European Journal of Control 85 (2025) 101336

Contents lists available at ScienceDirect

European Journal of Control

journal homepage: www.sciencedirect.com/journal/european-journal-of-control

Homogeneous predictor feedback for a 1D reaction–diffusion equation with input delay☆

Mericel Ayamou [a,*], Nicolas Espitia [b], Andrey Polyakov [a], Emilia Fridman [c]

[a] Univ. Lille, Inria, CNRS CRIStAL, Centrale Lille, FR-59000, France
[b] Univ. Lille, CNRS CRIStAL, Centrale Lille, FR-59000, France
[c] Department of Electrical Engineering and Systems, Tel Aviv University, Tel Aviv 69978, Israel

ARTICLE INFO

Recommended by T Parisini

Keywords:
Homogeneous control
1D reaction-diffusion equation
Delay control
Lyapunov function

ABSTRACT

This paper deals with nonlinear boundary stabilization of a 1D reaction–diffusion equation with input delay. Using the modal decomposition approach, we propose a *homogeneous*-based predictor feedback for stabilizing the unstable modes. We prove the stability of the closed-loop system via the construction of a suitable Lyapunov functional. We present numerical simulations to support the analytical results and compare our proposed controller to linear predictor feedback regarding closed-loop performance and peaking effect.

1. Introduction

Delay compensation in parabolic partial differential equations (PDEs) has become a critical topic, as control actions in complex systems (see Christofides & Chow, 2002, Curtain & Zwart, 2012) arising in fields such as biology, chemistry, and spatial ecology, can be significantly delayed. The presence of delays increases mathematical complexity, necessitating specialized techniques for stability analysis, numerical implementation, and control design (see Fridman, 2014). Control design for complex systems modeled by PDEs becomes even more challenging with input delays.

The boundary stabilization of the one-dimensional reaction–diffusion equation with input delay was first introduced and solved in Krstic (2009) using the *backstepping method*. An alternative approach for stabilizing parabolic PDEs is the *modal decomposition* approach (see Prieur & Trélat, 2019), which separates a finite-dimensional unstable component from a stable infinite-dimensional part of the PDE (see Coron & Trélat, 2004; Russell, 1978) and then designs a controller based on the unstable modes. Following this approach and using Artstein's transformation (Artstein, 1982), Prieur and Trélat (2019) proposed a predictor feedback based on the unstable modes and proved the stability of the entire system by constructing an appropriate Lyapunov function.

Notable extensions include, for example, Lhachemi, Prieur, and Trélat (2020), which addresses the output regulation of a one-dimensional reaction–diffusion equation with input delay. Observer-based control for one-dimensional reaction–diffusion PDEs with input/output delay has been investigated in Katz, Fridman, and Selivanov (2020). In contrast, delay compensation has been studied using classical predictors and sub-predictors (Katz & Fridman, 2021), with sub-predictors extended to the semilinear heat equation in Katz and Fridman (2022).

In these contributions, linear controllers are preferred due to their simplicity in control application and closed-loop analysis. However, they also have notable drawbacks, including the peaking effect and large overshoot (see Izmailov, 1987; Polyak & Smirnov, 2016). Achieving better convergence in the closed-loop system with linear control often results in significant deviation during the initial stabilization phase (peaking effect), leading to large overshoot—presenting practical challenges in real-world applications. A homogeneous controller (Polyakov, 2020; Polyakov & Krstic, 2023, 2025) can address these issues, achieving fast convergence without peaking and with minimal overshoot (see Polyakov, 2020, Chapter 1 for finite-dimensional systems and Ayamou, Espitia, Polyakov, and Fridman (2024) in the context of infinite-dimensional systems).

In this paper, using the modal decomposition approach and Artstein's transformation, we design a homogeneous predictor feedback from the unstable modes and prove the stability of the entire system by constructing an appropriate Lyapunov functional. Next, we investigate

☆ This work has been partially supported by the CPER RITMEA, ANR SLIMDISC, and ISF-NSCF grant no. 3054/23.

* Corresponding author.
E-mail addresses: mericel.ayamou@inria.fr (M. Ayamou), nicolas.espitia-hoyos@univ-lille.fr (N. Espitia), andrey.polyakov@inria.fr (A. Polyakov), emilia@tauex.tau.ac.il (E. Fridman).

https://doi.org/10.1016/j.ejcon.2025.101336
Received 9 June 2025; Accepted 9 July 2025
Available online 22 July 2025
0947-3580/

numerically whether the use of a homogeneous controller results in a smaller overshoot than a linear controller for a one-dimensional reaction–diffusion equation with input delay. To the best of our knowledge, the *homogeneous predictor feedback* has never been designed for reaction–diffusion PDEs.

The paper is organized as follows: Section 2 presents preliminary results on homogeneity. Section 3 deals with the problem statement and modal decomposition. Section 4 gives the main result. Section 5 presents numerical results.

Notation. $|z| = \sqrt{z^\top z}$ is Euclidean norm of $\mathbb{R}^n$; we write $P \succ 0$ if the symmetric matrix $P = P^\top \in \mathbb{R}^{n\times n}$ is positive definite ; we denote $\|x\|_P = \sqrt{x^\top P x}$ with the matrix $P \succ 0$; given $L > 0$, $\langle\cdot,\cdot\rangle$ is a scalar product on $L^2(0,L)$ such that $\forall f,g \in L^2(0,L)$, $\langle f,g\rangle = \int_0^L f(x)g(x)dx$, $H^k(0,L)$ is Sobolev space having k square integrable weak derivatives ; $H_0^1(0,L)$ is a subset of $H^1(0,L)$ composed of functions f such that $f(0) = f(L) = 0$. $C(\mathbb{R}_+)$ is a set of continuous function on $\mathbb{R}_+$. A function $\sigma : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be a class-$\mathcal{K}$ function if it is continuous, zero at zero, and strictly increasing. A class-$\mathcal{K}$ function $\sigma : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be a class-$\mathcal{K}_\infty$ function if it is unbounded with its argument. A continuous function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the class-$\mathcal{KL}$ if $\beta(\cdot,t) \in \mathcal{K}$ for each fixed $t \in \mathbb{R}_+$, and $\beta(r,\cdot)$ is decreasing and $\lim_{t\to+\infty} \beta(r,t) = 0$ for each fixed $r \in \mathbb{R}_+$.

