form of LMIs for the same values of $W_1 = W_2 = I$ we obtained a larger value of $\mu = 0.000\,08$.

Applying Theorem 3.1 and choosing $W_0 = W_1 = I$, we find from (38)–(40)

$$U_0(0) = \begin{bmatrix} 4 & 1 \\ 1 & 1 \end{bmatrix} \quad U(0) = \begin{bmatrix} 7 & 2 \\ 2 & 3 \end{bmatrix}$$
$$\mathcal{Q} = \begin{bmatrix} 42.8234 & 2.6938 \\ 2.6938 & 0.6103 \end{bmatrix}$$

and for $\mu = 0.12$ (41a) and (41b) are feasible. Hence, the system is asymptotically stable for essentially larger interval $[0.88, 1.12]$ for a wider class of delays (which may be not differentiable).

By descriptor approach of [3], the resulting interval is wider: $\tau(t) \in [0.73, 1.27]$ with $\mu = 0.27$. By descriptor approach the system is stable and thus conditions of [3] can be applied for $h \le 254$. In this example, the conditions of [8] and of Theorem 3.1 give reliable results till $h \le 22$, while for greater values of $h$ matrix $\mathcal{B}$ becomes ill-conditioned and the resulting $U_0(0)$ is not symmetric.

## IV. Conclusion

A new Lyapunov–Krasovskii technique is developed for stability of linear system with uncertain time-varying delay in the case when the nominal value of the delay is constant and nonzero: To a "complete" nominal LKF, which is appropriate to the system with the nominal value of the delay, terms are added that correspond to the perturbed system and that vanish when the delay perturbation approaches 0. The nominal "complete" LKF is considered, the derivative of which along the trajectories of the nominal system depends on both, the state and the state derivative. Given matrices $W_0$ and $W_1$, the stability sufficient conditions are reduced to linear algebraic operations, definite integral and to LMIs. The new method is applied to the case of multiple uncertain delays with one nonsmall delay. Similarly to "complete" LKF of [8], the new "complete" LKF can be applied in the case where the nondelayed system is not asymptotically stable, but it leads to simpler and less conservative conditions. Feasibility of the latter conditions is guaranteed for small perturbations of the delay.

The conditions derived are conservative since one have to choose first $W_0$ and $W_1$ in order to verify their feasibility. Less conservative conditions may be derived by choosing $\dot{V}_n$ to be a general negative–definite quadratic form of $x(t)$ and $\dot{x}(t)$.

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# Descriptor Discretized Lyapunov Functional Method: Analysis and Design

Emilia Fridman

***Abstract*—Stability and state-feedback stabilization of linear systems with uncertain coefficients and uncertain time-varying delays are considered. The system under consideration may be unstable without delay, but it becomes asymptotically stable for positive values of the delay. A new *descriptor discretized* Lyapunov–Krasovskii functional (LKF) method is introduced, which combines the application of the complete LKF and the discretization method of K. Gu with the descriptor model transformation. For the first time, the new method allows to apply the discretized LKF method to *synthesis* problems. Moreover, the analysis of systems with polytopic time-invariant uncertainties is less restrictive by the new discretized method. Sufficient conditions for robust stability and stabilization of uncertain neutral type systems are derived in terms of linear matrix inequalities (LMIs) via input–output approach to stability. Numerical examples illustrate the efficiency of the new method.**

***Index Terms*—Linear matrix inequality (LMI), Lyapunov–Krasovskii functional (LKF), robust stability, stabilization, time-delay.**

## I. Introduction

It is well known that the choice of an appropriate Lyapunov–Krasovskii functional (LKF) is crucial for deriving stability criteria and for obtaining a solution to various robust control problems

Manuscript received April 7, 2005; revised August 18, 2005. Recommended by Associate Editor I. Kolmanovsky. This work was supported by the Kamea Fund of Israel.

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Digital Object Identifier 10.1109/TAC.2006.872828

(see, e.g., [4] and [16]). The general (complete) form of this functional for systems with constant delays, which corresponds to necessary and sufficient stability conditions, was used by many authors (see [10], [14], [15], [17], and the references therein). The complete LKF leads to a complicated system of partial differential equations. Special (reduced) forms of LKFs lead to simpler finite dimensional conditions in terms of LMIs (see, e.g., [1], [10], [18], and [19]). The necessary condition for the application of the reduced LKFs is the asymptotic stability of the nondelayed system. If the latter conditions does not hold, the complete LKF should be applied.

LMI stability conditions via complete LKF and discretization were introduced by K. Gu [8] and appeared to be very efficient, leading in some examples to results close to analytical ones. The discretised LKF method of Gu has been applied to robust stability analysis of linear retarded and neutral type systems [9]–[11]. No design problems have been solved by this method. This is due to some terms in the LKF derivative condition, which arise after substitution of $\dot{x}(t)$ by the right hand side of the system. These terms are the products of the system matrices with the different decision variables of LMIs. Thus in the case of state-feedback design, the closed-loop system matrices depend on the unknown controller gain and the resulting matrix inequalities contain the products of the unknown gain with different decision variables.

Another method for robust stability of uncertain systems via complete LKF has been introduced in [14], [15]: Given a LKF derivative condition, construct LKF by solving a boundary value problem for linear ordinary differential equation, and use the resulting LKF for robust stability analysis. This method is not easy to apply even to analysis of single delay systems (due to the choice of the matrices in the LKF derivative condition) and has difficulties in treating the multiple delay case. Robust stability of linear systems with norm-bounded uncertainties and uncertain time-varying delays have been also analyzed in the frequency domain via input–output approach to stability [5], [10], [12], [13]. Therefore, the analysis of systems with *polytopic type uncertainties* and the *synthesis* are the main objectives for LKF-based methods. Robust design via complete LKF is an important problem, which can provide tools for such challenging topics as stabilizing of systems by inserting delays in the feedback [10], [19], [20]. The latter problem cannot be solved via reduced LKFs, since the nondelayed closed-loop system is unstable.

