants formula leading to explicit conditions in terms of simple inequalities with less conservative results in the most of numerical examples. Two numerical examples are provided to illustrate the efficiency of the method.

KEYWORDS

disturbed measurement, extremum seeking, Lie-brackets-based averaging, time-delay

1 | INTRODUCTION

Extremum seeking (ES), as a real-time model-free optimization approach, has received much attention during the past decades, see for example, References 1–4, starting with the rigorous proof of local convergence in Reference 5 and extension to semi-global convergence in Reference 6. ES controller was designed in Reference 7 for networked control systems using the sporadic packet transmissions. The behavior of ES in the presence of intermittent measurements was analyzed in Reference 8. In addition, input and output delays are unavoidable in practical applications.[9] Classical ES subject to a large known constant delay was studied in Reference 10 by using backstepping-based predictors and in Reference 11 by using sequential predictors. It is worthy mentioning that the aforementioned literature relies on the classical averaging method[12] and Lie brackets approximation.[13] By exploiting the converging trajectories property of the original system and the averaged system, the stability of the original system is guaranteed provided that the small parameter is small enough. However, till recently bounds on the small parameter could be found from simulations only, which is not reliable for the unknown systems. Some upper bounds on the small parameter were presented in Reference 14 in the context of averaging by using Lyapunov function where, however, the analytical upper bound was even not calculated in the example, and in Reference 15 for finite-time stabilization by ES controller where, however, the bound still employed approximations.

A new constructive time-delay approach to averaging of linear system was introduced in Reference 16. This approach allows, for the first time, to derive linear matrix inequalities (LMIs) for finding an efficient upper bound on the small parameter that ensures the stability and ISS of the original system. The time-delay approach to averaging was then successfully applied to power systems,[17] vibrational control,[18] L_2-gain analysis with stochastic extension,[19] and ES.[20,21] Very recently, a time-delay approach to Lie-brackets-based averaging of nonlinear affine systems was suggested in References 22 and 23, where stabilization under a bounded ES controller without/with time-varying measurement delay was considered, respectively. The input-to-state stability (ISS) analysis in References 22 and 23 was studied by using Lyapunov–Krasovskii (L-K) method that is complicated leading to conservative results in the examples. An improved and simplified analysis via time-delay approach to classical ES was recently suggested in Reference 24, where a further transformation of the time-delay system led to the nondelay one with delayed disturbances. The stability analysis was further provided by employing variation of constants formula that leads to simple inequalities. Inspired by Reference 24, we aim to present in this article a simpler analysis that leads to simple scalar inequalities and (in most of the examples) to less conservative results.

In practical applications, measurements may be subject to disturbances. Results for Lie-brackets approximation with discontinuous dithers were presented in Reference 25, where the disturbance derivative was assumed to be globally bounded and where the immunity to the additive, state-independent disturbances was shown. Recently, ISS-like properties of Lie-bracket approximations were presented in Reference 26 for the systems with Lipschitz continuous disturbances. Note also that constant disturbance that can be arbitrary large was considered in References 20,21,27, where the results were claimed to be applicable to the case of the differentiable disturbances with uniformly bounded derivatives. The existing results on ES with large amplitude and high frequency considered the differentiable measurement disturbances[25,27] with uniformly bounded derivatives or globally Lipschitz disturbances.[26] However, disturbances are usually discontinuous, for example, due to quantization of the measurement signal. Recently, robustness under discontinuous measurement disturbances was studied in Reference 28, where the results were qualitative (for fast enough oscillations) and the measurement delays were not considered. Up to now, qualitative and quantitative results for bounded ES in the presence of discontinuous in time/state measurement disturbances as well as small time-varying delays are missing in the literature, which motivates the present article.

The objective of this paper is to present the first quantitative results for stabilization of linear uncertain systems under a bounded ES controller with discontinuous disturbances and small measurement delays. We consider the measurements, that is, the state measurements and the state quadratic norm ones, subject to discontinuous disturbances. As a novelty, the disturbances possess not only the constant part as in Reference 27 but also small discontinuous part that may appear due to quantization. Our approach consists of two steps[23,24]: (1) system transformation and (2) stability analysis. The first step is challenging since the transformations in References 23,25–27 are not applicable to the discontinuous disturbances. We propose a novel time-delay transformation for the Lie-brackets-based averaging, where the transformed system does not depend on the disturbance derivative. The ISS of the resulting time-delay system guarantees the same for the original one. In the second step, we suggest variation of constants formula (as introduced in Reference 24 for perturbation-based ES) to derive explicit conditions in terms of simple inequalities. Moreover, we find that the constant part of the disturbances can be arbitrary large when using the state quadratic norm measurements. A conference version of this paper was presented in Reference 29, where the results were confined to the disturbance-free systems.

We now summarize the contribution as follows:

1. We consider, for the first time, the measurements subject to the time-delays and discontinuous disturbances (to be compared with the differentiable disturbances[25,27] and globally Lipschitz disturbances[26]). We propose a novel time-delay transformation leading to the transformed system that depends on the disturbance (and does not depend on its derivative as in Reference 23). The latter leads to a time-delay model with no terms depending on the disturbance derivative;
2. Inspired by Reference 24, we provide a simpler analysis and simpler stability conditions leading to less conservative results (i.e., larger uncertainties and larger bounds on the small parameter, time-delay as well as disturbed measurement) in the most of numerical examples compared to our previous (disturbance-free) result.[23]

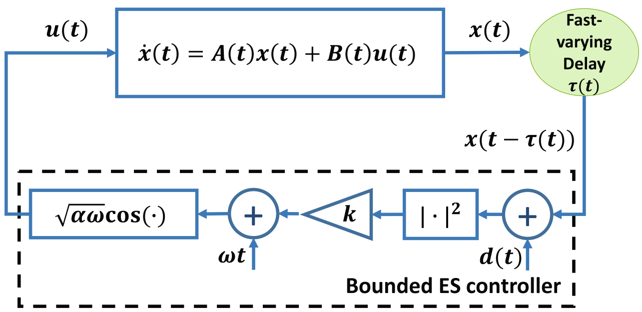

FIGURE 1 Stabilization of linear uncertain systems by a bounded ES controller.

Notation: Throughout the article, $\mathbb{R}^n$ denotes the n-dimensional Euclidean space with the vector norm $|\cdot|$, $\mathbb{R}^{n\times m}$ is the set of all $n \times m$ real matrices with the induced matrix norm $\|\cdot\|$. The notation $P > 0$, for $P \in \mathbb{R}^{n\times n}$, means that P is symmetric positive definite. Denote by $C[a, b]$ the Banach space of continuous functions $\phi : [a, b] \to \mathbb{R}^n$ with the norm $\|\phi\|_{C[a,b]} = \max_{\theta\in[a,b]} |\phi(\theta)|$. We use $a \pm b$ to denote $a + b - b$ (not the set $\{a + b, a - b\}$).

2 | STABILIZATION UNDER UNKNOWN CONTROL DIRECTIONS

In this section, we study stabilization of linear uncertain systems under unknown control directions using a bounded ES controller with discontinuous disturbances and transform the closed-loop system to a time-delay (neutral type) system by following the time-delay approach to Lie-Brackets-based averaging.[22,23]

As illustrated in Figure 1, the linear uncertain system is given by

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \quad t \geq 0, \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}$ is the input, the time-varying coefficients $A(t) \in \mathbb{R}^{n\times n}$ and $B(t) \in \mathbb{R}^n$ in (1) have the following form

$$A(t) = A_0 + \Delta A(t), \quad B(t) = B_0 + \frac{\sqrt{2\pi}}{\sqrt{\omega}}\Delta B(t). \tag{2}$$

Here A_0 is a constant matrix and B_0 is a known constant vector up to its sign. Since the sign of entries of B_0 is unknown, one cannot design for system (1) a classical PID type stabilizing controller.

We design for system (1) a bounded ES controller[30] with a measurement bias and a time-varying delay that appears due to delayed measurement of the state

$$u(t) = \sqrt{\alpha\omega}\cos\left(\omega t + k\left|x\left(t - \frac{2\pi}{\omega}\tau(t)\right) + d(t)\right|^2\right), \tag{3}$$

where ω is the frequency of the dither signal whose magnitude is $\sqrt{\alpha\omega}$ with $\alpha > 0$, and $k > 0$ is the controller gain. Moreover, the disturbance $d(t) \in \mathbb{R}^n$ is discontinuous (measurable in time) of the form

$$d(t) = d_0 + \Delta d(t). \tag{4}$$

Here d_0 is a constant vector and $\Delta d(t)$ denotes the measurement bias uncertainty that may stem from the quantization for example, in the network-based control system.

We now make the following assumptions:

A1 The delay $\tau(t)$ is supposed to be bounded, that is,

$$0 \leq \tau(t) \leq \tau_M, \quad t \geq 0, \tag{5}$$

and fast-varying (without any restrictions on its derivative).

A2 The uncertainties $\Delta A(t)$, $\Delta B(t)$ and $\Delta d(t)$ satisfy the following inequalities

$$\|\Delta A(t)\| \leq \Delta a, \quad |\Delta B(t)| \leq \Delta b, \quad |\Delta d(t)| \leq \Delta d^* \quad \forall t \geq 0 \tag{6}$$

with small constants $\Delta a \geq 0$, $\Delta b \geq 0$ and $\Delta d^* \geq 0$.

Note that the delay $\tau(t)$ includes sawtooth delays that model networked-based control. Assumption A2 implies

$$\|A(t)\| \leq a \quad \forall t \geq 0, \quad a = \|A_0\| + \Delta a, \quad |d(t)| \leq d^* \quad \forall t \geq 0, \quad d^* = |d_0| + \Delta d^*. \tag{7}$$

For simplicity we here consider that $\Delta A(t)$ and $\Delta B(t)$ depend on t only. Note that both uncertainties can be dependent on t and x provided that they satisfy (7) for all t and x and the solution of system (1), (3) is well-defined.

By letting $\omega = \frac{2\pi}{\varepsilon}$, we rewrite the closed-loop system (1), (3) in the following form

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + \frac{\sqrt{2\pi\alpha}}{\sqrt{\varepsilon}} B(t) \cos\left(\frac{2\pi t}{\varepsilon} + k|x(t-\varepsilon\tau(t)) + d(t)|^2\right) \\ &= A(t)x(t) + v(t) + \frac{\sqrt{2\pi\alpha}}{\sqrt{\varepsilon}} B_0 \cos\left(\frac{2\pi t}{\varepsilon} + k|x(t-\varepsilon\tau(t)) + d(t)|^2\right), \quad t \geq 0, \end{aligned} \tag{8}$$

where the initial condition is given by $x(\theta) = \phi(\theta)$, $\theta \in [-\varepsilon\tau_M, 0]$ with $\phi \in C[-\varepsilon\tau_M, 0]$, and

$$v(t) = \sqrt{2\pi\alpha}\Delta B(t) \cos\left(\frac{2\pi t}{\varepsilon} + k|x(t-\varepsilon\tau(t)) + d(t)|^2\right). \tag{9}$$

The averaged system that corresponds to system (8) with $\tau(t) = 0$, $\Delta B(t) = 0$ (i.e., $v(t) = 0$) and $d(t) = 0$ is given by the following Lie Brackets system[1–3,26]:

$$\dot{x}_{av}(t) = [A_{av} + \Delta A(t)]x_{av}(t), \quad x_{av}(t) \in \mathbb{R}^n, \tag{10}$$

where

$$A_{av} = A_0 - \alpha k B_0 B_0^T. \tag{11}$$

Here we assume that there exist constants α and k leading to Hurwitz A_{av} given by (11).