2. Preliminaries on homogeneity

A family of operators $\mathbf{d}(s) : \mathbb{R}^n \mapsto \mathbb{R}^n$ with $s \in \mathbb{R}$ is a continuous dilation in $\mathbb{R}^n$ if it satisfies:

- *group property*: $\mathbf{d}(0)x = x$, $\mathbf{d}(s)\circ\mathbf{d}(t)x = \mathbf{d}(s+t)x$, $\forall x \in \mathbb{R}^n, \forall s,t \in \mathbb{R}$;
- *continuity property*: the mapping $s \mapsto \mathbf{d}(s)x$ is continuous, $\forall x \in \mathbb{R}^n$;
- *limit property*: $\liminf_{s\to+\infty} \|\mathbf{d}(s)x\| = +\infty$ and $\limsup_{s\to-\infty} \|\mathbf{d}(s)x\| = 0$, $\forall x \neq 0_{n\times 1}$, where $\|\cdot\|$ is a norm of $\mathbb{R}^n$.

A dilation $\mathbf{d}$ is linear if $\mathbf{d}(s) \in \mathbb{R}^{n\times n}$ is linear. Any linear continuous dilation $\mathbf{d}$ in $\mathbb{R}^n$ admits the representation (Polyakov, 2019):

$$\mathbf{d}(s) = e^{sG_\mathbf{d}} = \sum_{j=0}^{\infty} \frac{s^j G_\mathbf{d}^j}{j!}, \quad s \in \mathbb{R}, \tag{1}$$

where $G_\mathbf{d} \in \mathbb{R}^{n\times n}$ is an anti-Hurwitz matrix called a generator of $\mathbf{d}$.

Definition 1. A *dilation* $\mathbf{d}$ is monotone if $s \mapsto \|\mathbf{d}(s)x\|$ is a monotone increasing function for any $x \neq 0$.

Proposition 1 (*Polyakov, 2020*). *A linear continuous dilation in $\mathbb{R}^n$ is monotone with respect to the weighted Euclidean norm $\|x\|_P$, $0 \prec P = P^\top \in \mathbb{R}^{n\times n}$ if and only if*

$$PG_\mathbf{d} + G_\mathbf{d}^\top P \succ 0, \quad P \succ 0. \tag{2}$$

Any linear continuous and monotone dilation in a normed vector space introduces also an alternative norm topology.

A function $\|\cdot\|_\mathbf{d} : \mathbb{R}^n \mapsto [0,+\infty)$ defined as follows: $\|0_{n\times 1}\|_\mathbf{d} = 0$ and

$$\|x\|_\mathbf{d} = e^{s_x}, \text{ where } s_x \in \mathbb{R} : \|\mathbf{d}(-s_x)x\|_P = 1, \quad x \neq 0_{n\times 1}, \tag{3}$$

is called a canonical $\mathbf{d}$-homogeneous norm in $\mathbb{R}^n$, where $\mathbf{d}$ is a linear continuous dilation being monotone with respect to the norm $\|\cdot\|_P$ and P is defined in (2). Note that for all $x \in \mathbb{R}^n$, one has:

$$\|\mathbf{d}(-\ln(\|x\|_\mathbf{d}))x\|_P = 1. \tag{4}$$

Dilation symmetry of system is introduced by the following definition.

Definition 2 (*Kawski, 1991*). Given a vector field $f : \mathbb{R}^n \mapsto \mathbb{R}^n$, a system $\dot{x} = f(x)$ is $\mathbf{d}$-homogeneous of degree $\mu \in \mathbb{R}$ if

$$f(\mathbf{d}(s)x) = e^{\mu s}\mathbf{d}(s)f(x), \quad \forall x \in \mathbb{R}^n, \quad \forall s \in \mathbb{R}. \tag{5}$$

3. Problem statement and modal decomposition

Let us consider the 1D reaction–diffusion equation

$$\partial_t z(t,x) = \partial_{xx} z(t,x) + qz(t,x), \tag{6}$$
$$z(t,0) = 0, \tag{7}$$
$$z(t,L) = U(t-r), \tag{8}$$
$$z(0,x) = z_0(x), \tag{9}$$

$(t,x) \in \mathbb{R}_+ \times [0,L]$, $z_0 \in H_0^1(0,L)$ the initial condition, where $z(t,\cdot)$ is the reaction–diffusion PDE state at time t, $q > 0$ is the reaction coefficient, $U(t-r) \in \mathbb{R}$ is the control input with $r > 0$ a constant delay where $U|_{[-r,0]} \equiv 0$.

This paper aims to design a homogeneous-based predictor feedback for the system (6)–(9) using the modal decomposition approach (Prieur & Trélat, 2019) and to investigate numerically if the use of a homogeneous feedback results in a smaller peaking than a linear feedback. More precisely we consider the following control problem: *given initial state $z_0 \in H_0^1(0,L)$, a stabilization precision $\epsilon > 0$ and a prescribed time $T_p > 0$, the closed-loop system has the desired stabilization precision:*

$$\|z(t,\cdot)\|_{H^1(0,L)} \le \epsilon, \quad \forall t > T_p, \tag{10}$$

with the restricted control magnitude

$$\sup_{t>0} |U(t-r)| \le \bar{U}, \tag{11}$$

for some maximal control amplitude $\bar{U} > 0$. It may happen that a high-gain linear controller ensures that the solution of the closed-loop system meets the criteria specified in (10), but it does not fulfill the control constraint outlined in (11). In contrast, the homogeneous control meets both criteria (10) and (11). Another situation that can be considered is when both the linear controller and the homogeneous one exhibit a similar overshoot value e.g.,. while verifying (11), but the linear controller may violate condition (10) whereas the homogeneous one does not. The advantage of the homogeneous controller lies then in its ability to achieve faster convergence. These considerations can be assessed numerically.

