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Constructive method for averaging-based stability via a delay free transformation☆

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ABSTRACT

We treat input-to-state stability-like (ISS-like) estimates for perturbed linear continuous-time systems with multiple time-scales, under the assumption that the averaged, unperturbed, system is exponentially stable. Such systems contain rapidly-varying, piecewise continuous and almost periodic coefficients with small parameters (time-scales). Our method relies on a novel delay-free system transformation in conjunction with a new system presentation, where the rapidly-varying coefficients are scalars that have zero average. We employ time-varying Lyapunov functions for ISS-like analysis. The analysis yields LMI conditions, leading to explicit bounds on the small parameters, decay rate and ISS-like gains. The novel system presentation plays a crucial role in the ISS-like analysis by allowing to derive essentially less conservative upper bounds on terms containing the small parameters. The obtained LMIs are accompanied by suitable feasibility guarantees. We further extend our approach to rapidly-varying systems subject to either discrete (constant/fast-varying) or distributed delays, where our approach decouples the effects of the delay and small parameters on the stability of the system, and leads to LMI conditions for stability of systems with non-small delays. Extensive numerical examples show that, compared to the existing results, our approach essentially enlarges the small parameter and delay bounds for which the ISS-like/stability property of the original system is preserved.

1. Introduction

Systems with almost periodic signals and/or excitations are central to physics and engineering. Applications of such systems include vibrational control (Cheng, Tan, & Mareels, 2018), power systems (Sandberg & Möllerstedt, 2001) and time-delay systems (Xie & Lam, 2018) (see also the references therein). Such systems often include components evolving over multiple time-scales (see e.g. Hek (2010) for applications to systems biology). Hence, it is not surprising that perturbation theory has played an essential part in the analysis of systems with rapidly time-varying coefficients and led to important results (Bogoliubov & Mitropolskij, 1961; Khalil, 2001),

The method of averaging is an important perturbation-based technique for the study of stability of systems with oscillatory control inputs (Bullo, 2002; Krstić & Wang, 2000; Meerkov, 1980) and switched systems (Caiazzo, Fridman, & Yang, 2023; Mostacciuolo, Trenn, & Vasca, 2022). The fundamental idea behind asymptotic averaging is that stability of the first-order averaged system guarantees stability of the original rapidly-varying system for small enough values of the time-scale parameter (see e.g. Murdock (1999)). However, it is often the case that asymptotic averaging provides only an existence result, without an efficient and explicit bound on the small parameter for which the stability of the original system is preserved. For singularly perturbed systems, such bounds were derived in, e.g., Kokotovic and Khalil (1986) and Fridman (2002) via a direct Lyapunov approach.

Recently, the first efficient quantitative methods for stability by averaging were suggested. A constructive time-delay approach to periodic averaging of a system with a single rapid time-scale was suggested in Fridman and Zhang (2020). The approach relies on backward integration of the system, which yields a neutral-type system presentation, where the delay magnitude is equal to the time-scale parameter. The stability and ISS of the delayed system were shown to guarantee the stability and ISS of the original system. Stability of the delayed system was analyzed via a direct Lyapunov–Krasovskii method, leading to LMI conditions which yield an efficient upper bound on the small parameter that preserves the stability of the original system. This method is also

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well suited for averaging of systems with time-varying delays, where the delay magnitude is of equal order to the time-scale parameter. These results were extended to L_2-gain analysis for periodic averaging and to stochastic systems in Zhang and Fridman (2022). However, the results of Fridman and Zhang (2020) were fairly conservative. Moreover, the Lyapunov–Krasovskii analysis for systems without delays was valid only for times greater than the small parameter. Hence, additional solution bounds on the first delay interval, where the time-delay model is invalid, are needed to complete the Lypunov analysis for times larger than the small parameter. Finally, the results of Fridman and Zhang (2020) were confined to one time-scale. The objective of the present paper is to present simpler analysis tools (i.e., Lyapunov functions that do not require additional bounding of solutions on the first interval, having length equal to the small parameter) with significantly improved results, as well as the extension to multi-scale systems.

We study ISS-like property of rapidly time-varying systems with multiple time-scales, under the assumption that the averages system satisfies an ISS-like property. We employ a novel presentation of the system, in conjunction with a novel delay-free transformation. The new presentation relies on two key ingredients: first, inspired by a similar presentation for systems with distributed delays and variable kernels (Solomon & Fridman, 2013), we present the rapidly-varying system matrices as linear combinations of constant matrices with rapidly-varying *scalar* coefficients. Second, we force the latter coefficients to have zero averages. We then employ a transformation leading to a system with stable nominal (averaged) part and time-varying perturbations of the order of the small parameters. The ISS-like property of the transformed system guarantees the ISS-like property of the original system. The ISS-like property of the transformed system is studied by employing time-varying Lyapunov functions and tight bounds on the scalar time-varying coefficients. The resulting LMIs are backed by theoretical feasibility guarantees.

We further extend the presented approach to rapidly-varying systems subject to delays. Classical results on averaging of time-delay systems can be found in Hale and Lunel (2002) and Lehman and Weibel (1999), whereas stability of linear systems with periodic coefficients and subject to constant or periodic delays was analyzed numerically in Butcher and Mann (2009) and Insperger and Stépán (2011). An eigenvalue-based method for stability analysis of such systems was presented in Michiels and Niculescu (2014). Complete Lyapunov–Krasovskii functionals were further employed for stability analysis of linear systems with continuous periodic coefficients and constant delays in Gomez, Ochoa, and Mondié (2016) and Letyagina and Zhabko (2009). Results on strict Lyapunov functions for rapidly time-varying nonlinear systems were presented in Mazenc and Malisoff (2017), Mazenc, Malisoff, and De Queiroz (2006). For rapidly-varying systems subject to fast-varying delays, the constructive approach in Fridman and Zhang (2020) is suitable for stability analysis provided the delay bound is of the order of the small parameter. The time-delay to averaging was recently extended to systems with non-small delays (Caiazzo et al., 2023), where the delayed state was multiplied by the constant matrix. Distributed delays with a *constant* kernel were treated in Griñó, Ortega, Fridman, Zhang, and Mazenc (2021) in the case of *scalar* systems. Our novel system presentation, together with the delay-free transformation lead to a unified constructive methodology for stability analysis of rapidly-varying systems subject to either discrete (i.e., constant/fast-varying) or distributed delays. Our approach decouples the effects of the delay and small parameters on the stability of the system and leads to LMI conditions for stability of systems with non-small delays, relative to the time-scale parameter. Extensive numerical examples show that, compared to the existing results, our approach significantly enlarges the small parameter and delay bounds for which the ISS-like/stability property of the original system is preserved.

Initial results on averaging via a delay free transformation, without the new system presentation were presented in IFAC WC 2023 (Katz, Mazenc, & Fridman, 2023), where results in the numerical examples are significantly more conservative than those of Fridman and Zhang (2020). Preliminary results with new system presentation confined to non-delayed systems were presented in the 62nd IEEE CDC conference 2023 (Katz, Fridman, & Mazenc, 2023).

Notations. Throughout the paper $\mathbb{R}^n$ denotes the n-dimensional Euclidean space with the vector norm $|\cdot|$, $\mathbb{R}^{n\times m}$ is the set of all $n\times m$ real matrices with the induced matrix norm $\|\cdot\|$. We also denote $\mathbb{Z}_+=\{0,1,2,\dots\}$ and $\mathbb{R}_{\geq 0}=[0,\infty)$. The superscript $\top$ denotes matrix transposition, and the notation $P>0$, for $P\in\mathbb{R}^{n\times n}$ means that P is symmetric and positive definite. The symmetric elements of the symmetric matrix are denoted by $*$. For $0<P\in\mathbb{R}^{n\times n}$ and $x\in\mathbb{R}^n$, we write $|x|_P^2=x^\top Px$. $\otimes$ denotes the Kronecker product. The standard lexicographic order on $\mathbb{R}^n$ is denoted by $\leq_{\text{lex}}$. We denote by $W([-h,0])$ the Banach space of a.e differentiable functions $\phi:[-h,0]\to\mathbb{R}^n$ with square integrable derivative. The norm on $W([-h,0])$ is given by the norm $\|\phi\|_W=\|\phi\|_W+\left\|\phi'\right\|_{L^2}$.

2. ISS-like estimates of rapidly time-varying systems

2.1. Problem formulation

The recent paper (Fridman & Zhang, 2020) considered the system with rapidly-varying coefficients

$$\dot{x}(t)=A\left(\tfrac{t}{\epsilon}\right)x(t)+B\left(\tfrac{t}{\epsilon}\right)d(t),\ \ t\geq 0 \tag{2.1}$$

where $x(t)\in\mathbb{R}^n$ for $t\geq 0$, $\epsilon>0$ is a small parameter defining a rapid time-scale, d is a piecewise continuous disturbance and $A:\mathbb{R}\to\mathbb{R}^{n\times n}$ and $B:\mathbb{R}\to\mathbb{R}^{n\times n_d}$ are piecewise continuous matrix functions, which are norm-bounded uniformly for $t\in[0,\infty)$. Under the assumption that there exist $0<T$ and matrices A_{av},B_{av}, such that

$$\begin{aligned}&T^{-1}\int_t^{t+T}B(s)ds=B_{av}+\Delta B(t),\\&T^{-1}\int_t^{t+T}A(s)ds=A_{av}+\Delta A(t),\quad\forall t\in\mathbb{R}\end{aligned} \tag{2.2}$$

with $\Delta A,\Delta B:\mathbb{R}\to\mathbb{R}^{n\times n}$ sufficiently small in norm, Fridman and Zhang (2020) proposed a novel time-delay transformation, leading to quantitative estimate on ϵ for which ISS of (2.1) is preserved.

Here we consider the generalized system with scalar time-varying zero average coefficients (see Assumption 1 below)

$$\begin{aligned}\dot{x}(t)=&\left[A_{av}+\textstyle\sum_{i=1}^N a_i\left(\tfrac{t}{\epsilon_i}\right)A_i\right]x(t)\\&+\left[B_{av}+\textstyle\sum_{i=1}^{N_d} b_i\left(\tfrac{t}{\epsilon_{d,i}}\right)B_i\right]d(t),\quad t\geq 0\end{aligned} \tag{2.3}$$

where $x(t)\in\mathbb{R}^n$ for $t\geq 0$, $d\in C^1([0,\infty))$, $N,N_d\in\mathbb{N}$, $\{\epsilon_i\}_{i=1}^N$ and $\left\{\epsilon_{d,i}\right\}_{i=1}^{N_d}$ are positive small parameters, $\{A_i\}_{i=1}^N\subseteq\mathbb{R}^{n\times n}$, $\{B_i\}_{i=1}^{N_d}\subseteq\mathbb{R}^{n\times n_d}$ are constant matrices, and $\{a_i\}_{i=1}^N$, $\{b_i\}_{i=1}^{N_d}$ are piecewise continuous scalar functions which are uniformly bounded on $[0,\infty)$. The arguments of the scalar functions may depend on *independent* time-scales. The matrices in (2.1) can be expanded in any two bases of $\mathbb{R}^{n\times n}$ and $\mathbb{R}^{n\times n_d}$, thereby yielding the presentation (2.3) with a single time-scale.

For simplicity of the presentation, we will proceed with the case $N=N_d=2$. The general case follows the same arguments (see Remark 2.6).

Assumption 1. The matrix A_{av} is Hurwitz, whereas for $\{a_i\}_{i=1}^2$ and $\{b_j\}_{j=1}^2$ there exist positive constants $\{T_i\}_{i=1}^2$, $\{T_{d,j}\}_{j=1}^2$ such that

$$T_i^{-1}\int_t^{t+T_i} a_i(s)ds =: \Delta a_i(t), \quad T_{d,j}^{-1}\int_t^{t+T_{d,j}} b_j(s)ds =: \Delta b_j(t), \quad \forall t\in\mathbb{R} \tag{2.4}$$

with $\{\Delta a_i\}_{i=1}^2, \{\Delta b_j\}_{j=1}^2$ satisfying

$$\sup_{\tau\in\mathbb{R}}\left|\Delta\xi_j(\tau)\right|^2 \le \Delta_{\xi_j,M}, \quad 1\le j\le 2,\ \xi\in\{a,b\}, \tag{2.5}$$

for some positive constants $\{\Delta_{a_i,M}\}_{i=1}^2, \{\Delta_{b_j,M}\}_{j=1}^2$.

Remark 2.1. System (2.1) can be presented as (2.3) by fixing $\epsilon_i=\epsilon_{d,j}=\epsilon,\ 1\le i\le N,\ 1\le j\le N_d$ and presenting $A\left(\frac{t}{\epsilon}\right), B\left(\frac{t}{\epsilon}\right)$ as linear combinations of constant matrices with time-varying coefficients. In this case $N, N_d\le n^2$.

We aim to derive efficient and constructive conditions which guarantee ISS-like estimates for (2.3), with respect to d and $\dot d$ (see Theorem 2.1).

2.2. System transformation and Lyapunov analysis

For clarity we begin with stability analysis of (2.3) with $d(t)\equiv 0$. Inspired by Mazenc et al. (2006), for $t\ge 0$, $1\le i\le 2$, let

$$\varrho_{\epsilon,i}(t) = -\frac{1}{\epsilon_i T_i}\int_t^{t+\epsilon_i T_i}(t+\epsilon_i T_i - s)\,a_i\left(\frac{s}{\epsilon_i}\right)ds \tag{2.6}$$

for which a simple computation yields

$$\sup_{t\in\mathbb{R}}\left|\varrho_{\epsilon,i}(t)\right| \le \epsilon_i T_i \sup_{t\in\mathbb{R}}|a_i(t)|. \tag{2.7}$$

Differentiating (2.6), we further have for $t\ge 0$

$$\dot\varrho_{\epsilon,i}(t) = a_i\left(\frac{t}{\epsilon_i}\right) - \Delta a_i\left(\frac{t}{\epsilon_i}\right). \tag{2.8}$$

We introduce the following transformation

$$z(t) = x(t) - \sum_{i=1}^2 \varrho_{\epsilon,i}(t)A_i x(t) \tag{2.9}$$

and the following assumption:

Assumption 2. $I_n - \sum_{i=1}^2\varrho_{\epsilon,i}(t)A_i$ is invertible for all $t\ge 0$ with

$$\sup_{t\ge 0}\left\|\left(I_n - \sum_{i=1}^2\varrho_{\epsilon,i}(t)A_i\right)^{-1}\right\| \le \delta_{1,x} < \infty.$$

Assumption 2 imposes a constraint on ϵ. Indeed, by (2.7), Assumption 2 holds if $\sum_{i=1}^2\epsilon_i T_i a_{i,M}\|A_i\| < 2$, where $a_{i,M} := \sup_{\tau\in\mathbb{R}}|a_i(\tau)|$. In this case, we have

$$\sup_{t\ge 0}\left\|\sum_{i=1}^2\varrho_{\epsilon,i}(t)A_i\right\| \le \frac{\sum_{i=1}^2\epsilon_i T_i a_{i,M}\|A_i\|}{2} =: \delta_{2,x} < 1. \tag{2.10}$$

By a Neumann series, the latter implies that we can take

$$\delta_{1,x} = \left(1-\delta_{2,x}\right)^{-1}. \tag{2.11}$$

Using (2.3) we obtain the following for $\dot z(t),\ t\ge 0$:

$$\begin{aligned}\dot z(t) &= A_{av}z(t) + \textstyle\sum_{i=1}^2\Delta a_i\left(\frac{t}{\epsilon_i}\right)A_i x(t)\\ &\quad + \textstyle\sum_{i=1}^2\varrho_{\epsilon,i}(t)W_i x(t)\\ &\quad - \textstyle\sum_{i,j=1}^2\varrho_{\epsilon,i}(t)a_j\left(\frac{t}{\epsilon_j}\right)A_iA_j x(t),\\ W_i &= A_{av}A_i - A_iA_{av}, \quad i=1,2.\end{aligned} \tag{2.12}$$

Considering (2.9), (2.12) is a system in the form of the averaged system perturbed by $O(\epsilon)$ and $O(\Delta a_{i,M})$ terms. This makes (2.12) amenable to Lyapunov analysis, which yields efficient estimates on ϵ_i that preserve stability.