Note that the Lie Brackets averaging[1–3,26] is an "approximate" method: it employs the averaged system (10) in terms of Lie Brackets to approximate the behavior of system (8) with $\tau(t) = 0$, $\Delta B(t) = 0$ and $d(t) = 0$. In contrast, a time-delay approach to Lie-Brackets-based averaging that has been recently proposed in References 20,22,23 does not use approximations. Inspired by References 20,22,23, we propose in this paper a novel time-delay approach to Lie-Brackets-based averaging for system (8) (see Appendix A) that allows to transform system (8) to the following time-delay system:

$$\frac{d}{dt}[x(t) + G(t)] = [A_{av} + \Delta A(t)]x(t) + \sum_{i=1}^{2}(Y_i(t) + Y_{\tau i}(t)) + \sum_{i=1}^{5} Y_{di}(t) + Y_v(t) + v(t), \quad t \geq \varepsilon + \varepsilon\tau_M, \tag{12}$$

where $\dot{x}(t)$ satisfies (8), A_{av} is defined by (11), $v(t)$ is given by (9), and

$$\begin{aligned} G(t) &= -\frac{\sqrt{2\pi\alpha}}{\varepsilon\sqrt{\varepsilon}} \int_{t-\varepsilon}^{t} (s-t+\varepsilon) B_0 \cos\left(\frac{2\pi s}{\varepsilon} + k|x(s) + d(t)|^2\right) ds, \\ Y_1(t) &= \frac{2\sqrt{2\pi\alpha}k}{\varepsilon\sqrt{\varepsilon}} \int_{t-\varepsilon}^{t}\int_{s}^{t} \sin\left(\frac{2\pi s}{\varepsilon} + k|x(\theta) + d(t)|^2\right) B_0 x^T(\theta) A(\theta) x(\theta) d\theta ds, \\ Y_2(t) &= -\frac{4\pi\alpha k}{\varepsilon^2} \int_{t-\varepsilon}^{t}\int_{s}^{t}\int_{\theta}^{t} \Big[2k \cos\left(\frac{2\pi}{\varepsilon}(s+\theta) + 2k|x(\xi) + d(t)|^2\right) x(\theta) x^T(\xi) \\ &\quad + \sin\left(\frac{2\pi s}{\varepsilon} + k|x(t) + d(t)|^2\right) \cos\left(\frac{2\pi\theta}{\varepsilon} + k|x(t) + d(t)|^2\right)\Big] B_0 B_0^T \dot{x}(\xi) d\xi d\theta ds, \end{aligned}$$

$$
\begin{aligned}
Y_{\tau1}(t) &= \frac{2\sqrt{2\pi\alpha}k}{\sqrt{\varepsilon}}\int_{t-\varepsilon\tau(t)}^{t}\sin\left(\frac{2\pi t}{\varepsilon}+k|x(s)+d(t)|^2\right)B_0x^T(s)\dot{x}(s)ds,\\
Y_{\tau2}(t) &= \frac{8\pi\alpha k^2}{\varepsilon^2}\int_{t-\varepsilon}^{t}\int_{s}^{t}\int_{\theta-\varepsilon\tau(\theta)}^{\theta}\sin\left(\frac{2\pi s}{\varepsilon}+k|x(\theta)+d(t)|^2\right)\sin\left(\frac{2\pi\theta}{\varepsilon}+k|x(\xi)+d(\theta)|^2\right)B_0B_0^Tx(\theta)x^T(\xi)\dot{x}(\xi)d\xi d\theta ds,\\
Y_{d1}(t) &= \frac{2\sqrt{2\pi\alpha}k}{\sqrt{\varepsilon}}\int_{t-\varepsilon\tau(t)}^{t}\sin\left(\frac{2\pi t}{\varepsilon}+k|x(s)+d(t)|^2\right)B_0d^T(t)\dot{x}(s)ds,\\
Y_{d2}(t) &= \frac{2\sqrt{2\pi\alpha}k}{\varepsilon\sqrt{\varepsilon}}\int_{t-\varepsilon}^{t}\int_{s}^{t}\sin\left(\frac{2\pi s}{\varepsilon}+k|x(\theta)+d(t)|^2\right)B_0d^T(t)\dot{x}(\theta)d\theta ds,\\
Y_{d3}(t) &= \frac{8\pi\alpha k^2}{\varepsilon^2}\int_{t-\varepsilon}^{t}\int_{s}^{t}\int_{\theta-\varepsilon\tau(\theta)}^{\theta}\sin\left(\frac{2\pi s}{\varepsilon}+k|x(\theta)+d(t)|^2\right)\sin\left(\frac{2\pi\theta}{\varepsilon}+k|x(\xi)+d(\theta)|^2\right)B_0B_0^Tx(\theta)d^T(\theta)\dot{x}(\xi)d\xi d\theta ds,\\
Y_{d4}(t) &= \frac{4\pi\alpha k^2}{\varepsilon^2}\int_{t-\varepsilon}^{t}\int_{s}^{t}\int_{|x(\theta)+d(\theta)|^2}^{|x(\theta)+d(t)|^2}\sin\left(\frac{2\pi s}{\varepsilon}+k|x(\theta)+d(t)|^2\right)\sin\left(\frac{2\pi\theta}{\varepsilon}+k\xi\right)B_0B_0^Tx(\theta)d\xi d\theta ds,\\
Y_{d5}(t) &= -\frac{8\pi\alpha k^2}{\varepsilon^2}\int_{t-\varepsilon}^{t}\int_{s}^{t}\int_{\theta}^{t}\cos\left(\frac{2\pi}{\varepsilon}(s+\theta)+2k|x(\xi)+d(t)|^2\right)B_0B_0^Tx(\theta)d^T(t)\dot{x}(\xi)d\xi d\theta ds,\\
Y_v(t) &= \frac{4\pi\alpha k}{\varepsilon\sqrt{\varepsilon}}\int_{t-\varepsilon}^{t}\int_{s}^{t}\sin\left(\frac{2\pi s}{\varepsilon}+k|x(\theta)+d(t)|^2\right)\cos\left(\frac{2\pi\theta}{\varepsilon}+k|x(\theta-\varepsilon\tau(\theta))+d(\theta)|^2\right)B_0\Delta B^T(\theta)x(\theta)d\theta ds.
\end{aligned} \tag{13}
$$

Clearly, $G(t)$, $Y_i(t)$ $(i=1,2)$ and $Y_v(t)$ are of the order of $O(\sqrt{\varepsilon})$, $Y_{\tau1}(t)$ is of the order of $O(\tau_M)$, $Y_{\tau2}(t)$ is of the order of $O(\sqrt{\varepsilon}\tau_M)$, $Y_{d1}(t)$ is of the order of $O(\tau_M d^*)$, $Y_{d2}(t)$ and $Y_{d4}(t)$ are of the order of $O(d^*)$, $Y_{d3}(t)$ is of the order of $O(\sqrt{\varepsilon}\tau_M d^*)$, and $Y_{d5}(t)$ is of the order of $O(\sqrt{\varepsilon}d^*)$ provided $\dot{x}(t)$ is of the order of $O(\frac{1}{\sqrt{\varepsilon}})$. Thus, it can be seen that system (12) is a perturbation of the stable averaged system (10). Note that the perturbations in (12) will vanish as $\varepsilon\to 0$, $\tau_M\to 0$ and $d^*\to 0$. If ε, τ_M and d^* increase, system (12) may become unstable. However, till recently bounds on ε could be found from simulations only, which is not reliable for the unknown systems. Thus, differently from the qualitative analysis in References 1–3,26,30, our objective is to find the first efficient quantitative upper bounds on ε, τ_M and d^* that ensure the stability.

Remark 1. Note that if one follows the transformation,[23] the resulting time-delay system will include the $\dot{d}(t)$-terms (i.e., the disturbances should be differentiable). In addition, an assumption on the derivative of $d(t)$ (which should be small) should be imposed such that the $\dot{d}(t)$-terms are small perturbations. Clearly, our transformation allows the disturbances to be discontinuous and remove the assumption on the derivative of $d(t)$.

To end this section, we present the relation between solutions of systems (8) and (12):

Proposition 1. *If $x(t)$ is a solution to system (8), then it satisfies the time-delay system (12) with notations (9) and (13), where $\dot{x}(t)$ is defined by (8).*

From Proposition 1 it follows that if solutions $x(t)$ of the time-delay system (12) for $t\geq\varepsilon+\varepsilon\tau_M$ satisfy some bound (e.g., ISS bound given by (21) below), then the same bound holds for solutions of system (8) for $t\geq\varepsilon+\varepsilon\tau_M$.

3 | MAIN RESULTS

In this section, we employ variation of constants formula to derive explicit conditions in terms of simple inequalities that guarantee the ISS of the original system. To proceed, inspired by References 24 and 29 we firstly denote

$$z(t)=x(t)+G(t). \tag{14}$$

Then system (12) can be further rewritten as

$$\dot{z}(t)=[A_{av}+\Delta A(t)](z(t)-G(t))+\sum_{i=1}^{2}(Y_i(t)+Y_{\tau i}(t))+\sum_{i=1}^{5}Y_{di}(t)+Y_v(t)+v(t),\quad t\geq\varepsilon+\varepsilon\tau_M. \tag{15}$$

It is clear that using (14) allows to explicitly present (12) in the form of ODE with delayed perturbations. Thus, one can employ variation of constants formula in the later stability analysis. This method will essentially simplify the stability analysis (that avoids L-K method) leading to simpler conditions, and greatly improve the robustness (including the uncertainties $\Delta A(t)$ and $\Delta B(t)$) comparatively to L-K method (see e.g., References 22 and 23).

For the sake of simplicity, we denote

$$\vartheta_1 = \sqrt{2\pi\alpha}|B_0|, \quad \vartheta_2 = k + 2k^2\sigma^2, \quad \vartheta_3 = a\sigma + \sqrt{2\pi\alpha}\Delta b, \quad \vartheta_4 = \sqrt{2\pi\alpha}\Delta b. \tag{16}$$

We are in a position to formulate the following main results proven in Appendix B:

Theorem 1. *Consider system (8) subject to A1 and A2 under the initial condition* $\|\phi\|_{C[-\varepsilon\tau_M,0]} \le \sigma_0$. *Let* α *and* k *lead to Hurwitz* A_{av} *given by (11). Given a tuning parameter* $\delta > 0$, *let there exist* $n \times n$ *matrix* $P > 0$ *and scalar* $p \ge 1$, $\lambda > 0$ *that satisfy the following inequalities*:

$$P - I \ge 0, \quad pI - P \ge 0, \tag{17}$$

$$\Xi = \begin{bmatrix} PA_{av} + A_{av}^T P + 2\delta P + \lambda(\Delta a)^2 & P \\ * & -\lambda I \end{bmatrix} \le 0. \tag{18}$$