3.1. Modal decomposition

We start by considering the following trigonometric change of variable (see e.g., Karafyllis, 2021; Katz & Fridman, 2023):

$$w(t,x) = z(t,x) - \kappa(x)U(t-r), \quad \kappa(x) = \sin(\sigma x), \tag{12}$$

with $\sigma = \frac{\pi}{2L}$ to obtain the following equivalent PDE:

$$\partial_t w(t,x) = \partial_{xx} w(t,x) + qw(t,x) - \kappa(x)\Big[\dot{U}(t-r) - (-\sigma^2+q)U(t-r)\Big], \tag{13}$$
$$w(t,0) = w(t,L) = 0, \tag{14}$$
$$w(0,x) = z_0(x). \tag{15}$$

By introducing a new control input $\xi(t)$ such that $\xi|_{[-r,0]} \equiv 0$ and

$$\xi(t) = \dot{U}(t) - (-\sigma^2+q)U(t), \quad \forall t \ge 0, \tag{16}$$

one obtains the following coupled ODE-PDE system:

$$\dot{U}(t-r) = (-\sigma^2+q)U(t-r) + \xi(t-r), \tag{17}$$
$$\partial_t w(t,x) = \partial_{xx} w(t,x) + qw(t,x) - \kappa(x)\xi(t-r), \tag{18}$$
$$w(t,0) = w(t,L) = 0,$$
$$w(0,x) = z_0(x). \tag{19}$$

The solution to (18) can be represented as follows:

$$w(t,x) = \sum_{n=1}^{\infty} w_n(t)\phi_n(x), \quad w_n(t) = \langle w(t,\cdot),\phi_n\rangle, \tag{20}$$

with $\phi_n(x) = \sqrt{\frac{2}{L}}\sin(\frac{n\pi x}{L})$, $n \geq 1$. Projecting onto the basis $\{\phi_n\}_{n=1}^{\infty}$, one has

$$\dot{U}(t-r)=(-\sigma^2+q)U(t-r)+\xi(t-r), \tag{21}$$

$$\dot{w}_n(t)=(-\lambda_n+q)w_n(t)+b_n\xi(t-r), \tag{22}$$

$$w_n(0)=\langle z_0,\phi_n\rangle, \quad n \geq 1, \tag{23}$$

where $n \geq 1$, $\lambda_n = \frac{n^2\pi^2}{L^2}$ is the eigenvalue of Sturm–Liouville eigenvalue problem

$$\phi''(x)+\lambda\phi(x)=0, \quad \phi(0)=\phi(L)=0, \tag{24}$$

corresponding to the eigenvector ϕ_n,

$$b_n = -\int_0^L \kappa(x)\phi_n(x)dx = \frac{(-1)^{n+2}\sqrt{2L}}{\pi}\left(\frac{1}{2n-1}+\frac{1}{2n+1}\right). \tag{25}$$

Since $\lambda_n \to +\infty$ when $n \to +\infty$, then there exists an integer $N \geq 1$ such that

$$-\lambda_n + q < -\delta, \quad \forall n \geq N+1, \tag{26}$$

with $\delta > 0$ some positive constant. We obtain from (21)–(23), the following system:

$$\dot{W}(t) = AW(t)+B\xi(t-r), \tag{27}$$

$$\dot{w}_n(t) = (-\lambda_n+q)w_n(t)+b_n\xi(t-r), n > N, \tag{28}$$

$$w_n(0) = \langle z_0,\phi_n\rangle, \quad n \geq 1, \tag{29}$$

with $W(t) = (U(t-r), w_1(t), \ldots, w_N(t))^{\top}$, $B = (1, b_1, \ldots, b_N)^{\top}$, $A = \begin{pmatrix} -\sigma^2+q & 0_{1\times N} \\ 0_{N\times 1} & \mathrm{diag}\{-\lambda_n+q\}_{n=1}^N \end{pmatrix}$.

By introducing the Artstein transformation (see Artstein (1982), Fridman (2014)) as follows

$$Z(t) = e^{Ar}W(t) + \int_{t-r}^{t} e^{A(t-s)}B\xi(s)ds, \tag{30}$$

one derives

$$\dot{Z}(t) = AZ(t)+B\xi(t), \quad Z(0)=e^{Ar}W(0). \tag{31}$$

4. Homogeneous stabilization of 1-D heat system by full-state feedback

In this part, we stabilize (31) using homogeneous control and next study the stability of the entire system (27)–(29).

4.1. Stabilization of the finite-dimensional part

Since $\{A,B\}$ is controllable (see Katz & Fridman, 2023, Karafyllis, 2021, Lemma 2.1) then according to Zimenko, Polyakov, Efimov, and Perruquetti (2020), Polyakov (2020), the linear algebraic equations

$$AG_0 - G_0A + BY_0 = A, \quad G_0B = 0, \tag{32}$$

have solutions $Y_0 \in \mathbb{R}^{1\times(N+1)}$, $G_0 \in \mathbb{R}^{(N+1)\times(N+1)}$ such that the matrix $G_0 - I_{N+1}$ is invertible, the matrix $G_{\mathbf{d}} := I_{N+1}+\mu G_0$ is anti-Hurwitz for any $\mu \in (-1,0)$ and the matrix $A_0 = A+BK_0$ with $K_0 = Y_0(G_0-I_{N+1})^{-1}$ satisfies the identity

$$A_0G_{\mathbf{d}} = (G_{\mathbf{d}}+\mu I_{N+1})A_0, \quad G_{\mathbf{d}}B = B. \tag{33}$$

Moreover, the linear matrix inequalities

$$\begin{aligned} &(A_0+\rho G_{\mathbf{d}})X+X(A_0+\rho G_{\mathbf{d}})^{\top}+BY+Y^{\top}B^{\top} \preceq 0,\\ &G_{\mathbf{d}}X+XG_{\mathbf{d}}^{\top} \succ 2(1+\mu)X, \qquad X=X^{\top}\succ 0, \end{aligned} \tag{34}$$

have solutions X, Y for any $\rho > 0$. Recall that $\mathbf{d}$ is the dilation generated by $G_{\mathbf{d}}$.

Proposition 2 (Polyakov, 2020). *For the* $\mathbf{d}$*-homogeneous norm* $\|\cdot\|_{\mathbf{d}}$ *induced by the norm* $\|\cdot\|_{X^{-1}}$ *one has for all* $x \in \mathbb{R}^n$,

$$\|x\|_{\mathbf{d}}^{\nu} \leq \|x\|_{X^{-1}} \leq \|x\|_{\mathbf{d}}^{\tau}, \qquad \|x\|_{X^{-1}} \leq 1; \tag{35}$$

$$\|x\|_{\mathbf{d}}^{\tau} \leq \|x\|_{X^{-1}} \leq \|x\|_{\mathbf{d}}^{\nu}, \qquad \|x\|_{X^{-1}} \geq 1, \tag{36}$$

with

$$\begin{aligned} \nu &= \frac{\lambda_{\max}(X^{-1/2}G_{\mathbf{d}}X^{1/2}+X^{1/2}G_{\mathbf{d}}^{\top}X^{-1/2})}{2},\\ \tau &= \frac{\lambda_{\min}(X^{-1/2}G_{\mathbf{d}}X^{1/2}+X^{1/2}G_{\mathbf{d}}^{\top}X^{-1/2})}{2}. \end{aligned} \tag{37}$$