Next, we aim to vectorize (2.12). For that purpose, recall that $\le_{\text{lex}}$ is the lexicographic order on $\mathbb{R}^n$ ($(i,j)\le_{\text{lex}}(k,l)$ iff $i<k$ or $i=k,\ j\le l$) and introduce the notations

$$\begin{aligned}&\Upsilon_\varrho(t) = \text{col}\left\{\varrho_{\epsilon,i}(t)x(t)\right\}_{i=1}^2,\\ &\Upsilon_{\varrho,a}(t) = \text{col}\left\{\varrho_{\epsilon,i}(t)a_k\left(\frac{t}{\epsilon_k}\right)x(t)\right\}_{\{(i,k)\}\le_{\text{lex}}},\\ &\Upsilon_{\Delta a}(t) = \text{col}\left\{\Delta a_i\left(\frac{t}{\epsilon_i}\right)x(t)\right\}_{i=1}^2,\\ &\mathbb{A} = \begin{bmatrix}A_1 & A_2\end{bmatrix},\ \mathbb{W} = \begin{bmatrix}W_1 & W_2\end{bmatrix},\\ &\mathbb{A}_1 = [A_1^2\ A_1A_2\ A_2A_1\ A_2^2].\end{aligned} \tag{2.13}$$

Employing (2.12) and (2.13), we obtain for $t\ge 0$:

$$\dot z(t) = A_{av}z(t) + \mathbb{A}\Upsilon_{\Delta a}(t) + \mathbb{W}\Upsilon_\varrho(t) - \mathbb{A}_1\Upsilon_{\varrho,a}(t). \tag{2.14}$$

For stability analysis of (2.14), let $\alpha > 0$ be a desired decay rate and $0<P\in\mathbb{R}^{n\times n}$. Introduce the Lyapunov function

$$V(t) = |z(t)|_P^2 \tag{2.15}$$

and the notation

$$Q_\alpha := PA_{av} + A_{av}^\top P + 2\alpha P. \tag{2.16}$$

Differentiating V along the solution to (2.14), we obtain

$$\begin{aligned}\dot V + 2\alpha V &= |z(t)|_{Q_\alpha}^2 + 2z^\top(t)P\left[\mathbb{A}\Upsilon_{\Delta a}(t) + \mathbb{W}\Upsilon_\varrho(t)\right]\\ &\quad - 2z^\top(t)P\mathbb{A}_1\Upsilon_{\varrho,a}(t).\end{aligned} \tag{2.17}$$

Substituting (2.9) and recalling (2.13), we have

$$|z(t)|_{Q_\alpha}^2 = |x(t)|_{Q_\alpha}^2 + \left|\Upsilon_\varrho(t)\right|_{\mathbb{A}^\top Q_\alpha\mathbb{A}}^2 - 2x^\top(t)Q_\alpha\mathbb{A}\Upsilon_\varrho(t). \tag{2.18}$$

Similarly,

$$\begin{aligned}&z^\top(t)P\left[\mathbb{A}\Upsilon_{\Delta a}(t) + \mathbb{W}\Upsilon_\varrho(t) - \mathbb{A}_1\Upsilon_{\varrho,a}(t)\right]\\ &= \left[z(t) - \mathbb{A}\Upsilon_\rho(t)\right]^\top P\left[\mathbb{A}\Upsilon_{\Delta a}(t) + \mathbb{W}\Upsilon_\varrho(t) - \mathbb{A}_1\Upsilon_{\varrho,a}(t)\right].\end{aligned} \tag{2.19}$$

To compensate $\Upsilon_\varrho(t)$, $\Upsilon_{\varrho,a}(t)$ and $\Upsilon_{\Delta a}(t)$ in the Lyapunov analysis, we will employ the S-procedure (Fridman, 2014). Let

$$H_\varrho = \text{col}\left\{\mathfrak{h}_\varrho^{(i)}\right\}_{i=1}^2,\quad H_{\varrho,a} = \text{col}\left\{\mathfrak{h}_{\varrho,a}^{(i,k)}\right\}_{\{(i,k)\}\le_{\text{lex}}} \tag{2.20}$$

with *nonnegative* entries such that $\forall i,k=1,2,\ t\ge 0$

$$(I)\quad \varrho_{\epsilon,i}^2(t)\le\mathfrak{h}_\varrho^{(i)},\quad (II)\quad \varrho_{\epsilon,i}^2(t)a_k^2\left(\frac{t}{\epsilon_k}\right)\le\mathfrak{h}_{\varrho,a}^{(i,k)}. \tag{2.21}$$

Uniformly for (small) $\epsilon_i,\epsilon_k>0$. The terms on the left-hand side of (2.21) are scalar-valued and can be efficiently bounded using tools from calculus. This is in contrast with Katz, Mazenc, and Fridman (2023), where bounds were derived on *matrix*-valued functions, using Jensen's inequalities, which result in much more conservative estimates.

Remark 2.2. Assuming that the averages of $a_i,\ i=1,2$ are zero is an important component of the system presentation and leads to essentially less conservative LMI conditions (see Remark 2.8 below). Note that this assumption poses no loss of generality, since we can always subtract the averages from the corresponding functions, while retaining Δa_i on the right-hand side of (2.5) and modifying the matrix A_{av}. This assumption leads to $\left\{a_i,\varrho_{\epsilon,i}\right\}_{i=1}^2$ having smaller L^∞ norms (whence the upper bounds in (2.21) will be of smaller magnitude) and plays a key role in achieving the less conservative LMIs (2.40) via the Lyapunov analysis.

By (2.21), let $\Lambda_{\Upsilon_\varrho}, \Lambda_{\Upsilon_{\Delta a}} \in \mathbb{R}^{2\times 2}$ and $\Lambda_{\Upsilon_{\varrho,a}} \in \mathbb{R}^{4\times 4}$ be positive *diagonal* matrices. We have

$$\begin{aligned}&\Upsilon_\varrho^\top(t)\left(\Lambda_{\Upsilon_\varrho}\otimes I_n\right)\Upsilon_\varrho(t) \le \left|\Lambda_{\Upsilon_\varrho}H_\varrho\right|_1 |x(t)|^2,\\ &\Upsilon_{\varrho,a}^\top(t)\left(\Lambda_{\Upsilon_{\varrho,a}}\otimes I_n\right)\Upsilon_{\varrho,a}(t) \le \left|\Lambda_{\Upsilon_{\varrho,a}}H_{\varrho,a}\right|_1 |x(t)|^2,\\ &\Upsilon_{\Delta a}^\top(t)\left(\Lambda_{\Upsilon_{\Delta a}}\otimes I_n\right)\Upsilon_{\Delta a}(t) \le \left|\Lambda_{\Upsilon_{\Delta a}}\Delta_{a,M}\right|_1 |x(t)|^2,\end{aligned} \tag{2.22}$$

where $\Delta_{a,M} = \mathrm{col}\left\{\Delta_{a_i,M}\right\}_{i=1}^2$. The matrices $\Lambda_{\Upsilon_\varrho}, \Lambda_{\Upsilon_{\Delta a}}$ and $\Lambda_{\Upsilon_{\varrho,a}}$ are *decision variables* in the LMI (2.26) below. Denoting

$$\eta(t) = \mathrm{col}\left\{x(t), \Upsilon_\varrho(t), \Upsilon_{\varrho,a}(t), \Upsilon_{\Delta a}(t)\right\} \tag{2.23}$$

(2.22) implies

$$\begin{aligned}&0 \le W = \eta^\top(t)\left[\Lambda_0 - \Lambda_1\right]\eta(t),\\ &\Lambda_0 = \mathrm{diag}\left\{\Lambda_0^{(1)}, 0, 0, 0\right\},\quad \Lambda_1 = \mathrm{diag}\left\{0, \Lambda_1^{(1)}\right\},\\ &\Lambda_0^{(1)} = \left(\left|\Lambda_{\Upsilon_\varrho}H_\varrho\right|_1 + \left|\Lambda_{\Upsilon_{\varrho,a}}H_{\varrho,a}\right|_1 + \left|\Lambda_{\Upsilon_{\Delta a}}\Delta_{a,M}\right|_1\right) I_n,\\ &\Lambda_1^{(1)} = \mathrm{diag}\left\{\Lambda_{\Upsilon_\varrho}\otimes I_n, \Lambda_{\Upsilon_{\varrho,a}}\otimes I_n, \Lambda_{\Upsilon_{\Delta a}}\otimes I_n\right\}.\end{aligned} \tag{2.24}$$

By (2.17)–(2.24) and the S-procedure (see e.g. Fridman (2014))

$$\dot V + 2\alpha V + W \le \eta^\top(t)\Psi_\epsilon\eta(t) \le 0, \tag{2.25}$$

provided

$$\begin{aligned}&\Psi_\epsilon = \left[\begin{array}{c|c|c} Q_\alpha + \Lambda_0^{(1)} & -Q_\alpha\mathbb{A} + P\mathbb{W} & \Psi_\epsilon^{(1)}\\ \hline * & \Psi_\epsilon^{(2)} & \Psi_\epsilon^{(3)}\\ \hline * & * & \Psi_\epsilon^{(4)}\end{array}\right] < 0,\\ &\Psi_\epsilon^{(1)} = \begin{bmatrix}-P\mathbb{A}_1 & P\mathbb{A}\end{bmatrix},\ \Psi_\epsilon^{(3)} = \begin{bmatrix}\mathbb{A}^\top P\mathbb{A}_1 & -\mathbb{A}^\top P\mathbb{A}\end{bmatrix}\\ &\Psi_\epsilon^{(2)} = -\left(\Lambda_{\Upsilon_\varrho}\otimes I_n\right) + \mathbb{A}^\top Q_\alpha\mathbb{A} - \mathbb{A}^\top P\mathbb{W} - \mathbb{W}^\top P\mathbb{A},\\ &\Psi_\epsilon^{(4)} = \begin{bmatrix}-\left(\Lambda_{\Upsilon_{\varrho,a}}\otimes I_n\right) & 0\\ 0 & -\left(\Lambda_{\Upsilon_{\Delta a}}\otimes I_n\right)\end{bmatrix}.\end{aligned} \tag{2.26}$$

We now modify the analysis for ISS-like estimates where $d \in C^1([0,\infty))$. First, introduce

$$\begin{aligned}\omega_{\epsilon_{d,j}}(t) = &-\tfrac{1}{\epsilon_{d,j}T_{d,j}}\\ &\times \int_t^{t+\epsilon_{d,j}T_{d,j}}\left(t+\epsilon_{d,j}T_{d,j}-s\right) b_j\left(\tfrac{s}{\epsilon_{d,j}}\right) ds.\end{aligned} \tag{2.27}$$

By arguments of (2.7), $\sup_{t\in\mathbb{R}}\left|\omega_{\epsilon_{d,j}}(t)\right| = O(\epsilon_j)$. We will further employ the notation

$$\delta_d := \sup_{t\ge 0}\left\|\textstyle\sum_{i=1}^n \omega_{\epsilon_{d,i}}(t)B_i\right\|. \tag{2.28}$$

Analogously to (2.10), we have

$$\delta_d \le \frac{1}{2}\sum_{i=1}^2 \epsilon_i T_i b_{i,M}\left\|B_i\right\|,\quad b_{i,M} := \sup_{\tau\in\mathbb{R}}\left|b_i(\tau)\right|. \tag{2.29}$$

Differentiating (2.27), we have for $t \ge 0$

$$\dot\omega_{\epsilon_{d,j}}(t) = b_j\left(\tfrac{t}{\epsilon_{d,j}}\right) - \Delta b_j\left(\tfrac{t}{\epsilon_{d,j}}\right). \tag{2.30}$$

For ISS-like estimates, the system transformation is

$$z(t) = x(t) - \sum_{i=1}^2 \varrho_{\epsilon,i}(t)A_i x(t) - \sum_{j=1}^2 \omega_{\epsilon,j}(t)B_j d(t). \tag{2.31}$$

Note that $d \in C^1([0,\infty))$ implies that $z \in C^1([0,\infty))$.

Remark 2.3. For the case of (2.1) with a single time-scale, the time-delay transformation employed in Fridman and Zhang (2020) has the form

$$\begin{aligned}&z(t) = x(t) - G(t),\\ &G(t) = \tfrac{1}{\epsilon T}\int_{t-\epsilon T}^t(\tau - t + \epsilon T)\left[A(s)x(\epsilon s) + B(s)d(\epsilon s)\right] ds,\end{aligned}$$

which leads to a neutral-type system. This transformation allows for ISS analysis which employs averaging of $B\left(\frac{t}{\epsilon}\right)$ for measurable functions d, whereas (2.31) allows ISS for non differentiable d without averaging of $B\left(\frac{t}{\epsilon}\right)$ only, which may be restrictive. Compared to Fridman and Zhang (2020), here we consider multiple rapid time-scales and unify the transformation in Katz, Mazenc, and Fridman (2023) with a novel system presentation. The non-delayed transformation (2.31) simplifies the Lyapunov-based analysis whereas the new system presentation (2.3) significantly improves the results in the numerical examples (see Section 2.3).

Let

$$\begin{aligned}&\mathcal{Z}_\omega(t) = \mathrm{col}\left\{\omega_{\epsilon_{d,j}}(t)d(t)\right\}_{j=1}^2,\\ &\mathcal{Z}_\varrho(t) = \mathrm{col}\left\{\varrho_{\epsilon,i}(t)d(t)\right\}_{i=1}^2,\\ &\Xi_\omega(t) = \mathrm{col}\left\{\omega_{\epsilon_{d,j}}(t)\dot d(t)\right\}_{j=1}^2,\\ &\mathcal{Z}_{\varrho,b}(t) = \mathrm{col}\left\{\varrho_{\epsilon,i}(t)b_j\left(\tfrac{t}{\epsilon_{d,j}}\right)d(t)\right\}_{\{(i,j)\}_{\le_{\mathrm{lex}}}},\\ &\mathcal{Z}_{\Delta b}(t) = \mathrm{col}\left\{\Delta b_j\left(\tfrac{t}{\epsilon_{d,j}}\right)d(t)\right\}_{j=1}^2,\\ &\mathbb{A}_2 = \left[A_1B_1\ A_1B_2\ A_2B_1\ A_2B_2\right],\ \mathbb{B} = \begin{bmatrix}B_1 & B_2\end{bmatrix}.\end{aligned} \tag{2.32}$$

Then, the new expression for $\dot z(t),\ t \ge 0$ is

$$\begin{aligned}\dot z(t) &= A_{av}z(t) + B_{av}d(t) + \mathbb{A}\Upsilon_{\Delta a}(t) + \mathbb{B}\mathcal{Z}_{\Delta b}(t)\\ &-\mathbb{A}\left(I_2\otimes B_{av}\right)\mathcal{Z}_\varrho(t) + \mathbb{W}\Upsilon_\varrho(t) - \mathbb{B}\Xi_\omega(t)\\ &+A_{av}\mathbb{B}\mathcal{Z}_\omega(t) - \mathbb{A}_1\Upsilon_{\varrho,a}(t) - \mathbb{A}_2\mathcal{Z}_{\varrho,b}(t).\end{aligned} \tag{2.33}$$

For Lyapunov ISS-like analysis we use (2.15) and arguments similar to (2.17)–(2.24). To employ the S-procedure, denote

$$H_\omega = \mathrm{col}\left\{\mathfrak{h}_\omega^{(j)}\right\}_{j=1}^2,\quad H_{\varrho,b} = \mathrm{col}\left\{\mathfrak{h}_{\varrho,b}^{(i,k)}\right\}_{\{(i,k)\}_{\le_{\mathrm{lex}}}} \tag{2.34}$$

be vectors with nonnegative entries such that

$$(III)\quad \omega_{\epsilon_{d,j}}^2(t) \le \mathfrak{h}_\omega^{(j)},\quad (IV)\quad \varrho_{\epsilon,i}^2(t)b_j^2\left(\tfrac{t}{\epsilon_{d,j}}\right) \le \mathfrak{h}_{\varrho,b}^{(i,j)} \tag{2.35}$$

$\forall i,j = 1,2,\ t \ge 0$, uniformly for (small) $\epsilon_{d,j} > 0$. Let $\Lambda_{\mathcal{Z}_\varrho}, \Lambda_{\mathcal{Z}_\omega}, \Lambda_{\Xi_\omega}, \Lambda_{\mathcal{Z}_{\Delta b}} \in \mathbb{R}^{2\times 2}$ and $\Lambda_{\mathcal{Z}_{\varrho,b}} \in \mathbb{R}^{4\times 4}$ be positive *diagonal* matrices (decision variables). We then have

$$\begin{aligned}&\mathcal{Z}_\varrho^\top(t)\left(\Lambda_{\mathcal{Z}_\varrho}\otimes I_{n_d}\right)\mathcal{Z}_\varrho(t) \le \left|\Lambda_{\mathcal{Z}_\varrho}H_\varrho\right|_1 |d(t)|^2,\\ &\mathcal{Z}_\omega^\top(t)\left(\Lambda_{\mathcal{Z}_\omega}\otimes I_{n_d}\right)\mathcal{Z}_\omega(t) \le \left|\Lambda_{\mathcal{Z}_\omega}H_\omega\right|_1 |d(t)|^2,\\ &\Xi_\omega^\top(t)\left(\Lambda_{\Xi_\omega}\otimes I_{n_d}\right)\Xi_\omega(t) \le \left|\Lambda_{\Xi_\omega}H_\omega\right|_1 \left|\dot d(t)\right|^2,\\ &\mathcal{Z}_{\varrho,b}^\top(t)\left(\Lambda_{\mathcal{Z}_{\varrho,b}}\otimes I_{n_d}\right)\mathcal{Z}_{\varrho,b}(t) \le \left|\Lambda_{\mathcal{Z}_{\varrho,b}}H_{\varrho,b}\right|_1 |d(t)|^2,\\ &\mathcal{Z}_{\Delta b}^\top(t)\left(\Lambda_{\mathcal{Z}_{\Delta b}}\otimes I_{n_d}\right)\mathcal{Z}_{\Delta b}(t) \le \left|\Lambda_{\mathcal{Z}_{\Delta b}}\Delta_{b,M}\right|_1 |d(t)|^2\end{aligned} \tag{2.36}$$

where $\Delta_{b,M} = \mathrm{col}\left\{\Delta_{b_j,M}\right\}_{j=1}^2$. Denoting

$$\begin{aligned}\eta(t) = \mathrm{col}\{&x(t), d(t), \dot d(t), \Upsilon_\varrho(t), \Upsilon_{\varrho,a}(t), \Upsilon_{\Delta a}(t)\\ &\mathcal{Z}_\varrho(t), \mathcal{Z}_{\varrho,b}(t), \mathcal{Z}_{\Delta b}(t), \mathcal{Z}_\omega(t), \Xi_\omega(t)\},\end{aligned} \tag{2.37}$$

we have the following upper bound

$$\begin{aligned}&0 \le W = \eta^\top(t)\left[\Lambda_0 - \Lambda_1\right]\eta(t),\\ &\Lambda_0 = \mathrm{diag}\left\{\Lambda_0^{(1)}, \Lambda_0^{(2)}, \Lambda_0^{(3)}, 0, 0, 0, 0, 0, 0, 0, 0\right\},\\ &\Lambda_1 = \mathrm{diag}\left\{0, 0, 0, \Lambda_1^{(1)}\right\},\ \Lambda_0^{(3)} = \left|\Lambda_{\Xi_\omega}H_\omega\right|_1 I_{n_d},\\ &\Lambda_0^{(1)} = \left(\left|\Lambda_{\Upsilon_\varrho}H_\varrho\right|_1 + \left|\Lambda_{\Upsilon_{\varrho,a}}H_{\varrho,a}\right|_1 + \left|\Lambda_{\Upsilon_{\Delta a}}\Delta_{a,M}\right|_1\right) I_n,\\ &\Lambda_0^{(2)} = \left(\left|\Lambda_{\mathcal{Z}_\varrho}H_\varrho\right|_1 + \left|\Lambda_{\mathcal{Z}_\omega}H_\omega\right|_1 + \left|\Lambda_{\mathcal{Z}_{\varrho,b}}H_{\varrho,b}\right|_1\right.\\ &\qquad\left.+\left|\Lambda_{\mathcal{Z}_{\Delta b}}\Delta_{b,M}\right|_1\right) I_{n_d},\\ &\Lambda_1^{(1)} = \mathrm{diag}\left\{\Lambda_{\Upsilon_\varrho}\otimes I_n, \Lambda_{\Upsilon_{\varrho,a}}\otimes I_n, \Lambda_{\Upsilon_{\Delta a}}\otimes I_n, \Lambda_{\mathcal{Z}_\varrho}\right.\\ &\left.\otimes I_{n_d}, \Lambda_{\mathcal{Z}_{\varrho,b}}\otimes I_{n_d}, \Lambda_{\mathcal{Z}_{\Delta b}}\otimes I_{n_d}, \Lambda_{\mathcal{Z}_\omega}\otimes I_{n_d}, \Lambda_{\Xi_\omega}\otimes I_{n_d}\right\}.\end{aligned} \tag{2.38}$$