If additionally, given tuning parameters $\Delta a \ge 0$, $\Delta b \ge 0$, $\varepsilon^* > 0$, $\tau_M > 0$, $d^* > 0$, *and* $0 < \sigma_0 < \sigma$, *the following inequality*

$$\begin{aligned} p\Bigg[& e^{a\varepsilon^*(1+\tau_M)}\left(\sigma_0 + \left(\sqrt{\varepsilon^*}\vartheta_1 + \varepsilon^*\sqrt{2\pi\alpha}\Delta b\right)(1+\tau_M)\right) + \frac{\sqrt{\varepsilon^*}}{2}\vartheta_1 \\ & + \frac{1}{\delta}\left(\vartheta_1\left(\sqrt{\varepsilon^*}\kappa_0 + \varepsilon^*\kappa_1 + 2\tau_M\sigma\left(\kappa_2 + \sqrt{\varepsilon^*}\kappa_3 + \varepsilon^*\kappa_4\right)\right.\right. \\ & \left.\left. + d^*\left(\kappa_5 + \sqrt{\varepsilon^*}\kappa_6 + \varepsilon^*\kappa_7 + d^*\kappa_8\right) + \kappa_9\right) + \vartheta_4\right)\Bigg]^2 < \left(\sigma - \frac{\sqrt{\varepsilon^*}}{2}\vartheta_1\right)^2, \end{aligned} \tag{19}$$

is valid, where

$$\begin{aligned} &\kappa_0 = \frac{1}{2}(\|A_{av}\| + \Delta a) + \frac{1}{3}\vartheta_1^2\vartheta_2 + k\sigma\vartheta_3, \quad \kappa_1 = \frac{1}{3}\vartheta_1\vartheta_2\vartheta_3, \quad \kappa_2 = k\vartheta_1, \\ &\kappa_3 = \sigma k^2\vartheta_1^2 + k\vartheta_3, \quad \kappa_4 = \sigma k^2\vartheta_1\vartheta_3, \quad \kappa_5 = k\vartheta_1(4\sigma^2 k + 2\tau_M + 1), \\ &\kappa_6 = 2\sigma k^2\vartheta_1^2\left(\tau_M + \frac{1}{3}\right) + k\vartheta_3(2\tau_M + 1), \quad \kappa_7 = 2\sigma k^2\vartheta_1\vartheta_3\left(\tau_M + \frac{1}{3}\right), \\ &\kappa_8 = 2\sigma k^2\vartheta_1, \qquad \kappa_9 = -2\sigma k^2\vartheta_1|d_0|(2\sigma + |d_0|) \end{aligned} \tag{20}$$

with a *defined in (7) and* ϑ_i $(i = 1, \dots, 4)$ *defined in (16), then for all* $\varepsilon \in (0, \varepsilon^*]$ *the solution of (8) starting from the initial condition* $\|\phi\|_{C[-\varepsilon\tau_M,0]} \le \sigma_0$ *satisfies*

$$\begin{aligned} |x(t)| &\le e^{at}\left(\|\phi\|_{C[-\varepsilon\tau_M,0]} + \left(\sqrt{\varepsilon}\vartheta_1 + \varepsilon\vartheta_4\right)(1+\tau_M)\right) < \sigma, \quad t \in [0, \varepsilon + \varepsilon\tau_M], \\ |x(t)| &< \sqrt{p}e^{-\delta(t-\varepsilon-\varepsilon\tau_M)}\left[e^{a(\varepsilon+\varepsilon\tau_M)}\left(\|\phi\|_{C[-\varepsilon\tau_M,0]} + (\sqrt{\varepsilon}\vartheta_1 + \varepsilon\vartheta_4)(1+\tau_M)\right) + \frac{\sqrt{\varepsilon}}{2}\vartheta_1\right] \\ &\quad + \frac{\sqrt{p}}{\delta}\left[\vartheta_1\left(\sqrt{\varepsilon}\kappa_0 + \varepsilon\kappa_1 + 2\tau_M\sigma\left(\kappa_2 + \sqrt{\varepsilon}\kappa_3 + \varepsilon\kappa_4 + d^*\left(\kappa_5 + \sqrt{\varepsilon}\kappa_6\right.\right.\right.\right. \\ &\quad \left.\left.\left.\left. + \ \varepsilon\kappa_7 + d^*\kappa_8\right) + \kappa_9\right) + \vartheta_4\right] + \frac{\sqrt{\varepsilon}}{2}\vartheta_1 < \sigma, \quad t \ge \varepsilon + \varepsilon\tau_M. \end{aligned} \tag{21}$$

Moreover, for all initial conditions $\|\phi\|_{C[-\varepsilon\tau_M,0]} \le \sigma_0$ *the ball*

$$\mathfrak{X} = \left\{ x \in \mathbb{R}^n : |x| \le \frac{\sqrt{p}}{\delta}\left[\vartheta_1\left(\sqrt{\varepsilon}\kappa_0 + \varepsilon\kappa_1 + 2\tau_M\sigma\left(\kappa_2 + \sqrt{\varepsilon}\kappa_3 + \varepsilon\kappa_4\right)\right.\right.\right.$$
$$\left.\left.\left. + d^*\left(\kappa_5 + \sqrt{\varepsilon}\kappa_6 + \varepsilon\kappa_7 + d^*\kappa_8\right) + \kappa_9\right) + \vartheta_4\right] + \frac{\sqrt{\varepsilon}}{2}\vartheta_1 \right\} \tag{22}$$

is exponentially attractive with a decay rate δ.

Remark 2. Note that inequalities (17) and (18) always hold provided matrix A_{av} is Hurwitz. Moreover, given any $\sigma_0 > 0$ and $\sigma > \sqrt{p}\sigma_0$, inequality (19) is always feasible for small enough $\Delta a > 0$, $\Delta b > 0$, $\varepsilon^* > 0$, $\tau_M > 0$ and $d^* > 0$. Therefore, the result is semi-global.

Remark 3. From (22) with notations (16) and (20), it follows that the ultimate bound depends upon α and k for small enough $\tau_M > 0$ and $d^* > 0$. A possible choice is $\alpha = O(\sqrt{\varepsilon})$ and $k = O(\frac{1}{\sqrt{\varepsilon}})$ that leads to the decay rate $\delta = O(1)$ and, thus, the ultimate bound is of the order of $O(\varepsilon^{\frac{1}{4}})$ for small enough $\tau_M > 0$ and $d^* > 0$. Moreover, it is easy to see that the larger $\tau_M > 0$ and $d^* > 0$ (i.e., larger measurement delays and disturbances) result in a larger ultimate bound.

Remark 4. In Reference 29, using (14) system (12) was transformed to

$$\dot{z}(t) = A_{av}z(t) + \Delta A(t)x(t) - A_{av}G(t) + \sum_{i=1}^{2}(Y_i(t) + Y_{\tau i}(t)) + \sum_{i=1}^{5}Y_{di}(t) + Y_v(t) + v(t), \quad t \ge \varepsilon + \varepsilon\tau_M. \tag{23}$$

In its corresponding stability analysis, the term $\Delta A(t)x(t)$ in (23) was treated as a “disturbance” that brings in the conservativeness. Clearly, in the current paper we have avoided this, which will lead to better results in the examples than in Reference 29.

We next consider that quadratic norm of the state is measured subject to a disturbance leading to the following bounded ES controller

$$u(t) = \sqrt{\alpha\omega}\cos\left(\omega t + k\left(\left|x\left(t - \frac{2\pi}{\omega}\tau(t)\right)\right|^2 + d(t)\right)\right), \tag{24}$$

where $d(t) \in \mathbb{R}$ satisfies (4). The resulting closed-loop system (1), (24) has the form

$$\begin{aligned}\dot{x}(t) &= A(t)x(t) + \frac{\sqrt{2\pi\alpha}}{\sqrt{\varepsilon}}B(t)\cos\left(\frac{2\pi t}{\varepsilon} + k\left(|x(t - \varepsilon\tau(t))|^2 + d(t)\right)\right) \\ &= A(t)x(t) + v(t) + \frac{\sqrt{2\pi\alpha}}{\sqrt{\varepsilon}}B_0\cos\left(\frac{2\pi t}{\varepsilon} + k\left(|x(t - \varepsilon\tau(t))|^2 + d(t)\right)\right), \quad t \ge 0,\end{aligned} \tag{25}$$

where

$$v(t) = \sqrt{2\pi\alpha}\Delta B(t)\cos\left(\frac{2\pi t}{\varepsilon} + k\left(|x(t - \varepsilon\tau(t))|^2 + d(t)\right)\right). \tag{26}$$

Assume also that A1 and A2 hold. By following the aforementioned time-delay approach to Lie-brackets-based averaging, we transform system (25) to the following time-delay system

$$\frac{d}{dt}[x(t) + G(t)] = [A_{av} + \Delta A(t)]x(t) + \sum_{i=1}^{2}(Y_i(t) + Y_{\tau i}(t)) + Y_d(t) + Y_v(t) + v(t), \quad t \ge \varepsilon + \varepsilon\tau_M, \tag{27}$$

where $\dot{x}(t)$ satisfies (25), A_{av} is defined by (11), and

$$
\begin{aligned}
G(t) &= -\frac{\sqrt{2\pi\alpha}}{\varepsilon\sqrt{\varepsilon}}\int_{t-\varepsilon}^{t}(s-t+\varepsilon)B_0\cos\Big(\frac{2\pi s}{\varepsilon}+k(|x(s)|^2+d(t))\Big)ds,\\
Y_1(t) &= \frac{2\sqrt{2\pi\alpha}k}{\varepsilon\sqrt{\varepsilon}}\int_{t-\varepsilon}^{t}\int_{s}^{t}\sin\Big(\frac{2\pi s}{\varepsilon}+k(|x(\theta)|^2+d(t))\Big)B_0x^T(\theta)A(\theta)x(\theta)d\theta ds,\\
Y_2(t) &= -\frac{4\pi\alpha k}{\varepsilon^2}\int_{t-\varepsilon}^{t}\int_{s}^{t}\int_{\theta}^{t}\Big[2k\cos\Big(\frac{2\pi}{\varepsilon}(s+\theta)+2k|x(\xi)|^2+2d(t)\Big)x(\theta)x^T(\xi)\\
&\quad+\sin\Big(\frac{2\pi s}{\varepsilon}+k|x(t)|^2+d(t)\Big)\cos\Big(\frac{2\pi\theta}{\varepsilon}+k|x(t)|^2+d(t)\Big)\Big]B_0B_0^T\dot{x}(\xi)d\xi d\theta ds,\\
Y_{\tau 1}(t) &= \frac{2\sqrt{2\pi\alpha}k}{\sqrt{\varepsilon}}\int_{t-\varepsilon\tau(t)}^{t}\sin\Big(\frac{2\pi t}{\varepsilon}+k|x(s)|^2+d(t)\Big)B_0x^T(s)\dot{x}(s)ds,\\
Y_{\tau 2}(t) &= \frac{8\pi\alpha k^2}{\varepsilon^2}\int_{t-\varepsilon}^{t}\int_{s}^{t}\int_{\theta-\varepsilon\tau(\theta)}^{\theta}\sin\Big(\frac{2\pi s}{\varepsilon}+k|x(\theta)|^2+d(t)\Big)\sin\Big(\frac{2\pi\theta}{\varepsilon}+k|x(\xi)|^2+d(\theta)\Big)B_0B_0^Tx(\theta)x^T(\xi)\dot{x}(\xi)d\xi d\theta ds,\\
Y_d(t) &= \frac{4\pi\alpha k^2}{\varepsilon^2}\int_{t-\varepsilon}^{t}\int_{s}^{t}\int_{d(\theta)}^{d(t)}\sin\Big(\frac{2\pi s}{\varepsilon}+k|x(\theta)|^2+d(t)\Big)\sin\Big(\frac{2\pi\theta}{\varepsilon}+k|x(\theta)|^2+k\xi\Big)B_0B_0^Tx(\theta)d\xi d\theta ds,\\
Y_v(t) &= \frac{4\pi\alpha k}{\varepsilon\sqrt{\varepsilon}}\int_{t-\varepsilon}^{t}\int_{s}^{t}\sin\Big(\frac{2\pi s}{\varepsilon}+k|x(\theta)|^2+d(t)\Big)\cos\Big(\frac{2\pi\theta}{\varepsilon}+k|x(\theta-\varepsilon\tau(\theta))|^2+d(\theta)\Big)B_0\Delta B^T(\theta)x(\theta)d\theta ds.
\end{aligned}
\tag{28}
$$

Following arguments of Theorem 1, we present the following results.