Moreover, from (34) *one has the following inequality*

$$1+\mu < \tau. \tag{38}$$

Lemma 1. *Let the canonical homogeneous norm* $\|Z(t)\|_{\mathbf{d}}$ *be induced by the weighted norm* $\|Z(t)\|_{X^{-1}}$. *Then the system* (31) *with the continuous feedback law*

$$\xi(Z(t)) = K_0Z(t)+\mathcal{N}(Z(t)), \tag{39}$$

$$\mathcal{N}(Z(t)) = \|Z(t)\|_{\mathbf{d}}^{1+\mu}K\mathbf{d}(-\ln(\|Z(t)\|_{\mathbf{d}}))Z(t), \tag{40}$$

where $K = YX^{-1}$, X, Y *being solutions of* (34) *for some* $\rho > 0$ *is* $\mathbf{d}$*-homogeneous of degree* μ *and globally finite-time stable*

$$Z(t)=0, \quad \forall t \geq T := \frac{\|Z(0)\|_{\mathbf{d}}^{-\mu}}{-\rho\mu}. \tag{41}$$

Proof. The proof follows the same lines as in the proof of Zimenko et al. (2020, Lemma 5) showing that the canonical homogeneous norm $\|\cdot\|_{\mathbf{d}}$ is a Lyapunov functional and satisfies

$$\frac{d\|Z(t)\|_{\mathbf{d}}}{dt} \leq -\rho\|Z(t)\|_{\mathbf{d}}^{1+\mu}, \tag{42}$$

along the solution of closed-loop system (31) and (39). ■

Since the control system (31) is uncontrolled for $t \leq 0$, one considers the following feedback law

$$\xi(t) = \begin{cases} 0, & \text{if} \quad t \leq 0,\\ K_0Z(t)+\mathcal{N}(Z(t)) & \text{if} \quad t > 0. \end{cases} \tag{43}$$

Then the closed-loop system (28),(29), (31) and (43) is:

$$\begin{aligned} \dot{Z}(t) &= AZ(t)+B\Big(K_0Z(t)+\mathcal{N}(Z(t))\Big),\\ \dot{w}_n(t) &= (-\lambda_n+q)w_n(t)+\chi_{(r,+\infty)}(t)b_n\Big(K_0Z(t-r)+\\ &\qquad \mathcal{N}(Z(t-r))\Big), \quad n > N,\\ Z(0) &= e^{Ar}W(0), \quad w_n(0)=\langle z_0,\phi_n\rangle, \quad n > N, \end{aligned} \tag{44}$$

with $\chi_{(r,+\infty)}(t)=0$ for $t \leq r$ and $\chi_{(r,+\infty)}(t)=1$ otherwise.

4.2. Stability of entire system

For H^1 stability analysis of the closed-loop system (44), we define the following Lyapunov functional:

$$\begin{aligned} V(t) = \gamma_1\Omega(\|Z(t)\|_{\mathbf{d}}) + \gamma_2\int_{\max\{t-r,0\}}^{t}\Psi(\|Z(s)\|_{\mathbf{d}})ds\\ +\sum_{n>N}\lambda_n w_n^2(t), \quad \forall t \geq 0, \end{aligned} \tag{45}$$

with

$$\begin{aligned} \Omega(s) &= \begin{cases} \frac{1}{2+\mu}s^{2+\mu}, & \text{if} \quad s \leq 1,\\ \frac{1}{2\nu-\mu}s^{2\nu-\mu}-\frac{1}{2\nu-\mu}+\frac{1}{2+\mu}, & \text{if} \quad s \geq 1, \end{cases}\\ \Psi(s) &= \begin{cases} s^{2(1+\mu)}, & \text{if} \quad s \leq 1,\\ s^{2\nu}, & \text{if} \quad s \geq 1. \end{cases} \end{aligned} \tag{46}$$

and $\gamma_1, \gamma_2 > 0$, ν is defined in (37). Note that $\Psi \in \mathcal{K}_{\infty}$ and using $1+\mu < \tau$ (from Proposition 2) then $2\nu-\mu > 2+\mu$ and $\Omega \in \mathcal{K}_{\infty}$.

Lemma 2. *There exists $\eta_1 \in \mathcal{K}_\infty$ such that for all $t \geq 0$,*

$$\eta_1(U^2(t-r) + \|w(t,\cdot)\|^2_{H^1(0,L)}) \leq V(t). \tag{47}$$

Proof. Since $w(t,\cdot) \in H^1_0(0,L)$ then there exists $C_1 > 0$ such that

$$\|w(t,\cdot)\|^2_{H^1(0,L)} \leq C_1 \sum_{n=1}^{\infty} \lambda_n w_n^2(t), \tag{48}$$

which implies that

$$U^2(t-r) + \|w(t,\cdot)\|^2_{H^1(0,L)} \leq C_2\Big(|W(t)|^2 + \sum_{n=N+1}^{\infty} \lambda_n w_n^2(t)\Big), \tag{49}$$

with $C_2 = \max\{C_1, C_1\lambda_{N+1} + 1\}$. From (30), one has

$$W(t) = e^{-Ar}Z(t) - \int_{\max\{t-r,0\}}^{t} e^{A(t-s-r)}B\Big(K_0 Z(s) + \mathcal{N}(Z(s))\Big)ds, \quad \forall t \geq 0. \tag{50}$$

From Proposition 2 (and using $1+\mu < \tau < \nu$), one has

$$\|Z(t)\|_{X^{-1}} \leq \Psi^{\frac{1}{2}}(\|Z(t)\|_{\mathbf{d}}). \tag{51}$$

Using (51) and the fact that $\|\mathbf{d}(-\ln\|Z(t)\|_{\mathbf{d}})Z(t)\|_{X^{-1}} = 1$, one derives for all $t \geq 0$,

$$|W(t)|^2 \leq M_1\Big(\Psi(\|Z(t)\|_{\mathbf{d}}) + \int_{\max\{t-r,0\}}^{t} \Psi(\|Z(s)\|_{\mathbf{d}})ds\Big), \tag{52}$$

with $M_1 = 2e^{2|A|r}\lambda_{min}(X^{-1})^{-1}\max\{1, r|B|^2 M_2\}$ and $M_2 = 4\max\{|K_0^\top|^2, |K^\top|^2\}$. This means for all $t \geq 0$,

$$|W(t)|^2 \leq M_1\Big(\Psi\circ\Omega^{-1}(\tfrac{1}{\gamma_1}V(t)) + \tfrac{1}{\gamma_2}V(t)\Big). \tag{53}$$