Letting $\gamma_1, \gamma_2 \in \mathbb{R}$ be *tuning parameters*, we obtain

$$\begin{aligned}&\dot V + 2\alpha V - \gamma_1^2|d(t)|^2 - \gamma_2^2\left|\dot d(t)\right|^2 + W\\ &\qquad\le \eta^\top(t)\Psi_{\epsilon,\epsilon_d}\eta(t) \le 0,\end{aligned} \tag{2.39}$$

provided

$$\Psi_{\epsilon,\epsilon_d}=\left[\begin{array}{c|c|c|c}\Psi^{(1)}_{\epsilon,\epsilon_d} & \Psi^{(2)}_{\epsilon,\epsilon_d} & \Psi^{(3)}_{\epsilon,\epsilon_d} & \Psi^{(4)}_{\epsilon,\epsilon_d}\\ \hline * & \Psi^{(5)}_{\epsilon,\epsilon_d} & \Psi^{(6)}_{\epsilon,\epsilon_d} & \Psi^{(7)}_{\epsilon,\epsilon_d}\\ \hline * & * & \Psi^{(8)}_{\epsilon,\epsilon_d} & \Psi^{(9)}_{\epsilon,\epsilon_d}\\ \hline * & * & * & \Psi^{(10)}_{\epsilon,\epsilon_d}\end{array}\right]<0 \tag{2.40}$$

with

$$\Psi^{(1)}_{\epsilon,\epsilon_d}=\begin{bmatrix}Q_\alpha+\Lambda_0^{(1)} & PB_{av} & 0\\ * & -\gamma_1^2 I_{n_d}+\Lambda_0^{(2)} & 0\\ * & * & -\gamma_2^2 I_{n_d}+\Lambda_0^{(3)}\end{bmatrix},$$

$$\Psi^{(2)}_{\epsilon,\epsilon_d}=\begin{bmatrix}-Q_\alpha\mathbb{A}+P\mathbb{W} & -P\mathbb{A}_1 & P\mathbb{A}\\ -B_{av}^\top P\mathbb{A} & 0 & 0\\ 0 & 0 & 0\end{bmatrix},$$

$$\Psi^{(3)}_{\epsilon,\epsilon_d}=\begin{bmatrix}-P\mathbb{A}\left(I_2\otimes B_{av}\right) & -P\mathbb{A}_2 & P\mathbb{B}\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix},$$

$$\Psi^{(4)}_{\epsilon,\epsilon_d}=\begin{bmatrix}-Q_\alpha\mathbb{B}+PA_{av}\mathbb{B} & -P\mathbb{B}\\ -B_{av}^\top P\mathbb{B} & 0\\ 0 & 0\end{bmatrix},$$

$$\Psi^{(5)}_{\epsilon,\epsilon_d}=\begin{bmatrix}\psi^{(1)}_{\epsilon,\epsilon_d} & \mathbb{A}^\top P\mathbb{A}_1 & -\mathbb{A}^\top P\mathbb{A}\\ * & -\left(\Lambda_{\Upsilon_{\varrho,a}}\otimes I_n\right) & 0\\ * & * & -\left(\Lambda_{\Upsilon_{\Delta a}}\otimes I_n\right)\end{bmatrix},$$

$$\Psi^{(6)}_{\epsilon,\epsilon_d}=\begin{bmatrix}\psi^{(4)}_{\epsilon,\epsilon_d} & \mathbb{A}^\top P\mathbb{A}_2 & -\mathbb{A}^\top P\mathbb{B}\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix},$$

$$\Psi^{(7)}_{\epsilon,\epsilon_d}=\begin{bmatrix}\psi^{(2)}_{\epsilon,\epsilon_d} & \mathbb{A}^\top P\mathbb{B}\\ \mathbb{A}_1^\top P\mathbb{B} & 0\\ -\mathbb{A}^\top P\mathbb{B} & 0\end{bmatrix},\quad \Psi^{(9)}_{\epsilon,\epsilon_d}=\begin{bmatrix}\psi^{(3)}_{\epsilon,\epsilon_d} & 0\\ \mathbb{A}_2^\top P\mathbb{B} & 0\\ -\mathbb{B}^\top P\mathbb{B} & 0\end{bmatrix},$$

$$\Psi^{(8)}_{\epsilon,\epsilon_d}=-\operatorname{diag}\left\{\Lambda_{\mathcal{Z}_\varrho},\Lambda_{\mathcal{Z}_{\varrho,b}},\Lambda_{\mathcal{Z}_{\Delta b}}\right\}\otimes I_n,$$

$$\Psi^{(10)}_{\epsilon,\epsilon_d}=\begin{bmatrix}-\left(\Lambda_{\mathcal{Z}_\omega}\otimes I_n\right)+2\alpha\mathbb{B}^\top P\mathbb{B} & \mathbb{B}^\top P\mathbb{B}\\ * & -\left(\Lambda_{\Xi_\omega}\otimes I_n\right)\end{bmatrix},$$

$$\psi^{(1)}_{\epsilon,\epsilon_d}=-\left(\Lambda_{\Upsilon_\varrho}\otimes I_n\right)+\mathbb{A}^\top Q_\alpha\mathbb{A}-\mathbb{A}^\top P\mathbb{W}-\mathbb{W}^\top P\mathbb{A},$$

$$\psi^{(2)}_{\epsilon,\epsilon_d}=\mathbb{A}^\top Q_\alpha\mathbb{B}-\mathbb{W}^\top P\mathbb{B}-\mathbb{A}^\top PA_{av}\mathbb{B},$$

$$\psi^{(3)}_{\epsilon,\epsilon_d}=\left(I_2\otimes B_{av}\right)^\top\mathbb{A}^\top P\mathbb{B},\quad \psi^{(4)}_{\epsilon,\epsilon_d}=\mathbb{A}^\top P\mathbb{A}\left(I_2\otimes B_{av}\right).$$

Remark 2.4. Differently from the preliminary analysis in Katz, Mazenc, and Fridman (2023), where $\eta(t)$ in (2.37) contained $z(t)$ instead of $x(t)$ and the inversion of the transformation (2.31) was used in the S-procedure, here the analysis is presented in terms of $x(t)$. This approach significantly reduces the conservatism of the derived LMIs (see examples below) and improves the derived bound on the small parameter.

Summarizing, we arrive at:

Theorem 2.1. *Consider (2.3) subject to Assumptions 1 and 2. Let $H_\varrho, H_\omega, H_{\varrho,a}, H_{\varrho,b}$ be given by (2.20) and (2.34). Given positive tuning parameters α, $\left\{\epsilon_i^*\right\}_{i=1}^2$, $\left\{\epsilon_{d,j}^*\right\}_{j=1}^2$, $\left\{\Delta_{a_i,M}\right\}_{i=1}^2$, $\left\{\Delta_{b_j,M}\right\}_{j=1}^2$ let there exist $0<P\in\mathbb{R}^{n\times n}$, positive diagonal matrices $\Lambda_{\Upsilon_\varrho},\Lambda_{\mathcal{Z}_\varrho}$, $\Lambda_{\Upsilon_{\Delta a}}\in\mathbb{R}^{2\times 2}$, $\Lambda_{\mathcal{Z}_\omega},\Lambda_{\Xi_\omega},\Lambda_{\mathcal{Z}_{\Delta b}}\in\mathbb{R}^{2\times 2}$ and $\Lambda_{\Upsilon_{\varrho,a}},\Lambda_{\mathcal{Z}_{\varrho,b}}\in\mathbb{R}^{4\times 4}$, and positive scalars γ_1^2,γ_2^2 such that $\Psi_{\epsilon^*,\epsilon_d^*}<0$, with $\Psi_{\epsilon,\epsilon_d}$ given by (2.40). Then for all $\epsilon\le\epsilon^*$ and $\epsilon_d\le\epsilon_d^*$, the solutions of (2.3) satisfy the ISS-like estimate*

$$\begin{aligned}|x(t)|^2 &\le \beta_1^2 e^{-2\alpha t}|x(0)|^2+\beta_2^2\max_{s\in[0,t]}|d(s)|^2\\ &\quad+\beta_3^2\max_{s\in[0,t]}\left|\dot d(s)\right|^2,\quad t\ge 0\end{aligned}\tag{2.41}$$

for some β_i, $i=1,2,3$. The LMI $\Psi_{\epsilon,\epsilon_d}<0$ is feasible for small enough α, ϵ_i, $\epsilon_{d,i}$, $\Delta_{a_i,M}$, $\Delta_{b_i,M}$, $i=1,2$ and large enough γ_i^2, $i=1,2$.

Proof. The fact that feasibility of (2.40) for some α, $\left\{\epsilon_i^*\right\}_{i=1}^2$, $\left\{\epsilon_{d,j}^*\right\}_{j=1}^2$ implies its feasibility for all $\epsilon_i<\epsilon_i^*$, $i=1,2$ and $\epsilon_{d,j}<\epsilon_{d,j}^*$, $j=1,2$ and the same α, γ_i, $i=1,2$ follows from monotonicity of (2.40) with respect to $\epsilon_i<\epsilon_i^*$, $i=1,2$ and $\epsilon_{d,j}<\epsilon_{d,j}^*$, $j=1,2$ (meaning that as the small parameters decrease, the eigenvalues of $\Psi_{\epsilon,\epsilon_d}$ are non-increasing).

Fix $\tau>0$. Feasibility of (2.40) implies that for all $t\in[0,\tau]$

$$\begin{aligned}&\dot V+2\alpha V-\gamma_1^2|d(t)|^2-\gamma_2^2\left|\omega_{\epsilon_d}(t)\dot d(t)\right|^2\le 0\\ &\Rightarrow V(t)\le e^{-2\alpha t}V(0)\\ &\qquad+\int_0^t e^{-2\alpha(t-s)}\left(\gamma_1^2|d(s)|^2+\gamma_2^2\left|\dot d(s)\right|^2\right)ds.\end{aligned}$$

Since $\lambda_{min}(P)|z(t)|^2\le V(t)\le\lambda_{max}(P)|z(t)|^2$ for all $t\ge 0$, we have

$$\begin{aligned}&|z(t)|^2\le\frac{\lambda_{max}(P)}{\lambda_{min}(P)}e^{-2\alpha t}|z(0)|^2+\frac{\gamma_1^2}{2\alpha\lambda_{min}(P)}\\ &\times\max_{s\in[0,\tau]}|d(s)|^2+\frac{\gamma_2^2}{2\alpha\lambda_{min}(P)}\max_{s\in[0,\tau]}\left|\dot d(s)\right|^2,\end{aligned}\tag{2.42}$$

meaning that (2.33) satisfies ISS-like estimates with respect to d and $\dot d$. To obtain ISS-like estimates for (2.3), we employ the transformation (2.31). By Assumption 2, (2.10), (2.28), Young's inequality and the triangle inequality

$$\begin{aligned}|z(0)|^2&\le 2\delta_{2,x}^2|x(0)|^2+2\delta_d^2\max_{s\in[0,\tau]}|d(s)|^2\\ |x(t)|^2&\le\delta_{1,x}^2\left|z(t)+\sum_{i=1}^2\omega_{\epsilon_{d,i}}(t)B_id(t)\right|^2\\ &\le 2\delta_{1,x}^2|z(t)|^2+2\delta_{1,x}^2\delta_d^2\max_{s\in[0,\tau]}|d(s)|^2.\end{aligned}$$

By combining the latter with (2.42), we obtain (2.41) with

$$\begin{aligned}&\beta_1^2=\frac{4\delta_{1,x}^2\delta_{2,x}^2\lambda_{max}(P)}{\lambda_{min}(P)},\quad\beta_3^2=\frac{2\delta_{1,x}^2\gamma_2^2}{2\alpha\lambda_{min}(P)},\\ &\beta_2^2=2\delta_{1,x}^2\left[\delta_d^2\frac{2\lambda_{max}(P)+\lambda_{min}(P)}{\lambda_{min}(P)}+\frac{\gamma_1^2}{2\alpha\lambda_{min}(P)}\right].\end{aligned}$$

For LMI feasibility guarantees, it is enough to consider the case when the small parameters satisfy $\epsilon_i=\epsilon_{d,j}=\epsilon$, $i,j=1,2$. Recall (2.21) and (2.35). It can be easily verified that there exists a constant $\mathcal{K}>0$ large enough, such that both hold when all entries of (2.20) and (2.34) are equal to $\mathcal{K}\epsilon^2$. Next, choose $\Lambda_{\Upsilon_\varrho},\Lambda_{\mathcal{Z}_\varrho},\Lambda_{\Upsilon_{\Delta a}},\Lambda_{\mathcal{Z}_\omega},\Lambda_{\Xi_\omega},\Lambda_{\mathcal{Z}_{\Delta b}}=\lambda I_2$ and $\Lambda_{\Upsilon_{\varrho,a}},\Lambda_{\mathcal{Z}_{\varrho,b}}=\lambda I_4$, where $\lambda>0$. Henceforth, we fix these choices. We begin by choosing $\alpha=0$, $0<P\in\mathbb{R}^n$ such that $Q_\alpha<0$ (see (2.16)). Fixing P and $\epsilon<1$ we look at the LMI (2.40). Considering the bottom-right 3×3 block submatrix (which we will henceforth denote as $\Xi_{\epsilon,\epsilon_d}$) we see that $\Xi_{\epsilon,\epsilon_d}<0$ for $\lambda>\lambda_*$ with $\lambda_*>0$ large enough (the diagonal elements are linear and negative in λ). Next, we apply Schur complement with respect to $\Xi_{\epsilon,\epsilon_d}$, to obtain the equivalent matrix inequality

$$\begin{aligned}&\Psi^{(1)}_{\epsilon,\epsilon_d}-\frac{1}{\lambda}\left[\Psi^{(2)}_{\epsilon,\epsilon_d}\quad\Psi^{(3)}_{\epsilon,\epsilon_d}\quad\Psi^{(4)}_{\epsilon,\epsilon_d}\right]\left(\lambda^{-1}\Xi_{\epsilon,\epsilon_d}\right)^{-1}\\ &\qquad\times\left[\Psi^{(2)}_{\epsilon,\epsilon_d}\quad\Psi^{(3)}_{\epsilon,\epsilon_d}\quad\Psi^{(4)}_{\epsilon,\epsilon_d}\right]^\top<0.\end{aligned}\tag{2.43}$$

Note that $\left(\lambda^{-1}\Xi_{\epsilon,\epsilon_d}\right)^{-1}$ is bounded as $\lambda\to\infty$ (converges to the identity matrix), whereas $\left[\Psi^{(2)}_{\epsilon,\epsilon_d}\quad\Psi^{(3)}_{\epsilon,\epsilon_d}\quad\Psi^{(4)}_{\epsilon,\epsilon_d}\right]$ is independent of λ. On the other hand, for any $\lambda>0$, we can always find $\epsilon>0$ small enough and $\gamma_i>0$, $i=1,2$ large enough so that $\Psi^{(1)}_{\epsilon,\epsilon_d}<0$. Indeed, by choosing $\gamma_i=\lambda^2$, $i=1,2$, $\epsilon=\frac{1}{\lambda^2}$, we obtain that (2.43) holds for $\lambda>0$ large enough, whence feasibility of (2.40) follows. $\square$

Remark 2.5. Recall (2.21) and (2.35). In the Lyapunov analysis above we assume the scalar bounds on the right-hand side of both are identical for all $t\ge 0$. Assume that there exists a partition of $[0,\infty)$ into intervals such that every interval in the partition belongs to one of finitely many classes (types), denoted by $\left\{\mathcal{I}_j\right\}_{j=1}^{\zeta}$. As an example, consider Example 3.1 below, where we treat a switched system with two functioning modes. In this case $\zeta = 2$ and $\mathcal{I}_1$ corresponds to subintervals where $A(\tau) \equiv A_1$, whereas $\mathcal{I}_2$ corresponds to subintervals where $A(\tau) \equiv A_2$. Assume that for each $1 \leq j \leq \zeta$, there exist vectors $H_{\varrho,j}, H_{\omega,j}, H_{\varrho,a,j}, H_{\varrho,b,j}$ whose entries serve as upper bounds in (2.21) and (2.35) whenever $t \geq 0$ belongs to an interval of type $\mathcal{I}_j$ (the vectors may vary between classes). In this case our proposed approach can be applied to each of the classes separately and will yield ζ LMIs of the form (2.40) (one for each class). Feasibility of the LMIs can then be verified simultaneously with *the same* P and γ_i, $i = 1, 2$. Note that the decision matrices $\Lambda_{\Upsilon_\varrho}, \Lambda_{\mathcal{Z}_\varrho}, \Lambda_{\Upsilon_{\Delta a}}, \Lambda_{\mathcal{Z}_\omega}, \Lambda_{\Xi_\omega}, \Lambda_{\mathcal{Z}_{\Delta b}}, \Lambda_{\Upsilon_{\varrho,a}}, \Lambda_{\mathcal{Z}_{\varrho,b}}$ may differ between LMIs corresponding to different classes. This approach is expected to yield less conservative results than choosing bounds in (2.21) and (2.35) which hold uniformly for all $t \geq 0$, and verifying feasibility of a single LMI (2.40).