Theorem 2. *Consider system (25) subject to A1 and A2 under the initial condition $\|\phi\|_{C[-\varepsilon\tau_M,0]} \le \sigma_0$. Let α and k lead to Hurwitz A_{av} given by (11). Given a tuning parameter $\delta > 0$, let there exist $n \times n$ matrix $P > 0$ and scalar $p \ge 1$, $\lambda > 0$ that satisfy (17) and (18). If additionally, given tuning parameters $\Delta a \ge 0$, $\Delta b \ge 0$, $\varepsilon^* > 0$, $\tau_M > 0$, $\Delta d^* > 0$, and $0 < \sigma_0 < \sigma$, inequality (19) is valid, where*

$$
d^* \text{ is changed by } \Delta d^*, \quad \kappa_i\ (i=0,\dots,4) \text{ are from (20)}, \quad \kappa_5 = 2\sigma k^2\vartheta_1, \quad \kappa_i = 0\ (i=6,\dots,9) \tag{29}
$$

with ϑ_1 defined in (16), then for all $\varepsilon \in (0, \varepsilon^*]$ the solution of (25) starting from the initial condition $\|\phi\|_{C[-\varepsilon\tau_M,0]} \le \sigma_0$ satisfies (21) with notation (29). Moreover, for all initial conditions $\|\phi\|_{C[-\varepsilon\tau_M,0]} \le \sigma_0$ the ball (22) with notation (29) is exponentially attractive with a decay rate δ.

Remark 5. It is clear that the conditions of Theorem 2 depend on Δd^* (that may model the errors due to the quantization) only and are independent of $|d_0|$. Thus, if one uses controller (24) to stabilize system (1), $|d_0|$ can be arbitrary large as in References [4,20,27] (to be compared with that $|d_0|$ is small when using the controller (3) to stabilize system (1)).

Remark 6. Note that in the stability analysis of systems (15) and (27), we employed variation of constants formula, where the perturbation terms are treated as disturbances by using the direct upper bounding. The latter may introduce some conservativeness (see the comparisons in Example 1 below). In the future, the results may be improved for example, by advanced Lyapunov-based method with appropriate L-K functionals.

4 | NUMERICAL EXAMPLES

In this section, we present two numerical examples to illustrate the efficiency of the proposed method.

Example 1. Consider the scalar system (1) with

$$
|A(t)| = |\Delta A(t)| \le \Delta a, \quad B_0 \in \{-1, 1\}, \quad \Delta B(t) = 0. \tag{30}
$$

It is clear that $a = \Delta a$, $\Delta b = 0$ and $|B_0| = 1$.

TABLE 1 Solutions via different methods.

	Δa	Δb	Reference 23 ε^*	Reference 23 τ_M	Reference 23 UB	Reference 29 ε^*	Reference 29 τ_M	Reference 29 UB	Theorem 1 ε^*	Theorem 1 τ_M	Theorem 1 UB
Example 1	0	0	$0.53 \cdot 10^{-4}$	–	1.2572	$0.20 \cdot 10^{-4}$	–	0.9836	$0.20 \cdot 10^{-4}$	–	0.9838
	0.1	0	$0.28 \cdot 10^{-4}$	–	1.2589	$0.07 \cdot 10^{-4}$	–	0.9959	$0.19 \cdot 10^{-4}$	–	0.9847
	0.19	0	–	–	–	$0.01 \cdot 10^{-4}$	–	0.9921	$0.18 \cdot 10^{-4}$	–	0.9828
Example 2	0	0	$0.10 \cdot 10^{-4}$	0.0045	1.3176	$0.30 \cdot 10^{-4}$	0.0050	1.4963	$0.30 \cdot 10^{-4}$	0.0050	1.4964
	0.002	0.002	$0.10 \cdot 10^{-4}$	0.0037	1.3523	$0.18 \cdot 10^{-4}$	0.0041	1.4960	$0.30 \cdot 10^{-4}$	0.0048	1.4940
	0.032	0.05	–	–	–	$0.02 \cdot 10^{-4}$	0.0008	1.4992	$0.30 \cdot 10^{-4}$	0.0020	1.4960
	0.032	0.1	–	–	–	–	–	–	$0.25 \cdot 10^{-4}$	0.0008	1.4958

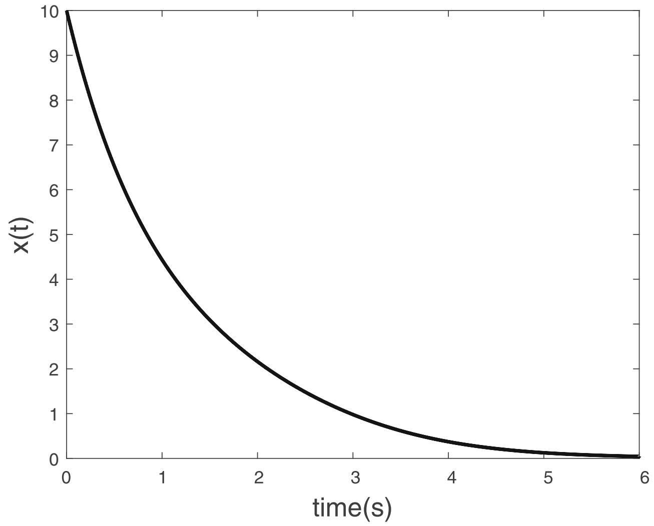

FIGURE 2 Example 1: state responseunder the bounded ES controller (3).

Consider first system (1), (30) under the bounded ES controller (3) with $\alpha = 0.1$ and $k = 9$. Then we have $A_{av} = -0.9$. Let the desired decay rate be $\delta = 0.5$, $\sigma_0 = 1$ and $\sigma = 2$. We consider the following two cases:

i) $d(t) = 0$ (i.e., $d^* = 0$). We verify the inequalities of Theorem 1 with $\tau_M = 0$ and different values of Δathat lead to upper bound on ε^* (that preserves the ISS for all $\varepsilon \in (0, \varepsilon^*]$) and the resulting ultimate bound (UB), see lines 3–5 in Table 1. Clearly, Theorem 1 improves the results of Reference 29 and, however, leads to more conservative upper bounds on ε^* than in Reference 23. The latter may be due to that the perturbation terms are treated as disturbances by using the direct upper bounding. However, Theorem 1 allows larger uncertainty $|\Delta A(t)| \leq \Delta a$ than in Reference 23.

ii) $d(t) \neq 0$ (that was not considered in Reference 23 and 29). By verifying Theorem 1 with $\Delta a = 0.19$, $\varepsilon^* = 0.1 \cdot 10^{-4}$, $\tau_M = 0.004$ and $|d_0| = 0.002$, we find a quantitative upper bound $\Delta d^* = 0.3 \cdot 10^{-4}$.

Consider next system (1), (30) under the bounded ES controller (24) with the same $\alpha = 0.1$ and $k = 9$ leading to $A_{av} = -0.9$. For $\delta = 0.5$, $\sigma_0 = 1$, $\sigma = 2$, $\Delta a = 0.19$, $\varepsilon^* = 0.1 \cdot 10^{-4}$, $\tau_M = 0.004$ and any $d_0 \in \mathbb{R}$, by verifying Theorem 2 we find a quantitative upper bound $\Delta d^* = 1.9 \cdot 10^{-4}$.

For the numerical simulations, under the initial condition $\phi(t) = 10$ for $t \leq 0$, $A(t) = 0.19\sin(t)$, $\varepsilon = 0.1 \cdot 10^{-4}$, $\tau(t) = 0.004$, $\alpha = 0.1$, $k = 9$ and $B_0 = 1$, Figure 2 plots the state responses of system (1), (30) under the bounded ES controller (3) with $d(t) = 0.002 + 0.3\cos(t) \cdot 10^{-4}$. Note that the averaged $|x(t)|$ for $t \in [5.998, 6]$ is 0.0437. Thus, we obtain the practical decay rate as $\frac{1}{6}\ln(\frac{10}{0.0437}) = 0.9053$ that is larger than the theoretical decay rate 0.5 illustrating the conservatism of the proposed method.

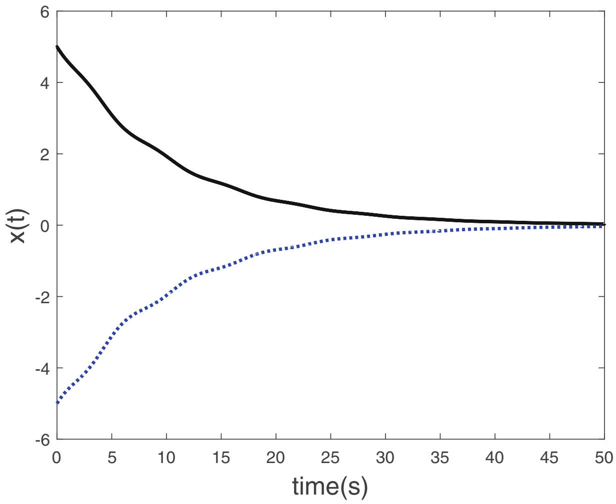

FIGURE 3 Example 2: state responses under the bounded ES controller (24).

Example 2. Consider system (1), where

$$A_0 = \begin{bmatrix} 0 & 0.1 \\ 0.1 & 0 \end{bmatrix}, \quad B_0 \in \left\{ \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}, \tag{31}$$

and

$$\|\Delta A(t)\| \leq \Delta a, \quad \Delta B(t) = \begin{bmatrix} \Delta b_1(t) & \Delta b_2(t) \end{bmatrix}^T, \quad \sum_{i=1}^{2} |\Delta b_i(t)|^2 \leq (\Delta b)^2, \quad t \geq 0. \tag{32}$$

It is clear that $a = \Delta a + 0.1$ and $|B_0| = \sqrt{2}$.

Consider first system (1), (31), (32) under the bounded ES controller (3) with $\alpha = 0.01$ and $k = 10$. Here (11) has the form $A_{av} = -0.1I$. Let the desired decay rate be $\delta = 0.067$, $\sigma_0 = 0.5$ and $\sigma = 2$. We consider the following two cases:

i) $d(t) = 0$ (i.e., $d^* = 0$). The solutions via Theorem 1 are shown in lines 6–9 of Table 1 for different values of Δa and Δb. Comparatively to Reference 23 and 29, Theorem 1 provides larger upper bounds on ε^* and τ_M, which allows larger parameter uncertainties $\|\Delta A(t)\| \leq \Delta a$ and $|\Delta B(t)| \leq \Delta b$.

ii) $d(t) \neq 0$ (that was not considered in Reference 23 and 29). For $\Delta a = 0.032$, $\Delta b = 0.05$, $\varepsilon^* = 0.2 \cdot 10^{-4}$, $\tau_M = 0.003$ and $|d_0| = 3 \cdot 10^{-4}$, by verifying Theorem 1 we find a quantitative upper bound $\Delta d^* = 0.4 \cdot 10^{-4}$.