Using (49) and (53), the proof is complete. ■

Let for all $t \geq 0$,

$$V_1(t) = \|Z(t)\|_{\mathbf{d}}, \quad V_2(t) = \sum_{n>N} \lambda_n w_n^2(t), \tag{54}$$

$$V_3(t) = \chi_{(r,+\infty)}(t)\|Z(t-r)\|_{\mathbf{d}}. \tag{55}$$

Computing the time-derivatives along the solutions of closed-loop system (44), for all $t > r$, one has from (42) and (46)

$$\frac{d\Omega(V_1(t))}{dt} \leq -\rho\Psi(V_1(t)), \tag{56}$$

and

$$\frac{d}{dt}\int_{t-r}^{t} \Psi(V_1(s))ds = \Psi(V_1(t)) - \Psi(V_3(t)). \tag{57}$$

On other hand, along solution of closed-loop system (44) for all $t > r$,

$$\dot{V}_2(t) = 2\sum_{n>N} \lambda_n w_n(t)\Big((-\lambda_n + q)w_n(t) + b_n\xi(t-r)\Big). \tag{58}$$

Using the Young inequality, for all $\gamma_3 > \frac{\lambda_{N+1}}{2(\lambda_{N+1}-q)}$ the equality (58) implies for all $t > r$,

$$\dot{V}_2(t) \leq 2\sum_{n>N} \lambda_n\Big((-1 + \tfrac{1}{2\gamma_3})\lambda_n + q\Big)w_n^2(t) + \gamma_3 M_3 \Psi(V_3(t)), \tag{59}$$

with $M_3 = \lambda_{min}(X^{-1})^{-1} M_2\Big(\sum_{n>N}|b_n|^2\Big)$.

Using (52), one derives for all $t > r$,

$$\dot{V}_2(t) \leq -2\theta_{N,1}V_2(t) + \gamma_3 M_3\Psi(V_3(t)), \tag{60}$$

with

$$\theta_{N,1} = (1 - \tfrac{1}{2\gamma_3})\lambda_{N+1} - q > 0. \tag{61}$$

From (45), along the solution of closed loop system (44), one has for all $t > r$,

$$\dot{V}(t) = \gamma_1 \frac{d\Omega(V_1(t))}{dt} + \gamma_2 \frac{d}{dt}\int_{t-r}^{t}\Psi(V_1(s))ds \qquad + \dot{V}_2(t). \tag{62}$$

Then using (56),(57) and (60) one derives for all $t > r$,

$$\dot{V}(t) \leq -2\theta_{N,1}V_2(t) + (\gamma_3 M_3 - \gamma_2)\Psi(V_3(t)) + (\gamma_2 - \gamma_1\rho)\Psi(V_1(t)). \tag{63}$$

Since from (42) the map $t \mapsto V_1(t)$ is decreasing on $[0,+\infty)$ then for all $t > r$, and for all $s \in [t-r, t]$,

$$\Psi(V_1(t)) \leq \Psi(V_1(s)) \leq \Psi(V_3(t)). \tag{64}$$

This implies that for all $t > r$

$$\Psi(V_3(t)) \geq \frac{1}{r}\int_{t-r}^{t}\Psi(V_1(s))ds. \tag{65}$$

In addition, for $\gamma_2 > \gamma_3 M_3$ and $\gamma_1 > \frac{\gamma_2}{\rho}$, one has

$$\tilde{\rho} = \min\{2\theta_{N,1}, \gamma_1\rho - \gamma_2, \tfrac{\gamma_2 - \gamma_3 M_3}{r}\} > 0, \tag{66}$$

and then the inequality (63) implies that for all $t > r$

$$\dot{V}(t) \leq -\tilde{\rho}\Big(V_2(t) + \Psi(V_1(t)) + \int_{t-r}^{t}\Psi(V_1(s))ds\Big). \tag{67}$$

By using Lemma 3 (in the Appendix), one has the following inequality

$$\int_{t-r}^{t}\Psi(V_1(s))ds + V_2(t) + \Psi\circ\Omega^{-1}(\Omega(V_1(t))) \geq \eta_2(V(t)), \tag{68}$$

where η_2 is the $\mathcal{K}_\infty$ function given by

$$\eta_2(s) = \min\{\tfrac{s}{3}, \Psi\circ\Omega^{-1}\Big(\tfrac{s}{3\gamma_1}\Big), \tfrac{s}{3\gamma_2}\}, \quad \forall s \geq 0. \tag{69}$$

Finally, one concludes that, for all $t > r$,

$$\dot{V}(t) \leq -\tilde{\rho}\eta_2(V(t)), \tag{70}$$

and then there exists $\beta_1 \in \mathcal{KL}$ such that, for all $t \geq r$

$$V(t) \leq \beta_1(V(r), t-r). \tag{71}$$

Consider now the case $t < r$. Along the solution of closed-loop system (44), for all $t \in [0,r)$ one has:

$$\dot{V}(t) \leq (-\rho\gamma_1 + \gamma_2)\Psi(V_1(t)) + 2(-\lambda_{N+1} + q)\sum_{n>N}\lambda_n w_n^2(t). \tag{72}$$

Since from (66) $\rho\gamma_1 > \gamma_2$, one derives, for all $t \in [0,r)$

$$\dot{V}(t) \leq 0, \tag{73}$$

which implies that for all $t \in [0,r]$,

$$V(t) \leq \gamma_1\Omega(\|Z(0)\|_{\mathbf{d}}) + \sum_{n>N}\lambda_n w_n^2(0). \tag{74}$$

Since $\Omega \in \mathcal{K}_\infty$, using Proposition 2 and the fact that $Z(0) = e^{Ar}W(0)$ there exists $\eta_3 \in \mathcal{K}_\infty$ such that for all $t \in [0,r]$

$$V(t) \leq \eta_3(|W(0)|^2 + \sum_{n>N}\lambda_n w_n^2(0)). \tag{75}$$

Using the fact that $z_0 \in H^1_0(0,L)$ and $U(-r) = 0$, one has

$$\min\{\lambda_1, 1\}\Big(|W(0)|^2 + \sum_{n>N}\lambda_n w_n^2(0)\Big) \leq \|z_0\|^2_{H^1(0,L)}. \tag{76}$$