In this note, we introduce a new *descriptor discretized* LKF method, which combines the application of the complete LKF and the discretization procedure of Gu [8] with the descriptor model transformation [1]. In the descriptor approach both $x(t)$ and $\dot{x}(t)$ are the state variables, which allows to avoid the mentioned above terms in the LKF derivative condition (since $\dot{x}(t)$ is not substituted everywhere by the right hand part of the system). The new method can be easily adopted to design problems. Moreover, due to the absence of the mentioned above terms, the new method has advantages in the case of systems with polytopic time-invariant uncertainties. In this note, we develop the discretized LKF method for systems with a single constant delay. Robust stability of neutral type systems with uncertain time-varying delays from given segments is studied next via application of input–output approach to stability [10], [12], [22]. State-feedback stabilization is solved. An example of using delay for static output-feedback stabilization of uncertain double integrator illustrates the advantages of the new method.

**Notation**: Throughout this note, the superscript "$T$" stands for matrix transposition, $\mathcal{R}^n$ denotes the $n$-dimensional Euclidean space with vector norm $\|\cdot\|$, $\mathcal{R}^{n\times m}$ is the set of all $n\times m$ real matrices, and the notation $P>0$, for $P\in\mathcal{R}^{n\times n}$ means that $P$ is symmetric and positive definite. The symmetric elements of the symmetric matrix will be denoted by $*$. $L_2$ is the space of square integrable functions $v:[0,\infty)\to C^n$ with the norm $\|v\|_{L_2}=\left[\int_0^\infty\|v(t)\|^2dt\right]^{1/2}$.

## II. Descriptor Discretized LKF Method: Constant Delay Case

### A. Descriptor Complete LKF

Consider a linear system

$$\dot{x}(t)=A_0x(t)+A_1x(t-r) \tag{1}$$

where $x(t)\in\mathcal{R}^n$, $r>0$ is constant time-delay, and $A_0$ and $A_1$ are constant matrices.

We apply a complete LKF of the same form as in [10]

$$\begin{aligned}V(x_t)&=x^T(t)P_1x(t)+2x^T(t)\int_{-r}^0Q(\xi)x(t+\xi)d\xi\\&+\int_{-r}^0\int_{-r}^0x^T(t+s)R(s,\xi)dsx(t+\xi)d\xi\\&+\int_{-r}^0x^T(t+\xi)S(\xi)x(t+\xi)d\xi,\quad P_1>0\end{aligned} \tag{2}$$

where $Q(\xi)\in R^{n\times n}$, $R(\xi,\eta)=R^T(\eta,\xi)\in R^{n\times n}$, $S(\xi)=S^T(\xi)\in R^{n\times n}$, and $Q$, $R$, and $S$ are continuous matrix-functions.

The novelty of our complete LKF is in the derivative condition

$$\dot{V}(x_t)\le-\varepsilon_0\|x(t)\|^2-\varepsilon_1\|\dot{x}(t)\|^2 \tag{3}$$

where $\varepsilon_0>0$ and $\varepsilon_1>0$ are some constants. The second term in the right-hand side of (3) has been taken to be zero in the existing literature (see, e.g., [10], [14], and [15]), but it is exactly this term that leads to simple design algorithms. Such derivative condition appears naturally if one applies to (1) the descriptor model transformation [1]

$$\begin{aligned}&E\frac{d}{dt}\begin{bmatrix}x(t)\\\dot{x}(t)\end{bmatrix}=\begin{bmatrix}0&I\\A_0&-I\end{bmatrix}\begin{bmatrix}x(t)\\\dot{x}(t)\end{bmatrix}+\begin{bmatrix}0\\A_1\end{bmatrix}x(t-r),\\&E=\begin{bmatrix}I&0\\0&0\end{bmatrix}\end{aligned} \tag{4}$$

and the descriptor type LKF, where the first term of (2) is represented in the form

$$x^T(t)P_1x(t)=\begin{bmatrix}x(t)\\\dot{x}(t)\end{bmatrix}^TEP\begin{bmatrix}x(t)\\\dot{x}(t)\end{bmatrix}\qquad P=\begin{bmatrix}P_1&0\\P_2&P_3\end{bmatrix}. \tag{5}$$

The existence of descriptor complete LKF (with $S\equiv0$) is a necessary and sufficient condition for the asymptotic stability of (1) [3].

Differentiating LKF (2) along (1), we have

$$\begin{aligned}\dot{V}(x_t)&=2\dot{x}^T(t)\left[P_1x(t)+\int_{-r}^0Q(\xi)x(t+\xi)d\xi\right]\\&+2x^T(t)\int_{-r}^0Q(\xi)\dot{x}(t+\xi)d\xi\\&+2\int_{-r}^0\int_{-r}^0\dot{x}^T(t+s)R(s,\xi)dsx(t+\xi)d\xi\\&+2\int_{-r}^0\dot{x}^T(t+\xi)S(\xi)x(t+\xi)d\xi.\end{aligned} \tag{6}$$

Integrating by parts in (6) and representing the first term of (6) in the form of (7), as shown at the bottom of the next page, we find

$$\begin{aligned}\dot{V}(x_t)&=\zeta^T\Xi\zeta+2\dot{x}^T(t)\int_{-r}^0Q(\xi)x(t+\xi)d\xi\\&-\int_{-r}^0\int_{-r}^0x^T(t+\xi)\left(\frac{\partial}{\partial\xi}R(\xi,\theta)+\frac{\partial}{\partial\theta}R(\xi,\theta)\right)\\&\times x(t+\theta)d\theta d\xi\\&+2x^T(t)\int_{-r}^0[-\dot{Q}(\xi)+R(0,\xi)]x(t+\xi)d\xi\\&-2x^T(t-r)\int_{-r}^0R(-r,\theta)x(t+\theta)d\theta\\&-\int_{-r}^0x^T(t+\xi)\dot{S}(\xi)x(t+\xi)d\xi\end{aligned} \tag{8}$$

where (9a) and (9b), as shown at the bottom of the next page, hold.

### *B. Discretization and Stability Criterion*

We apply the discretization of Gu [8]. Divide the delay interval $[-r, 0]$ into $N$ segments $[\theta_p, \theta_{p-1}]$, $p = 1, \ldots, N$ of equal length $h = r/N$, where $\theta_p = -ph$. This divides the square $[-r, 0] \times [-r, 0]$ into $N \times N$ small squares $[\theta_p, \theta_{p-1}] \times [\theta_q, \theta_{q-1}]$. Each small square is further divided into two triangles.