Consider next system (1), (31), (32) under the bounded ES controller (24) with $\alpha = 0.01$ and $k = 10$ leading to $A_{av} = -0.1I$. For $\delta = 0.067$, $\sigma_0 = 0.5$, $\sigma = 2$, $\Delta a = 0.032$, $\Delta b = 0.05$, $\varepsilon^* = 0.2 \cdot 10^{-4}$, $\tau_M = 0.003$ and any $d_0 \in \mathbb{R}$, by verifying Theorem 2 we find a quantitative upper bound $\Delta d^* = 1.8 \cdot 10^{-4}$.

For the numerical simulations, under the initial condition $\phi(t) = [5, -5]^T$ for $t \leq 0$, $\Delta A(t) = 0.032\sin(t)I$, $\Delta B(t) = [0.05\sin(t), 0]^T$, $\varepsilon = 0.2 \cdot 10^{-4}$, $\tau(t) = 0.003$, $\alpha = 0.1$, $k = 9$ and $B_0 = [1, 1]^T$, Figure 3 plots the state responses of system (1), (30) under the bounded ES controller (24) with any constant $d_0 \in \mathbb{R}$ and $d(t) = 1.8\cos(t) \cdot 10^{-4}$. Note that the averaged $|x(t)|$ for $t \in [59.99, 60]$ is 0.0676. Thus, we obtain the practical decay rate as $\frac{1}{50}\ln(\frac{5\sqrt{2}}{0.0676}) = 0.093$ that is larger than the theoretical decay rate 0.067 illustrating the conservatism of the proposed method.

5 | CONCLUSIONS

This paper has studied stabilization of linear uncertain systems under unknown control directions using a bounded ES controller with a time-varying measurement delay. We have considered two types of measurements, that is, the state measurements and the state quadratic norm ones, subject to discontinuous disturbances, where as a main novelty the disturbances possess not only the constant part but also small discontinuous part that may appear due to quantization. Simple ISS analysis in terms of explicit simple inequalities with less conservativeness has been presented via a time-delay approach to ES controller. Future work will focus on consideration of applications to the practical engineering.[31]

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CONFLICT OF INTEREST STATEMENT

The authors declare no potential conflict of interests.

DATA AVAILABILITY STATEMENT

Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

ORCID

Jin Zhang https://orcid.org/0000-0002-6043-309X

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APPENDIX A. NOVEL TRANSFORMATION VIA TIME-DELAY APPROACH

First, similar to Reference 23 we present

$$\begin{aligned}
&\frac{\sqrt{2\pi\alpha}}{\sqrt{\varepsilon}}B_0\cos\Big(\frac{2\pi t}{\varepsilon}+k|x(t-\varepsilon\tau(t))+d(t)|^2\Big)\\
&=\frac{\sqrt{2\pi\alpha}}{\sqrt{\varepsilon}}B_0\Big[\cos\Big(\frac{2\pi t}{\varepsilon}+k|x(t-\varepsilon\tau(t))+d(t)|^2\Big)\pm\cos\Big(\frac{2\pi t}{\varepsilon}+k|x(t)+d(t)|^2\Big)\Big]\\
&=\frac{\sqrt{2\pi\alpha}}{\sqrt{\varepsilon}}B_0\left[\cos\Big(\frac{2\pi t}{\varepsilon}+k|x(t)+d(t)|^2\Big)+2k\int_{t-\varepsilon\tau(t)}^{t}\sin\Big(\frac{2\pi t}{\varepsilon}+k|x(s)+d(t)|^2\Big)(x^T(s)+d^T(t))\dot{x}(s)ds\right].
\end{aligned}$$

Using $Y_{\tau 1}(t)$ and $Y_{d1}(t)$ defined in (13), we rewrite system (8) as

$$\dot{x}(t)=A(t)x(t)+Y_{\tau 1}(t)+Y_{d1}(t)+v(t)+\frac{\sqrt{2\pi\alpha}}{\sqrt{\varepsilon}}B_0\cos\Big(\frac{2\pi t}{\varepsilon}+k|x(t)+d(t)|^2\Big),\quad t\geq 0. \tag{A1}$$

Inspired by References 16 and 19, we integrate both sides of system (A1) over $[t-\varepsilon,t]$ for $t\geq\varepsilon+\varepsilon\tau_M$, that is,

$$\begin{aligned}
\frac{x(t)-x(t-\varepsilon)}{\varepsilon}=\frac{1}{\varepsilon}\int_{t-\varepsilon}^{t}\Bigg[&A(s)x(s)+Y_{\tau 1}(s)+Y_{d1}(s)+v(s)\\
&+\frac{\sqrt{2\pi\alpha}}{\sqrt{\varepsilon}}B_0\cos\Big(\frac{2\pi s}{\varepsilon}+k|x(s)+d(s)|^2\Big)\Bigg]ds,\quad t\geq\varepsilon+\varepsilon\tau_M.
\end{aligned}\tag{A2}$$

If we follow the existing time-delay transformation,[19,27,32] we can present the left-hand side of (A2) as

$$\begin{aligned}
\frac{x(t)-x(t-\varepsilon)}{\varepsilon}=\frac{d}{dt}\Big[&x(t)+\tilde{G}(t)]-A(t)x(t)-Y_{\tau 1}(t)-Y_{d1}(t)-v(t)\\
&+\frac{1}{\varepsilon}\int_{t-\varepsilon}^{t}[A(s)x(s)+Y_{\tau 1}(s)+Y_{d1}(s)+v(s)\Big]ds,\quad t\geq\varepsilon+\varepsilon\tau_M,
\end{aligned}\tag{A3}$$

where

$$\tilde{G}(t) = -\frac{\sqrt{2\pi\alpha}}{\varepsilon\sqrt{\varepsilon}}\int_{t-\varepsilon}^{t}(s-t+\varepsilon)B_0\cos\left(\frac{2\pi s}{\varepsilon}+k|x(s)+d(s)|^2\right)ds. \tag{A4}$$

This choice of $\tilde{G}(t)$ leads to $\dot{d}$ in the analysis. Indeed, using (A2) and (A3) one obtains

$$\frac{d}{dt}[x(t)+\tilde{G}(t)] = A(t)x(t) + \frac{\sqrt{2\pi\alpha}}{\varepsilon\sqrt{\varepsilon}}\int_{t-\varepsilon}^{t}B_0\cos\left(\frac{2\pi s}{\varepsilon}+k|x(s)+d(s)|^2\right)ds$$
$$+ Y_{\tau 1}(t) + Y_{d1}(t) + v(t), \quad t \geq \varepsilon + \varepsilon\tau_M. \tag{A5}$$

It is clear that when using $\tilde{G}(t)$ defined by (A4), one will get the second term on the right-hand side of (A5). For this term, similar to Reference 27 one may further present

$$\begin{aligned}
&\frac{\sqrt{2\pi\alpha}}{\varepsilon\sqrt{\varepsilon}}\int_{t-\varepsilon}^{t}B_0\cos\left(\frac{2\pi s}{\varepsilon}+k|x(s)+d(s)|^2\right)ds\\
&= \frac{\sqrt{2\pi\alpha}}{\varepsilon\sqrt{\varepsilon}}\int_{t-\varepsilon}^{t}B_0\left[\cos\left(\frac{2\pi s}{\varepsilon}+k|x(s)+d(s)|^2\right) \pm \cos\left(\frac{2\pi s}{\varepsilon}+k|x(t)+d(t)|^2\right)\right]ds\\
&= \frac{2\sqrt{2\pi\alpha}k}{\varepsilon\sqrt{\varepsilon}}\int_{t-\varepsilon}^{t}\int_{s}^{t}B_0\sin\left(\frac{2\pi s}{\varepsilon}+k|x(\theta)+d(\theta)|^2\right)\left(x^T(\theta)+d^T(\theta)\right)\left(\dot{x}(\theta)+\dot{d}(\theta)\right)d\theta ds.
\end{aligned} \tag{A6}$$

Thus, as in References 26 and 27 one needs to impose an additional assumption on the derivative of $d(t)$ (which should be small) such that the $\dot{d}(t)$-perturbation does not ruin the stability.

To avoid $\dot{d}$ in the stability analysis, we modify $\tilde{G}(t)$ with $d(s)$ inside of integral to $G(t)$ in (13) with $d(t)$ inside of integral. By simple calculations, we obtain

$$\begin{aligned}
\frac{d}{dt}[x(t)+\tilde{G}(t)] &= \frac{d}{dt}[x(t)+\tilde{G}(t)\pm G(t)]\\
&= \frac{d}{dt}[x(t)+G(t)] + \frac{\sqrt{2\pi\alpha}}{\varepsilon\sqrt{\varepsilon}}\int_{t-\varepsilon}^{t}B_0\left[\cos\left(\frac{2\pi s}{\varepsilon}+k|x(s)+d(s)|^2\right) - \cos\left(\frac{2\pi s}{\varepsilon}+k|x(s)+d(t)|^2\right)\right]ds.
\end{aligned}$$

The latter together with (A5) yields

$$\frac{d}{dt}[x(t)+G(t)] = A(t)x(t) + \frac{\sqrt{2\pi\alpha}}{\varepsilon\sqrt{\varepsilon}}\int_{t-\varepsilon}^{t}B_0(s)\cos\left(\frac{2\pi s}{\varepsilon}+k|x(s)+d(t)|^2\right)ds$$
$$+ Y_{\tau 1}(t) + Y_{d1}(t) + v(t), \quad t \geq \varepsilon + \varepsilon\tau_M. \tag{A7}$$

It is seen that using the novel term $G(t)$ leads to the second term on the right-hand side of (A7) (and thus, (A8) below) with no terms depending on the derivative of $d(\cdot)$ (to be compared the $\dot{d}$-terms in (A6) when using $\tilde{G}(t)$).

We next present

$$\begin{aligned}
&\frac{\sqrt{2\pi\alpha}}{\varepsilon\sqrt{\varepsilon}}\int_{t-\varepsilon}^{t}B_0\cos\left(\frac{2\pi s}{\varepsilon}+k|x(s)+d(t)|^2\right)ds\\
&= \frac{\sqrt{2\pi\alpha}}{\varepsilon\sqrt{\varepsilon}}\int_{t-\varepsilon}^{t}B_0\left[\cos\left(\frac{2\pi s}{\varepsilon}+k|x(s)+d(t)|^2\right) \pm \cos\left(\frac{2\pi s}{\varepsilon}+k|x(t)+d(t)|^2\right)\right]ds\\
&= \frac{2\sqrt{2\pi\alpha}k}{\varepsilon\sqrt{\varepsilon}}\int_{t-\varepsilon}^{t}\int_{s}^{t}\sin\left(\frac{2\pi s}{\varepsilon}+k|x(\theta)+d(t)|^2\right)B_0x^T(\theta)\dot{x}(\theta)d\theta ds + Y_{d2}(t)\\
&= \frac{4\pi\alpha k}{\varepsilon^2}\int_{t-\varepsilon}^{t}\int_{s}^{t}\sin\left(\frac{2\pi s}{\varepsilon}+k|x(\theta)+d(t)|^2\right)\cos\left(\frac{2\pi\theta}{\varepsilon}+k|x(\theta)+d(\theta)|^2\right)\\
&\quad\times B_0B_0^Tx(\theta)d\theta ds + Y_1(t) + Y_{\tau 2}(t) + Y_{d2}(t) + Y_{d3}(t) + Y_v(t),
\end{aligned} \tag{A8}$$

where in the third equality we used $\int_{t-\varepsilon}^{t} \cos(\frac{2\pi s}{\varepsilon} + k|x(t) + d(t)|^2)ds = 0$ and in the fourth equality we substituted the right-hand side of (A1) for $\dot{x}(t)$.