One concludes that there exists $\eta_4 \in \mathcal{K}_\infty$ such that

$$\forall t \in [0,r], \quad V(t) \leq \eta_4(\|z_0\|^2_{H^1(0,L)}). \tag{77}$$

From (71) and (77), there exists $\beta_2 \in \mathcal{KL}$ such that

$$\forall t \geq 0, \quad V(t) \leq \beta_2(\|z_0\|^2_{H^1(0,L)}, t). \tag{78}$$

Summarizing, one arrives at the following main result:

Theorem 1. *For any initial condition $z_0 \in H^1_0(0,L)$, there exists $\beta \in \mathcal{KL}$ such that the solution of closed-loop system (6)–(8) with the homogeneous feedback law (17) and (43) where Z is given by (27) and (30) satisfies:*

$$\|z(t,\cdot)\|^2_{H^1(0,L)} \leq \beta(\|z_0\|^2_{H^1(0,L)}, t), \quad \forall t \geq 0. \tag{79}$$

Moreover, for all $t \geq T_r$, the control input U and z satisfy

$$U(t-r) = 0, \quad \langle z(t,\cdot), \phi_n \rangle = 0, \quad n = 1,\dots,N, \tag{80}$$

$$\|z(t,\cdot)\|^2_{H^1(0,L)} \leq M e^{-2\delta(t-T_r)} \|z(T_r,\cdot)\|^2_{H^1(0,L)}, \tag{81}$$

with $T_r := \frac{\|e^{Ar} z_0^N\|_{\mathbf{d}}^{-\mu}}{-\rho\mu} + r$, $z_0^N = (0, \langle z_0, \phi_1 \rangle, \dots, \langle z_0, \phi_N \rangle)^\top$, *with some positive* $M \geq 1$.

Proof. The inequality (78) together with Lemma 2 imply that there exists $\beta_3 \in \mathcal{KL}$ such that

$$U^2(t-r) + \|w(t,\cdot)\|^2_{H^1(0,L)} \leq \beta_3(\|z_0\|^2_{H^1(0,L)}, t), \forall t \geq 0. \tag{82}$$

From (12), one has for all $t \geq 0$,

$$\begin{aligned} \|z(t,\cdot)\|^2_{H^1(0,L)} \leq & 2\max\{1, \|\kappa\|^2_{H^1(0,L)}\} \\ & \times (\|w(t,\cdot)\|^2_{H^1(0,L)} + U^2(t-r)), \end{aligned} \tag{83}$$

then one concludes that there exists $\beta \in \mathcal{KL}$ such that

$$\|z(t,\cdot)\|^2_{H^1(0,L)} \leq \beta(\|z_0\|^2_{H^1(0,L)}, t), \quad \forall t \geq 0. \tag{84}$$

Since from Lemma 1 for all $t \geq T$, $Z(t) = 0$ then using (30) one obtains,

$$W(t) = 0, \quad \forall t \geq T_r := T + r. \tag{85}$$

Thus using (12), for all $t \geq T_r$,

$$\begin{aligned} & U(t-r) = 0, \quad \langle z(t,\cdot), \phi_n \rangle = 0, \quad n = 1,\dots,N \\ & \tfrac{d}{dt}\langle z(t,\cdot), \phi_n \rangle = (-\lambda_n + q)\langle z(t,\cdot), \phi_n \rangle, \quad n > N. \end{aligned} \tag{86}$$

Using (26), one derives for all $t \geq T_r$,

$$\sum_{n>N} \lambda_n |\langle z(t,\cdot), \phi_n \rangle|^2 \leq e^{-2\delta(t-T_r)} \sum_{n>N} \lambda_n |\langle z(T_r,\cdot), \phi_n \rangle|^2, \tag{87}$$

which together the fact that for all $t \geq T_r$, $z(t,\cdot) \in H^1_0(0,L)$, the proof is complete. ■

Remark 1. For $\mu = 0$, the homogeneous feedback control $\xi(Z(t)) = (K_0 + K)Z(t)$ is a linear . In this case, $\mathbf{d}(s) = e^s I_{N+1}$, $G_{\mathbf{d}} = I_{N+1}$ and then $\nu = \tau = 1$. One recovers the classical Lyapunov functional considered in the linear case (Prieur & Trélat, 2019):

$$V(t) = \tfrac{\gamma_1}{2} Z(t)^\top X^{-1} Z(t) + \sum_{n>N} \lambda_n w_n^2(t) \tag{88}$$

$$+\gamma_2 \int_{\max\{t-r,0\}}^{t} Z(s)^\top X^{-1} Z(s) ds. \tag{89}$$

5. Numerical simulations

We perform numerical simulations on the system (6)–(9), by using system (17)–(19) and transformation (12). The initial condition and parameters are: $L = \pi$, $z_0(x) = \frac{x}{L}(L-x)$, $r = 2$, $q = 1.25$ and $N = 1$. We consider (20) with $M = 10$ truncated basis. We use the control toolbox on Matlab to compute the linear control gains:

$$\begin{aligned} K_{l_1} &= (-26.6667 - 17.3114), \\ K_{l_2} &= (-34.3467 - 23.4025), \end{aligned} \tag{90}$$

with which we compute the linear control (Katz & Fridman, 2022) given by

$$U(t-r) = \chi_{(r,+\infty)} K_l \int_0^{t-r} e^{(-\sigma^2+q)(t-s-r)} Z(s) ds. \tag{91}$$

We use the Homogeneous Control Systems (HCS) Toolbox for MATLAB (Polyakov, 2023) to compute $\|\cdot\|_{\mathbf{d}}, K_0, K, G_{\mathbf{d}}$:

$$\begin{aligned} & \mu = -0.2, \tau = 1, \nu = 1.2, K_0 = (-1.3333, -0.0783), \\ & K = (-25.3333, -17.2331), G_{\mathbf{d}} = \begin{pmatrix} 0.9333 & -0.0627 \\ 0.2837 & 1.2667 \end{pmatrix}. \end{aligned} \tag{92}$$

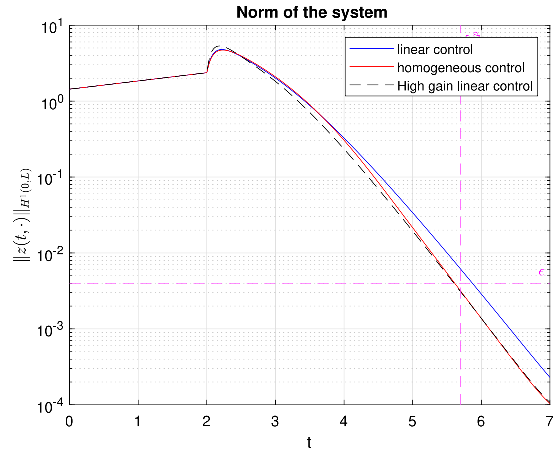

Fig. 1. Evolution of $\|z(t,\cdot)\|_{H^1(0,L)}$ in a logarithmic scale of the closed-loop system (6)–(8) with linear control (91) (blue and black dashed) and homogeneous control (17) and (43) (red).