Remark 2.6. For general $N, N_d \in \mathbb{N}$, the proposed approach requires only minor modifications, which are related to the dimensions of the matrices. In particular, in (2.13) and (2.32) the dimensions of the vectors require changing, whereas the matrices now having the form

$$\begin{aligned}
&\mathbb{A} = \begin{bmatrix} A_1 & \dots & A_N \end{bmatrix}, \ \mathbb{B} = \begin{bmatrix} B_1 & \dots & B_{N_d} \end{bmatrix}, \\
&\mathbb{A}_1 = [A_1^2 \dots A_1 A_N \dots A_N A_1 \dots A_N^2], \\
&\mathbb{A}_2 = [A_1 B_1 \dots A_1 B_{N_d} \dots A_N B_1 \dots A_N B_{N_d}], \\
&\mathbb{W} = \begin{bmatrix} W_1 & \dots & W_N \end{bmatrix}, \\
&W_i = A_{av} A_i - A_i A_{av}, \ \ 1 \leq i \leq N.
\end{aligned} \tag{2.44}$$

The system (2.33) (and derived LMIs) will have the same form with I_2 replaced by I_{N_d}. Thus, the Lyapunov analysis and LMIs of Section 2.3, subject to the changes in (2.44) and I_2 replaced by I_{N_d}, will guarantee (2.41) for (2.3).

Remark 2.7. Instead of the ISS-like estimates (2.41), we are also able to obtain standard ISS bounds (i.e., with respect to d only) for (2.3). Consider the system (2.3). In order to avoid introducing the disturbance derivative one can simply not use averaging for $\left[\sum_{i=1}^{2} b_i \left(\frac{t}{\epsilon_{d,i}}\right) B_i\right] d(t)$. Instead, one can treat this term as a norm bounded time-varying matrix-valued function which multiplies the disturbance. In this case the presentation of this matrix valued function as a linear combination is obviously not needed and (2.31) will be replaced with $z(t) = x(t) - \sum_{i=1}^{2} \varrho_{\epsilon,i}(t) A_i x(t)$. The norm bound on $\sum_{i=1}^{2} b_i \left(\frac{t}{\epsilon_{d,i}}\right) B_i$ will be employed in a standard ISS analysis. This approach is expected to result in larger estimates on the ISS gains.

2.3. Numerical examples

Example 2.1 (*Stabilization by Fast Switching I*). We consider a switched linear system with two unstable modes (see Fridman and Zhang (2020, Example 2.2)), defined by

$$A_1 = \begin{bmatrix} 0.1 & 0.3 \\ 0.6 & -0.2 \end{bmatrix}, \ A_2 = \begin{bmatrix} -0.13 & -0.16 \\ -0.33 & 0.03 \end{bmatrix}. \tag{2.45}$$

Given $\tau \in [k, k+1)$, $k \in \mathbb{Z}_+$, let

$$A(\tau) = \chi_{[k,k+0.4)}(\tau) A_1 + \left[1 - \chi_{[k+0.4,k+1)}(\tau)\right] A_2, \tag{2.46}$$

where $\chi_{[k,k+0.4)}$ is the indicator function of the interval $[k, k+0.4)$. Note that $A(\tau)$ is 1-periodic.

We present the switched system $\dot{x}(t) = A\left(\frac{t}{\epsilon}\right) x(t)$ as (2.3) with $\epsilon_i = \epsilon > 0, T_i = 1, \ i = 1, 2, B_{av} = B_1 = B_2 = 0$,

$$A_{av} = \begin{bmatrix} -0.038 & 0.024 \\ 0.042 & -0.062 \end{bmatrix}, \tag{2.47}$$

Table 1
Switched I - maximum value ϵ^* preserving LMI feasibility.

	$\alpha = 0$	$\alpha = \frac{1}{200}$	$\alpha = \frac{1}{100}$
Zhang and Fridman (2022)	0.192	0.13	Unchecked
Katz, Mazenc, and Fridman (2023)	0.061	0.037	Unchecked
No zero avg.	0.156	0.105	0.041
Theorem 2.1	0.433	0.3	0.166

Table 2
Switched I - ISS gains: (β_1, β_2).

	$\epsilon = 0.002$	$\epsilon = 0.16$
$\alpha = 0.005$	$(0.0054, 73.503)$	$(0.5147, 99.266)$
$\alpha = 0.01$	$(0.006, 76.48)$	$(0.7126, 389.89)$

which is Hurwitz, and

$$\begin{aligned}
&a_1(\tau) = \begin{cases} 0.6, & \tau \in [k, k+0.4), \ k \in \mathbb{Z}_+ \\ -0.4, & \tau \in [k+0.4, k+1), \ k \in \mathbb{Z}_+, \end{cases} \\
&a_2(\tau) = -a_1(\tau).
\end{aligned} \tag{2.48}$$

Note that the latter functions are 1-periodic, meaning that $\Delta_{a_i,M} = 0$, $i = 1, 2$. Let $t \in [m\epsilon, (m+1)\epsilon)$, $m \in \mathbb{Z}_+$ and denote $w = t - m\epsilon \in [0, \epsilon)$, $m \in \mathbb{Z}_+$. An explicit computation of $\varrho_{\epsilon,i}(t)$, $i = 1, 2$ yields the bounds $\varrho_{\epsilon,1}^2(t) \leq 0.0144\epsilon^2$ and $\varrho_{\epsilon,2}^2(t) \leq 0.0144\epsilon^2$. We then use the fact that $a_1(\tau)$, $a_2(\tau)$ are indicator functions to separate the analysis into two cases

$$\begin{aligned}
&a_1\left(\tfrac{t}{\epsilon}\right) \varrho_{\epsilon_j}(t) = \begin{cases} 0.6\varrho_{\epsilon,j}(t), & w \in [0, 0.4\epsilon) \\ -0.4\varrho_{\epsilon,j}(t), & w \in [0.4\epsilon, \epsilon) \end{cases} \\
&a_2\left(\tfrac{t}{\epsilon}\right) \varrho_{\epsilon,j}(t) = -a_1\left(\tfrac{t}{\epsilon}\right) \varrho_{\epsilon_j}(t)
\end{aligned}$$

and obtain tight upper bounds in (2.21) for each of the cases. Thus, we separate the analysis into the two subintervals $0 \leq w < 0.4\epsilon$ and $0.4\epsilon \leq w < \epsilon$. For each subinterval (and its corresponding bounds (2.21)) we obtain an LMI of the form (2.40) (see Remark 2.5). We verify feasibility for both LMIs with the same α and P.

We consider $\alpha \in \{0, 0.005, 0.01\}$ and verify the LMIs of Theorem 2.1 to obtain the maximal value ϵ^* which preserves feasibility of the LMIs. Note that ϵ^* guarantees internal exponential stability (and thus the ISS-like bounds) of (2.3). The values of ϵ^* are given in Table 1, where we further compare our results to the bounds in the recent work (Zhang & Fridman, 2022). We further check the proposed approach without ensuring zero average of a_i, $i = 1, 2$, as well as compare it to results of Katz, Mazenc, and Fridman (2023), where the transformation was used with matrix averaging (i.e., without the new system presentation). It is seen that our results essentially improve the results of Zhang and Fridman (2022) with a value of ϵ^* larger by 2.5 times. Moreover, guaranteeing that a_i, $i = 1, 2$ have zero average has significant impact on the conservatism of the results.

Next, we set $B_{av} = [0 \ 1]^\top$ and $B_1 = B_2 = 0_{2\times 1}$ and verify feasibility of (2.40) in order to guarantee (2.41). Note that in this case the transformation (2.31) will not result in terms involving $\dot{d}$. Hence, we obtain classical ISS estimates (i.e., we have $\gamma_2 = 0$ in (2.39) $\beta_3 = 0$ in (2.41)). Table 2 presents several pairs (β_1, β_2) (see proof of Eq. (2.41)) for different choices of α and ϵ. Note that in this case $\delta_{1,x}$ and $\delta_{2,x}$ were computed using the bounds (2.10) and (2.11).

Example 2.2 (*Stabilization by Fast Switching II*). We consider a switched linear system with three unstable modes (see Albea and Seuret (2021) and Caiazzo et al. (2023)), defined by the matrices

$$A_1 = \begin{bmatrix} 0 & 0.5 \\ 0 & -1 \end{bmatrix}, \ A_2 = \begin{bmatrix} 0.1 & 0 \\ -1 & -1 \end{bmatrix}, \ A_3 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{2.49}$$

Table 3
Switched II - maximum value ϵ^* preserving LMI feasibility.

	$\alpha = 0$	$\alpha = 0.005$	$\alpha = 0.25$
Theorem 2.1	0.4341	0.4177	0.0591

Set

$$A(\tau) = \begin{cases} A_1, & \tau \in [k, k+0.4),\ k \in \mathbb{Z}_+ \\ A_2, & \tau \in [k+0.4, k+0.87), \\ A_3, & \tau \in [k+0.87, k+1).. \end{cases} \quad (2.50)$$

Note that $A(\tau)$ is 1-periodic and can be presented as a linear combination of $A_i,\ i = 1, 2, 3$ with indicator coefficients, similarly to (2.46).

We present the switched system $\dot{x}(t) = A\left(\frac{t}{\epsilon}\right)x(t)$ as (2.3) with $\epsilon_i = \epsilon > 0, T_i = 1,\ i = 1, 2, 3, B_{av} = B_1 = B_2 = B_3 = 0$,

$$A_{av} = \begin{bmatrix} 0.047 & 0.33 \\ -0.6 & -0.87 \end{bmatrix}, \quad (2.51)$$

which is Hurwitz, and

$$\begin{aligned} a_1(\tau) &= \chi_{[k,k+0.4)}(\tau) - 0.4, \quad k \in \mathbb{Z}_+, \\ a_2(\tau) &= \chi_{[k+0.4,k+0.87)}(\tau) - 0.47, \\ a_3(\tau) &= \chi_{[k+0.87,k+1)}(\tau) - 0.13. \end{aligned}$$

Note that the latter functions are 1-periodic, meaning that $\Delta_{a_i,M} = 0,\ i = 1, 2, 3$. Similarly to Example 2.1, an explicit computation of $\varrho_{\epsilon,i}(t),\ i = 1, 2$ yields the bounds $\varrho^2_{\epsilon,1}(t) \le 0.0144\epsilon^2$, $\varrho^2_{\epsilon,2}(t) \le 0.0155127\epsilon^2$ and $\varrho^2_{\epsilon,3}(t) \le 0.0031979\epsilon^2$. We then use the fact that $a_1(\tau)$, $a_2(\tau)$ and $a_3(\tau)$ are indicator functions to separate the analysis into three cases, corresponding to the subintervals in (2.50). For each subinterval (and corresponding bounds (2.21)) we obtain an LMI of the form (2.40) (see Remarks 2.5, 2.6). We verify feasibility for both LMIs with the same α and P.

We consider $\alpha \in \{0, 0.005, 0.25\}$ and verify the LMIs of Theorem 2.1 to obtain the maximal value ϵ^* which preserves feasibility of the LMI. Note that ϵ^* guarantees internal exponential stability (and thus ISS-like bounds) of (2.3). The values of ϵ^* are given in Table 3.

Remark 2.8. In examples 2.1 and 2.2, presenting the systems as (2.3) with $A_{av} = 0$ and

Example 2.1: $a_1(\tau) = \chi_{[k,k+0.4)}(\tau)$,
$a_2(\tau) = \chi_{[k+0.4,k+1)}(\tau)$,
Example 2.2: $a_1(\tau) = \chi_{[k,k+0.4)}(\tau)$,
$a_2(\tau) = \chi_{[k+0.4,k+0.87)}(\tau)$,
$a_3(\tau) = \chi_{[k+0.87,k+1)}(\tau)$.

with non-zero averages of $a_i(\tau)$ leads to essentially smaller ϵ^*. For example, for $\alpha = 0$ we find $\epsilon^* = 0.1566$ (compared to 0.4332) in Example 2.1 and $\epsilon^* = 0.141$ (compared to 0.4341) in Example 2.1. The reason for the significantly improved results is that $\varrho_{\epsilon,i}$ become essentially smaller when the averages $a_{av,i}$ are zero, thereby decreasing the bounds required on the right-hand side of (2.21).

Example 2.3 (*Control of a Pendulum*)**.** We consider a suspended pendulum with the suspension point that is subject to vertical vibrations of small amplitude and high frequency (see Khalil (2001, Example 10.10) and Fridman and Zhang (2020, Example 2.1)). The model linearized at the upper equilibrium position is given by $\dot{x}(t) = A\left(\frac{t}{\epsilon}\right)x(t)$ with $\epsilon > 0$ and

$$A(\tau) = \begin{bmatrix} \cos(\tau) & 1 \\ 0.04 - \cos^2(\tau) & -0.2 - \cos(\tau) \end{bmatrix}, \quad \tau = \tfrac{t}{\epsilon}. \quad (2.52)$$

Note that $A(\tau)$ is 2π periodic. Employing the identity $2\cos^2(\tau) = 1 + \cos(2\tau)$, we present the system as (2.3) with $\epsilon_i = \epsilon, T_i = 2\pi,\ i = 1, 2, B_{av} = B_1 = B_2 = 0$ and

$$A_{av} = \begin{bmatrix} 0 & 1 \\ -0.46 & -0.2 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0 & 0 \\ -0.5 & 0 \end{bmatrix}, \quad a_1(\tau) = \cos(\tau),\ a_2(\tau) = \cos(2\tau).$$

Note that $a_i(\tau),\ i = 1, 2$ are 2π-periodic, whence $\Delta_{a_i,M} = 0,\ i = 1, 2$. An explicit computation of $\varrho_{\epsilon,i}(t),\ i = 1, 2$ yields

$$\begin{aligned} &\varrho_{\epsilon,1}(t) = \epsilon\sin(\tau),\ a_2(\tau)\varrho_{\epsilon,2}(t) = \tfrac{\epsilon}{4}\sin(4\tau), \\ &\varrho_{\epsilon,2}(t) = a_1(\tau)\varrho_{\epsilon,1}(t) = \epsilon\cos(\tau)\sin(\tau), \\ &a_2(\tau)\varrho_{\epsilon,1}(t) = (2\cos^2(\tau) - 1)\varrho_{\epsilon,1}(t), \\ &a_1(\tau)\varrho_{\epsilon,2}(t) = \cos^2(\tau)\varrho_{\epsilon,1}(t),\ \tau = \tfrac{t}{\epsilon} \end{aligned}$$

which are used to derive the upper bounds in (2.21). Differently from the previous examples, here we obtain only one LMI of the form (2.40).

We consider $\alpha \in \left\{0, \frac{1}{10\pi}\right\}$ and verify the LMIs of Theorem 2.1 to obtain the maximal value ϵ^* which preserves the LMI feasibility. Note that ϵ^* guarantees internal exponential stability (and thus the ISS-like bounds) of (2.3). The values of ϵ^* are given in Table 4, where we further compare our results to the bounds in the recent work (Zhang & Fridman, 2022). Finally, we consider this example subject to uncertainty. For that purpose, we replace $a_2(\tau) = \cos(2\tau)$ with $a_2(\tau) = \cos(2\tau) + 0.4g(\tau)$, where $\|g\|_\infty \le 0.1$. In this case we obtain a nonzero $\Delta a_2(t)$ in (2.4), satisfying $\|\Delta a_2\|_\infty \le 0.04 =: \Delta_{a_2,M}$. We consider $\alpha \in \left\{0, \frac{1}{10\pi}\right\}$ and verify the LMIs of Theorem 2.1 to obtain the maximal value ϵ^* which preserves feasibility of the LMI. The results are given in Table 5. Our results essentially improve the results of Zhang and Fridman (2022).

Table 4
Pendulum - maximum value ϵ^* preserving LMI feasibility.

	$\alpha = 0$	$\alpha = (10\pi)^{-1}$
Zhang and Fridman (2022)	0.0074	0.005
Theorem 2.1	0.0457	0.0321

Table 5
Pendulum - maximum value ϵ^* preserving LMI feasibility.