We have

$$
\begin{aligned}
&\frac{4\pi\alpha k}{\varepsilon^2}\int_{t-\varepsilon}^{t}\int_{s}^{t}\sin\Big(\frac{2\pi s}{\varepsilon}+k|x(\theta)+d(t)|^2\Big)\cos\Big(\frac{2\pi\theta}{\varepsilon}+k|x(\theta)+d(\theta)|^2\Big)B_0B_0^Tx(\theta)d\theta ds\\
&=\frac{4\pi\alpha k}{\varepsilon^2}\int_{t-\varepsilon}^{t}\int_{s}^{t}\sin\Big(\frac{2\pi s}{\varepsilon}+k|x(\theta)+d(t)|^2\Big)\Big[\cos\Big(\frac{2\pi\theta}{\varepsilon}+k|x(\theta)+d(\theta)|^2\Big)\\
&\pm\cos\Big(\frac{2\pi\theta}{\varepsilon}+k|x(\theta)+d(t)|^2\Big)\Big]B_0B_0^Tx(\theta)d\theta ds\\
&=\frac{4\pi\alpha k}{\varepsilon^2}\int_{t-\varepsilon}^{t}\int_{s}^{t}\sin\Big(\frac{2\pi s}{\varepsilon}+k|x(\theta)+d(t)|^2\Big)\cos\Big(\frac{2\pi\theta}{\varepsilon}+k|x(\theta)+d(t)|^2\Big)B_0B_0^Tx(\theta)d\theta ds+Y_{d4}(t),
\end{aligned}
\tag{A9}
$$

where $Y_{d4}(t)$ is given in (13). Note that the following holds:

$$
\begin{aligned}
&\sin\Big(\frac{2\pi s}{\varepsilon}+k|x(\theta)+d(t)|^2\Big)\cos\Big(\frac{2\pi\theta}{\varepsilon}+k|x(\theta)+d(t)|^2\Big)\\
&=\sin\Big(\frac{2\pi s}{\varepsilon}+k|x(\theta)+d(t)|^2\Big)\cos\Big(\frac{2\pi\theta}{\varepsilon}+k|x(\theta)+d(t)|^2\Big)\\
&\quad\pm\sin\Big(\frac{2\pi s}{\varepsilon}+k|x(t)+d(t)|^2\Big)\cos\Big(\frac{2\pi\theta}{\varepsilon}+k|x(t)+d(t)|^2\Big)\\
&=\sin\Big(\frac{2\pi s}{\varepsilon}+k|x(t)+d(t)|^2\Big)\cos\Big(\frac{2\pi\theta}{\varepsilon}+k|x(t)+d(t)|^2\Big)\\
&\quad-2k\int_{\theta}^{t}\Big[\cos\Big(\frac{2\pi s}{\varepsilon}+k|x(\xi)+d(t)|^2\Big)\cos\Big(\frac{2\pi\theta}{\varepsilon}+k|x(\xi)+d(t)|^2\Big)\\
&\quad-\sin\Big(\frac{2\pi s}{\varepsilon}+k|x(\xi)+d(t)|^2\Big)\sin\Big(\frac{2\pi\theta}{\varepsilon}+k|x(\xi)+d(t)|^2\Big)\Big](x^T(\xi)+d^T(t))\dot{x}(\xi)d\xi\\
&=\sin\Big(\frac{2\pi s}{\varepsilon}+k|x(t)+d(t)|^2\Big)\cos\Big(\frac{2\pi\theta}{\varepsilon}+k|x(t)+d(t)|^2\Big)\\
&\quad-2k\int_{\theta}^{t}\cos\Big(\frac{2\pi}{\varepsilon}(s+\theta)+2k|x(\xi)+d(t)|^2\Big)(x^T(\xi)+d^T(t))\dot{x}(\xi)d\xi,
\end{aligned}
\tag{A10}
$$

$$
x(\theta)=x(\theta)\pm x(t)=x(t)-\int_{\theta}^{t}\dot{x}(\xi)d\xi,
\tag{A11}
$$

$$
\begin{aligned}
&\frac{4\pi}{\varepsilon^2}\int_{t-\varepsilon}^{t}\int_{s}^{t}\sin\Big(\frac{2\pi s}{\varepsilon}+k|x(t)+d(t)|^2\Big)\cos\Big(\frac{2\pi\theta}{\varepsilon}+k|x(t)+d(t)|^2\Big)d\theta ds\\
&=\frac{2}{\varepsilon}\int_{t-\varepsilon}^{t}\sin\Big(\frac{2\pi s}{\varepsilon}+k|x(t)+d(t)|^2\Big)\Big[\sin\Big(\frac{2\pi t}{\varepsilon}+k|x(t)+d(t)|^2\Big)-\sin\Big(\frac{2\pi s}{\varepsilon}+k|x(t)+d(t)|^2\Big)\Big]ds\\
&=-\frac{2}{\varepsilon}\int_{t-\varepsilon}^{t}\sin^2\Big(\frac{2\pi s}{\varepsilon}+k|x(t)+d(t)|^2\Big)ds\\
&=-\frac{1}{\varepsilon}\int_{t-\varepsilon}^{t}\Big(1-\cos\Big(\frac{4\pi s}{\varepsilon}+2k|x(t)+d(t)|^2\Big)ds=-1.
\end{aligned}
\tag{A12}
$$

Then, using $Y_2(t)$ and $Y_{d5}(t)$ given in (13) we further present the first term in the last equality of (A9) as

$$
\begin{aligned}
&\frac{4\pi\alpha k}{\varepsilon^2}\int_{t-\varepsilon}^{t}\int_{s}^{t}\sin\Big(\frac{2\pi s}{\varepsilon}+k|x(\theta)+d(t)|^2\Big)\cos\Big(\frac{2\pi\theta}{\varepsilon}+k|x(\theta)+d(t)|^2\Big)B_0B_0^Tx(\theta)d\theta ds\\
&\overset{(A10)}{=}\frac{4\pi\alpha k}{\varepsilon^2}\int_{t-\varepsilon}^{t}\int_{s}^{t}B_0B_0^Tx(\theta)\Big[\sin\Big(\frac{2\pi s}{\varepsilon}+k|x(t)+d(t)|^2\Big)\cos\Big(\frac{2\pi\theta}{\varepsilon}+k|x(t)+d(t)|^2\Big)\\
&-2k\int_{\theta}^{t}\cos\Big(\frac{2\pi}{\varepsilon}(s+\theta)+2k|x(\xi)+d(t)|^2\Big)x^T(\xi)\dot{x}(\xi)d\xi\Big]d\theta ds+Y_{d5}(t)
\end{aligned}
$$

$$\overset{(A11)}{=} -\frac{8\pi\alpha k^2}{\varepsilon^2}\int_{t-\varepsilon}^{t}\int_{s}^{t}\int_{\theta}^{t}\cos\Big(\frac{2\pi}{\varepsilon}(s+\theta)+2k|x(\xi)+d(t)|^2\Big)B_0B_0^Tx(\theta)x^T(\xi)\dot{x}(\xi)d\xi d\theta ds$$

$$+\frac{4\pi\alpha k}{\varepsilon^2}\int_{t-\varepsilon}^{t}\int_{s}^{t}\sin\Big(\frac{2\pi s}{\varepsilon}+k|x(t)+d(t)|^2\Big)\cos\Big(\frac{2\pi\theta}{\varepsilon}+k|x(t)+d(t)|^2\Big)B_0B_0^T\left[x(t)-\int_{\theta}^{t}\dot{x}(\xi)d\xi\right]d\theta ds+Y_{d5}(t)$$

$$\overset{(A12)}{=} -\alpha kB_0B_0^Tx(t)+Y_2(t)+Y_{d5}(t). \tag{A13}$$

Substituting (A13), into (A9), into (A8) and further into (A7), we transform (8) to the time-delay system (12).

APPENDIX B. PROOF OF THEOREM 1

Assume as in References 20 and 23 that

$$|x(t)|<\sigma \quad \forall t\geq 0 \tag{B1}$$

holds for solutions of system (8). Denote $x_t(\theta)=x(t+\theta)$, $\theta\in[-\varepsilon\tau_M,0]$. From (8), it follows that

$$x_t(\theta)=\begin{cases}\phi(t+\theta), & t+\theta<0,\\ \phi(0)+\int_0^{t+\theta}[A(s)x(s)+\frac{\sqrt{2\pi\alpha}}{\sqrt{\varepsilon}}B(s)\cos(\frac{2\pi s}{\varepsilon}+k|x(s-\varepsilon\tau(s))+d(s)|^2)ds, & t+\theta\geq 0.\end{cases}$$

The latter together with (6), (7) and (16) implies

$$\begin{aligned}\|x_t\|_{C[-\varepsilon\tau_M,0]} &\leq \|\phi\|_{C[-\varepsilon\tau_M,0]}+\left(\frac{1}{\sqrt{\varepsilon}}\vartheta_1+\vartheta_4\right)t+a\int_0^t|x(s)|ds\\ &\leq \|\phi\|_{C[-\varepsilon\tau_M,0]}+(\sqrt{\varepsilon}\vartheta_1+\varepsilon\vartheta_4)(1+\tau_M)+a\int_0^t\|x_s\|_{C[-\varepsilon\tau_M,0]}ds, \quad t\in[0,\varepsilon+\varepsilon\tau_M],\end{aligned}$$

which by Gronwall's inequality yields

$$|x(t)|\leq\|x_t\|_{C[-\varepsilon\tau_M,0]}\leq e^{at}[\|\phi\|_{C[-\varepsilon\tau_M,0]}+(\sqrt{\varepsilon}\vartheta_1+\varepsilon\vartheta_4)(1+\tau_M)], \quad t\in[0,\varepsilon+\varepsilon\tau_M]. \tag{B2}$$

Then under the initial condition $\|\phi\|_{C[-\varepsilon\tau_M,0]}\leq\sigma_0$, inequality (21) follows from (B2) since (19) implies

$$e^{a\varepsilon(1+\tau_M)}[\sigma_0+(\sqrt{\varepsilon}\vartheta_1+\varepsilon\vartheta_4)(1+\tau_M)]<\sigma$$

for all $\varepsilon\in(0,\varepsilon^*]$.