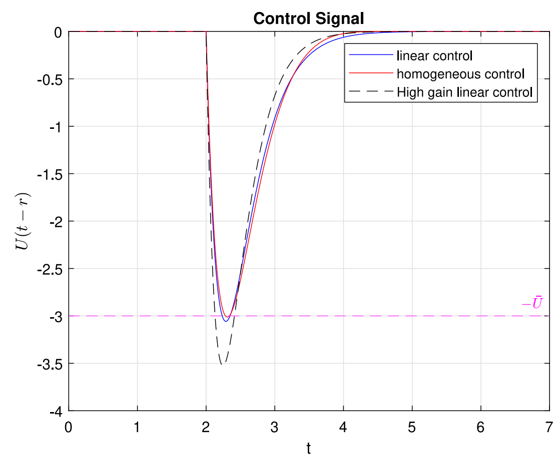

Fig. 2. Time-evolution of the control signal $U(t-r)$ of the linear control (91) (black and blue line) and homogeneous control (17) and (43) (red line).

Fig. 1 shows the simulations in logarithmic scale of the norm $\|z(t,\cdot)\|_{H^1(0,L)}$ of closed-loop system (6)–(8) with linear control (91) (with K_{l_1}) in blue line, with high-gain linear control (91) (with K_{l_2}) in black dashed line and with homogeneous control (17) and (43) in red line. Fig. 4 shows the solution of closed-loop system (6)–(8) with homogeneous control (17) and (43).

We can observe the performance of the closed-loop system under both a high-gain linear control and homogeneous control while achieving a prescribed precision,e.g.,

$$\|z(t,\cdot)\|_{H^1(0,L)} \leq \epsilon := 4.10^{-3}, \quad \forall t \geq T_p := 5.7. \tag{93}$$

with the control restriction

$$\sup_{t>0} |U(t-r)| \leq \bar{U} := 3. \tag{94}$$

However, from Fig. 2 we can observe that only the homogeneous predictor feedback allows to achieve (93) with the control restriction (94). Indeed, one can also observe in Fig. 3 and Fig. 2 the price to pay of using high-gain linear control: achieving (93) implies a large deviation of solution closed-loop system (a peaking) and an overshoot of the control signal during the initial phase of stabilization. Additionally we can observe in Fig. 2 that both the linear controller and the homogeneous one exhibit a similar overshoot value while verifying (94), but the linear controller violates condition (93) whereas the homogeneous one does not.

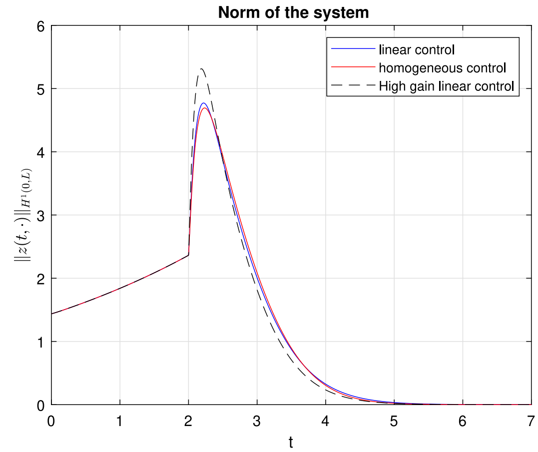

Fig. 3. Evolution of $\|z(t,\cdot)\|_{H^1(0,L)}$ of the closed-loop system (6)–(8) with linear control (91) (blue and black dashed) and homogeneous control (17) and (43) (red) during the initial phase of stabilization.

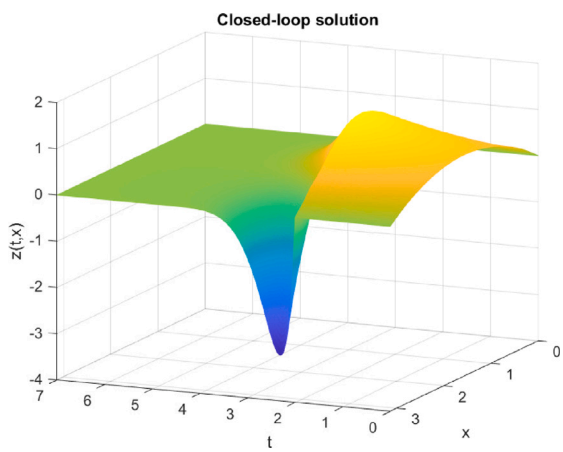

Fig. 4. The solution $z(t,x)$ of closed-loop system (6)–(8) with homogeneous control (17) and (43).

6. Conclusion

In this paper, we have designed a homogeneous boundary control for a 1D reaction–diffusion equation with input delay. We construct a suitable Lyapunov functional to prove the stability of the closed-loop system. The simulations showed that the homogeneous controller makes it possible to obtain faster convergence without peaking and with less overshoot of the controller. Future work will involve developing a homogeneous output feedback controller for 1D reaction–diffusion PDEs with input delay. Additionally, we aim to investigate the inversion of the Artstein transform (see Bresch-Pietri, Prieur, and Trélat (2018), Prieur and Trélat (2019) in the case of linear feedback) to express the constructed Lyapunov function $V(t)$ in terms of $W(t)$ and the homogeneous stabilizing control ξ defined in (43) directly as a feedback of $W(t)$.

CRediT authorship contribution statement

Mericel Ayamou: Investigation, Writing – original draft, Writing – review & editing. **Nicolas Espitia:** Investigation, Writing – original draft, Writing – review & editing. **Andrey Polyakov:** Investigation, Writing – original draft, Writing – review & editing. **Emilia Fridman:** Investigation, Writing – original draft, Writing – review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Appendix

The following Lemma is similar to Postoyan, Tabuada, Nešić, and Anta (2014, Lemma 2) and is useful for lower bounding Lyapunov functions.