	$\alpha = 0$	$\alpha = (10\pi)^{-1}$
Zhang and Fridman (2022)	0.0058	0.0034
Theorem 2.1	0.0204	0.0146

3. Rapidly-varying systems with discrete delays

3.1. Systems with constant delay

In this section we consider the system

$$\begin{aligned} &\dot{x}(t) = A_0x(t) + \left[A_h + \textstyle\sum_{i=1}^2 a_i\left(\tfrac{t}{\epsilon}\right)A_i\right]x(t-h),\ t \ge 0, \\ &x(t) = \phi(t), \quad t \in [-h, 0] \end{aligned} \quad (3.1)$$

where $x(t) \in \mathbb{R}^n$ for $t \ge 0$, $A_h, A_0, A_1, A_2 \in \mathbb{R}^{n\times n}$, $h, \epsilon > 0$ and $\phi \in W([-h, 0], \mathbb{R}^n)$ (see Fridman (2014)). Note that the delayed term $x(t-h)$ is multiplied by a linear combination of constant matrices, with the rapidly-varying coefficients $a_i(t/\epsilon),\ i = 1, 2$. The coefficients are assumed to satisfy Assumptions 1 and 2, where now $\epsilon_1 = \epsilon_2 = \epsilon$, $T_1 = T_2 = T$, and $A_{av} := A_0 + A_h$ is assumed to be *Hurwitz*.

Recall $\varrho_{\epsilon,i}(t),\ i = 1, 2$ in (2.6), where now we set $\epsilon_1 = \epsilon_2 = \epsilon$ and $T_1 = T_2 = T$. Introduce the following transformation

$$z(t) = x(t) - \textstyle\sum_{i=1}^2 \varrho_{\epsilon,i}(t)A_ix(t-h), \quad t \ge h. \quad (3.2)$$

Note that $z(t)$ is differentiable for $t \ge h$.

Remark 3.1. For simplicity only in sections 3 and 4 we consider one small parameter ϵ. We can easily consider the more general system

$$\dot{x}(t) = A_0x(t) + \left[\sum_{i=1}^{2} a_i\left(\frac{t}{\epsilon_i}\right)A_i\right]x(t) + \left[A_h + \sum_{i=1}^{2} a_i^h\left(\frac{t}{\epsilon_{i,h}}\right)A_i^h\right]x(t-h), \quad t \geq 0$$

with different small parameters $\epsilon_i, \epsilon_{i,h} > 0$, $i = 1, 2$. In this case, the transformation below will be replaced by

$$z(t) = x(t) - \sum_{i=1}^{2} \varrho_{\epsilon,i}(t)A_ix(t) - \sum_{i=1}^{2} \varrho_{\epsilon,i}^h(t)A_i^hx(t-h).$$

Differentiating $z(t)$ we obtain for $\dot{z}(t)$, $t \geq h$:

$$\begin{aligned}\dot{z}(t) &= A_{av}z(t) - A_h\xi_h(t) + \textstyle\sum_{i=1}^{2} A_i\Delta a_i\left(\frac{t}{\epsilon}\right)x(t-h) \\ &+ \textstyle\sum_{i=1}^{n} \overline{W}_i\varrho_{\epsilon,i}(t)x(t-h) - \sum_{i=1}^{2}\varrho_{\epsilon,i}(t)A_iA_hx(t-2h) \\ &- \textstyle\sum_{i,j=1}^{2} A_iA_j\varrho_{\epsilon,i}(t)a_j\left(\frac{t-h}{\epsilon}\right)x(t-2h)\end{aligned} \tag{3.3}$$

with $\xi_h(t)$ and $\overline{W}_i$, $i = 1, 2$ given by

$$\xi_h(t) = x(t) - x(t-h), \ \overline{W}_i = A_{av}A_i - A_iA_0, \ i = 1, 2.$$

Note that $x(t-2h)$ is obtained by differentiating $x(t-h)$.

In order to vectorize (3.3) (cf. (2.32)) we first introduce

$$\begin{aligned}&\Upsilon_\varrho^h(t) = \mathrm{col}\left\{\varrho_{\epsilon,i}(t)x(t-h)\right\}_{i=1}^2, \\ &\Upsilon_\varrho^{2h}(t) = \mathrm{col}\left\{\varrho_{\epsilon,i}(t)x(t-2h)\right\}_{i=1}^2, \\ &\Upsilon_{\Delta a}^h(t) = \mathrm{col}\left\{\Delta a_i\left(\tfrac{t}{\epsilon}\right)x(t-h)\right\}_{i=1}^2, \\ &\Upsilon_{\varrho,a}^h(t) = \mathrm{col}\left\{\varrho_{\epsilon,i}(t)a_k\left(\tfrac{t-h}{\epsilon}\right)x(t-2h)\right\}_{\{(i,k)\}\leq_{\mathrm{lex}}}, \\ &\Upsilon_a^h(t) = \mathrm{col}\left\{a_i\left(\tfrac{t}{\epsilon}\right)x(t-h)\right\}_{i=1}^2, \\ &\overline{\mathbb{W}} := \begin{bmatrix}\overline{W}_1 & \overline{W}_2\end{bmatrix}, \quad \mathbb{A}_h = \begin{bmatrix}A_1A_h & A_2A_h\end{bmatrix}.\end{aligned} \tag{3.4}$$

Recalling (2.32), (3.3) can be presented as

$$\begin{aligned}\dot{z}(t) &= A_{av}z(t) - A_h\xi_h(t) + \mathbb{A}\Upsilon_{\Delta a}^h(t) \\ &+ \overline{\mathbb{W}}\Upsilon_\varrho^h(t) - \mathbb{A}_h\Upsilon_\varrho^{2h}(t) - \mathbb{A}_1\Upsilon_{\varrho,a}^h(t), \quad t \geq h.\end{aligned} \tag{3.5}$$

For exponential stability of (3.5), let $0 < P, S_i \in \mathbb{R}^{n\times n}$, $i = 1, 2$ and $0 < \alpha \in \mathbb{R}$. We introduce the Lyapunov functional

$$V(t) = |z(t)|_P^2 + \sum_{i=1}^{2} V_{S_i}(t) + V_{R_1}(t),$$

where

$$\begin{aligned}V_{S_i}(t) &= \textstyle\int_{t-ih}^{t} e^{-2\alpha(t-s)}|x(s)|_{S_i}^2 ds, \quad i = 1, 2 \\ V_{R_1}(t) &= h\textstyle\int_{-h}^{0}\int_{t+\theta}^{t} e^{-2\alpha(t-s)}|\dot{x}(s)|_{R_1}^2 ds d\theta\end{aligned}$$

will compensate the delayed terms $x(t-h)$ and $x(t-2h)$. Using (2.16) and differentiating $|z(t)|_P^2$ along (3.5), we find

$$\begin{aligned}&\tfrac{d}{dt}|z(t)|_P^2 + 2\alpha|z(t)|_P^2 = |z(t)|_{Q_\alpha}^2 - 2z^\top(t)PA_h\xi_h(t) \\ &+ 2z^\top(t)P\overline{\mathbb{W}}\Upsilon_\varrho^h(t) + 2z^\top(t)P\mathbb{A}\Upsilon_{\Delta a}^h(t) \\ &- 2z^\top(t)P\mathbb{A}_h\Upsilon_\varrho^{2h}(t) - 2z^\top(t)P\mathbb{A}_1\Upsilon_{\varrho,a}^h(t).\end{aligned} \tag{3.6}$$

Similarly to (2.17)–(2.24), we have

$$\begin{aligned}|z(t)|_{Q_\alpha}^2 &= \left|x(t) - \mathbb{A}\Upsilon_\varrho^h(t)\right|_{Q_\alpha}^2 = |x(t)|_{Q_\alpha}^2 \\ &+ \left|\Upsilon_\varrho^h(t)\right|_{\mathbb{A}^\top Q_\alpha\mathbb{A}}^2 - 2x^\top(t)Q_\alpha\mathbb{A}\Upsilon_\varrho^h(t)\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}&z^\top(t)P\left[-A_h\xi_h(t) + \overline{\mathbb{W}}\Upsilon_\varrho^h(t) + \mathbb{A}\Upsilon_{\Delta a}^h(t) - \mathbb{A}_1\Upsilon_{\varrho,a}^h(t)\right. \\ &\left.-\mathbb{A}_h\Upsilon_\varrho^{2h}(t)\right] = \left[x(t) - \mathbb{A}\Upsilon_\varrho^h(t)\right]^\top P\left[-A_h\xi_h(t)\right. \\ &\left.+\overline{\mathbb{W}}\Upsilon_\varrho^h(t) + \mathbb{A}\Upsilon_{\Delta a}^h(t) - \mathbb{A}_1\Upsilon_{\varrho,a}^h(t) - \mathbb{A}_h\Upsilon_\varrho^{2h}(t)\right].\end{aligned} \tag{3.8}$$

Differentiating $V_{S_i}(t)$, $i = 1, 2$ we have

$$\begin{aligned}\dot{V}_{S_1} + 2\alpha V_{S_1} &= \left(1 - e^{-2\alpha h}\right)|x(t)|_{S_1}^2 - e^{-2\alpha h}|\xi_h(t)|_{S_1}^2 \\ &+ 2e^{-2\alpha h}x^\top(t)S_1\xi_h(t), \\ \dot{V}_{S_2} + 2\alpha V_{S_2} &= |x(t)|_{S_2}^2 - e^{-4\alpha h}|x(t-2h)|_{S_2}^2.\end{aligned} \tag{3.9}$$

For $V_{R_1}(t)$ we employ Jensen's inequality (Fridman, 2014) to obtain

$$\dot{V}_{R_1} + 2\alpha V_{R_1} \leq h^2|\dot{x}(t)|_{R_1}^2 - e^{-2\alpha h}|\xi_h(t)|_{R_1}^2. \tag{3.10}$$

We now employ the S-procedure. Let

$$\begin{aligned}&H_\varrho^h = \mathrm{col}\left\{\mathfrak{h}_{\varrho,h}^{(i)}\right\}_{i=1}^2, \ H_{\varrho,a}^h = \mathrm{col}\left\{\mathfrak{h}_{\varrho,a,h}^{(i,k)}\right\}_{\{(i,k)\}\leq_{\mathrm{lex}}}, \\ &H_a^h = \mathrm{col}\left\{\mathfrak{h}_{a,h}^{(i)}\right\}_{i=1}^2\end{aligned} \tag{3.11}$$

with nonnegative entries such that $\forall 1 \leq i, k \leq 2$, $t \geq h$ and (small) $\epsilon > 0$ the following conditions hold

$$\begin{aligned}&(I)\quad \varrho_{\epsilon,i}^2(t) \leq \mathfrak{h}_{\varrho,h}^{(i)}, \quad (II)\quad \varrho_{\epsilon,i}^2(t)a_k^2\left(\tfrac{t-h}{\epsilon}\right) \leq \mathfrak{h}_{\varrho,a,h}^{(i,k)}, \\ &(III)\quad a_i^2\left(\tfrac{t}{\epsilon}\right) \leq \mathfrak{h}_{a,h}^{(i)}.\end{aligned} \tag{3.12}$$

Note that all the inequalities involve scalar functions. Let $\Lambda_{\Upsilon_\varrho^h}$, $\Lambda_{\Upsilon_\varrho^{2h}}$, $\Lambda_{\Upsilon_{\Delta a}^h}$, $\Lambda_{\Upsilon_a^h} \in \mathbb{R}^{2\times 2}$ and $\Lambda_{\Upsilon_{\varrho,a}^h} \in \mathbb{R}^{4\times 4}$ be positive *diagonal* matrices and recall (3.4). By (2.5) and (3.12) we have

$$\begin{aligned}&\left(\Upsilon_\varrho^h(t)\right)^\top\left(\Lambda_{\Upsilon_\varrho^h}\otimes I_n\right)\Upsilon_\varrho^h(t) \leq \left|\Lambda_{\Upsilon_\varrho^h}H_\varrho^h\right|_1|x(t-h)|^2, \\ &\left(\Upsilon_{\varrho,a}^h(t)\right)^\top\left(\Lambda_{\Upsilon_{\varrho,a}^h}\otimes I_n\right)\Upsilon_{\varrho,a}^h(t) \\ &\qquad\leq \left|\Lambda_{\Upsilon_{\varrho,a}^h}H_{\varrho,a}^h\right|_1|x(t-2h)|^2, \\ &\left(\Upsilon_\varrho^{2h}(t)\right)^\top\left(\Lambda_{\Upsilon_\varrho^{2h}}\otimes I_n\right)\Upsilon_\varrho^{2h}(t) \leq \left|\Lambda_{\Upsilon_\varrho^{2h}}H_\varrho^h\right|_1|x(t-2h)|^2, \\ &\left(\Upsilon_{\Delta a}^h(t)\right)^\top\left(\Lambda_{\Upsilon_{\Delta a}^h}\otimes I_n\right)\Upsilon_{\Delta a}^h(t) \\ &\qquad\leq \left|\Lambda_{\Upsilon_{\Delta a}^h}\Delta_{a,M}\right|_1|x(t-h)|^2, \\ &\left(\Upsilon_a^h(t)\right)^\top\left(\Lambda_{\Upsilon_a^h}\otimes I_n\right)\Upsilon_a^h(t) \\ &\qquad\leq \left|\Lambda_{\Upsilon_a^h}H_a^h\right|_1|x(t-h)|^2\end{aligned} \tag{3.13}$$

where $\Delta_{a,M} = \mathrm{col}\left\{\Delta_{a_i,M}\right\}_{i=1}^2$. Let

$$\begin{aligned}\eta(t) = \mathrm{col}&\left\{x(t), \xi_h(t), x(t-2h), \Upsilon_\varrho^h(t),\right. \\ &\left.\Upsilon_\varrho^{2h}(t), \Upsilon_a^h(t), \Upsilon_{\Delta a}^h(t), \Upsilon_{\varrho,a}^h(t)\right\}.\end{aligned} \tag{3.14}$$

Then

$$\begin{aligned}|\dot{x}(t)|_{R_1}^2 &= \left|A_{av}x(t) - A_h\xi_h(t) + \mathbb{A}\Upsilon_a^h(t)\right|_{R_1}^2 \\ &= \eta^\top(t)\mathcal{L}^\top R_1\mathcal{L}\eta(t), \\ \mathcal{L} &= \begin{bmatrix}A_{av} & -A_h & 0 & 0 & 0 & \mathbb{A} & 0 & 0\end{bmatrix}.\end{aligned} \tag{3.15}$$

By employing (3.13), we obtain

$$\begin{aligned}&0 \leq W_1 = -\eta^\top(t)\Pi\eta(t) + \left[\left|\Lambda_{\Upsilon_\varrho^{2h}}H_\varrho^h\right|_1 + \left|\Lambda_{\Upsilon_{\varrho,a}^h}H_{\varrho,a}^h\right|_1\right] \\ &\qquad\times|x(t-2h)|^2 + \left[\left|\Lambda_{\Upsilon_\varrho^h}H_\varrho^h\right|_1 + \left|\Lambda_{\Upsilon_{\Delta a}^h}\Delta_{a,M}\right|_1\right. \\ &\qquad\left.+\left|\Lambda_{\Upsilon_a^h}H_a^h\right|_1\right]|x(t) - \xi_h(t)|^2 \\ &\Pi = \mathrm{diag}\left\{0, 0, 0, \Pi^{(1)}\right\}, \\ &\Pi^{(1)} = \mathrm{diag}\left\{\Lambda_{\Upsilon_\varrho^h}, \Lambda_{\Upsilon_\varrho^{2h}}, \Lambda_{\Upsilon_a^h}, \Lambda_{\Upsilon_{\Delta a}^h}, \Lambda_{\Upsilon_{\varrho,a}^h}\right\}\otimes I_n\end{aligned} \tag{3.16}$$

By (3.6)–(3.16) and the S-procedure (see Fridman (2014))

$$\dot{V} + 2\alpha V + W_1 \leq \eta^\top(t)\Theta_{\epsilon,h}\eta(t) \leq 0, \tag{3.17}$$

provided

$$\Theta_{\epsilon,h} = \left[\begin{array}{c|c} \Theta^{(1)}_{\epsilon,h} & \Theta^{(2)}_{\epsilon,h} \\ \hline * & \Theta^{(3)}_{\epsilon,h} \end{array}\right] + h^2\mathcal{L}^\top R_1 \mathcal{L} < 0 \tag{3.18}$$

with

$$\Theta^{(1)}_{\epsilon,h} = \begin{bmatrix} \varphi & e^{-2\alpha h}S_1 - PA_h - \lambda_h I_n & 0 \\ * & -e^{-2\alpha h}(S_1+R_1) + \lambda_h I_n & 0 \\ * & * & -e^{-4\alpha h}S_2 + \lambda_{2h} I_n \end{bmatrix},$$
$$\Theta^{(2)}_{\epsilon,h} = \begin{bmatrix} -Q_\alpha \mathbb{A} + P\overline{\mathbb{W}} & -P\mathbb{A}_h & 0 & P\mathbb{A} & -P\mathbb{A}_1 \\ A_h^\top P\mathbb{A} & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{3.19}$$

$$\Theta^{(3)}_{\epsilon,h} = -\Pi^{(1)} + \operatorname{diag}\{\theta, 0, 0, 0, 0\},$$
$$+ \begin{bmatrix} 0 & \mathbb{A}^\top P\mathbb{A}_h & 0 & -\mathbb{A}^\top P\mathbb{A} & \mathbb{A}^\top P\mathbb{A}_1 \\ * & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 \\ * & * & * & 0 & 0 \\ * & * & * & * & 0 \end{bmatrix},$$
$$\varphi = Q_\alpha + \left(1 - e^{-2\alpha h}\right)S_1 + S_2 + \lambda_h I_n,$$
$$\lambda_h = \left|\Lambda_{\Upsilon^h_\varrho} H^h_\varrho\right|_1 + \left|\Lambda_{\Upsilon^h_{\Delta a}} \Delta_{a,M}\right|_1 + \left|\Lambda_{\Upsilon^h_a} H^h_a\right|_1,$$
$$\lambda_{2h} = \left|\Lambda_{\Upsilon^{2h}_\varrho} H^h_\varrho\right|_1 + \left|\Lambda_{\Upsilon^h_{\varrho,a}} H^h_{\varrho,a}\right|_1,$$
$$\theta = -\mathbb{A}^\top P\overline{\mathbb{W}} - \overline{\mathbb{W}}^\top P\mathbb{A} + \mathbb{A}^\top Q_\alpha \mathbb{A}.$$