The continuous matrix functions $Q(\xi)$ and $S(\xi)$ are chosen to be linear within each segment and the continuous matrix function $R(\xi, \theta)$ is chosen to be linear within each triangular, as shown in (10) at the bottom of the page. Thus, the LKF is completely determined by $P_1, Q_p, S_p, R_{pq}, p, q = 0, 1, \ldots, N$.

The LKF condition $V(x_t) \geq \varepsilon \|x(t)\|^2$, $\varepsilon > 0$ is satisfied ([10, p. 185]) if $S_p > 0$, $p = 0, 1, \ldots, N$ and

$$\begin{bmatrix} P_1 & \tilde{Q} \\ * & \tilde{R} + \tilde{S} \end{bmatrix} > 0, \tag{11}$$

where

$$\tilde{Q} = [Q_0 \ Q_1 \ \ldots Q_N] \quad \tilde{S} = \operatorname{diag}\left\{\frac{1}{h}S_0, \frac{1}{h}S_1, \ldots, \frac{1}{h}S_N\right\}$$
$$\tilde{R} = \begin{bmatrix} R_{00} & R_{01} & \ldots & R_{0N} \\ R_{10} & R_{11} & \ldots & R_{1N} \\ \ldots & \ldots & \ldots & \ldots \\ R_{N0} & R_{N1} & \ldots & R_{NN} \end{bmatrix}. \tag{12}$$

To derive the LKF derivative condition, we note that

$$\begin{aligned} \dot{S}(\xi) &= \frac{1}{h}(S_{p-1} - S_p) \\ \dot{Q}(\xi) &= \frac{1}{h}(Q_{p-1} - Q_p) \\ \frac{\partial}{\partial \xi}R(\xi, \theta) + \frac{\partial}{\partial \theta}R(\xi, \theta) &= \frac{1}{h}(R_{p-1,q-1} - R_{pq}). \end{aligned} \tag{13}$$

Thus

$$\begin{aligned} & 2\dot{x}^T(t) \int_{-r}^{0} Q(\xi) x(t+\xi) d\xi \\ & = 2\dot{x}^T(t) \sum_{p=1}^{N} h \int_0^1 [(1-\alpha)Q_p + \alpha Q_{p-1}] x \\ & \quad \times (t + \theta_p + \alpha h) d\alpha \\ & = 2\dot{x}^T(t) \sum_{p=1}^{N} h \int_0^1 \left[(1-\alpha)\left(Q_p^s + Q_p^a\right) + \alpha\left(Q_p^s - Q_p^a\right)\right] x \\ & \quad \times (t + \theta_p + \alpha h) d\alpha \end{aligned} \tag{14}$$

where $Q_p^s = (Q_{p-1} + Q_p)/2, Q_p^a = (Q_p - Q_{p-1})/2$.

Equations (8), (9), and (13) imply (cf. [10, (5.146)–(5.164)])

$$\begin{aligned} \dot{V}(x_t) = \zeta^T \bar{\Xi} \zeta & - \int_0^1 \phi^T(\alpha) S_d \phi(\alpha) d\alpha \\ & - \int_0^1 \left[\int_0^1 \phi^T(\alpha) R_d \phi(\beta) d\alpha\right] d\beta \\ & + 2\zeta^T \int_0^1 [D^s + (1 - 2\alpha) D^a] \phi(\alpha) d\alpha \end{aligned} \tag{15}$$

where $\zeta$ is given by (9a) and (16a)–(16i), as shown at the bottom of the next page.

Applying [10, Prop. 5.21] to (15) we conclude that $\dot{V}(x_t) < 0$ if the following LMI holds:

$$\begin{bmatrix} \bar{\Xi} & D^s & D^a \\ * & -R_d - S_d & 0 \\ * & * & -3S_d \end{bmatrix} < 0. \tag{17}$$

Moreover, (17) implies that $S_0 > S_1 > \cdots S_N > 0$ (see [10, Prop. 5.22]). Hence, (17) guarantees $V(x_t) \geq \varepsilon \|x(t)\|^2$, $\varepsilon > 0$. We thus proved

*Theorem 2.1:* System (1) is asymptotically stable if there exist $n \times n$ matrices $P_1 > 0$, $P_2$, $P_3$, $S_p = S_p^T$, $Q_p$, $R_{pq} = R_{qp}^T$, $p = 0, 1, \ldots, N$, $q = 0, 1, \ldots, N$ such that LMIs (11) and (17) are satisfied with notations defined in (5), (12), and (16).

*Remark 2.1:* The descriptor complete LKF leads to LMIs, which do not contain the terms $A_0^T Q_p$ and $A_1^T Q_p$, $p = 1, \ldots, N$. Such terms appear in $D^s$ and $D^a$ of discretized LKF method of Gu (see [10, (5.159)–(5.164)]).

- The latter terms essentially complicate the *design procedure*.
- In the case of system with $A_0$ and $A_1$ from the uncertain time-invariant polytope

$$\Omega = \sum_{j=1}^{M} f_j \Omega_j \quad \text{for some} \quad 0 \leq f_j \leq 1 \quad \sum_{j=1}^{M} f_j = 1$$
$$\Omega_j = \left[\, A_0^{(j)} \quad A_1^{(j)} \,\right] \tag{18}$$

by the descriptor discretized method one have to solve the LMIs (11) and (17) simultaneously for all the $M$ vertices $\Omega_j$, applying the same matrices $P_2$ and $P_3$ and solving for the $M$ vertices. By the method of Gu not only $P_1$, but also $Q_p, p = 1, \ldots, N$ should be common for the $M$ vertices.