Lemma 3. *For any* α_1, α_2 *and* $\alpha_3 \in \mathcal{K}_\infty$ *one has*

$$\alpha_1(s_1) + \alpha_2(s_2) + \alpha_3(s_3) \geq \bar{\alpha}(s_1 + s_2 + s_3), \tag{95}$$

for any $s_1, s_2, s_3 \geq 0$ *where* $\bar{\alpha} \in \mathcal{K}_\infty$ *is given by*

$$\bar{\alpha}(s) = \min\{\alpha_1(\tfrac{s}{3}), \alpha_2(\tfrac{s}{3}), \alpha_3(\tfrac{s}{3})\}, \tag{96}$$

for all $s \geq 0$.

Proof. Let $s_1, s_2, s_3 \geq 0$. By using the fact that $\bar{\alpha} \in \mathcal{K}_\infty$ one has the following inequality

$$\bar{\alpha}(s_1 + s_2 + s_3) \leq \bar{\alpha}(3s_1) + \bar{\alpha}(3s_2) + \bar{\alpha}(3s_3), \tag{97}$$

which combined with (96) implies that

$$\bar{\alpha}(s_1 + s_2 + s_3) \leq \sum_{i=1}^{3} \min\{\alpha_1(s_i), \alpha_2(s_i), \alpha_3(s_i)\}$$

and then

$$\bar{\alpha}(s_1 + s_2 + s_3) \leq \alpha_1(s_1) + \alpha_2(s_2) + \alpha_3(s_3). \tag{98}$$

The proof is complete. ■

References

Artstein, Z. (1982). Linear systems with delayed controls: A reduction. *IEEE Transactions on Automatic Control*, *27*(4), 869–879.

Ayamou, M., Espitia, N., Polyakov, A., & Fridman, E. (2024). Finite-dimensional homogeneous boundary control for a 1d reaction-diffusion equation. In *63rd IEEE conference on decision and control*.

Bresch-Pietri, D., Prieur, C., & Trélat, E. (2018). New formulation of predictors for finite-dimensional linear control systems with input delay. *Systems & Control Letters*, *113*, 9–16.

Christofides, P., & Chow, J. (2002). Nonlinear and robust control of PDE systems: Methods and applications to transport-reaction processes. *Applied Mechanics Reviews*, *55*(2), B29–B30.

Coron, J.-M., & Trélat, E. (2004). Global steady-state controllability of one-dimensional semilinear heat equations. *SIAM Journal on Control and Optimization*, *43*(2), 549–569.

Curtain, R., & Zwart, H. (2012). *An introduction to infinite-dimensional linear systems theory*: *Vol. 21*, Springer Science & Business Media.

Fridman, E. (2014). Introduction to time-delay systems. *Analysis and Control. Birkhäuser*, *75*.

Izmailov, R. (1987). Peak effect in stationary linear-systems in scalar inputs and outputs. *Automation and Remote Control*, *48*(8), 1018–1024.

Karafyllis, I. (2021). Lyapunov-based boundary feedback design for parabolic PDEs. *International Journal of Control*, *94*(5), 1247–1260.

Katz, R., & Fridman, E. (2021). Sub-predictors and classical predictors for finite-dimensional observer-based control of parabolic PDEs. *IEEE Control Systems Letters*, *6*, 626–631.

Katz, R., & Fridman, E. (2022). Global finite-dimensional observer-based stabilization of a semilinear heat equation with large input delay. *Systems & Control Letters*, *165*, Article 105275.

Katz, R., & Fridman, E. (2023). Global stabilization of a 1D semilinear heat equation via modal decomposition and direct Lyapunov approach. *Automatica*, *149*, Article 110809.

Katz, R., Fridman, E., & Selivanov, A. (2020). Boundary delayed observer-controller design for reaction–diffusion systems. *IEEE Transactions on Automatic Control*, *66*(1), 275–282.

Kawski, M. (1991). Families of dilations and asymptotic stability. In *Analysis of controlled dynamical systems: proceedings of a conference held in lyon, France, July 1990* (pp. 285–294). Springer.

Krstic, M. (2009). Control of an unstable reaction–diffusion PDE with long input delay. *Systems & Control Letters*, *58*(10–11), 773–782.

Lhachemi, H., Prieur, C., & Trélat, E. (2020). PI regulation of a reaction–diffusion equation with delayed boundary control. *IEEE Transactions on Automatic Control*, *66*(4), 1573–1587.

Polyak, B. T., & Smirnov, G. (2016). Large deviations for non-zero initial conditions in linear systems. *Automatica*, *74*, 297–307.

Polyakov, A. (2019). Sliding mode control design using canonical homogeneous norm. *International Journal of Robust and Nonlinear Control*, *29*(3), 682–701.

Polyakov, A. (2020). *Generalized homogeneity in systems and control*. Springer.

Polyakov, A. (2023). *Homogeneous systems control toolbox (HCS toolbox) for MATLAB*: *Tech. rep.*, https://gitlab.inria.fr/polyakov/hcs-toolbox-for-matlab.

Polyakov, A., & Krstic, M. (2023). Finite-and fixed-time nonovershooting stabilizers and safety filters by homogeneous feedback. *IEEE Transactions on Automatic Control*.

Polyakov, A., & Krstic, M. (2025). Homogeneous control systems on cones and nonovershooting finite-time stabilizers. *SIAM Journal on Control and Optimization*, *63*(3), 1590–1615.

Postoyan, R., Tabuada, P., Nešić, D., & Anta, A. (2014). A framework for the event-triggered stabilization of nonlinear systems. *IEEE Transactions on Automatic Control*, *60*(4), 982–996.

Prieur, C., & Trélat, E. (2019). Feedback stabilization of a 1-D linear reaction–diffusion equation with delay boundary control. *IEEE Transactions on Automatic Control*, *64*(4), 1415–1425.

Russell, D. L. (1978). Controllability and stabilizability theory for linear partial differential equations: recent progress and open questions. *Siam Review*, *20*(4), 639–739.

Zimenko, K., Polyakov, A., Efimov, D., & Perruquetti, W. (2020). Robust feedback stabilization of linear mimo systems using generalized homogenization. *IEEE Transactions on Automatic Control*, *65*(12), 5429–5436.