Summarizing, we arrive at:

Theorem 3.1. *Consider (3.1) subject to Assumptions 1 and 2. Let $H^h_\varrho, H^h_{\varrho,a}, H^h_a$ be given by (3.11) and satisfying (3.12). Given $A_0, A_1, A_2, A_h \in \mathbb{R}^{n\times n}$ such that $A_{av} = A_0 + A_h$ is Hurwitz, and positive tuning parameters α, ϵ^*, h^* and $\Delta_{a_i,M}$, $i = 1, 2$ let there exist $0 < P, S_i, R_1 \in \mathbb{R}^{n\times n}$, $i = 1, 2$ and positive diagonal matrices $\Lambda_{\Upsilon^h_\varrho}, \Lambda_{\Upsilon^{2h}_\varrho}, \Lambda_{\Upsilon^h_{\Delta a}}, \Lambda_{\Upsilon^h_a} \in \mathbb{R}^{2\times 2}$ and $\Lambda_{\Upsilon^h_{\varrho,a}} \in \mathbb{R}^{4\times 4}$ such that (3.18), where $\epsilon = \epsilon^*$, $h = h^*$, and $\delta_{2,x} < e^{-\alpha h^*}$ (see (2.10)) hold. Then, for all $\epsilon \le \epsilon^*$ and $h \le h^*$ the system (3.1) is exponentially stable with decay rate $\alpha > 0$. The LMI (3.18) and $\delta_{2,x} < e^{-\alpha h}$ are feasible for small enough α, ϵ, h and $\Delta_{a_i,M}$, $i = 1, 2$.*

Proof. Feasibility of the LMI (3.18) implies

$$\dot V + 2\alpha V \le 0 \Rightarrow V(t) \le e^{-2\alpha(t-h)}V(h), \quad t \ge h.$$

Now,

$$V(h) = |z(h)|^2 + \int_0^h e^{-2\alpha(h-s)}\, |x(s)|^2_{S_1+S_2}\, ds + \int_{-h}^0 e^{-2\alpha(h-s)}\, |\phi(s)|^2_{S_2}\, ds$$

whereas

$$V(t) \ge \sigma_{min}(P)\, |z(t)|^2, \quad t \ge h.$$

Using variation of constants and (3.2), it can be easily verified that there exists a constant $0 < M$ such that $M_\phi := M\, \|\phi\|_{C([-h,0])}$ satisfies

$$|z(t)| \le M_\phi e^{-\alpha(t-h)}, \quad t \ge h. \tag{3.20}$$

To conclude the same for the solution $x(t)$ of the system (3.1), for any $k \in \mathbb{N}$, we denote $X_k = \sup_{\tau\in[kh,(k+1)h)} |x(\tau)|$. From (2.10), (3.2) and (3.20), we find that $X_{k+1} \le M_\phi e^{-\alpha kh} + \delta_{2,x}X_k$, $k \in \mathbb{N}$. Consider the linear difference equation

$$Y_{k+1} = M_\phi e^{-\alpha kh} + \delta_{2,x}Y_k, \ k \in \mathbb{N}. \tag{3.21}$$

By using induction, we have $X_k \le Y_k$ for all $k \in \mathbb{N}$, provided $Y_1 \ge X_1 \ge 0$. Setting $Y_1 = X_1$, it can be easily verified that the solution of (3.21) with initial condition $Y_1 = X_1$ is given by $Y_k = (X_1 - \mu_h)\,\delta^{k-1}_{2,x} + \mu_h e^{-\alpha(k-1)h}$, $k \in \mathbb{N}$, where $\mu_h = \frac{M_\phi e^{-\alpha h}}{e^{-\alpha h} - \delta_{2,x}}$. Let $t \ge h$ and $k \in \mathbb{N}$ such that $t \in [kh, (k+1)h)$. Then

$$|x(t)| \le X_k \le \left(\frac{X_1 - \mu_h}{\delta_{2,x}} + \mu_h e^{\alpha h}\right) e^{-\alpha(t-h)}$$

where the last step follows from the assumption $\delta_{2,x} < e^{-\alpha h}$. Applying the step method and variation of constants on $t \in [0, 2h]$ there clearly exists a constant $M_1 > 0$ such that $|x(t)| \le M_1\, \|\phi\|_{C([-h,0])} \le M_1 e^{2\alpha h}\, \|\phi\|_{C([-h,0])}\, e^{-\alpha t}$, the exponential stability of (3.1) follows. Proof of feasibility of (3.18) and $\delta_{2,x} < e^{-\alpha h}$ follows by arguments similar to Theorem 2.1 and is omitted due to space limitations. □

3.2. Systems with fast-varying delay

In this section we consider the system for $t \ge 0$

$$\begin{aligned} \dot x(t) &= A_0 x(t) + \left[A_h + \textstyle\sum_{i=1}^2 a_i\left(\frac{t}{\epsilon}\right) A_i\right] x(t - h(t)), \\ x(\theta) &= \phi(\theta), \ \theta \in [-h_M, 0]. \end{aligned} \tag{3.22}$$

Here $x(t) \in \mathbb{R}^n$ for $t \ge 0$, $\epsilon > 0$, $A_0, A_1, A_2, A_h \in \mathbb{R}^n$. Furthermore, $h : [0,\infty) \to \mathbb{R}$ is a piecewise-continuous time-varying delay, which is unknown and satisfies

$$h(t) \le h_M, \quad t \ge 0 \tag{3.23}$$

for some *known* $0 < h_M$, whereas $\phi \in W([-h_M, 0])$. Let $a_i\left(\frac{t}{\epsilon}\right)$, $i = 1, 2$ satisfy Assumptions 1 and 2, whereas $A_{av} = A_0 + A_h$ is assumed to be *Hurwitz*.

We begin by presenting the system (3.22) as

$$\begin{aligned} \dot x(t) &= A_{av}x(t) + \textstyle\sum_{i=1}^2 a_i\left(\frac{t}{\epsilon}\right) A_i x(t) + A_h\xi(t), \\ &\quad + \textstyle\sum_{i=1}^2 a_i\left(\frac{t}{\epsilon}\right) A_i\xi(t), \quad t \ge 0, \\ \xi(t) &= x(t - h(t)) - x(t). \end{aligned} \tag{3.24}$$

Recalling $\varrho_{\epsilon_i}(t)$, $i = 1, 2$ in (2.6) and subject to (2.8), we introduce the transformation

$$z(t) = x(t) - \textstyle\sum_{i=1}^2 \varrho_{\epsilon,i}(t) A_i x(t). \tag{3.25}$$

Remark 3.2. Differently from (3.2), we do not employ here the transformation

$$z(t) = x(t) - \sum_{i=1}^2 \varrho_{\epsilon,i}(t) A_i x(t - h(t)).$$

The latter transformation cannot be differentiated, since the delay $h(t)$ is assumed to only be piecewise continuous.

Employing (2.8) and (3.25), we obtain the following:

$$\begin{aligned} \dot z(t) = {} & A_{av}z(t) + \textstyle\sum_{i=1}^2 A_{av}A_i\varrho_{\epsilon,i}(t)x(t) + A_h\xi(t) \\ & + \textstyle\sum_{i=1}^2 a_i\left(\frac{t}{\epsilon}\right) A_i\xi(t) + \sum_{i=1}^2 \Delta a_i\left(\frac{t}{\epsilon}\right) A_i\xi(t) \\ & - \left[\varrho_{\epsilon,1}(t)A_1 + \varrho_{\epsilon,2}(t)A_2\right]\dot x(t), \quad t \ge \tau_M. \end{aligned} \tag{3.26}$$

To vectorize (3.26), recall $\Upsilon_\varrho(t)$, $\Upsilon_{\varrho,a}(t)$, $\Upsilon_{\Delta a}(t)$, $\mathbb{A}$, $\mathbb{A}_1$ and $\mathbb{W}$ in (2.13), where we set $\epsilon_1 = \epsilon_2 = \epsilon$. We introduce

$$\begin{aligned} &\mathcal{Z}_\varrho(t) = \operatorname{col}\left\{\varrho_{\epsilon,i}(t)\xi(t)\right\}_{i=1}^2, \ \ \mathbb{A}_h = \begin{bmatrix} A_1A_h & A_2A_h \end{bmatrix}, \\ &\mathcal{Z}_{\varrho,a}(t) = \operatorname{col}\left\{\varrho_{\epsilon,i}(t)a_k\left(\tfrac{t}{\epsilon}\right)\xi(t)\right\}_{\{(i,k)\}_{\le \text{lex}}}, \\ &\Upsilon_a(t) = \operatorname{col}\left\{a_i\left(\tfrac{t}{\epsilon}\right)x(t)\right\}_{i=1}^2, \\ &\mathcal{Z}_a(t) = \operatorname{col}\left\{a_i\left(\tfrac{t}{\epsilon}\right)\xi(t)\right\}_{i=1}^2, \\ &\mathcal{Z}_{\Delta a}(t) = \operatorname{col}\left\{\Delta a_j\left(\tfrac{t}{\epsilon}\right)\xi(t)\right\}_{j=1}^2. \end{aligned} \tag{3.27}$$

Then, (3.25) and (3.26) can be presented as

$$\begin{aligned} &z(t) = x(t) - \mathbb{A}\Upsilon_\varrho(t), \\ &\dot z(t) = A_{av}z(t) + A_h\xi(t) + \mathbb{W}\Upsilon_\varrho(t) - \mathbb{A}_h\mathcal{Z}_\varrho(t) \\ &- \mathbb{A}_1\Upsilon_{\varrho,a}(t) - \mathbb{A}_1\mathcal{Z}_{\varrho,a}(t) + \mathbb{A}\mathcal{Z}_a(t) + \mathbb{A}\Upsilon_{\Delta a}(t), \ t \ge \tau_M, \end{aligned} \tag{3.28}$$

whereas by (3.24) we have

$$\dot{x}(t)=A_{av}x(t)+A_h\xi(t)+\mathbb{A}\Upsilon_a(t)+\mathbb{A}\mathcal{Z}_a(t),\ t\geq 0. \tag{3.29}$$

For exponential stability analysis of (3.28), let $0<P,S,R\in\mathbb{R}^n$ and $0<\alpha\in\mathbb{R}$. We introduce the following Lyapunov functional for $t\geq h_M$:

$$\begin{array}{l} V(t)=|z(t)|_P^2+V_R(t)+V_S(t),\\ V_R(t)=h_M\int_{-h_M}^0\int_{t+\theta}^t e^{-2\alpha(t-\tau)}\,|\dot{x}(\tau)|_R^2\,d\tau d\theta,\\ V_S(t)=\int_{t-h_M}^t e^{-2\alpha(t-\tau)}\,|x(\tau)|_S^2\,d\tau \end{array} \tag{3.30}$$

where $V_S(t)$ and $V_R(t)$ will compensate the delay error $\xi(t)$.

Differentiating $|z(t)|_P^2$ along the solution to (3.28), we have

$$\begin{array}{l} \frac{d}{dt}|z(t)|_P^2+2\alpha\,|z(t)|_P^2=|z(t)|_{Q_\alpha}^2+2z^\top(t)P\,[A_h\xi(t)\\ \quad+\mathbb{W}\Upsilon_\varrho(t)-\mathbb{A}_h\mathcal{Z}_\varrho(t)-\mathbb{A}_1\Upsilon_{\varrho,a}(t)-\mathbb{A}_1\mathcal{Z}_{\varrho,a}(t)\\ \quad+\mathbb{A}\mathcal{Z}_a(t)+\mathbb{A}\Upsilon_{\Delta a}(t)],\ t\geq h_M \end{array} \tag{3.31}$$

where Q_α is given in (2.16). Employing (3.28), we then have

$$|z(t)|_{Q_\alpha}^2=|x(t)|_{Q_\alpha}^2+\left|\Upsilon_\varrho(t)\right|^2_{\mathbb{A}^\top Q_\alpha\mathbb{A}}-2x^\top(t)Q_\alpha\mathbb{A}\Upsilon_\varrho(t) \tag{3.32}$$

and

$$\begin{array}{l} 2z^\top(t)P\left[A_h\xi(t)+\mathbb{W}\Upsilon_\varrho(t)-\mathbb{A}_h\mathcal{Z}_\varrho(t)-\mathbb{A}_1\Upsilon_{\varrho,a}(t)\right.\\ \left.-\mathbb{A}_1\mathcal{Z}_{\varrho,a}(t)+\mathbb{A}\mathcal{Z}_a(t)+\mathbb{A}\Upsilon_{\Delta a}(t)\right]=2\left[x(t)-\mathbb{A}\Upsilon_\rho(t)\right]^\top\\ \times P\left[A_h\xi(t)+\mathbb{W}\Upsilon_\varrho(t)-\mathbb{A}_h\mathcal{Z}_\varrho(t)-\mathbb{A}_1\Upsilon_{\varrho,a}(t)\right.\\ \left.-\mathbb{A}_1\mathcal{Z}_{\varrho,a}(t)+\mathbb{A}\mathcal{Z}_a(t)+\mathbb{A}\Upsilon_{\Delta a}(t)\right]. \end{array} \tag{3.33}$$

Differentiating $V_S(t)$ along the solution to (3.28), we have

$$\begin{array}{l} \frac{d}{dt}V_S(t)+2\alpha V_S(t)=|x(t)|_S^2-e^{-2\alpha h_M}\,|x(t)+\xi(t)+\nu(t)|_S^2,\\ \nu(t)=x(t-h_M)-x(t-h(t)). \end{array} \tag{3.34}$$

Let $G\in\mathbb{R}^n$ satisfy

$$\begin{bmatrix} R & G\\ * & R\end{bmatrix}\geq 0. \tag{3.35}$$

Differentiating $V_R(t)$ along the solution to (3.28) and employing the Jensen and Park inequalities (see Fridman (2014))

$$\begin{array}{l} \frac{d}{dt}V_R(t)+2\alpha V_R(t)\leq -e^{-2\alpha h_M}\begin{bmatrix}\xi(t)\\ \nu(t)\end{bmatrix}^\top\begin{bmatrix}R & G\\ * & R\end{bmatrix}\begin{bmatrix}\xi(t)\\ \nu(t)\end{bmatrix}\\ \quad +h_M^2\,|A_{av}x(t)+A_h\xi(t)+\mathbb{A}\Upsilon_a(t)+\mathbb{A}\mathcal{Z}_a(t)|_R^2. \end{array} \tag{3.36}$$

To employ the S-procedure, recall H_ϱ and $H_{\varrho,a}$ in (2.20) and introduce $H_a=\text{col}\left\{\mathfrak{h}_a^{(k)}\right\}_{k=1}^2$. Let H_a have nonnegative entries such that (2.35) and

$$a_k^2\,(t/\epsilon)\leq\mathfrak{h}_a^{(k)} \tag{3.37}$$

hold for all $1\leq i,k\leq 2$ and $t\geq 0$, uniformly in (small) $\epsilon>0$. Let $\Lambda_{\Upsilon_\varrho},\Lambda_{\mathcal{Z}_\varrho},\Lambda_{\Upsilon_{\Delta a}}\in\mathbb{R}^{2\times 2}$, $\Lambda_{\mathcal{Z}_{\Delta a}},\Lambda_{\Upsilon_a},\Lambda_{\mathcal{Z}_a}\in\mathbb{R}^{2\times 2}$ and $\Lambda_{\Upsilon_{\varrho,a}},\Lambda_{\mathcal{Z}_{\varrho,a}}\in\mathbb{R}^{4\times 4}$ be positive *diagonal* matrices (decision variables). By (2.5), (2.35) and (3.37), we have

$$\begin{array}{l} \Upsilon_\varrho^\top(t)\left(\Lambda_{\Upsilon_\varrho}\otimes I_n\right)\Upsilon_\varrho(t)\leq\left|\Lambda_{\Upsilon_\varrho}H_\varrho\right|_1|x(t)|^2,\\ \mathcal{Z}_\varrho^\top(t)\left(\Lambda_{\mathcal{Z}_\varrho}\otimes I_n\right)\mathcal{Z}_\varrho(t)\leq\left|\Lambda_{\mathcal{Z}_\varrho}H_\varrho\right|_1|\xi(t)|^2,\\ \Upsilon_a^\top(t)\left(\Lambda_{\Upsilon_a}\otimes I_n\right)\Upsilon_a(t)\leq\left|\Lambda_{\Upsilon_a}H_a\right|_1|x(t)|^2,\\ \mathcal{Z}_a^\top(t)\left(\Lambda_{\mathcal{Z}_a}\otimes I_n\right)\mathcal{Z}_a(t)\leq\left|\Lambda_{\mathcal{Z}_a}H_a\right|_1|\xi(t)|^2,\\ \Upsilon_{\varrho,a}^\top(t)\left(\Lambda_{\Upsilon_{\varrho,a}}\otimes I_n\right)\Upsilon_{\varrho,a}(t)\leq\left|\Lambda_{\Upsilon_{\varrho,a}}H_{\varrho,a}\right|_1|x(t)|^2,\\ \mathcal{Z}_{\varrho,a}^\top(t)\left(\Lambda_{\mathcal{Z}_{\varrho,a}}\otimes I_n\right)\mathcal{Z}_{\varrho,a}(t)\leq\left|\Lambda_{\mathcal{Z}_{\varrho,a}}H_{\varrho,a}\right|_1|\xi(t)|^2,\\ \Upsilon_{\Delta a}^\top(t)\left(\Lambda_{\Upsilon_{\Delta a}}\otimes I_n\right)\Upsilon_{\Delta a}(t)\leq\left|\Lambda_{\Upsilon_{\Delta a}}\Delta_{a,M}\right|_1|x(t)|^2,\\ \mathcal{Z}_{\Delta a}^\top(t)\left(\Lambda_{\mathcal{Z}_{\Delta a}}\otimes I_n\right)\mathcal{Z}_{\Delta a}(t)\leq\left|\Lambda_{\mathcal{Z}_{\Delta a}}\Delta_{a,M}\right|_1|\xi(t)|^2. \end{array} \tag{3.38}$$