---

$$\begin{aligned} 2\dot{x}^T(t) P_1 x(t) = 2x^T(t) P_1 \dot{x}(t) &= 2 \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}^T P^T E \frac{d}{dt} \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} \\ &= 2 \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}^T P^T \left[ \begin{pmatrix} 0 & I \\ A_0 & -I \end{pmatrix} \begin{pmatrix} x(t) \\ \dot{x}(t) \end{pmatrix} + \begin{pmatrix} 0 \\ A_1 \end{pmatrix} x(t-r) \right] \end{aligned} \tag{7}$$

---

$$\zeta^T = \left[\, x^T(t) \quad \dot{x}^T(t) \quad x^T(t-r) \,\right] \tag{9a}$$

$$\Xi = \begin{bmatrix} P^T \begin{bmatrix} 0 & I \\ A_0 & -I \end{bmatrix} + \begin{bmatrix} 0 & A_0^T \\ I & -I \end{bmatrix} P + \begin{bmatrix} Q(0) + Q^T(0) + S(0) & 0 \\ 0 & 0 \end{bmatrix} & P^T \begin{bmatrix} 0 \\ A_1 \end{bmatrix} - \begin{bmatrix} Q(-r) \\ 0 \end{bmatrix} \\ * & -S(-r) \end{bmatrix} \tag{9b}$$

---

$$Q(\theta_p + \alpha h) = (1-\alpha)Q_p + \alpha Q_{p-1}, \ S(\theta_p + \alpha h) = (1-\alpha)S_p + \alpha S_{p-1}, \ \alpha \in [0, 1],$$
$$R(\theta_p + \alpha h, \theta_q + \beta h) = \begin{cases} (1-\alpha)R_{pq} + \beta R_{p-1,q-1} + (\alpha - \beta)R_{p-1,q}, & \alpha \geq \beta, \\ (1-\beta)R_{pq} + \alpha R_{p-1,q-1} + (\beta - \alpha)R_{p,q-1}, & \alpha < \beta \end{cases} \tag{10}$$

*Remark 2.2:* Stability of linear system with multiple delays

$$\dot{x}(t) = A_0 x(t) + \sum_{i=1}^{2} A_i x(t - r_i), \qquad r_i > 0$$

may be analyzed either via application of the corresponding complete LKF (see [10, (7.30)]) or by the mixed complete-descriptor LKF

$$\begin{aligned} V(x_t) &= x^T(t)P_1 x(t) + 2x^T(t)\int_{-r_1}^{0} Q(\xi)x(t+\xi)d\xi \\ &\quad + \int_{-r_1}^{0}\int_{-r_1}^{0} x^T(t+s)R(s,\xi)ds\,x(t+\xi)d\xi \\ &\quad + \int_{-r_1}^{0} x^T(t+\xi)S(\xi)x(t+\xi)d\xi \\ &\quad + \left[\int_{-r_2}^{0}\int_{t+\theta}^{t} \dot{x}^T(s)R^{(2)}\dot{x}(s)ds\,d\theta \right. \\ &\qquad \left. + \int_{t-r_2}^{t} x^T(s)S^{(2)}x(s)ds\right], \\ R^{(2)} &> 0, \quad S^{(2)} > 0, \quad P_1 > 0. \end{aligned} \tag{19}$$

The necessary condition for the application of the mixed LKF is the asymptotic stability of the system with $r_2 = 0$

$$\dot{x}(t) = (A_0 + A_2)x(t) + A_1 x(t - r_1).$$

The case of multiple delays $r_i > 0$ will not be considered in this note.

*Example 2.1:* Consider (1) with $A_0$ and $A_1$ from the uncertain polytope (18) with the vertices given by

$$\begin{aligned} A_0^{(1)} &= \begin{bmatrix} 0 & 1 \\ -2 & 0.1 \end{bmatrix}, \quad A_1^{(1)} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}; \\ A_0^{(2)} &= \begin{bmatrix} 0 & 1 \\ -2 & -0.1 \end{bmatrix}, \quad A_1^{(2)} = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}. \end{aligned} \tag{20}$$

The system with the matrices from the first vertex has been considered in ([10, pp. 288–289]). This system is unstable for $r = 0$. The system in the second vertex is not asymptotically stable for $r = 0$. Therefore, the reduced-order LKFs are not applicable in each vertex. We find estimates on the stability interval $r \in [r_{\min}, r_{\max}]$ for robust asymptotic stability of (20) inside the polytope $\Omega$ by applying two discretized LKF methods: The method of Gu [9] and the descriptor method of Theorem 2.1 (see Remark 2.1). The resulting estimates by the descriptor method are less restrictive (see Table I).

TABLE I
EXAMPLE 2.1

| $\Omega$ | N | 1 | 2 | 3 | N | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|---|
| Gu (2001) | $r_{min}$ | 0.43 | 0.35 | 0.32 | $r_{max}$ | 1.09 | 1.46 | 1.55 |
| Theorem 2.1 | $r_{min}$ | 0.12 | 0.12 | 0.11 | $r_{max}$ | 1.24 | 1.51 | 1.59 |

## III. Stability of Uncertain Neutral Type Systems

We consider the following linear system with uncertain coefficients and uncertain delays:

$$\begin{aligned} &\dot{x}(t) - C\dot{x}(t-g) \\ &\quad = (A_0 + H\Delta E_0)x(t) + (A_1 + H\Delta E_1)x(t - \tau(t)) \end{aligned} \tag{21}$$

where $x(t) \in \mathcal{R}^n$ is the system state, $A_i, E_j$ $i = 0, 1, C$ and $H$ are constant matrices of appropriate dimensions and $\Delta(t)$ is a time-varying uncertain $n \times n$ matrix that satisfies

$$\Delta^T(t)\Delta(t) \le I_n. \tag{22}$$

The uncertain delay $\tau(t)$ is *piecewise-continuous* function of the form

$$\tau(t) = r + \eta(t), \qquad r > 0, \quad |\eta(t)| \le \mu \le r \tag{23}$$

with the known upper bound $\mu$.

Equation (21) is a neutral type system. Our results will be independent on $g$ and dependent on $r$ and $\mu$. For e.g., $g = \tau(t) = r$ one can apply the results with $\mu \to 0$.

To guarantee the asymptotic stability of the difference equation $x(t) - Cx(t-g) = 0$ we assume that the eigenvalues of $C$ are inside the unit circle. Similarly to the stability conditions via reduced descriptor LKF [1], the feasibility of our LMIs for stability of (21) will guarantee the stability of the difference equation.