We next prove the first inequality of (21). The solution of system (15) is given by

$$\begin{aligned}z(t)=&\,e^{\int_{\varepsilon+\varepsilon\tau_M}^{t}(A_{av}+\Delta A(\theta))d\theta}z(\varepsilon+\varepsilon\tau_M)+\int_{\varepsilon+\varepsilon\tau_M}^{t}e^{\int_s^t(A_{av}+\Delta A(\theta))d\theta}\Bigg[-(A_{av}+\Delta A(s))G(s)\\&+\sum_{i=1}^{2}(Y_i(s)+Y_{\tau i}(s))+\sum_{i=1}^{5}Y_{di}(s)+Y_v(s)+v(s)\Bigg]ds, \quad t\geq\varepsilon+\varepsilon\tau_M\end{aligned}$$

leading to

$$\begin{aligned}|z(t)|\leq&\,\|e^{\int_{\varepsilon+\varepsilon\tau_M}^{t}(A_{av}+\Delta A(\theta))d\theta}\||z(\varepsilon+\varepsilon\tau_M)|+\int_{\varepsilon+\varepsilon\tau_M}^{t}\|e^{\int_s^t(A_{av}+\Delta A(\theta))d\theta}\|\Bigg[|(A_{av}+\Delta A(s))G(s)|\\&+\sum_{i=1}^{2}(|Y_i(s)|+|Y_{\tau i}(s)|)+\sum_{i=1}^{5}|Y_{di}(s)|+|Y_v(s)|+|v(s)|\Bigg]ds, \quad t\geq\varepsilon+\varepsilon\tau_M.\end{aligned} \tag{B3}$$

Note that from (6)–(8), (16) and (B1), it follows that

$$
\begin{aligned}
|\dot{x}(t)| &= \left| A(t)x(t) + \frac{\sqrt{2\pi\alpha}}{\sqrt{\varepsilon}} B(t)\cos\left(\frac{2\pi t}{\varepsilon} + k|x(t-\varepsilon\tau(t))|^2\right)\right| \\
&< a\sigma + \frac{\sqrt{2\pi\alpha}}{\sqrt{\varepsilon}}(|B_0| + \sqrt{\varepsilon}\Delta b) \\
&= \frac{1}{\sqrt{\varepsilon}}\vartheta_1 + \vartheta_3, \quad t \geq 0.
\end{aligned} \tag{B4}
$$

By using (5)–(7), (9), (13), (16), (B1), and (B4), we obtain for all $t \geq \varepsilon + \varepsilon\tau_M$

$$
\begin{aligned}
|(A_{av} + \Delta A(t))G(t)| &= \frac{\sqrt{2\pi\alpha}}{\varepsilon\sqrt{\varepsilon}}\left|\int_{t-\varepsilon}^{t}(s-t+\varepsilon)(A_{av} + \Delta A(t))B_0\cos\left(\frac{2\pi s}{\varepsilon} + k|x(s) + d(t)|^2\right)ds\right| \\
&\leq \frac{\sqrt{2\pi\alpha}}{\varepsilon\sqrt{\varepsilon}}(\|A_{av}\| + \Delta a)|B_0|\int_{t-\varepsilon}^{t}(s-t+\varepsilon)ds \\
&= \frac{\sqrt{\varepsilon}}{2}\vartheta_1(\|A_{av}\| + \Delta a),
\end{aligned} \tag{B5}
$$

$$
\begin{aligned}
|Y_1(t)| &= \frac{2\sqrt{2\pi\alpha}k}{\varepsilon\sqrt{\varepsilon}}\left|\int_{t-\varepsilon}^{t}\int_{s}^{t}\sin\left(\frac{2\pi s}{\varepsilon} + k|x(\theta) + d(t)|^2\right)B_0x^T(\theta)A(\theta)x(\theta)d\theta ds\right| \\
&< \frac{2\sqrt{2\pi\alpha}ka\sigma^2}{\varepsilon\sqrt{\varepsilon}}|B_0|\int_{t-\varepsilon}^{t}\int_{s}^{t}d\theta ds \\
&= \sqrt{\varepsilon}ka\sigma^2\vartheta_1,
\end{aligned} \tag{B6}
$$

$$
\begin{aligned}
|Y_2(t)| &= \frac{4\pi\alpha k}{\varepsilon^2}\left|\int_{t-\varepsilon}^{t}\int_{s}^{t}\int_{\theta}^{t}\left[2k\cos\left(\frac{2\pi}{\varepsilon}(s+\theta) + 2k|x(\xi) + d(t)|^2\right)x(\theta)x^T(\xi)\right.\right. \\
&\quad \left.\left. + \sin\left(\frac{2\pi s}{\varepsilon} + k|x(t) + d(t)|^2\right)\cos\left(\frac{2\pi\theta}{\varepsilon} + k|x(t) + d(t)|^2\right)\right]B_0B_0^T\dot{x}(\xi)d\xi d\theta ds\right| \\
&< \frac{4\pi\alpha k}{\varepsilon^2\sqrt{\varepsilon}}|B_0|^2(1 + 2k\sigma^2)(\vartheta_1 + \sqrt{\varepsilon}\vartheta_3)\int_{t-\varepsilon}^{t}\int_{s}^{t}\int_{\theta}^{t}d\xi d\theta ds \\
&= \frac{\sqrt{\varepsilon}}{3}\vartheta_1^2\vartheta_2(\vartheta_1 + \sqrt{\varepsilon}\vartheta_3),
\end{aligned} \tag{B7}
$$

$$
\begin{aligned}
|Y_{\tau 1}(t)| &= \frac{2\sqrt{2\pi\alpha}k}{\sqrt{\varepsilon}}\left|\int_{t-\varepsilon\tau(t)}^{t}\sin\left(\frac{2\pi t}{\varepsilon} + k|x(s) + d(t)|^2\right)B_0x^T(s)\dot{x}(s)ds\right| \\
&< \frac{2\sqrt{2\pi\alpha}k\sigma}{\varepsilon}|B_0|(\vartheta_1 + \sqrt{\varepsilon}\vartheta_3)\int_{t-\varepsilon\tau_M}^{t}ds \\
&= 2\tau_M k\sigma\vartheta_1(\vartheta_1 + \sqrt{\varepsilon}\vartheta_3),
\end{aligned} \tag{B8}
$$

$$
\begin{aligned}
|Y_{\tau 2}(t)| &= \frac{8\pi\alpha k^2}{\varepsilon^2}\left|\int_{t-\varepsilon}^{t}\int_{s}^{t}\int_{\theta-\varepsilon\tau(\theta)}^{\theta}\sin\left(\frac{2\pi s}{\varepsilon} + k|x(\theta) + d(t)|^2\right)\sin\left(\frac{2\pi\theta}{\varepsilon} + k|x(\xi) + d(\theta)|^2\right)B_0B_0^Tx(\theta)x^T(\xi)\dot{x}(\xi)d\xi d\theta ds\right| \\
&< \frac{8\pi\alpha k^2\sigma^2}{\varepsilon^2\sqrt{\varepsilon}}|B_0|^2(\vartheta_1 + \sqrt{\varepsilon}\vartheta_3)\int_{t-\varepsilon}^{t}\int_{s}^{t}\int_{\theta-\varepsilon\tau_M}^{\theta}d\xi d\theta ds \\
&= 2\sqrt{\varepsilon}\tau_M k^2\sigma^2\vartheta_1^2(\vartheta_1 + \sqrt{\varepsilon}\vartheta_3),
\end{aligned} \tag{B9}
$$

$$
|Y_{d1}(t)| = \frac{2\sqrt{2\pi\alpha}k}{\sqrt{\varepsilon}}\left|\int_{t-\varepsilon\tau(t)}^{t}\sin\left(\frac{2\pi t}{\varepsilon} + k|x(s) + d(t)|^2\right)B_0d^T(t)\dot{x}(s)ds\right|
$$

$$
\begin{aligned}
&< \frac{2d^*\sqrt{2\pi\alpha}k}{\varepsilon}|B_0|(\vartheta_1+\sqrt{\varepsilon}\vartheta_3)\int_{t-\varepsilon\tau_M}^{t}ds \\
&= 2\tau_M d^* k\vartheta_1(\vartheta_1+\sqrt{\varepsilon}\vartheta_3),
\end{aligned} \tag{B10}
$$

$$
\begin{aligned}
|Y_{d2}(t)| &= \frac{2\sqrt{2\pi\alpha}k}{\varepsilon\sqrt{\varepsilon}}\left|\int_{t-\varepsilon}^{t}\int_{s}^{t}\sin\left(\frac{2\pi s}{\varepsilon}+k|x(\theta)+d(t)|^2\right)B_0 d^T(t)\dot{x}(\theta)d\theta ds\right| \\
&< \frac{2d^*k}{\varepsilon^2}\vartheta_1\left(\vartheta_1+\sqrt{\varepsilon}\vartheta_3\right)\int_{t-\varepsilon}^{t}\int_{s}^{t}dsd\theta \\
&= d^*k\vartheta_1\left(\vartheta_1+\sqrt{\varepsilon}\vartheta_3\right),
\end{aligned} \tag{B11}
$$

$$
\begin{aligned}
|Y_{d3}(t)| &= \frac{8\pi\alpha k^2}{\varepsilon^2}\left|\int_{t-\varepsilon}^{t}\int_{s}^{t}\int_{\theta-\varepsilon\tau(\theta)}^{\theta}\sin\left(\frac{2\pi s}{\varepsilon}+k|x(\theta)+d(t)|^2\right)\sin\left(\frac{2\pi\theta}{\varepsilon}+k|x(\xi)+d(\theta)|^2\right)B_0B_0^Tx(\theta)d^T(\theta)\dot{x}(\xi)d\xi d\theta ds\right| \\
&< \frac{4d^*\sigma k^2}{\varepsilon^2\sqrt{\varepsilon}}\vartheta_1^2\left(\vartheta_1+\sqrt{\varepsilon}\vartheta_3\right)\int_{t-\varepsilon}^{t}\int_{s}^{t}\int_{\theta-\varepsilon\tau_M}^{\theta}d\xi d\theta ds \\
&= 2\sqrt{\varepsilon}\tau_M d^*\sigma k^2\vartheta_1^2\left(\vartheta_1+\sqrt{\varepsilon}\vartheta_3\right),
\end{aligned} \tag{B12}
$$

$$
\begin{aligned}
|Y_{d4}(t)| &= \frac{4\pi\alpha k^2}{\varepsilon^2}\left|\int_{t-\varepsilon}^{t}\int_{s}^{t}\int_{|x(\theta)+d(\theta)|^2}^{|x(\theta)+d(t)|^2}\sin\left(\frac{2\pi s}{\varepsilon}+k|x(\theta)+d(t)|^2\right)\sin\left(\frac{2\pi\theta}{\varepsilon}+k\xi\right)B_0B_0^Tx(\theta)d\xi d\theta ds\right| \\
&< \frac{4\pi\alpha\sigma k^2}{\varepsilon^2}|B_0|^2\left|\int_{t-\varepsilon}^{t}\int_{s}^{t}\int_{|x(\theta)+d(\theta)|^2}^{|x(\theta)+d(t)|^2}d\xi d\theta ds\right| \\
&< 2\sigma k^2\vartheta_1^2(d^*-|d_0|)(2\sigma+|d_0|+d^*),
\end{aligned} \tag{B13}
$$

$$
\begin{aligned}
|Y_{d5}(t)| &= \frac{8\pi\alpha k^2}{\varepsilon^2}\left|\int_{t-\varepsilon}^{t}\int_{s}^{t}\int_{\theta}^{t}\cos\left(\frac{2\pi}{\varepsilon}(s+\theta)+2k|x(\xi)+d(t)|^2\right)B_0B_0^Tx(\theta)d^T(t)\dot{x}(\xi)d\xi d\theta ds\right| \\
&< \frac{8d^*\pi\alpha\sigma k^2}{\varepsilon^2\sqrt{\varepsilon}}|B_0|^2\left(\vartheta_1+\sqrt{\varepsilon}\vartheta_3\right)\int_{t-\varepsilon}^{t}\int_{s}^{t}\int_{\theta}^{t}d\xi d\theta ds \\
&= \frac{2\sqrt{\varepsilon}d^*\sigma k^2}{3}\vartheta_1^2(\vartheta_1+\sqrt{\varepsilon}\vartheta_3),
\end{aligned} \tag{B14}
$$

$$
\begin{aligned}
|Y_v(t)| &= \frac{4\pi\alpha k}{\varepsilon\sqrt{\varepsilon}}\left|\int_{t-\varepsilon}^{t}\int_{s}^{t}\sin\left(\frac{2\pi s}{\varepsilon}+k|x(\theta)+d(t)|^2\right)\cos\left(\frac{2\pi\theta}{\varepsilon}+k|x(\theta-\varepsilon\tau(\theta))+d(\theta)|^2\right)B_0\Delta B^T(\theta)x(\theta)d\theta ds\right| \\
&< \frac{4\pi\alpha k\sigma\Delta b}{\varepsilon\sqrt{\varepsilon}}|B_0|\times\int_{t-\varepsilon}^{t}\int_{s}^{t}d\theta ds \\
&= \sqrt{\varepsilon}k\sigma\vartheta_1\vartheta_4,
\end{aligned} \tag{B15}
$$

$$
|v(t)| = \sqrt{2\pi\alpha}\left|\Delta B(t)\cos\left(\frac{2\pi t}{\varepsilon}+k|x(t-\varepsilon\tau(t))+d(t)|^2\right)\right| \leq \vartheta_4. \tag{B16}
$$