Let

$$\begin{array}{l} \eta(t)=\text{col}\left\{x(t),\,\xi(t),\,\nu(t),\,\Upsilon_\varrho(t),\,\Upsilon_a(t),\,\Upsilon_{\Delta a}(t),\right.\\ \quad\left.\Upsilon_{\varrho,a}(t),\,\mathcal{Z}_\varrho(t),\,\mathcal{Z}_a(t),\,\mathcal{Z}_{\Delta a}(t),\,\mathcal{Z}_{\varrho,a}(t)\right\}. \end{array} \tag{3.39}$$

Recalling (3.38), we have

$$\begin{array}{l} 0\leq W_3=\eta^\top(t)\left[\Sigma_0-\Sigma_1\right]\eta(t)\\ \Sigma_1=\text{diag}\left\{0,0,0,-\Lambda_{\Upsilon_\varrho},-\Lambda_{\Upsilon_a},-\Lambda_{\Upsilon_{\Delta a}},-\Lambda_{\Upsilon_{\varrho,a}}\right.\\ \quad\left.,-\Lambda_{\mathcal{Z}_\varrho},-\Lambda_{\mathcal{Z}_a},-\Lambda_{\mathcal{Z}_{\Delta a}},-\Lambda_{\mathcal{Z}_{\varrho,a}}\right\}\otimes I_n,\\ \Sigma_0=\text{diag}\left\{\Sigma_0^{(1)},\,\Sigma_0^{(2)},0,0,0,0,0,0,0,0,0\right\},\\ \Sigma_0^{(1)}=\left(\left|\Lambda_{\Upsilon_\varrho}H_\varrho\right|_1+\left|\Lambda_{\Upsilon_a}H_a\right|_1+\left|\Lambda_{\Upsilon_{\varrho,a}}H_{\varrho,a}\right|_1,\right.\\ \quad\left.\left|\Lambda_{\Upsilon_{\Delta a}}\Delta_{a,M}\right|_1\right)I_n,\\ \Sigma_0^{(2)}=\left(\left|\Lambda_{\mathcal{Z}_\varrho}H_\varrho\right|_1+\left|\Lambda_{\mathcal{Z}_a}H_a\right|_1+\left|\Lambda_{\mathcal{Z}_{\varrho,a}}H_{\varrho,a}\right|_1,\right.\\ \quad\left.\left|\Lambda_{\mathcal{Z}_{\Delta a}}\Delta_{a,M}\right|_1\right)I_n. \end{array} \tag{3.40}$$

By (3.31)–(3.40) and the S-procedure (Fridman, 2014)

$$\dot{V}+2\alpha V\leq\dot{V}+2\alpha V+W_3\leq\eta^\top(t)\Phi_{\epsilon,h}\eta(t)\leq 0, \tag{3.41}$$

provided

$$\Phi_{\epsilon,h}=\left[\begin{array}{c|c|c}\Phi_{\epsilon,h}^{(1)} & \Phi_{\epsilon,h}^{(2)} & \Phi_{\epsilon,h}^{(3)}\\ \hline * & \Phi_{\epsilon,h}^{(4)} & \Phi_{\epsilon,h}^{(5)}\\ \hline * & * & \Phi_{\epsilon,h}^{(6)}\end{array}\right]+h_M^2\mathcal{L}^\top R\mathcal{L}<0 \tag{3.42}$$

where

$$\begin{array}{l} \Phi_{\epsilon,h}^{(1)}=\begin{bmatrix} Q_\alpha & PA_h-\epsilon_M S & -\epsilon_M S\\ * & -\epsilon_M(S+R) & -\epsilon_M(S+G)\\ * & * & -\epsilon_M(S+R)\end{bmatrix}\\ \quad+\text{diag}\left\{\Sigma_0^{(1)},\,\Sigma_0^{(2)},0\right\}+\text{diag}\left\{(1-\epsilon_M)S,0,0\right\},\\ \Phi_{\epsilon,h}^{(2)}=\begin{bmatrix} -Q_\alpha\mathbb{A}+P\mathbb{W} & 0 & P\mathbb{A} & -P\mathbb{A}_1\\ -A_h^\top P\mathbb{A} & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix},\\ \Phi_{\epsilon,h}^{(3)}=\begin{bmatrix} -P\mathbb{A}_h & P\mathbb{A} & 0 & -P\mathbb{A}_1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix},\\ \Phi_{\epsilon,h}^{(4)}=\begin{bmatrix} \mathbb{A}^\top Q_\alpha\mathbb{A}-\mathbb{A}^\top P\mathbb{W}-\mathbb{W}^\top P\mathbb{A} & 0 & -\mathbb{A}^\top P\mathbb{A} & \mathbb{A}^\top P\mathbb{A}_1\\ * & 0 & 0 & 0\\ * & * & 0 & 0\\ * & * & * & 0\end{bmatrix}\\ \quad+\text{diag}\left\{-\Lambda_{\Upsilon_\varrho}\otimes I_n,-\Lambda_{\Upsilon_a}\otimes I_n,\right.\\ \quad\quad\left.-\Lambda_{\Upsilon_{\Delta a}}\otimes I_n,-\Lambda_{\Upsilon_{\varrho,a}}\otimes I_n\right\},\\ \Phi_{\epsilon,h}^{(6)}=\text{diag}\left\{-\Lambda_{\mathcal{Z}_\varrho}\otimes I_n,-\Lambda_{\mathcal{Z}_a}\otimes I_n,\right.\\ \quad\quad\left.-\Lambda_{\mathcal{Z}_{\Delta a}}\otimes I_n,-\Lambda_{\mathcal{Z}_{\varrho,a}}\otimes I_n\right\},\\ \Phi_{\epsilon,h}^{(5)}=\begin{bmatrix} \mathbb{A}^\top P\mathbb{A}_h & -\mathbb{A}^\top P\mathbb{A} & 0 & \mathbb{A}^\top P\mathbb{A}_1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix},\\ \mathcal{L}=[A_{av}\ A_h\ 0\ 0\ \mathbb{A}\ 0\ 0\ 0\ \mathbb{A}\ 0\ 0],\ \epsilon_M=e^{-2\alpha h_M}. \end{array} \tag{3.43}$$

Summarizing, we arrive at:

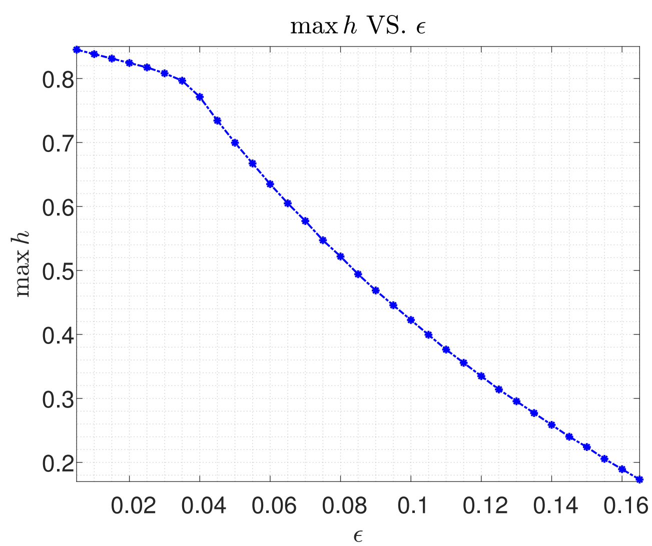

Fig. 1. Theorem 3.1 - max constant delay h which preserves the exponential stability of the switched delayed system with decay rate $\alpha = 0.005$.

Theorem 3.2. *Consider (3.22) where $\epsilon > 0$, $A_0, A_1, A_2, A_h \in \mathbb{R}^n$ and $h(t)$ is a piecewise continuous delay, subject to (3.23). Let the rapidly-varying coefficients $a_i\left(\frac{t}{\epsilon}\right)$, $i = 1, 2$ satisfy Assumptions 1 and 2 for some $T > 0$. Assume further that $A_{av} := A_0 + A_h$ is Hurwitz. Let H_ϱ, $H_{\varrho,a}$ and H_a be vectors with nonnegative entries such that (2.35) and (3.37) hold. Given positive tuning parameters $\alpha, \epsilon^*, h_M^*, \Delta_{a_1,M}, \Delta_{a_2,M}$, let there exist $0 < P, R, S \in \mathbb{R}^n$, $G \in \mathbb{R}^n$ and positive diagonal matrices $\Lambda_{\Upsilon_\varrho}, \Lambda_{\Xi_\varrho}, \Lambda_{\Upsilon_{\Delta a}} \in \mathbb{R}^{2\times 2}$, $\Lambda_{\Xi_{\Delta a}}, \Lambda_{\Upsilon_a}, \Lambda_{\Xi_a} \in \mathbb{R}^{2\times 2}$ and $\Lambda_{\Upsilon_{\varrho,a}}, \Lambda_{\Xi_{\varrho,a}} \in \mathbb{R}^{4\times 4}$ such that (3.35) and (3.42) hold with $\epsilon = \epsilon^*$ and $h_M = h_M^*$. Then, for all $\epsilon \le \epsilon^*$ and $h_M \le h_M^*$ system 3.2 is exponentially stable with decay rate $\alpha > 0$. The LMIs (3.35) and (3.42) are feasible for small enough $\alpha, \epsilon, h_M, \Delta_{a_i,M}$, $i = 1, 2$.*

Proof. The proof is similar to the proof of Theorem 3.1 and is omitted due to space constraints. Note that (2.10) implies invertibility of (3.25) (see Assumption 2). Hence, exponential stability of (3.22) follows from exponential decay of $z(t)$ (which is guaranteed by (3.35) and (3.42)). □

3.3. Numerical example

Delayed stabilization by fast switching

We consider the delayed Example 2.1 of the previous section $\dot{x}(t) = A(\frac{t}{\epsilon})x(t-h)$ with A given by (2.45) and (2.46). This system can be presented as (3.1) with $A_0 = 0_{2\times 2}$, A_1 and A_2 given in (2.45) and a_i defined in (2.48). We further set $A_h = A_{av}$, where A_{av} is given in (2.47). The upper bounds in (3.12) are obtained using the explicit description of $a_i(\tau)$, $i = 1, 2$ and the bounds on $\varrho^2_{\epsilon,i}(t)$, $i = 1, 2$ appearing in Example 2.3.1. We consider both constant delay and general time-varying delays. For the case of constant delay, we fix $\alpha = 0.0075$, and verify the feasibility of (3.18) and $\delta_{2,x} < e^{-\alpha h}$, given in Theorem 3.1, for $\epsilon \in [0.005, 0.0165]$. For each ϵ in the latter range, the conditions of Theorem 3.1 were verified to obtain the largest delay h which preserves feasibility of (3.18) and $\delta_{2,x} < e^{-\alpha h}$. The results are given in Fig. 1. Note that decreasing ϵ leads to an increase of $\max h$.

Next, we consider the case of fast-varying delays and compare our approach with the results of Fridman and Zhang (2020, Example 5.1). Let $\alpha \in \{0, 0.005, 0.01\}$ and $\epsilon = 0.05$. We verify the LMIs of Theorem 3.2 to obtain the maximal value of the delay bound τ_M which preserves feasibility of the LMIs. The results are given in Table 6. Our results improve the results of Fridman and Zhang (2020). In particular, the results for $\alpha = 0.005$ present an improvement of 34.75% over the corresponding case in Fridman and Zhang (2020). We further consider the case $\epsilon = 0.25$ for which the method of Fridman and Zhang (2020) fails. The results are given in Table 7.

Table 6
Switched system with fast-varying delay - maximum h_M preserving LMI feasibility.

$\epsilon = 0.05$	$\alpha = 0$	$\alpha = \frac{1}{200}$	$\alpha = \frac{1}{100}$
Fridman and Zhang (2020)	0.0516	0.0259	Unchecked
Theorem 3.2	0.054	0.0349	0.0161

Table 7
Switched system with fast-varying delay - maximum h_M preserving LMI feasibility.

$\epsilon = 0.25$	$\alpha = 0$	$\alpha = 0.0025$	$\alpha = 0.005$
Theorem 3.2	0.0252	0.0161	0.0069

4. Rapidly-varying systems with distributed delays

In this section we consider the system

$$\begin{aligned} &\dot{x}(t) = A_0x(t) + A_D\left(\tfrac{t}{\epsilon}\right)\int_{-h}^{0}\varpi(\theta)x(t+\theta)d\theta, \quad t \ge 0,\\ &x(t) = \phi(t), \quad t \in [-h, 0] \end{aligned} \tag{4.1}$$

where $x(t) \in \mathbb{R}^n$ for $t \ge 0$, $A_D(\tau) = A_h + a_1(\tau)A_1$, $\tau \in \mathbb{R}$, $A_h, A_0, A_1 \in \mathbb{R}^{n\times n}$, $h, \epsilon > 0$ and $\phi \in W([-h,0], \mathbb{R}^n)$. The weight function $\varpi \in L^1([-h, 0])$ satisfies $\varpi(t) > 0$ a.e. in $[-h, 0]$. The rapidly-varying coefficient $a_1\left(\frac{t}{\epsilon}\right)$ satisfies Assumptions 1 and 2. We assume that either A_0 or $A_{av} := A_0 + \|\varpi\|_{L^1}\cdot A_h$ is Hurwitz (see Fridman (2014, Section 3)).

Recalling $\varrho_{\epsilon_1}(t)$ in (2.6) and (2.8), we introduce the transformation

$$z(t) = x(t) - \varrho_{\epsilon,1}(t)A_1\xi(t) - \|\varpi\|_{L^1}\cdot\varrho_{\epsilon,1}(t)A_1x(t) \tag{4.2}$$

where

$$\xi(t) = \int_{-h}^{0}\varpi(\theta)\left[x(t+\theta) - x(t)\right]d\theta. \tag{4.3}$$

Employing (2.6) and (4.2), we obtain the following expression for $\dot{z}(t)$, $t \ge h$:

$$\begin{aligned} \dot{z}(t) = \;& A_{av}x(t) + A_h\xi(t) + \Delta a_1\left(\tfrac{t}{\epsilon}\right)A_1\xi(t)\\ &+ \|\varpi\|_{L^1}\,\Delta a_1\left(\tfrac{t}{\epsilon}\right)A_1x(t) - \varrho_{\epsilon,1}(t)A_1\Xi(t),\\ \Xi(t) = \;& \int_{-h}^{0}\varpi(\theta)\dot{x}(t+\theta)d\theta. \end{aligned} \tag{4.4}$$

To further vectorize (4.4), we introduce

$$\begin{aligned} &\Upsilon_\varrho(t) = \varrho_{\epsilon,1}(t)\,\mathrm{col}\{x(t), \xi(t)\},\\ &\Upsilon_{\Delta a_1}(t) = \Delta a_1\left(\tfrac{t}{\epsilon}\right)\mathrm{col}\{x(t), \xi(t)\},\\ &\Upsilon_{a_1}(t) = a_1\left(\tfrac{t}{\epsilon}\right)\mathrm{col}\{x(t), \xi(t)\}, \quad \mathbb{A}_1 = \left[\|\varpi\|_{L^1}\cdot A_1 \quad A_1\right]. \end{aligned} \tag{4.5}$$

Then, (4.2)–(4.4) can be presented as

$$\begin{aligned} z(t) &= x(t) - \mathbb{A}_1\Upsilon_\rho(t),\\ \dot{z}(t) &= A_{av}z(t) + A_h\xi(t) + \mathbb{A}_1\Upsilon_{\Delta a}(t)\\ &\quad + A_{av}\mathbb{A}_1\Upsilon_\varrho(t) - \varrho_{\epsilon,1}(t)A_1\Xi(t), \quad t \ge \mathfrak{h}. \end{aligned} \tag{4.6}$$

For stability analysis of (4.6), let $0 < P, R_\xi, R_\Xi, Z_\xi \in \mathbb{R}^n$ and decay rate $0 < \alpha \in \mathbb{R}$. We introduce the following Lyapunov functional for $t \geq h$ (cf. Fridman (2014, Section 4.5)):

$$\begin{aligned} &V(t) = |z(t)|_P^2 + V_{R_\xi}(t) + V_{Z_\xi}(t) + V_{R_\Xi}(t), \\ &V_{R_\xi}(t) = h \int_{-h}^0 \int_{t+\theta}^t \varpi(\theta) e^{-2\alpha(t-\tau)} |x(\tau)|_{R_\xi}^2 \, d\tau d\theta, \\ &V_{Z_\xi}(t) = \tfrac{h^2}{2} \int_{-h}^0 \int_\theta^0 \int_{t+\lambda}^t \varpi(\theta) e^{-2\alpha(t-\tau)} |\dot{x}(\tau)|_{Z_\xi}^2 \, d\tau d\lambda d\theta, \\ &V_{R_\Xi}(t) = h \int_{-h}^0 \int_{t+\theta}^t \varpi(\theta) e^{-2\alpha(t-\tau)} |\dot{x}(\tau)|_{R_\Xi}^2 \, d\tau d\theta \end{aligned} \tag{4.7}$$

where we recall that $\varpi \in L^1([-h, 0])$ is positive a.e. in $[-h, 0]$. The components $V_{R_\xi}(t)$, $V_{Z_\xi}(t)$ and $V_{R_\Xi}(t)$ are introduced to compensate $\xi(t)$ and $\Xi(t)$ in (4.6).