Representing

$$x(t - \tau(t)) = x(t-r) - \int_{t-r-\eta(t)}^{t-r} \dot{x}(s)ds$$

---

$$\phi^T(\alpha) = [x^T(t - h + \alpha h)\; x^T(t - 2h + \alpha h)\; \ldots\; x^T(t - Nh + \alpha h)] \tag{16a}$$

$$\bar{\Xi} = \begin{bmatrix} P^T \begin{bmatrix} 0 & I \\ A_0 & -I \end{bmatrix} + \begin{bmatrix} 0 & A_0^T \\ I & -I \end{bmatrix} P + \begin{bmatrix} Q_0 + Q_0^T + S_0 & 0 \\ 0 & 0 \end{bmatrix} & P^T \begin{bmatrix} 0 \\ A_1 \end{bmatrix} - \begin{bmatrix} Q_N \\ 0 \end{bmatrix} \\ * & -S_N \end{bmatrix} \tag{16b}$$

$$S_d = \mathrm{diag}\{S_0 - S_1, S_1 - S_2, \ldots, S_{N-1} - S_N\} \tag{16c}$$

$$R_d = \begin{bmatrix} R_{d11} & R_{d12} & \ldots & R_{d1N} \\ R_{d21} & R_{d22} & \ldots & R_{d2N} \\ \ldots & \ldots & \ldots & \ldots \\ R_{dN1} & R_{dN2} & \ldots & R_{dNN} \end{bmatrix} \tag{16d}$$

$$R_{dpq} = h(R_{p-1,q-1} - R_{pq}) \tag{16e}$$

$$D^s = [\,D_1^s \quad D_2^s \quad \ldots \quad D_N^s\,] \tag{16f}$$

$$D^a = [\,D_1^a \quad D_2^a \quad \ldots \quad D_N^a\,] \tag{16g}$$

$$D_p^s = \begin{bmatrix} (R_{0,p-1} + R_{0p}) - (Q_{p-1} - Q_p) \\ \frac{h}{2}(Q_{p-1} + Q_p) \\ -\frac{h}{2}(R_{N,p-1} + R_{Np}) \end{bmatrix} \tag{16h}$$

$$D_p^a = \begin{bmatrix} -\frac{h}{2}(R_{0,p-1} - R_{0p}) \\ -\frac{h}{2}(Q_{p-1} - Q_p) \\ \frac{h}{2}(R_{N,p-1} - R_{Np}) \end{bmatrix} \tag{16i}$$

and applying the input–output approach (see [10] and the references therein), we consider the following forward system:

$$\begin{aligned} \dot{x}(t) &= A_0 x(t) + A_1 x(t-r) + \mu A_1 v_1(t) \\ &\quad + C v_2(t) + H v_3(t) && \text{(24a)} \\ y_1(t) &= \dot{x}(t) && \text{(24b)} \\ y_2(t) &= \dot{x}(t) && \text{(24c)} \\ y_3(t) &= E_0 x(t) + E_1 x(t-r) + \mu E_1 v_1(t) && \text{(24d)} \end{aligned}$$

with the feedback of

$$\begin{aligned} v_1(t) &= -\frac{1}{\sqrt{2}\mu} \int_{-r-\eta(t)}^{-r} y_1(t+s) ds, \\ v_2(t) &= y_2(t-g), \\ v_3(t) &= \Delta y_3(t). \end{aligned} \tag{25}$$

Note that in the case of retarded system with $C = 0$ the input–output model (24), (25) has been introduced in [5].

Let $v^T = [\, v_1^T \quad v_2^T \quad v_3^T \,]$, $y^T = [\, y_1^T \quad y_2^T \quad y_3^T \,]$. Assume that $y_i(t) = 0, \forall t \le 0, i = 1, 2, 3$. The following holds for $n \times n$ matrices $R_a > 0, U > 0$ and a scalar $\rho > 0$ [5], [10]:

$$\begin{aligned} \|\sqrt{R_a} v_1\|_{L_2} &\le \sqrt{2} \|\sqrt{R_a} y_1\|_{L_2} \\ \|\sqrt{U} v_2\|_{L_2} &= \|\sqrt{U} y_2\|_{L_2} \\ \|\rho v_3\|_{L_2} &\le \|\rho y_3\|_{L_2}. \end{aligned} \tag{26}$$

Let $V$ be LKF (2). Due to (26), the following condition along (24):

$$\begin{aligned} \mathcal{W} &\triangleq \dot{V}(t) + 2\mu y_1^T(t) R_a y_1(t) + y_2^T(t) U y_2(t) + \rho y_3^T(t) y_3(t) \\ &\quad - \mu v_1^T(t) R_a v_1(t) - v_2^T(t) U v_2(t) - \rho v_3^T(t) v_3(t) \\ &< -\varepsilon(\|x(t)\|^2 + \|\dot{x}(t)\|^2 + \|v(t)\|^2), \quad \varepsilon > 0 \end{aligned} \tag{27}$$

guarantees the asymptotic stability of (21) [10].

Differentiating $V(x_t)$ along the trajectories of (24) we obtain that $\dot{V}$ is given by (6), where

$$\begin{aligned} 2\dot{x}^T(t) P_1 x(t) = 2 & \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}^T \\ & \times P^T \Bigg[ \begin{pmatrix} 0 & I \\ A_0 & -I \end{pmatrix} \begin{pmatrix} x(t) \\ \dot{x}(t) \end{pmatrix} \\ & \quad + \begin{pmatrix} 0 \\ A_1 \end{pmatrix} x(t-r) + \mu \begin{pmatrix} 0 \\ A_1 \end{pmatrix} v_1(t) \\ & \quad + \begin{pmatrix} 0 \\ C \end{pmatrix} v_2(t) + \begin{pmatrix} 0 \\ H \end{pmatrix} v_3(t) \Bigg]. \end{aligned} \tag{28}$$

Therefore, choosing $Q$, $S$ and $R$ to be piecewise-linear of the form (10), we find similarly to the previous section that

$$\begin{aligned} \mathcal{W} &= \zeta_v^T \Xi_v \zeta_v - \int_0^1 \phi^T(\alpha) S_d \phi(\alpha) d\alpha \\ &\quad - \int_0^1 \left[ \int_0^1 \phi^T(\alpha) R_d \phi(\beta) d\alpha \right] d\beta \\ &\quad + 2\zeta^T \int_0^1 [D^s + (1-2\alpha) D^a] \phi(\alpha) h d\alpha \end{aligned} \tag{29}$$

with the notations defined in (16) and

$$\zeta_v^T = [\, x^T(t) \quad \dot{x}^T(t) \quad x^T(t-r) \quad v_1^T(t) \quad v_2^T(t) \quad v_3^T(t) \,]$$

$$\begin{aligned} \Xi_v &= \begin{bmatrix} \bar{\Xi} & \mu P^T \begin{bmatrix} 0 \\ A_1 \end{bmatrix} & P^T \begin{bmatrix} 0 \\ C \end{bmatrix} & P^T \begin{bmatrix} 0 \\ H \end{bmatrix} \\ & 0 & 0 & 0 \\ * & -\mu R_a & 0 & 0 \\ * & * & -U & 0 \\ * & * & * & -\rho I_n \end{bmatrix} \\ &\quad + \rho \begin{bmatrix} E_0^T \\ 0 \\ E_1^T \\ \mu E_1^T \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} E_0^T \\ 0 \\ E_1^T \\ \mu E_1^T \\ 0 \\ 0 \end{bmatrix}^T \\ &\quad + 2\mu [0 \quad I_n \quad 0 \quad 0 \quad 0 \quad 0]^T \\ &\quad \times R_a [0 \quad I_n \quad 0 \quad 0 \quad 0 \quad 0] \\ &\quad + [0 \quad I_n \quad 0 \quad 0 \quad 0 \quad 0]^T U [0 \quad I_n \quad 0 \quad 0 \quad 0 \quad 0]. \end{aligned}$$