By using (B3), (B5)–(B16), we obtain

$$
\begin{aligned}
|z(t)| < {}&\|e^{\int_{\varepsilon+\varepsilon\tau_M}^{t}(A_{av}+\Delta A(\theta))d\theta}\||z(\varepsilon+\varepsilon\tau_M)| + [\vartheta_1(\sqrt{\varepsilon}\kappa_0+\varepsilon\kappa_1+2\tau_M\sigma(\kappa_2+\sqrt{\varepsilon}\kappa_3+\varepsilon\kappa_4) \\
&+ d^*(\kappa_5+\sqrt{\varepsilon}\kappa_6+\varepsilon\kappa_7+d^*\kappa_8)+\kappa_9)+\vartheta_4]\int_{\varepsilon+\varepsilon\tau_M}^{t}\|e^{\int_s^t(A_{av}+\Delta A(\theta))d\theta}\|ds, \quad t\geq\varepsilon+\varepsilon\tau_M
\end{aligned} \tag{B17}
$$

where κ_i $(i=0,\ldots,10)$ are given by (20). Assuming as in Reference 24 that there exist scalars $\delta>0$ and $p>1$ satisfying

$$
\|e^{\int_s^t(A_{av}+\Delta A(\theta))d\theta}\| \leq \sqrt{p}e^{-\delta(t-s)} \quad \forall t\geq s\geq\varepsilon+\varepsilon\tau_M, \tag{B18}
$$

from (B17) we obtain

$$\begin{aligned}|z(t)| &< \sqrt{p}e^{-\delta(t-\varepsilon-\varepsilon\tau_M)}|z(\varepsilon+\varepsilon\tau_M)| + \Big[\vartheta_1\Big(\sqrt{\varepsilon}\kappa_0+\varepsilon\kappa_1+2\tau_M\sigma\Big(\kappa_2+\sqrt{\varepsilon}\kappa_3 \\ &\quad + \varepsilon\kappa_4)+d^*\Big(\kappa_5+\sqrt{\varepsilon}\kappa_6+\varepsilon\kappa_7+d^*\kappa_8\Big)+\kappa_9\Big)+\vartheta_4\Big]\int_{\varepsilon+\varepsilon\tau_M}^{t}\sqrt{p}e^{-\delta(t-s)}ds \\ &\le \sqrt{p}e^{-\delta(t-\varepsilon-\varepsilon\tau_M)}|z(\varepsilon+\varepsilon\tau_M)| + \frac{\sqrt{p}}{\delta}\Big[\vartheta_1\Big(\sqrt{\varepsilon}\kappa_0+\varepsilon\kappa_1+2\tau_M\sigma(\kappa_2 \\ &\quad + \sqrt{\varepsilon}\kappa_3+\varepsilon\kappa_4\Big)+d^*\Big(\kappa_5+\sqrt{\varepsilon}\kappa_6+\varepsilon\kappa_7+d^*\kappa_8\Big)+\kappa_9\Big)+\vartheta_4\Big], \quad t\ge\varepsilon+\varepsilon\tau_M. \end{aligned} \tag{B19}$$

Moreover, the following holds:

$$|x(t)| \overset{(14)}{=} |z(t)-G(t)| \le |z(t)|+|G(t)| \le |z(t)|+\frac{\sqrt{\varepsilon}}{2}\vartheta_1, \quad t\ge\varepsilon+\varepsilon\tau_M, \tag{B20}$$

$$|z(t)| \overset{(14)}{=} |x(t)+G(t)| \le |x(t)|+|G(t)| \le |x(t)|+\frac{\sqrt{\varepsilon}}{2}\vartheta_1, \quad t\ge\varepsilon+\varepsilon\tau_M. \tag{B21}$$

Thus, we arrive at

$$\begin{aligned}|x(t)| &\overset{(B20)}{<} \sqrt{p}e^{-\delta(t-\varepsilon-\varepsilon\tau_M)}|z(\varepsilon+\varepsilon\tau_M)| + \frac{\sqrt{p}}{\delta}\Big[\vartheta_1\Big(\sqrt{\varepsilon}\kappa_0+\varepsilon\kappa_1+2\tau_M\sigma(\kappa_2 \\ &\quad + \sqrt{\varepsilon}\kappa_3+\varepsilon\kappa_4\Big)+d^*\Big(\kappa_5+\sqrt{\varepsilon}\kappa_6+\varepsilon\kappa_7+d^*\kappa_8\Big)+\kappa_9\Big)+\vartheta_4\Big]+\frac{\sqrt{\varepsilon}}{2}\vartheta_1 \\ &\overset{(B21)}{<} \sqrt{p}e^{-\delta(t-\varepsilon-\varepsilon\tau_M)}\left(|x(\varepsilon+\varepsilon\tau_M)|+\frac{\sqrt{\varepsilon}}{2}\vartheta_1\right) + \frac{\sqrt{p}}{\delta}\Big[\vartheta_1\Big(\sqrt{\varepsilon}\kappa_0+\varepsilon\kappa_1+2\tau_M\sigma(\kappa_2 \\ &\quad + \sqrt{\varepsilon}\kappa_3+\varepsilon\kappa_4\Big)+d^*\Big(\kappa_5+\sqrt{\varepsilon}\kappa_6+\varepsilon\kappa_7+d^*\kappa_8\Big)+\kappa_9\Big)+\vartheta_4\Big]+\frac{\sqrt{\varepsilon}}{2}\vartheta_1 \\ &\overset{(B2)}{<} \sqrt{p}e^{-\delta(t-\varepsilon-\varepsilon\tau_M)}\left[e^{\alpha\varepsilon(1+\tau_M)}\Big(\|\phi\|_{C[-\varepsilon\tau_M,0]}+\Big(\sqrt{\varepsilon}\vartheta_1+\varepsilon\sqrt{2\pi\alpha}\Delta b\Big)(1+\tau_M)\Big)+\frac{\sqrt{\varepsilon}}{2}\vartheta_1\right] \\ &\quad + \frac{\sqrt{p}}{\delta}\Big[\vartheta_1\Big(\sqrt{\varepsilon}\kappa_0+\varepsilon\kappa_1+2\tau_M\sigma\Big(\kappa_2+\sqrt{\varepsilon}\kappa_3+\varepsilon\kappa_4\Big)+d^*\Big(\kappa_5+\sqrt{\varepsilon}\kappa_6+\varepsilon\kappa_7 \\ &\quad + d^*\kappa_8)+\kappa_9)+\vartheta_4\Big]+\frac{\sqrt{\varepsilon}}{2}\vartheta_1, \quad t\ge\varepsilon+\varepsilon\tau_M. \end{aligned} \tag{B22}$$

This implies the second inequality of (21) for all $\varepsilon \in (0,\varepsilon^*]$ if under the initial condition $\|\phi\|_{C[-\varepsilon\tau_M,0]} \le \sigma_0$ the following holds

$$\begin{aligned}&\sqrt{p}\left[e^{\alpha\varepsilon^*(1+\tau_M)}\Big(\sigma_0+\Big(\sqrt{\varepsilon^*}\vartheta_1+\varepsilon^*\vartheta_4\Big)(\tau_M+1)\Big)+\frac{\sqrt{\varepsilon^*}}{2}\vartheta_1\right. \\ &\quad +\frac{1}{\delta}\Big(\vartheta_1\Big(\sqrt{\varepsilon^*}\kappa_0+\varepsilon^*\kappa_1+2\tau_M\sigma\Big(\kappa_2+\sqrt{\varepsilon^*}\kappa_3+\varepsilon^*\kappa_4\Big) \\ &\quad \left.+ d^*\Big(\kappa_5+\sqrt{\varepsilon^*}\kappa_6+\varepsilon^*\kappa_7+d^*\kappa_8\Big)+\kappa_9\Big)+\vartheta_4\Big)\right] < \sigma-\frac{\sqrt{\varepsilon^*}}{2}\vartheta_1. \end{aligned}$$

The latter, by squaring both sides, is equivalent to (19).

To prove (B18), we consider the following system

$$\dot{z}(t) = [A_{av}+\Delta A(t)]z(t), \quad t\ge\varepsilon+\varepsilon\tau_M. \tag{B23}$$

Choose a Lyapunov function

$$V(t) = z^T(t)Pz(t), \quad t \geq \varepsilon + \varepsilon\tau_M, \tag{B24}$$

where matrix P satisfies (17). Differentiating $V(t)$ along (B23) we obtain

$$\dot{V}(t) = z^T(t)P[A_{av} + \Delta A(t)]z(t), \quad t \geq \varepsilon + \varepsilon\tau_M. \tag{B25}$$

To compensate $\Delta A(t)z(t)$ in (B25) we apply S-procedure: we add to $\dot{V}(t)$ the left-hand part of

$$\lambda[(\Delta a)^2|z(t)|^2 - |\Delta A(t)z(t)|^2] > 0 \tag{B26}$$

with some $\lambda > 0$. Then we have

$$\dot{V}(t) + 2\delta V(t) \leq \zeta^T(t)\Xi\zeta(t) \leq 0, \quad t \geq \varepsilon + \varepsilon\tau_M, \tag{B27}$$

where $\zeta(t) = [z^T(t), z^T(t)\Delta A^T(t)]^T$ and Ξ is given by (18). Thus, $V(t) \leq e^{-2\delta(t-s)}V(s)$ for $t \geq s \geq \varepsilon + \varepsilon\tau_M$. Using the fact $|z(t)|^2 \leq V(t) \leq p|z(t)|^2$ for $t \geq \varepsilon + \varepsilon\tau_M$, we obtain

$$|z(t)| \leq \sqrt{p}e^{-\delta(t-s)}|z(s)|, \quad t \geq s \geq \varepsilon + \varepsilon\tau_M.$$

In addition, from (B23) it follows that $z(t) = e^{\int_s^t (A_{av}+\Delta A(\theta))d\theta}z(s)$, $t \geq s \geq \varepsilon + \varepsilon\tau_M$ leading to

$$|z(t)| = |e^{\int_s^t (A_{av}+\Delta A(\theta))d\theta}z(s)|, \quad t \geq s \geq \varepsilon + \varepsilon\tau_M. \tag{B28}$$

By the norm's definition, we obtain

$$\|e^{\int_s^t (A_{av}+\Delta A(\theta))d\theta}\| = \max_{|z(s)|=1} |e^{\int_s^t (A_{av}+\Delta A(\theta))d\theta}z(s)| = \max_{|z(s)|=1} |z(t)| \leq \sqrt{p}e^{-\delta(t-s)}, \quad t \geq s \geq \varepsilon + \varepsilon\tau_M,$$

that is, (B18) holds.

Finally, by following the contradiction-based arguments in Reference 20, it can be proved that inequality (19) results in (B1). This completes the proof.