Differentiating $|z(t)|_P^2$ along the solution to (4.6), we have

$$\begin{aligned} &\tfrac{d}{dt} |z(t)|_P^2 + 2\alpha |z(t)|_P^2 = |z(t)|_{Q_\alpha}^2 + 2z^\top(t) P \left[A_h \xi(t) \right. \\ &\left. + A_{av} \mathbb{A}_1 \Upsilon_\varrho(t) + \mathbb{A}_1 \Upsilon_{\Delta a}(t) - \varrho_{\epsilon,1}(t) A_1 \Xi(t)\right] \end{aligned} \tag{4.8}$$

where Q_α is given in (2.16). Employing (4.6), we then have

$$\begin{aligned} |z(t)|_{Q_\alpha}^2 &= |x(t)|_{Q_\alpha}^2 \\ &+ \left|\Upsilon_\varrho(t)\right|^2_{\mathbb{A}_1^\top Q_\alpha \mathbb{A}_1} - 2x^\top(t) Q_\alpha \mathbb{A}_1 \Upsilon_\varrho(t) \end{aligned} \tag{4.9}$$

and

$$\begin{aligned} &2z^\top(t) P \left[A_h \xi(t) + \mathbb{A}_1 \Upsilon_{\Delta a}(t) - \varrho_{\epsilon,1}(t) A_1 \Xi(t)\right] \\ &= 2\left[x(t) - \mathbb{A}_1 \Upsilon_\rho(t)\right]^\top P \left[A_h \xi(t) + \mathbb{A}_1 \Upsilon_{\Delta a}(t)\right. \\ &\quad \left. - \varrho_{\epsilon,1}(t) A_1 \Xi(t) + A_{av} \mathbb{A}_1 \Upsilon_\varrho(t)\right]. \end{aligned} \tag{4.10}$$

Differentiating $V_{R_\xi}(t)$ along the solution to (4.6) and employing Jensen's inequality, we have

$$\begin{aligned} &\tfrac{d}{dt} V_{R_\xi}(t) + 2\alpha V_{R_\xi}(t) \leq h \|\varpi\|_{L^1} \cdot |x(t)|_{R_\xi}^2 \\ &- e^{-2\alpha h} h \int_{-h}^0 \varpi(\theta) |x(t+\theta)|_{R_\xi}^2 \, d\theta \leq -\tfrac{e^{-2\alpha h} h}{\|\varpi\|_{L^1}} |\xi(t)|_{R_\xi}^2 \\ &- 2e^{-2\alpha h} h x^\top(t) R_\xi \xi(t) + h \|\varpi\|_{L^1} \left(1 - e^{-2\alpha h}\right) |x(t)|_{R_\xi}^2. \end{aligned} \tag{4.11}$$

By applying similar arguments to $V_{R_\Xi}(t)$, we have

$$\begin{aligned} &\tfrac{d}{dt} V_{R_\Xi}(t) + 2\alpha V_{R_\Xi}(t) \leq -\tfrac{e^{-2\alpha h} h}{\|\varpi\|_{L^1}} |\Xi(t)|_{R_\Xi}^2 \\ &+ h \|\varpi\|_{L^1} \cdot \left|A_{av} x(t) + A_h \xi(t) + \mathbb{A}_1 \Upsilon_{a_1}(t)\right|_{R_\Xi}^2. \end{aligned} \tag{4.12}$$

Differentiating $V_{Z_\xi}(t)$ along the solution to (4.6) and employing Jensen's inequality, we have

$$\begin{aligned} &\tfrac{d}{dt} V_{Z_\Xi}(t) + 2\alpha V_{Z_\Xi}(t) \leq \tfrac{h^2}{2} \varphi_\varpi |\dot{x}(t)|_{Z_\xi}^2 \\ &- \tfrac{e^{-2\alpha h} h^2}{2} \int_{-h}^0 \int_{t+\theta}^t \varpi(\theta) |\dot{x}(\tau)|_{Z_\xi}^2 \, d\tau d\theta \leq -\tfrac{e^{-2\alpha h} h^2}{2\varphi_\varpi} |\xi(t)|_{Z_\xi}^2 \\ &+ \tfrac{h^2}{2} \varphi_\varpi \left|A_{av} x(t) + A_h \xi(t) + \mathbb{A}_1 \Upsilon_{a_1}(t)\right|_{Z_\xi}^2, \\ &\varphi_\varpi = -\int_{-h}^0 \theta \varpi(\theta) d\theta. \end{aligned} \tag{4.13}$$

Remark 4.1. The normalizing constants appearing prior to the integrals in $V_{R_\xi}(t)$, $V_{R_\Xi}(t)$ and $V_{Z_\xi}(t)$ in (4.7) were chosen so that for the case $\varpi(\theta) \equiv 1$, we have $\|\varpi\|_{L^1} = h$ and $\varphi_\varpi = \frac{h^2}{2}$, whence the compensating negative terms in the bounds (4.11) and (4.13) are multiplied by $e^{-2\alpha h}$.

To employ the S-procedure, let $\mathfrak{h}_\varrho, \mathfrak{h}_{a_1} > 0$ be positive scalars such that $\forall t \geq h$ and (small) $\epsilon > 0$:

$$(I) \quad \varrho_{\epsilon,1}^2(t) \leq \mathfrak{h}_\varrho, \quad (II) \quad a_1^2(t/\epsilon) \leq \mathfrak{h}_{a_1}. \tag{4.14}$$

Let $\Lambda_{\Upsilon_\varrho}, \Lambda_{\Upsilon_{\Delta a_1}}, \Lambda_{\Upsilon_{a_1}} \in \mathbb{R}^{2\times 2}$ be positive *diagonal* matrices (decision variables) and recall (4.5). By (2.5) and (4.14), we have

$$\begin{aligned} &\left(\Upsilon_\varrho(t)\right)^\top \left(\Lambda_{\Upsilon_\varrho} \otimes I_n\right) \Upsilon_\varrho(t) \\ &\qquad \leq \mathfrak{h}_\varrho \begin{bmatrix} x(t) \\ \xi(t) \end{bmatrix}^\top \left(\Lambda_{\Upsilon_\varrho} \otimes I_n\right) \begin{bmatrix} x(t) \\ \xi(t) \end{bmatrix}, \\ &\left(\Upsilon_{\Delta a_1}(t)\right)^\top \left(\Lambda_{\Upsilon_{\Delta a_1}} \otimes I_n\right) \Upsilon_{\Delta a_1}(t) \\ &\qquad \leq \Delta_{a_1,M} \begin{bmatrix} x(t) \\ \xi(t) \end{bmatrix}^\top \left(\Lambda_{\Upsilon_{\Delta a_1}} \otimes I_n\right) \begin{bmatrix} x(t) \\ \xi(t) \end{bmatrix}, \\ &\left(\Upsilon_{a_1}(t)\right)^\top \left(\Lambda_{\Upsilon_{a_1}} \otimes I_n\right) \Upsilon_{a_1}(t) \\ &\qquad \leq \mathfrak{h}_{a_1} \begin{bmatrix} x(t) \\ \xi(t) \end{bmatrix}^\top \left(\Lambda_{\Upsilon_{a_1}} \otimes I_n\right) \begin{bmatrix} x(t) \\ \xi(t) \end{bmatrix}. \end{aligned} \tag{4.15}$$

Define

$$\begin{aligned} \eta(t) = \mathrm{col}\left\{x(t), \xi(t), \Xi(t), \Upsilon_\varrho(t), \Upsilon_{a_1}(t),\right. \\ \left. \Upsilon_{\Delta a_1}(t), \varrho_{\epsilon,1}\Xi(t)\right\}. \end{aligned} \tag{4.16}$$

Recalling (4.15) and letting $0 < \mu \in \mathbb{R}$, we have

$$\begin{aligned} &0 \leq W_2 = \eta^\top(t) \left[\Gamma_0 - \Gamma_1\right] \eta(t) \\ &\Gamma_1 = \mathrm{diag}\left\{0, 0, 0, -\Lambda_{\Upsilon_\varrho} \otimes I_n, -\Lambda_{\Upsilon_{a_1}} \otimes I_n \right. \\ &\qquad \left. -\Lambda_{\Upsilon_{\Delta a_1}} \otimes I_n, -\mu I_n\right\}, \\ &\Gamma_0 = \mathrm{diag}\left\{\Gamma_0^{(1)}, \mu \mathfrak{h}_\varrho I_n, 0, 0, 0, 0\right\}, \\ &\Gamma_0^{(1)} = \mathfrak{h}_\varrho \left(\Lambda_{\Upsilon_\varrho} \otimes I_n\right) + \Delta_{a_1,M} \left(\Lambda_{\Upsilon_{\Delta a_1}} \otimes I_n\right) \\ &\qquad + \mathfrak{h}_{a_1} \left(\Lambda_{\Upsilon_{a_1}} \otimes I_n\right). \end{aligned} \tag{4.17}$$

By (4.8)–(4.17) and the S-procedure (Fridman, 2014)

$$\dot{V} + 2\alpha V \leq \dot{V} + 2\alpha V + W_2 \leq \eta^\top(t) \Omega_{\epsilon,h} \eta(t) \leq 0, \tag{4.18}$$

provided

$$\Omega_{\epsilon,h} = \left[\begin{array}{c|c|c} \Omega_{\epsilon,h}^{(1)} & \begin{matrix} 0 \\ 0 \end{matrix} & \Omega_{\epsilon,h}^{(2)} \\ \hline * & -\frac{e^{-2\alpha h} h}{\|\varpi\|_{L^1}} R_\Xi + \mu \mathfrak{h}_\varrho I_n & 0 \\ \hline * & * & \Omega_{\epsilon,h}^{(3)} \end{array}\right] < 0 \tag{4.19}$$

where

$$\begin{aligned} &\Omega_{\epsilon,h}^{(1)} = \begin{bmatrix} \omega_1 & PA_h - e^{-2\alpha h} h R_\xi + A_{av}^\top M_{\xi,\Xi} A_h \\ * & \omega_2 + A_h^\top M_{\xi,\Xi} A_h \end{bmatrix} + \Gamma_0^{(1)}, \\ &\Omega_{\epsilon,h}^{(2)} = \begin{bmatrix} -Q_\alpha \mathbb{A}_1 + PA_{av} \mathbb{A}_1 & A_{av}^\top M_{\xi,\Xi} \mathbb{A}_1 & P\mathbb{A}_1 & -PA_1 \\ -A_h^\top P \mathbb{A}_1 & A_h^\top M_{\xi,\Xi} \mathbb{A}_1 & 0 & 0 \end{bmatrix}, \\ &\Omega_{\epsilon,\mathfrak{h}}^{(3)} = \begin{bmatrix} 2\alpha \mathbb{A}_1^\top P \mathbb{A}_1 & 0 & -\mathbb{A}_1^\top P \mathbb{A}_1 & \mathbb{A}_1^\top P A_1 \\ * & \mathbb{A}_1^\top M_{\xi,\Xi} \mathbb{A}_1 & 0 & 0 \\ * & * & 0 & 0 \\ * & * & * & 0 \end{bmatrix} \\ &\qquad - \mathrm{diag}\left\{\Lambda_{\Upsilon_\varrho} \otimes I_n, \Lambda_{\Upsilon_{a_1}} \otimes I_n, \Lambda_{\Upsilon_{\Delta a_1}} \otimes I_n, \mu I_n\right\}, \\ &\omega_1 = Q_\alpha + h \|\varpi\|_{L^1} \cdot \left(1 - e^{-2\alpha h}\right) R_\xi + A_{av}^\top M_{\xi,\Xi} A_{av}, \\ &\omega_2 = -\tfrac{e^{-2\alpha h} h}{\|\varpi\|_{L^1}} R_\xi - \tfrac{e^{-2\alpha h} h^2}{2\varphi_\varpi} Z_\xi, \\ &M_{\xi,\Xi} = h \|\varpi\|_{L^1} \cdot R_\Xi + \tfrac{h^2}{2} \varphi_\varpi Z_\xi. \end{aligned} \tag{4.20}$$

Summarizing, we arrive at:

Theorem 4.1. *Consider the system (4.1) where $A_D(\tau) = A_h + a_1(\tau) A_1$, $A_h, A_0, A_1 \in \mathbb{R}^{n\times n}$ and $\varpi \in L^1([-h, 0])$ satisfying $\varpi(t) > 0$ a.e. in $[-h, 0]$. Let the rapidly-varying coefficient $a_1\left(\frac{t}{\epsilon}\right)$ satisfies Assumptions 1 and 2 for some $T > 0$. Assume further that either A_0*

or $A_{av} := A_0 + \|\varpi\|_{L^1} \cdot A_h$ is Hurwitz. Let $\mathfrak{h}_\varrho, \mathfrak{h}_{a_1} > 0$ be positive scalars such that for any $t \geq h$ and $\epsilon > 0$ (4.14) holds. Given tuning parameters $\epsilon^*, h^*, \Delta_{a_1,M} > 0$, let there exist $0 < P, R_\xi, R_\Xi, Z_\xi \in \mathbb{R}^n$, positive diagonal *matrices $\Lambda_{\Upsilon_\varrho}, \Lambda_{\Upsilon_{\Delta a_1}}, \Lambda_{\Upsilon_{a_1}} \in \mathbb{R}^{2\times 2}$, and $0 < \mu \in \mathbb{R}$ such that (4.19) and $\delta_{2,x} \|\varpi\|_{L^1} < e^{-\alpha h}$ hold with $\epsilon = \epsilon^*$ and $h = h^*$, where $\delta_{2,x}$ is defined by (2.10). Then, for all $\epsilon \leq \epsilon^*$ and $h \leq h^*$ system (4.1) is exponentially stable with decay rate $\alpha > 0$. The LMI (4.19) and $\delta_{2,x} \|\varpi\|_{L^1} < e^{-\alpha h}$ are feasible for small enough ϵ, h, $\Delta_{a_1,M}$.*

Proof. The proof is similar to the proof of Theorem 3.1 and is omitted due to space constraints. □

Example 4.1: Single phase AC system

In Griñó et al. (2021), the authors considered the following scalar system:

$$\dot{x}(t) = -\frac{k_i}{h} v^2(t) \int_{t-h}^t x(\theta) d\theta, \quad v(t) = \sqrt{2} V \sin\left(\frac{2\pi}{h} t\right),$$

which can be rewritten as

$$\dot{x}(t) = \left[-\frac{k_i V^2}{h} + \frac{k_i V^2}{h} \cos\left(\frac{4\pi t}{h}\right)\right] \int_{t-h}^t x(\theta) d\theta.$$

Note that in the latter, $h > 0$ appears in the denominator of the cosine. In order to apply our results to this system, we modify it as follows:

$$\dot{x}(t) = \left[-\frac{k_i V^2}{h} + \frac{k_i V^2}{h} \cos\left(\frac{4\pi t}{\epsilon}\right)\right] \int_{t-h}^t x(\theta) d\theta,$$

decoupling $\epsilon > 0$ and h. Here $V = 230$ is the RMS value of the voltage and the stabilizing gain $k_i > 0$ is to be maximized. This system can be presented as (4.1) with $T = 0.5$, $A_0 = 0$, $A_h = -\frac{k_i}{h} V^2$, $A_1 = \frac{k_i}{h} V^2$, $\varpi(\theta) \equiv 1$ and $a_1(\tau) = \cos(4\pi\tau)$, which leads to $\Delta a_i(t) \equiv 0$. In particular, note that $A_0 + \|\varpi\|_{L^1} A_h < 0$ for all $h > 0$ and $k_i > 0$.

We set $\alpha = 0$ and verify the feasibility of Theorem 4.1 conditions (i.e., inequalities (4.19) and $\delta_{2,x} \|\varpi\|_{L^1} < e^{-\alpha h}$) for two cases. Note that feasibility of the strict inequalities of Theorem 4.1 with $\alpha = 0$ imply their feasibility with some $\alpha > 0$, meaning that the system is exponentially stable with a small enough decay rate. First, we set $k_i = 3.1077 \cdot 10^{-4}$, $\epsilon = 0.02$ and obtain the largest value of h which preserves the stability. The result is given by $\max h = 0.0627$. Second, to apply our results to the setting of Griñó et al. (2021), we fix $\epsilon = 0.02$, $h = 0.02$ and verify the conditions of Theorem 4.1 to maximize k_i which preserves the stability. The result is $\max k_i = 6.96 \cdot 10^{-4}$, which is 2.24 times larger than $\max k_i = 3.1077 \cdot 10^{-4}$, obtained in Griñó et al. (2021).

5. Conclusions

We introduced a novel quantitative methodology for deriving ISS-like/stability properties for linear continuous-time systems. The presented methodology relies on a new system presentation, in conjunction with a delay-free system transformation. Compared to the recent time-delay approach to averaging, the new method presents a simpler ISS analysis of the transformed non-delayed system that employs Lyapunov functions and does not need additional solution bounds for times smaller than the time-scale parameter, and significantly improve the results in the numerical examples. However, the time-delay approach is applicable not just to classical averaging as considered in the present paper, but also to Lie-brackets-based averaging (Zhang & Fridman, 2023), Zhu and Fridman (2022) where application of the non-delay transformation seems to be questionable. Future work may include applications of the method to control problems that employ averaging.

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