Applying [10, Prop. 5.21] to (29) and Schur complements to the last three terms of $\Xi_v$ we conclude that $\dot{V}(x_t) < 0$ if the LMI shown in (30) at the bottom of the page holds.

Thus, we obtained the following.

*Theorem 3.1:* System (21) is asymptotically stable for all delays satisfying (23), if there exist $n \times n$ matrices $0 < P_1, P_2, P_3, R_a, U$ $S_p = S_p^T, Q_p, R_{pq} = R_{qp}^T, p = 0, 1, \ldots, N, q = 0, 1, \ldots, N$ and a scalar $\rho > 0$ such that LMIs (11), (30) are satisfied with notations defined in (5), (12) and (16b)–(16i).

*Remark 3.1:* In the case when the delay $\tau(t)$ of the form (23) satisfies the additional constraint $\dot{\tau}(t) \le 1$, the following inequality holds [5]:

$$\|\sqrt{R_a} u_1\|_{L_2} \le \|\sqrt{R_a} y_1\|_{L_2}$$

which leads to LMIs (11), (30), where in the latter LMI the coefficient 2, multiplying $\mu R_a$, should be deleted.

$$\begin{bmatrix} & & & \mu P_2^T A_1 & P_2^T C & 0 & 0 & P_2^T H & \rho E_0^T \\ \bar{\Xi} & D^s & D^a & \mu P_3^T A_1 & P_3^T C & 2\mu R_a & U & P_3^T H & 0 \\ & & & 0 & 0 & 0 & 0 & 0 & \rho E_1^T \\ * & -R_d - S_d & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & -3S_d & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -\mu R_a & 0 & 0 & 0 & 0 & \rho\mu E_1^T \\ * & * & * & * & -U & 0 & 0 & 0 & 0 \\ * & * & * & * & * & -2\mu R_a & 0 & 0 & 0 \\ * & * & * & * & * & * & -U & 0 & 0 \\ * & * & * & * & * & * & * & -\rho I_n & 0 \\ * & * & * & * & * & * & * & * & -\rho I_n \end{bmatrix} < 0 \tag{30}$$

*Remark 3.2:* Consider (21) with $H = 0$ and with $A_0$, $A_1$ and $C$ from the uncertain time-invariant polytope given by (18) where the $M$ vertices of the polytope are described by $\Omega_j = [\, A_0^{(j)} \quad A_1^{(j)} \quad C^{(j)} \,]$. By the descriptor discretized method one have to solve the LMIs (11), (30) (with the deleted sixth and ninth rows and columns) simultaneously for all the $M$ vertices, applying the same matrices $P_2$ and $P_3$ and solving for the $M$ vertices only.

## IV. Robust Stabilization

Given the following system:

$$\begin{aligned}\dot{x}(t) - C\dot{x}(t-g) &= (A_0 + H\Delta E_0)x(t)\\ &\quad + (A_1 + H\Delta E_1)x(t-\tau(t)) + (B + H\Delta E_2)u(t) \qquad (31)\end{aligned}$$

where $x(t) \in \mathcal{R}^n$ is the system state vector, $u(t) \in \mathcal{R}^m$ is the control input, $A_i, C, H, B, E_i, E_2, i = 0, 1$ are constant matrices, time-delay $\tau(t)$ is a piecewise-continuous function, satisfying (23). We are looking for a stabilizing state-feedback

$$u(t) = K_0 x(t) + K_1 x(t - \tau(t)). \qquad (32)$$

Note that for $K_1 = 0$ the above state-feedback is instantaneous. For $K_0 = 0$ it is a delayed controller. The closed-loop system (31), (32) has the form

$$\begin{aligned}\dot{x}(t) - C\dot{x}(t-g) &= (A_0 + BK_0 + H\Delta(E_0 + E_2K_0))x(t)\\ &\quad + (A_1 + BK_1 + H\Delta(E_1 + E_2K_1))x(t-\tau(t)). \qquad (33)\end{aligned}$$

Following [21] we choose $P_3 = \delta P_2$, $\delta \in R$, where $\delta$ is a tuning scalar parameter (which may be restrictive). Note that $P_2$ is nonsingular due to the fact that the only matrix which can be negative definite in the second block on the diagonal of (30) is $-\delta\left(P_2 + P_2^T\right)$. Defining (34), as shown at the bottom of the page, we multiply (11) by $\mathrm{diag}\{\bar{P}, \ldots, \bar{P}\}$ and its transpose, from the right and the left, respectively. Multiplying further (30) by $\mathrm{diag}\{\bar{P}, \ldots, \bar{P}, \bar{\rho}I_n, \bar{\rho}I_n\}$ and its transpose, from the right and the left we obtain the following.

*Theorem 4.1:* Consider (31) with a piecewise-continuous delay $\tau$ given by (23). Under the state-feedback law (32) the system is asymptotically stable if for some tuning scalar parameter $\delta$ there exist $n \times n$ matrices $0 < \bar{P}_1, \bar{P}, \bar{R}_a, \bar{U}, \bar{S}_p = \bar{S}_p^T, \bar{Q}_p, \bar{R}_{pq} = \bar{R}_{qp}^T$, $p = 0, 1, \ldots, N$, $q = 0, 1, \ldots, N$, a scalar $\bar{\rho} > 0$ and $m \times n$-matrices $Y_i$, $i = 0, 1$ such that the LMIs, shown in (35) and (36) at the bottom of the page, are satisfied. where (37), as shown at the bottom of the page, holds, and where $\tilde{R}, \tilde{Q}, \tilde{S}$ and $D^s$, $D^a$, $R_d$, $S_d$ are given by (12) and (16) correspondingly with bars over $R_{pq}, Q_p, S_p, p = 1, \ldots, N$, $q = 1, \ldots, N$.

The state-feedback gains are given by $K_i = Y_i\bar{P}^{-1}$, $i = 0, 1$. To design the state-feedback with $K_i = 0$ for some $i = 0$, 1, one have to set $Y_i = 0$ in (36).

*Remark 4.1:* Consider (31) with $H = 0$ and with $A_0$, $A_1$, $C$ and $B$ from the uncertainty polytope given by (18), where $\Omega_j = [\, A_0^{(j)} \quad A_1^{(j)} \quad C^{(j)} \quad B^{(j)} \,]$. To stabilize the system inside the polytope one have to solve LMIs (35) and (36) simultaneously for all the $M$ vertices, applying the same matrices $\bar{P}$ and $Y_i$, $i = 0, 1$.

$$\begin{aligned}\bar{P} &= P_2^{-1},\\ [\,\bar{P}_1 \quad \bar{Q}_p \quad \bar{S}_p \quad \bar{R}_{pq} \quad \bar{R}_a \quad \bar{U}\,] &= \bar{P}^T [\,P_1\bar{P} \quad Q_p\bar{P} \quad S_p\bar{P} \quad R_{pq}\bar{P} \quad R_a\bar{P} \quad U\bar{P}\,]\\ Y_i &= K_i\bar{P}, \quad i = 0, 1, \;\; p = 1, \ldots, N,\\ q &= 1, \ldots, N, \;\; \bar{\rho} = \frac{1}{\rho} \end{aligned} \qquad (34)$$

$$\begin{bmatrix} \bar{P}_1 & \tilde{Q} \\ * & \tilde{R} + \tilde{S} \end{bmatrix} > 0 \qquad (35)$$

$$\begin{bmatrix} \hat{\Xi} & D^s & D^a & \begin{matrix} A_1\bar{P} + BY_1 \\ \mu\,\delta(A_1\bar{P} + BY_1) \\ 0 \end{matrix} & \begin{matrix} C\bar{P} \\ \delta C\bar{P} \\ 0 \end{matrix} & \begin{matrix} 0 \\ 2\mu\bar{R}_a \\ 0 \end{matrix} & \begin{matrix} 0 \\ \bar{U} \\ 0 \end{matrix} & \begin{matrix} \bar{\rho}H \\ \delta\bar{\rho}H \\ 0 \end{matrix} & \begin{matrix} \bar{P}^T E_0^T + Y_0^T E_2^T \\ 0 \\ \bar{P}^T E_1^T + Y_1^T E_2^T \end{matrix} \\ * & -R_d - S_d & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & -3S_d & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -\mu\bar{R}_a & 0 & 0 & 0 & 0 & \mu\bar{P}E_1^T \\ * & * & * & * & -\bar{U} & 0 & 0 & 0 & 0 \\ * & * & * & * & * & -2\mu\bar{R}_a & 0 & 0 & 0 \\ * & * & * & * & * & * & -\bar{U} & 0 & 0 \\ * & * & * & * & * & * & * & -\bar{\rho}I_n & 0 \\ * & * & * & * & * & * & * & * & -\bar{\rho}I_n \end{bmatrix} < 0 \qquad (36)$$

$$\hat{\Xi} = \begin{bmatrix} A_0\bar{P} + \bar{P}^T A_0^T + BY_0 + Y_0^T B^T + \bar{Q}_0 + \bar{Q}_0^T + \bar{S}_0 & \bar{P}_1 - \bar{P} + \delta\bar{P}^T A_0^T + \delta Y_0^T B^T & A_1\bar{P} + BY_1 - \bar{Q}_N \\ * & -\delta\bar{P} - \delta\bar{P}^T & \delta(A_1\bar{P} + BY_1) \\ * & * & -\bar{S}_N \end{bmatrix} \qquad (37)$$

*Example 4.1:* [18] We address the problem of finding a state-feedback controller for (31) with known system matrices $(H = 0)$, where

$$A_0 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} -1 & -1 \\ 0 & -0.9 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{38}$$

We compare the results obtained by application of Theorem 4.1 with the existing results via reduced LKFs. Even for $N = 1$ the new results are essentially less conservative. We give below the new results for $N = 1$, noting that for $N > 1$ some further improvement can be achieved.

In the case of retarded-type system $(C = 0)$ and constant delay $\tau(t) \equiv r$ it was shown in [7] via reduced descriptor LKF that the system is stabilizable by the instantaneous state-feedback $u(t) = K_0x(t)$ for $r \in [0, 3.2]$. By Theorem 4.1, where $N = 1$ and $\delta = 100$, we find that the system is stabilizable for $r \in [0, 1500]$. For higher values of $r$ the controller becomes high-gain. Thus, for $r = 1500$ the resulting gain is $K_0 = -10^9 \cdot [1.6694\ 1.6696]$.

In the case of neutral type system with $C = \mathrm{diag}\{-0.1,\ -0.2\}$ and constant delay $\tau(t) \equiv r$ it was found in [4] by reduced descriptor LKF that the system is stabilizable by $u(t) = K_0x(t)$ for $r \le 1.2$. By applying Theorem 4.1 with $N = 1$ and $\delta = 100$ we find that the system is stabilizable by $u(t) = K_0x(t)$ for $r \le 768$.

Consider next the time-varying delay $\tau(t) = r + \eta(t)$ and $C = 0$. For $r = 2$, it was found in [2] that the system is stabilizable by $u(t) = K_0x(t)$ for all $|\eta(t)| \le 0.2$, where $K_0 = -[74.8\ 105.5]$. Applying Theorem 4.1 with $r = 2$, $N = 1$ and $\delta = 1$, we find that the system is stabilizable for all delays from a wider segment with $|\eta(t)| \le 0.22$ and the resulting controller has a lower gain: $u(t) = -[20.5108\ 34.6753]x(t)$. Note that the controllers obtained by the reduced-order descriptor LKF stabilizes the system for all $r \le 2$, while the controller designed by the descriptor discretized method stabilizes the system for $r = 2$ only.

*Example 4.2: Using delay for robust static output-feedback stabilization*. Given the following system:

$$\dot{x}(t) = A_0x(t) + Bu(t), \qquad y(t) = [1\ 0]x(t), \quad x(t) \in R^2 \tag{39}$$

with $B = [0\ 1]^T$ and $A_0$ from the uncertain polytope (18), where

$$\begin{aligned} &\Omega_j = A_0^{(j)}, \qquad j = 1, 2 \\ &A_0^{(1)} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad A_0^{(2)} = \begin{bmatrix} 0 & 1 \\ -1 & 0.1 \end{bmatrix}. \end{aligned} \tag{40}$$

The system with $A_0 = A_0^{(1)}$ is a double integrator. The system with $A_0 = A_0^{(2)}$ has been considered in ([10, p. 156]). Both systems in the vertices are known to be not stabilizable by the nondelayed output-feedback $u(t) = K_0y(t)$. Following [10], [19], and [20], we are looking for a stabilizing time-delayed output-feedback

$$u(t) = K_0y(t) + K_1y(t - r), \qquad r > 0. \tag{41}$$

The closed-loop system (39), (41) has the form

$$\dot{x}(t) = (A_0 + BK_0[1\ 0])x(t) + BK_1[1\ 0]x(t - r). \tag{42}$$

Since for $r = 0$ (42) is unstable, the existing LKF-based design methods are not applicable.

Differently from the state-feedback case (cf. (34)), we have here $Y_i = K_i[1\ 0]\bar{P}$. Therefore, we assume that for some tuning parameter $\delta_1$

$$\bar{P} = \begin{bmatrix} P_{11} & \delta_1 P_{11} \\ P_{21} & P_{22} \end{bmatrix}, \quad Y_i = [Y_{1i}\ \delta_1 Y_{1i}], \quad i = 0, 1 \tag{43}$$

where $Y_{1i} = K_iP_{11}$. We thus verify LMIs of Theorem 4.1 simultaneously for two vertices, applying the same matrices $\bar{P}$ and $Y_i$ of the form (43). The output-feedback gains are given by $K_i = Y_{1i}P_{11}^{-1}$, $i = 0, 1$. The resulting LMIs have two tuning parameters $\delta$ and $\delta_1$.

Choosing (for simplicity) $N = 1$ and $\delta = \delta_1 = 1$ we find that the latter LMIs are feasible (and thus the uncertain system is robustly stabilizable by the feedback of (41)) for all $r \in [0.1, 2.5]$. Thus, for $r = 1$ the resulting gains are given by $K_0 = -0.7947$, $K_1 = 0.3067$. Considering further the closed-loop system (42) with the previous gains and unknown $r > 0$, we verify for this system the conditions of Theorem 2.1 (and Remark 2.1) with $N = 3$. We find that the feedback $u(t) = -0.7947y(t) + 0.3067y(t - r)$ robustly stabilizes (39), (40) for all $r \in [0.34, 1.82]$.

## V. Conclusion

Stability and state-feedback stabilization of linear neutral type systems with uncertain time-varying delays from given segments and either norm-bounded or polytopic type uncertainties are studied. The system under consideration may be unstable without delay, but it becomes asymptotically stable for positive values of the delay. Such systems can not be treated via the reduced LKFs (delay-independent or delay-dependent, corresponding to different model transformations). The new discretised LKF method is introduced, which combines the discretized LKF method of Gu with the descriptor model transformation. The descriptor approach allows to solve for the first time the *synthesis* problems via discretized LKF. The new method essentially improves the existing design results. It leads to less restrictive results for robust stability of time-delay systems with polytopic type uncertainties.

The introduced method provide new tools for the important design problems, such as stabilization of systems by using delays in the feedback, where the existing LMI methods are not applicable (since the nondelayed closed-loop system is not stable). A simple example of static output-feedback stabilization of uncertain second-order system by using delay is given in this note. Stabilization of more general systems by using delays as well as different robust control problems for time-delay systems are the topics for the future research.

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# A Note on Spectral Conditions for Positive Realness of Single-Input–Single-Output Systems With Strictly Proper Transfer Functions

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***Abstract*—Necessary and sufficient conditions for strict positive realness and positive realness of strictly proper functions are derived. The conditions are expressed in terms of eigenvalues of the state matrices representation of the system. Previous results rendered conditions which were significantly more complex than those for proper (but not strictly proper) functions. The present conditions for strictly proper functions are simpler than the ones for proper functions, which is consistent with intuition in this case. Illustrative numerical examples are provided.**

***Index Terms*—Circle criterion, eigenvalues locations, positive real (PR) conditions, state–space representation, time varying systems.**

## I. Introduction

The concept of positive realness (PR) and strict positive realness (SPR) of a rational function appears frequently in various aspects of system theory. In particular, in control theory, positive realness plays a central role in adaptive control [1], and in stability theory [2], [3]. Similarly, the passivity of electrical networks is also strongly related to positive realness, as are other fundamental concepts in circuit and VLSI design [4].

Roughly speaking, checking whether a dynamic system is positive real amounts to testing whether a certain matrix valued function of a frequency variable is positive definite for all frequencies. Exhaustive numerical checking of such matrices for all frequencies is expensive for large dimensional systems. Consequently, several authors over the past two decades have sought to derive easily verifiable conditions for checking whether a given transfer function is PR; see [4]–[6] and the references therein for a review of some of this work. Recently, compact conditions for checking whether a single-input–single-output (SISO) transfer function is PR (or SPR) were derived [7]. These conditions, which amount to checking whether or not a $n$-dimensional matrix has an eigenvalue on the negative real axis, can be easily applied to determine the strict positive realness of $n$-dimensional SISO systems that are described in state–space form. Unfortunately, while the conditions derived in [7] for testing strict positive realness and positive realness of a proper transfer function are simple and transparent, and also provide new insights into the meaning of strict positive realness, the sister conditions for checking positive realness of a strictly proper transfer function are more involved and rather less transparent. Our objective in this paper is to revisit this problem and to derive more satisfactory conditions for the case of strictly proper transfer functions.

## II. Definitions

In the remainder of this note we use the following common definitions for PR and SPR, which appear in almost any textbook on the synthesis of passive networks.

Manuscript received January 17, 2005; revised July 25, 2005. Recommended by Associate Editor W. X. Zheng.
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Digital Object Identifier 10.1109/TAC.2006.872829